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HIGGS BUNDLES, THE TOLEDO INVARIANT AND THE CAYLEY CORRESPONDENCE

OLIVIER BIQUARD, OSCAR GARCÍA-PRADA, AND ROBERTO RUBIO

ABSTRACT. Motivated by the study of the topology of the character variety for a non-compact Lie group of Hermitian type G, we undertake a uniform approach, independent of classification theory of Lie groups, to the study of the moduli space of G-Higgs bundles over a compact Riemann surface. We give an intrinsic definition of the Toledo invariant of a G-Higgs bundle which relies on the Jordan algebra structure of the isotropy representation for groups defining a symmetric space of tube type, and prove a general Milnor–Wood type bound of this invariant when the G-Higgs bundle is semistable. Finally, we prove rigidity results when the Toledo invariant is maximal, establishing in particular a Cayley correspondence when G is of tube type, which reveals new topological invariants only seen in particular cases from the character variety viewpoint.

1. Introduction

Non-abelian Hodge theory establishes a homeomorphism between the character variety or moduli of representations of the fundamental group of a compact Riemann surface X in a real non-compact reductive Lie group G and the moduli space of G-Higgs bundles over X. One of the most successful applications of this correspondence is to the study of the topology of the character variety by means of Morse theory and other localization methods on the moduli space of Higgs bundles, taking advantage of the fact that the moduli space of G-Higgs bundles is a complex algebraic variety. In this paper, we look at groups of Hermitian type from the Higgs bundles viewpoint and show that the algebraic structure of the isotropy representation can be used to give a simple and intrinsic definition of the Toledo invariant, and is responsible for a Milnor-Wood type inequality and rigidity phenomena in the moduli space. Our results provide the starting point for a systematic general study of the topology of the moduli spaces, of which very little is known besides some specific examples.

To briefly explain the basics of Higgs bundle theory over a compact Riemann surface X of genus $g \geq 2$, let G be a real reductive Lie group and $H \subset G$ a maximal compact subgroup. Fixing an invariant metric on the Lie algebra \mathfrak{g} of G, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H. From the isotropy representation $H \to \operatorname{Aut}(\mathfrak{m})$ we obtain the representation $\operatorname{Ad}: H^{\mathbb{C}} \to \operatorname{Aut}(\mathfrak{m}^{\mathbb{C}})$. A G-Higgs

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bundle on X is a pair (E, φ) consisting of a holomorphic principal $H^{\mathbb{C}}$ -bundle E and a holomorphic section φ (the Higgs field) of the bundle $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the $\mathfrak{m}^{\mathbb{C}}$ -bundle associated to E via the representation Ad, and K is the canonical line bundle of X. We will also consider E-twisted E-Higgs bundles, replacing E by an arbitrary line bundle E over E. There are natural notions of stability, semistability, and polystability for these objects, leading to corresponding moduli spaces (see [18]).

In this paper we study the case of a connected non-compact real simple Lie group G of Hermitian type with finite centre. In this situation the centre \mathfrak{z} of \mathfrak{h} is isomorphic to \mathbb{R} , and the adjoint action of a special element $J \in \mathfrak{z}$ defines an almost complex structure on $\mathfrak{m} = T_o(G/H)$, where $o \in G/H$ corresponds to the coset H, making the symmetric space G/H into a Kähler manifold. The almost complex structure $\mathrm{ad}(J)$ gives a decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$ in $\pm i$ -eigenspaces, which is $H^{\mathbb{C}}$ -invariant. An immediate consequence of this decomposition for a G-Higgs bundle (E, φ) is that it gives a bundle decomposition $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^+) \oplus E(\mathfrak{m}^-)$ and hence the Higgs field decomposes as $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^+ \in H^0(X, E(\mathfrak{m}^+) \otimes K)$ and $\varphi^- \in H^0(X, E(\mathfrak{m}^-) \otimes K)$. Groups of Hermitian type fall into two classes: tube and non-tube type, depending whether their Harish-Chandra realization as a bounded domain is biholomorphic or not to a tube-type domain (see [36]). The isotropy representation of a tube-type group naturally carries a Jordan algebra structure whose determinant is semi-equivariant by the action of the group $H^{\mathbb{C}}$. This semi-equivariance is described by a character of $\mathfrak{h}^{\mathbb{C}}$, which is the base for our introduction (Definition 2.4) of the Toledo character $\chi_T : \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$:

$$\chi_T(Y) = \frac{1}{N} \langle -iJ, Y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on \mathfrak{g} , and N is the dual Coxeter number. Moreover, maybe up to multiplication by an integer, χ_T lifts to a character $\tilde{\chi}_T$ of $H^{\mathbb{C}}$.

