AN ESTIMATE OF THE DIRICHLET EIGENVALUES IN CURVED SPACES

HANG CHEN

Abstract. Considering an $n$-dimensional Riemannian manifold $M$ whose sectional curvature is bounded above by $\kappa$ and the Ricci curvature is bounded below by $(n-1)K$, we obtain an isoperimetric inequality for the first $n$ Dirichlet eigenvalues of the Laplace operator for domains contained in $M$. This generalizes the result for domains in the Euclidean space ([Wang-Xia 2021]) to the curved manifold, including the hemisphere and the hyperbolic space.

1. Introduction

Let $(M^n, g)$ be an $n (\geq 2)$-dimensional complete Riemannian manifold. For a bounded domain $\Omega \subset M$ with smooth boundary $\partial \Omega$, one can consider the Dirichlet eigenvalues of the Laplacian $\Delta$ on $\Omega$, which are determined by the eigenvalue equation with the Dirichlet boundary condition

$$\begin{cases}
\Delta u = -\lambda u, & \text{in } \Omega; \\
 u = 0, & \text{on } \partial \Omega.
\end{cases}$$

(1.1)

It is well known that the Dirichlet eigenvalues are discrete and satisfies

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty.$$

The study of the Dirichlet eigenvalues has a long history, and it is closely related to some isoperimetric inequalities. For the unit ball $B^n$ in the Euclidean space $\mathbb{R}^n$, it is well known that

$$\lambda_1(B^n) = j_2^{2, 1, 1}, \quad \lambda_2(B^n) = \cdots = \lambda_{n+1}(B^n) = j_2^{2, -1, 1},$$

where $j_{p,k}$ denotes the $k$-th positive zero of the Bessel function $J_p$ of the first kind of order $p$. Noting that $\lambda$ has the dimensions of a length to the power $-2$, the celebrated Faber–Krahn inequality ([13, 17]) says that

$$\lambda_1(\Omega) \geq \left(\frac{|\mathbb{B}^n|}{|\Omega|}\right)^{\frac{2}{n}} \lambda_1(\mathbb{B}^n) = \lambda_1(\Omega^*),$$

(1.2)

with the equation if and only if $\Omega = \Omega^*$, where $\Omega^*$ is a geodesic ball of $\mathbb{R}^n$ with the same volume of $\Omega$.

For the second eigenvalue, after a series of work by many mathematicians (e.g., [6, 10, 11, 14, 17, 18, 21]), Ashbaugh and Benguria [1] settled the long-standing Payne-Pólya-Weinberger (PPW) Conjecture ([17, 18, 21]), which says

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)},$$

(1.3)

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with equality if and only if $\Omega = \Omega^*$.

Noting that (1.3) is equivalent to
\begin{equation}
\frac{\lambda_1(\Omega)}{\lambda_2(\Omega) - \lambda_1(\Omega)} \geq \frac{\lambda_1(\Omega^*)}{\lambda_2(\Omega^*) - \lambda_1(\Omega^*)},
\end{equation}

Ashbaugh and Benguria [2, 8] proposed the following conjecture.

**Conjecture 1.1.** For any bounded domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, we have
\begin{equation}
\frac{\lambda_1(\Omega)}{\lambda_2(\Omega) - \lambda_1(\Omega)} + \cdots + \frac{\lambda_1(\Omega)}{\lambda_{n+1}(\Omega) - \lambda_1(\Omega)} \geq \frac{n\lambda_1(\Omega^*)}{\lambda_2(\Omega^*) - \lambda_1(\Omega^*)},
\end{equation}

where $\Omega^*$ is a geodesic ball in $\mathbb{R}^n$ such that $|\Omega^*| = |\Omega|$. Moreover, the equality holds if and only if $\Omega$ is isometric to $\Omega^*$.

In [2] the authors proved the left side of (1.3) is not less than $\frac{2j^2_{2,1} + n(n-4)}{6}$, which is very close to the right side of (1.5) by the asymptotic analysis. Conjecture 1.1 would be true if the following conjecture (also called PPW conjecture) were to hold:

**Conjecture 1.2.**
\begin{equation}
\frac{\lambda_2(\Omega) + \cdots + \lambda_{n+1}(\Omega)}{\lambda_1(\Omega)} \leq \frac{n\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}
\end{equation}

with the equality holds if and only if $\Omega$ is isometric to $\Omega^*$.

Conjecture 1.2 is still unsolved completely, and some non-optimal upper bounds have been obtained, see [1, 8, 14, 16, 17] for $n = 2$ and [2, 21] for general $n \geq 2$.