Our intrinsic new definition of the Toledo invariant (Definition 4.1) is

$$\tau = \tau(E) := \deg(E(\tilde{\chi}_T)).$$

This still makes sense if only an integral multiple χ_T lifts to $H^{\mathbb{C}}$.

The Toledo invariant τ is a topological invariant attached to a G-Higgs bundle (E, φ) which is key to the study of the moduli space. Another very important feature of the Hermitian condition is that the stability criterion depends on an element $\alpha \in i\mathfrak{z}$, hence basically a real number. We then define the moduli space $\mathcal{M}^{\alpha}(G)$ of α -polystable G-Higgs bundles over X.

The case when $\alpha = 0$, which will be denoted by $\mathcal{M}(G)$ and referred to as polystable bundles, is of special significance as $\mathcal{M}(G)$ is homeomorphic, by non-abelian Hodge theory (see [18]), to the moduli space $\mathcal{R}(G)$ of reductive representations of the fundamental group of X in G. Although the moduli spaces $\mathcal{M}^{\alpha}(G)$ for $\alpha \neq 0$ are not a priori related with representations of the fundamental group, they turn out to play an important role in the study of the topology of $\mathcal{M}(G)$ and hence $\mathcal{R}(G)$. This is a powerful motivation for us to consider the study of α -semistable G-Higgs bundles and prove one of the main results of this paper (Theorem 4.5).

Theorem 1.1. Let $\alpha \in i\mathfrak{z}$ such that $\alpha = i\lambda J$ for $\lambda \in \mathbb{R}$. Let $(E, \varphi^+, \varphi^-)$ be an α -semistable G-Higgs bundle. Then, the Toledo invariant of E satisfies:

$$-\operatorname{rk}(\varphi^+)(2g-2) - \left(\frac{\dim\mathfrak{m}}{N} - \operatorname{rk}(\varphi^+)\right)\lambda \leq \tau \leq \operatorname{rk}(\varphi^-)(2g-2) - \left(\frac{\dim\mathfrak{m}}{N} - \operatorname{rk}(\varphi^-)\right)\lambda,$$

where N is the dual Coxeter number. In the tube case, this simplifies to:

$$-\operatorname{rk}(\varphi^+)(2g-2) - (r - \operatorname{rk}(\varphi^+))\lambda \le \tau \le \operatorname{rk}(\varphi^-)(2g-2) - (r - \operatorname{rk}(\varphi^-))\lambda.$$

The ranks $\operatorname{rk}(\varphi^+)$ and $\operatorname{rk}(\varphi^-)$ can be defined in the tube case using the Jordan algebra structure, and reducing the non-tube case to the tube situation, by means of the maximal subspace of tube type that always exists. In many of the classical cases, the spaces \mathfrak{m}^{\pm} are spaces of matrices and these ranks coincide with the familiar rank of a matrix. The maximum value of these ranks is given by the rank of the symmetric space $\operatorname{rk}(G/H)$. For $\alpha=0$ one obtains as a consequence the Milnor-Wood inequality for semistable G-Higgs bundles,

$$|\tau| \le \operatorname{rk}(G/H)(2g - 2),$$

proved for representations in [9]. This inequality is therefore being extended in two ways: finding more accurate bounds and considering α -semistability for a parameter α .

We then focus on the study of G-Higgs bundles for which the Toledo invariant attains the bound in the Milnor-Wood inequality, that is, $\tau = \pm \operatorname{rk}(G/H)(2g-2)$. We call these, by analogy with the terminology applied to surface group representations, maximal G-Higgs bundles. Maximal representations — and hence maximal Higgs bundles — have special significance in the context of 'higher Teichmüller theory' since they provide examples of Anosov representations, and are related to geometric structures of various kinds, in a similar way to that of Hitchin representations of the fundamental group of the surface in a split real form (see e.g. [23, 8, 29, 37, 22, 9, 5]).

In our study, the tube-type condition plays a fundamental role. If G is of tube type we construct a bijective correspondence between maximal G-Higgs bundles and K^2 -twisted H^* -Higgs bundles over X, where $H^* \subset H^{\mathbb{C}}$ is the non-compact dual of H, as defined in Definition 2.1. Our main result is Theorem 5.2.

Theorem 1.2 (Cayley correspondence). Let G be a connected non-compact real simple Hermitian Lie group of tube type with finite centre. Let H be a maximal compact subgroup of G and H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element in \mathfrak{z} (the centre of \mathfrak{h}) defining the almost complex structure on \mathfrak{m} . If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides (2g-2), then there is an isomorphism of complex algebraic varieties

$$\mathcal{M}_{\max}(G) \cong \mathcal{M}_{K^2}(H^*).$$

It is useful to observe that the hypothesis on J is always satisfied for the adjoint group.