Recently, Wang and Xia [22] considered the first $n$ Dirichlet eigenvalues and proved the following result, which makes an important step toward the proof of Conjecture 1.1.

**Theorem 1.3** ([22, Theorem 1.1]). Let $\Omega$ a bounded domain with smooth boundary in $\mathbb{R}^n$. Let $\Omega^*$ be a geodesic ball in $\mathbb{R}^n$ such that $|\Omega^*| = |\Omega|$. Then we have
\begin{equation}
\frac{\lambda_1(\Omega)}{\lambda_2(\Omega) - \lambda_1(\Omega)} + \cdots + \frac{\lambda_1(\Omega)}{\lambda_{n}(\Omega) - \lambda_1(\Omega)} \geq \frac{(n - 1)\lambda_1(\Omega^*)}{\lambda_2(\Omega^*) - \lambda_1(\Omega^*)}.
\end{equation}

Moreover, the equality holds if and only if $\Omega$ is isometric to $\Omega^*$.

Considering the space forms instead of $\mathbb{R}^n$, (1.3) has been extended to the hemisphere $\mathbb{S}^n_+$ by Ashbaugh-Benguria [1] and hyperbolic space $\mathbb{H}^n$ by Benguria-Linde [5], respectively.

**Theorem 1.4** ([4, Theorem 1.1], [5, Theorem 1.1]). Let $M = \mathbb{S}^n_+$ or $\mathbb{H}^n$. For a bounded domain $\Omega$ with smooth boundary in $M$, let $\Omega^*_\lambda$ be a geodesic ball in $M$ such that $\lambda_1(\Omega) = \lambda_1(\Omega^*_\lambda)$. Then
\begin{equation}
\lambda_2(\Omega) \leq \lambda_2(\Omega^*_\lambda)
\end{equation}

with equality if and only if $\Omega$ is itself a geodesic ball in $M$.

We remark that, a normalization of the geodesic ball by $\lambda_1$ is required in curved spaces, which is different from $\mathbb{R}^n$, where the ratio $\frac{\lambda_2}{\lambda_1}$ is scale-invariant.

In this paper, we consider a Riemannian manifold $M$ with certain bounded curvatures. Precisely, we consider the following setting throughout this paper.

**Settings:** Let $M^n$ be an $n$-dimensional, complete, simply connected Riemannian manifold, whose sectional curvature and Ricci curvature satisfy $\text{Sect}_M \leq \kappa$ and $\text{Ric}_M \geq (n - 1)K$, respectively. Let $\Omega$ be a bounded domain with smooth boundary in $M$. If $\kappa \leq 0$, then $\Omega$ has a closed geodesic convex hull, written as $\text{hull}(\Omega)$, satisfying $\text{diam}(\Omega) = \text{diam}(\text{hull} \Omega)$. If $\kappa > 0$, then $\Omega$ has an open geodesic convex hull, written as $\text{hull}(\Omega)$, satisfying $\text{diam}(\Omega) = \text{diam}(\text{hull} \Omega)$.
If $\kappa > 0$, we assume additionally that $\Omega$ is contained in certain strongly convex closed set $\text{hull}(\Omega)$ satisfying the following two conditions:

(A) $\text{diam}(\Omega) = \text{diam}(\text{hull}(\Omega)) < \min\left\{\frac{\pi}{2\sqrt{\kappa}}, \text{injectivity radius of } M\right\}$;

(B) $|\text{hull}(\Omega)|_M \leq \frac{|\mathbb{M}_n|_\kappa}{2}$.

We briefly introduce the motivation to consider the above settings. We know that Faber-Krahn inequality holds for any Riemannian manifold where the isoperimetric inequality exists (cf. [7, Chapter IV]). For any bounded domain $\Omega \subset \mathbb{M}_n^\kappa$, we have the isoperimetric inequality (cf. [19])

$$|\partial \Omega| \geq A_{n,\kappa} |\Omega|,$$

with equality if and only if $\Omega$ is a geodesic ball. The constant $A_{n,\kappa}$ is usually called the isoperimetric profile.

Generally, for a manifold $M$ with $\text{Sect}_M \leq \kappa$, there exists some $\alpha \leq 1$ such that

$$|\partial \Omega| \geq \alpha A_{n,\kappa} |\Omega|$$

for any bounded domain $\Omega \subset M$.

When $\kappa > 0$, Conditions (A) and (B) makes us work in a hemisphere, which is crucial for the monotonicity of some functions.

On the other hand, when the Ricci curvatures have a lower bound, we have the volume comparison theorem. This makes us to estimate some geometric quantities.

Recently, Edelen [12] gave an upper bound of $\lambda_2 - \lambda_1$ for domains in $M$.

**Theorem 1.5** ([12, Theorem 1.4]). Let $M$ and $\Omega$ satisfy the above settings. Let $\Omega^*_\alpha$ be the geodesic ball in $\mathbb{M}_n^\kappa$ satisfying $\lambda_1(\Omega^*_\alpha) = \alpha^{-2} \lambda_1(\Omega)$. Then

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq C^2 (\lambda_2(\Omega^*_\alpha) - \lambda_1(\Omega^*_\alpha))$$

with

$$C = \left(\frac{\text{sn}_K(\text{diam}(\Omega))}{\text{sn}_\kappa(\text{diam}(\Omega))}\right)^{n-1}.$$

In this paper, we prove the following theorem.

**Theorem 1.6.** Settings as in Theorem 1.5. Then we have

$$\frac{1}{\lambda_2(\Omega) - \lambda_1(\Omega)} + \cdots + \frac{1}{\lambda_n(\Omega) - \lambda_1(\Omega)} \geq \frac{1}{C^2} \frac{n-1}{\lambda_2(\Omega^*_\alpha) - \lambda_1(\Omega^*_\alpha)},$$

where $C$ is given by (1.10).

**Remark 1.7.** To ensure Theorem 1.5 and Theorem 1.6 hold, we just need $\text{Ric} \geq (n-1)K$ on $\text{hull}(\Omega)$ rather than $\text{Ric} \geq (n-1)K$ on the whole $M$ (cf. [12]).

For Cartan-Hadamard manifolds, we have a more explicit estimate.

**Corollary 1.8.** Suppose $k = 0$, and $\Omega$ is a bounded domain in $M^n$ so that $\text{Ric} \geq (n-1)K$ on $\text{hull}(\Omega)$ for some $K < 0$. Then

$$\frac{\lambda_1(\Omega)}{\lambda_2(\Omega) - \lambda_1(\Omega)} + \cdots + \frac{\lambda_1(\Omega)}{\lambda_n(\Omega) - \lambda_1(\Omega)} \geq \alpha^2 \left(\frac{\sqrt{-K} \times \text{diam}(\Omega)}{\sinh(\sqrt{-K} \times \text{diam}(\Omega))}\right)^{2n-2} \frac{n-1}{\frac{\lambda_2(\mathbb{B}^n)}{\lambda_1(\mathbb{B}^n)} - 1}.$$

Noting that $C \equiv 1$ for the space forms, we can prove
Theorem 1.9. Let $M = S^n_1$ or $\mathbb{H}^n$. For a bounded domain $\Omega$ with smooth boundary in $M$, let $\Omega^*_\lambda$ be the geodesic ball in $M$ satisfying $\lambda_1(\Omega^*_\lambda) = \lambda_1(\Omega)$. Then we have
\begin{equation}
\frac{1}{\lambda_2(\Omega) - \lambda_1(\Omega)} + \cdots + \frac{1}{\lambda_n(\Omega) - \lambda_1(\Omega)} \geq \frac{n-1}{\lambda_2(\Omega^*_\lambda) - \lambda_1(\Omega^*_\lambda)}.
\end{equation}
Moreover, the equality only holds for the geodesic balls.

2. Preliminaries

In this section, we introduce some notations and recall some well-known facts. Although most of them can be found in previous literatures, we still list them (and give the sketch of the proofs if needed) for the convenience of the readers and the integrity of the article.

2.1. Notations. Let $M$ be an $n$-dimensional Riemannian manifold. For a fixed point $p \in M$, let $r(x) = d(p, x)$ be the distance function. Using exponential polar coordinate $(t, \theta)$ around $p$, we denote the volume element on $M$ by $d\text{vol} = A(t, \theta)dt \wedge d\theta$. Let $\psi_i(\theta)$ are the restrictions of the linear coordinate functions on $\mathbb{S}^{n-1}$, precisely,
\begin{equation}
\psi_i(\theta) = \frac{x_i}{r}
\end{equation}
under the normal coordinate $(x_1, \cdots, x_n)$ at $p$.

For the space form $\mathbb{M}^n_K$, the volume element $d\text{vol}_K$ is a radial function and does not depend on the choice of the point $p$. Precisely, and can be written as $d\text{vol}_K = A_K dt \wedge d\theta$ (when $K > 0$, generally $t < \pi/\sqrt{K}$ is required), where
\begin{equation}
A_K(t) = A_K(t, \cdot) = \text{sn}_K^{n-1}(t),
\end{equation}
\begin{equation}
\text{sn}_K(t) = \begin{cases} 
\frac{1}{\sqrt{K}} \sin(\sqrt{K}t), & \text{for } K > 0; \\
t, & \text{for } K = 0; \\
\frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t), & \text{for } K < 0.
\end{cases}
\end{equation}

We list the following notations used throughout the paper.
- $B(p, r)$: the geodesic ball of radius $r$ centered at $p$ in $M$.
- $|B(p, r)|_M$: the $n$-dimensional volume in $M$.
- $B_{\kappa}(q_\kappa, r)$: the geodesic ball of radius $r$ centered at $q_\kappa$ in $\mathbb{M}^n_K$.
- $m_K(r) = |B_{\kappa}(q_\kappa, r)|_{\mathbb{M}^n_K}$: the volume of $B_{\kappa}(q_\kappa, r)$ in $\mathbb{M}^n_K$, which is an increasing function of $r$ and don’t depend on the center $q_\kappa$.
- $\Omega^*_\alpha = B_\alpha(q_\kappa, R_\alpha)$: the geodesic ball in $\mathbb{M}^n_K$ with the same volume as $\Omega$.
- $\Omega^*_\alpha = B_\alpha(q_\kappa, R_\alpha)$: the geodesic ball in $\mathbb{M}^n_K$ such that $\lambda_1(\Omega^*_\alpha) = \alpha^{-2}\lambda_1(\Omega)$.

Since we always calculate the volume of a domain by using the corresponding volume form of the manifold which contains the domain, we will omit the subscript of the volume functional $\cdot$ except that we want to emphasize it.

2.2. Eigenfunctions on space forms. According to Sect 2.1, $q_\kappa$ and $R_\alpha$ denote the center and the radius of $\Omega^*_\alpha = B_\alpha(q_\kappa, R_\alpha)$ in $\mathbb{M}^n_K$, respectively.

By the simplicity of the first Dirichlet eigenvalue, we know that the first eigenfunction on $\Omega^*_\alpha$ is a radial function and strictly decreasing to 0 on $[0, R_\alpha]$.

Let $u_1$ and $z = z(r)$ denote the first Dirichlet eigenfunction on $\Omega$ and $\Omega^*_\alpha$, respectively, which are normalized such that
\begin{equation}
\|u_1\|_{L^2(\Omega)} = \|z\|_{L^2(\Omega^*_\alpha)}.
\end{equation}
Let $J = J(r)$ be the radial component of the second Dirichlet eigenfunction of $\Omega^*_\alpha$, and then we define (cf. Eq. (2.11) of [1], Sect. 3 of [4], Sect. 3 of [5])

$$(2.3) \quad h(t) = \begin{cases} 
\frac{J(t)}{z(t)}, & \text{for } 0 \leq t < R_\alpha; \\
\lim_{s \to R_\alpha^-} h(s), & \text{for } t \geq R_\alpha.
\end{cases}$$

$$(2.4) \quad F(t) = h'(t)^2 + (n - 1) \frac{h^2(t)}{sn^2(t)}.$$ 

Lemma 2.1. $h(t)$ is increasing while $F(t)$ is decreasing.

**Proof.** See Corollary 3.4 in [1], Theorem 4.1 in [4] and Lemma 7.1 in [5] for $\kappa = 0, \kappa > 0$ and $\kappa < 0$, respectively. □

Lemma 2.2.

$$h'(t)^2 - \frac{h^2(t)}{sn^2(t)} \leq 0.$$ 

**Proof.** See Theorem 3.3 in [1], Theorem 4.1 in [4] and Eq. (40) in [5] for $\kappa = 0, \kappa > 0$ and $\kappa < 0$, respectively. □

2.3. **Spherical symmetrization.** For any non-negative real function $f(x)$ defined on $\Omega$, denote

$$(2.5) \quad m_f(t) = |x \in \Omega : f(x) \geq t|_M.$$

The spherical symmetrization of $f$ are defined as follows.

**Definition 2.3** (cf. [12]). Let $|x|$ be the distance from $q_\kappa$ to $x \in \Omega^*_\kappa$. Define

$$(2.6) \quad f^* : \Omega^*_\kappa \to \mathbb{R},$$

$$(2.7) \quad f^*(x) = m_f^{-1}(|B_\kappa(q_\kappa, |x|)|);$$

$$(2.8) \quad f_* : \Omega^*_\kappa \to \mathbb{R},$$

$$(2.9) \quad f_*(x) = m_f^{-1}(|\Omega^*_\kappa| - |B_\kappa(q_\kappa, |x|)|).$$

We call $f^*$ and $f_*$ the spherical decreasing and increasing symmetrization of $f$, respectively.

By using the Fubini's theorem, it is not hard to show that

**Proposition 2.4** ([12, Proposition 2.2]). For any $s \geq 1$, we have

$$(2.10) \quad \|f\|_{L^s(\Omega)} = \|f^*\|_{L^s(\Omega^*_\kappa)} = \|f_*\|_{L^s(\Omega^*_\kappa)}.$$ 

**Proposition 2.5** ([12, Proposition 2.4]). For functions $f, g : D \to \mathbb{R}_+$, we have

$$(2.11) \quad \int_{\Omega^*_\kappa} f^* g_* \leq \int_D f g \leq \int_{\Omega^*_\kappa} f^* g_*.$$ 

2.4. **Tools for the proof.** We recall some inequalities which will used in the proof. The details of the proofs can be found in [12]; we mention the ideas of the proof.

**Theorem 2.6** ([12, Theorems 3.1 and 3.2]). For a bounded $\Omega$ in $M$, let $u_1, v_1$ and $z = z(r)$ denote the first Dirichlet eigenfunction on $\Omega, \Omega^*_\kappa$ and $\Omega^*_\alpha$, respectively, satisfying the normalization condition

$$\|u_1\|_{L^2(\Omega)} = \|v_1\|_{L^2(\Omega^*_\kappa)} = \|z\|_{L^2(\Omega^*_\alpha)}.$$

Then we have
We give some comments on this theorem. Assert (1) can be viewed as a weak version of Faber-Krahn inequality. The proof depends on the isoperimetric inequalities (1.8); the computation involves the co-area formula. \( \lambda_1 \) is introduced by using Stokes’ formula since the Dirichlet boundary condition; the equality can be contained by using the fact that the 1st eigenvalue is simple. Moreover, by the monotonicity of eigenvalues for geodesic balls, the inequality implies \( \Omega^*_\alpha \subset \Omega^*_\kappa \), equivalently, \( R_\alpha \leq R_\kappa \). If \( \Omega^*_\alpha = \Omega^*_\kappa \), then we must have \( z = v_1 = u^*_1 \). For this reason, we can extend \( z \) by zero to \( [R_\alpha, R_\kappa] \).

Assert (2) theorem is a weak version of Chiti’s theorem. It can be proven by contradiction. The normalization condition is necessary in the proof. The other version of Chiti was obtained in the author’s recent paper [9], where the arguments can be used in this theorem.

Combining with Proposition 2.4, the weak Chiti (2.12) implies the following

**Corollary 2.7** ([12, Corollary 3.3]). If \( g : \Omega^*_\kappa \to \mathbb{R}_+ \) is a radial decreasing function, then

\[
\int_{\Omega^*_\kappa} (u^*_1)^2 g \leq \int_{\Omega^*_\kappa} z^2 g.
\]

If \( g : \Omega^*_\kappa \to \mathbb{R}_+ \) is a radial increasing function, then

\[
\int_{\Omega^*_\kappa} (u^*_1)^2 g \geq \int_{\Omega^*_\kappa} z^2 g.
\]

3. **Proof of Theorem 1.6**

In this section, we prove Theorem 1.6. First, we make some preparations.

### 3.1. Test functions.

Recall that the variational characterization of \( \lambda_{i+1}(\Omega)(i \geq 1) \) (cf. [20, Chapter 3])

\[
\lambda_{i+1}(\Omega) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \left\{ \frac{\int_{\Omega} |
abla u|^2}{\int_{\Omega} u^2} \right\} \left\{ \int_{\Omega} uu_j = 0, j = 1, \ldots, i \right\},
\]

where \( \{u_i\}_{i \geq 1} \) is an orthonormal set of eigenfunctions satisfying \( \Delta u_i = -\lambda_i u_i \) with the Neumann boundary condition. We will write \( \lambda_i(\Omega) \) as \( \lambda_i \) for short without confusion.

For a given \( p \in \text{hull} \, \Omega \), define \( P_p : \text{hull} \, \Omega \to T_p M \) by

\[
P_p = \frac{\exp_{p}^{-1}(x)}{r_p} h(\sigma_p(r_p(x))).
\]

Here \( h \) is defined by (2.3), and \( \sigma_p \) is a function defined by

\[
|B(q_\kappa, \sigma_p(r))| = |B(p, r) \cap \text{hull} \, \Omega|_M,
\]

that is, \( \sigma_p(r) = m_\kappa^{-1}(|B(p, r) \cap \text{hull} \, \Omega|_M) \). We also write \( \sigma_p \) as \( \sigma \) for short.

Now there exist \( p \in \text{hull} \, \Omega \) such that

\[
\int_{\Omega} P_p u_1^2 = 0,
\]

and such \( p \) is usually called the center of mass of \( \Omega \) (see [12, Lemma 4.1] for details).
For any orthonormal basis \( \{e_i\} \) of \( T_pM \), we define \( P_i = e_i \cdot P_p(x) \). Eq. \((3.3)\) means that \( P_i u_1 \) is \( L^2 \)-orthogonal to the first eigenfunction \( u_1 \) of \( \Omega \).

We claim that we can adjust the orthonormal basis \( \{e_i\} \) such that

\[
\int_{\Omega} (P_i u_1) u_{j+1} = 0, \text{ for } 1 \leq j < i \leq n. \tag{3.4}
\]

To see this, we consider the QR-decomposition of \( A = (a_{ij})_{n \times n} \) via the Gram–Schmidt process, where

\[
a_{ij} = \int_{\Omega} (P_i u_1) u_{j+1} = 0, \text{ for } 1 \leq i, j \leq n. \tag{3.5}
\]

Now we have \( A = QR \), where \( Q = (q_{ij}) \in O(n) \) and \( R \) is an upper triangular matrix. This is equivalent to \( R = Q^T A \). Hence, if we take \( (e'_1, \cdots, e'_n) = (e_1, \cdots, e_n) Q^T \), then

\[
\int_{\Omega} (e'_i \cdot P_p(x)) u_{j+1} = 0, \text{ for } 1 \leq j < i \leq n. \tag{3.6}
\]

### 3.2. Proof of Theorem 1.6.

We take \( B = h(\sigma(r_p)) \psi_i(\theta) u_1(1 \leq i \leq n) \) in Sect. 3.1 as the test functions, which satisfy Eqs. \((3.3)\) and \((3.4)\). For simplicity, we denote \( Q_i = h(\sigma(r_p)) \psi_i(\theta) \). By the variational characterization \((3.1)\), for each \( i = 1, \cdots, n \), we have

\[
\lambda_{i+1} \int_{\Omega} Q_i^2 u_1^2 = \lambda_{i+1} \int_{\Omega} P_i^2 \leq \int_{\Omega} |\nabla P_i|^2 = - \int_{\Omega} (Q_i u_1) \Delta (Q_i u_1) = - \int_{\Omega} (Q_i^2 u_1^2) \Delta u_1 - \int_{\Omega} 2(Q_i u_1) \langle \nabla Q_i, \nabla u_1 \rangle
\]

\[
= \int_{\Omega} \langle \nabla (Q_i u_1^2), \nabla Q_i \rangle + \lambda_1 \int_{\Omega} Q_i^2 u_1^2 - \int_{\Omega} 2(Q_i u_1) \langle \nabla Q_i, \nabla u_1 \rangle
\]

\[
= \int_{\Omega} \langle \nabla Q_i |^2 u_1^2 + \lambda_1 \int_{\Omega} Q_i^2 u_1^2,
\]

that is,

\[
\int_{\Omega} Q_i^2 u_1^2 \leq \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} |\nabla Q_i|^2 u_1^2. \tag{3.7}
\]

Notice that \( \kappa \leq \kappa \), by the Rauch comparison theorem (see [12, pp. 860–861] for details), we have

\[
|\nabla Q_i|^2 = |h'(\sigma(r_p)) \sigma'(r_p) |\psi_i^2(\theta) + h^2(\sigma(r_p)) |\nabla^{\Sigma r} \psi_i(\theta)|^2
\]

\[
\leq |h'(\sigma(r_p)) \sigma'(r_p) |\psi_i^2(\theta) + \frac{h^2(\sigma(r_p))}{\text{sn}_k(r_p)} |\nabla^{\Sigma n-1} \psi_i(\theta)|^2. \tag{3.8}
\]

Recall \( \psi_i = \frac{r_i}{r} \), one has

\[
|\nabla^{\Sigma n-1} \psi_i(\theta)|^2 = 1 - \psi_i^2. \tag{3.9}
\]

We also have (cf. [12, pp. 863])

\[
\max \left\{ \sigma'(r_p), \frac{\text{sn}_k(\sigma_p(r_p))}{\text{sn}_k(r_p)} \right\} \leq C, \tag{3.10}
\]

where \( C \) is given by \((1.10)\).
Since \( \sum_{i=1}^{n} \psi_i^2(\theta) = 1 \), from (3.7), (3.8), (3.9) and (3.10), by denoting \( s = \sigma(r_p) \), we obtain
\[
\int_{\Omega} h(s)^2 u_1^2 \leq \sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} |\nabla Q_i|^2 u_1^2
\]
\[
\leq C^2 \sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} h'(s)^2 \psi_i^2(\theta) u_1^2 + C^2 \sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} h^2(s) (1 - \psi_i(\theta))^2 u_1^2
\]
(3.11)
\[
=C^2 \sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right) \psi_i^2(\theta) u_1^2 + C^2 \sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} \frac{h^2(s)}{\kappa^2(s)} u_1^2.
\]

Considering Lemma 2.2, we have
\[
\sum_{i=1}^{n} \frac{1}{\lambda_{i+1} - \lambda_1} \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right) \psi_i^2(\theta) u_1^2
\]
\[
= \sum_{i=1}^{n-1} \frac{1}{\lambda_{i+1} - \lambda_1} \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right) \psi_i^2(\theta) + \frac{1}{\lambda_{n+1} - \lambda_1} \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right) (1 - \sum_{i=1}^{n-1} \psi_i^2(\theta))
\]
\[
= \sum_{i=1}^{n-1} \left( \frac{1}{\lambda_{i+1} - \lambda_1} - \frac{1}{\lambda_{n+1} - \lambda_1} \right) \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right) \psi_i^2(\theta) + \frac{1}{\lambda_{n+1} - \lambda_1} \left( h'(s)^2 - \frac{h^2(s)}{\kappa^2(s)} \right)
\]
\[
\leq \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\lambda_{i+1} - \lambda_1} \frac{h'(s)^2}{\kappa^2(s)} - \frac{1}{\lambda_{n+1} - \lambda_1} \frac{h^2(s)}{\kappa^2(s)}
\]

Putting this into (3.11), we obtain
\[
(3.12) \quad \int_{\Omega} h(s)^2 u_1^2 \leq \frac{C^2}{n-1} \sum_{i=1}^{n-1} \frac{1}{\lambda_{i+1} - \lambda_1} \int_{\Omega} \left( h'(s)^2 + (n - 1) \frac{h^2(s)}{\kappa^2(s)} \right) u_1^2.
\]

Proof of Theorem 3.6. Considering the monotonicity in Lemma 2.1, we apply Corollary 2.7 combining with Proposition 2.5 for \( h(t) \) and \( F(t) \), respectively.
\[
\int_{\Omega} h(\sigma_p(r_p))^2 u_1^2 \geq \int_{\Omega^*_{\alpha}} (h^2(\sigma_p(r_p)))^*(u_1^2)^*
\]
\[
= \int_{\Omega^*_{\alpha}} h^2(r)(u_1^*)^2
\]
\[
\geq \int_{\Omega^*_{\alpha}} h^2(r)z^2.
\]

Similarly,
\[
\int_{\Omega} \left( h'(\sigma_p(r_p))^2 + (n - 1) \frac{h^2(\sigma_p(r_p))}{\kappa^2(\sigma_p(r_p))} \right) u_1^2 \leq \int_{\Omega^*_{\alpha}} (F^2(\sigma_p(r_p)))^*(u_1^2)^*
\]
\[
= \int_{\Omega^*_{\alpha}} F^2(r)(u_1^*)^2
\]
\[
\leq \int_{\Omega^*_{\alpha}} \left( h'(r)^2 + (n - 1) \frac{h^2(r)}{\kappa^2(r)} \right) z^2.
\]

[1] Precisely, one should consider the symmetrization on hull \( \Omega \) after extending \( u_1 \) by zero to hull \( \Omega \), see [12], pp. 862].
Combining these with (3.12), we derive

\[
(3.13) \quad \sum_{i=1}^{n-1} \frac{1}{\lambda_i(\Omega) - \lambda_1(\Omega)} \geq \frac{n-1}{C^2} \int_{\Omega^+} h(r)^2 \left( h^2(r) + \frac{(n-1)h^2(r)}{\sin^2(r)} \right) = \frac{n-1}{C^2 \lambda_2(\Omega^+)}.
\]

We complete the proof. \qed

**Proof of Theorem 1.9.** For $\mathbb{H}^n$, it is a direct corollary from Theorem 1.6, and the equality can follow from the equalities in Theorem 2.6 and Corollary 2.7.

However, Theorem 1.6 implies that the inequality holds for $M \subset S^n$ being a spherical cap of geodesic radius less than $\pi/4$ rather than a hemisphere $S^o_n$. In order to show that it is valid for $S^o_n$, we need to modify the proof of Theorem 1.6.

First, we remark that the test functions in Sect. 3.1 remain valid for $\Omega \subset S^n$ which is contained in a hemisphere. By choosing Cartesian coordinates $(y^1, \ldots, y^{n+1})$ of $\mathbb{R}^{n+1}$, we can assume $S^n$ is centered at the origin, and the center of mass $p$ of $\Omega$ is the north pole.

Now $\sigma_p(r(x)) = r(x), x \in \Omega$, where $r$ is the geodesic distance function from $p$. But it is possible that $r(x) \geq \pi/2$ for some $x \in \Omega$, since we cannot ensure that $\Omega$ is contained in the northern hemisphere $S^o_n = \{y \in S^n \mid y^{n+1} > 0\}$. To overcome this issue, we use the argument in [4, Sect. 6].

We modify $h(t)$ and $F(t)$ such that both $h$ and $F$ are symmetric with respect to $t = \frac{\pi}{2}$.

\[
(3.14) \quad h(t) = \begin{cases} \frac{\psi(t)}{z(t)}, & \text{for } 0 \leq t < R_\alpha; \\ \lim_{s \to R^-} h(s), & \text{for } R_\alpha \leq \frac{\pi}{2}; \\ h(\pi - t), & \text{for } \frac{\pi}{2} \leq t \leq \pi. \end{cases}
\]

\[
F(t) = h'(t)^2 + (n - 1) \frac{h^2(t)}{\sin^2(t)}.
\]

Due to the symmetry of $h(t)$, it follows from Lemma 2.2 that

\[
\frac{h'(t)^2}{\sin^2(t)} \leq \frac{h^2(t)}{\sin^2(t)} \quad \text{for } t \in [0, \pi].
\]

Hence, by doing the same computations as in Sect. 3.2, we have

\[
\int_{\Omega} h(r)^2 u_1^2 \leq \sum_{i=1}^{n} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} |\nabla Q_1|^2 u_1^2
\]

\[
= \sum_{i=1}^{n} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} h'(r)^2 \psi_1^2(\theta) u_1^2 + \sum_{i=1}^{n} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} \frac{h^2(r)}{\sin^2(r)} (1 - \psi_1(\theta))^2 u_1^2
\]

\[
= \sum_{i=1}^{n} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} \left( h'(r)^2 - \frac{h^2(r)}{\sin^2(r)} \right) \psi_1^2(\theta) u_1^2 + \sum_{i=1}^{n} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} \frac{h^2(r)}{\sin^2(r)} u_1^2
\]

\[
\leq \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} \left( h'(r)^2 + (n - 1) \frac{h^2(r)}{\sin^2(r)} \right) u_1^2
\]

\[
= \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\lambda_i+1 - \lambda_1} \int_{\Omega} F(r) u_1^2.
\]

Clearly, $-\Omega \subset S^n \setminus \Omega$ since $\Omega$ is contained is a hemisphere. Define $\Omega_+ = \{x \in \Omega \mid \pm y^{n+1}(x) > 0\}$, then we have $(-\Omega) \cap \Omega_+ = \emptyset$. 


Due to the symmetry of $h(t)$ and $F(t)$, if we transplant $u_1$ along with $\Omega_-$ to $-\Omega_-$, and denote by $\tilde{u}_1$ the transplanted function of $u_1$ to $\tilde{\Omega}$, then we have
\[
\int_{\tilde{\Omega}} h(r) \tilde{u}_1^2 = \int_{\Omega} h(r) u_1^2, \quad \int_{\tilde{\Omega}} F(r) \tilde{u}_1^2 = \int_{\Omega} F(r) u_1^2.
\]
We have moved the whole problem to the domain $\tilde{\Omega}$ which is contained in the northern hemisphere. Then we can complete the proof by using the monotonicity of $h$ and $F$ on $[0, \pi/2]$, and the rearrangements inequalities. \hfill $\square$

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**References**


(Hang Chen) School of Mathematics and Statistics, Northwestern Polytechnical University, Xi’an 710129, P. R. China, email: chenhang86@nwpu.edu.cn