One of the immediate consequences of Theorem 5.2 is the existence of other invariants attached to a maximal G-Higgs bundle in the tube case. These are the topological invariants of the corresponding Cayley partner. These 'hidden' invariants are not apparent from the point of view of the corresponding maximal representation and, as it

has been seen for classical groups, play a crucial role in the computation of connected components of $\mathcal{M}_{\max}(G)$.

Maximal Higgs bundles in the non-tube case present also very interesting rigidity phenomena. Our main result in this case is Theorem 6.1, where \mathfrak{g}_T is the maximal tube subalgebra of \mathfrak{g} .

Theorem 1.3. Let G be a simple Hermitian group of non-tube type and let H be its maximal compact subgroup. Then, there are no stable G-Higgs bundles with maximal Toledo invariant. In fact, every polystable maximal G-Higgs bundle reduces to a stable $N_G(\mathfrak{g}_T)_0$ -Higgs bundle, where $N_G(\mathfrak{g}_T)_0$ is the identity component of the normalizer of \mathfrak{g}_T in G.

In particular, the dimension of the moduli space of maximal G-Higgs bundles is smaller than expected — this rigidity phenomenon is very rare in the context of surface groups, and is more frequent for representations of the fundamental group of higher-dimensional Kähler manifolds. This theorem implies that the moduli space fibers over the moduli space of maximal G_T^{Ad} -Higgs bundles, where G_T^{Ad} is the adjoint group of the maximal subgroup of tube type $G_T \subset G$, where the fibre is a connected moduli space of bundles for a certain reductive complex Lie group (see Theorem 6.2). In particular this allows us to obtain results on the connectedness of $\mathcal{M}_{max}(G)$.

A brief description of the sections of the paper is as follows. In Section 2 we review some basic facts about groups of Hermitian type, the Cayley transform and define the Toledo character. In Section 3 we introduce the basics of non-abelian Hodge theory relating Higgs bundles over a compact Riemann surface to representations of the fundamental group of the surface. In Section 4 we initiate the study of G-Higgs bundles for a group G of Hermitian type. We define the Toledo invariant of a G-Higgs bundle and prove the Milnor-Wood inequality. We also study involutions defining isomorphisms between moduli space of G-Higgs bundles with opposite Toledo invariant. In Section 5 we consider maximal G-Higgs bundles when G is of tube type and establish the Cayley correspondence. Finally, in Section 6 we study maximal G-Higgs bundles when G is not of tube type and prove the rigidity phenomena taking place.

As mentioned above, our results should provide the starting point for an intrinsic study of the topology and geometry of the moduli spaces of G-Higgs bundles when G is a group of Hermitian type, in particular for the counting of connected components of the moduli space. This had been carried out to some extent for some of the classical groups on a case by case basis ([27, 24, 20, 3, 4, 6, 19]), making use of the classification theorem of Lie groups, but no general principle emerged. Our present intrinsic approach offers a new understanding of the Toledo invariant, Milnor-Wood bound and rigidity phenomena, which are fundamental to the topological study of the moduli space. A preliminary version of some of our results is in [42].

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2. Groups of Hermitian type and the Toledo Character

The results surveyed in the first part of this section can be found in Chapter VIII of [25] and Part III of [17].

2.1. Hermitian symmetric spaces and Cayley transform. Let G/H be an irreducible Hermitian symmetric space of non-compact type, where G is a connected, non-compact real simple Lie group of Hermitian type with finite centre. Such a group is characterized by the fact that the centre Z(H) of a maximal compact subgroup H is isomorphic to U(1). Note that the same symmetric space is obtained by starting with the adjoint group of G, which acts effectively on G/H, or any of its finite coverings.

We denote by $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the corresponding Cartan decomposition and by θ the Cartan involution, so we have $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$, which lifts to the isotropy representation $\mathrm{Ad}: H \to \mathrm{Aut}(\mathfrak{m})$. Let $H^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ be the complexifications of H, \mathfrak{h} and \mathfrak{m} respectively. The almost complex structure J_0 on $\mathfrak{m} = T_o(G/H)$, where $o \in G/H$ corresponds to the coset H, is induced by the adjoint action of an element $J \in \mathfrak{z}(\mathfrak{h})$, so $J_0 = \mathrm{ad}(J)|_{\mathfrak{m}}$. Since $J_0^2 = -\mathrm{Id}$, we decompose $\mathfrak{m}^{\mathbb{C}}$ into $\pm i$ -eigenspaces for J_0 : $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$. Both \mathfrak{m}^+ and \mathfrak{m}^- are abelian, $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{m}^{\pm}] \subset \mathfrak{m}^{\pm}$, and there are $\mathrm{Ad}(H)$ -equivariant isomorphisms $\mathfrak{m} \cong \mathfrak{m}^{\pm}$ given by $X \mapsto \frac{1}{2}(X \mp iJ_0X)$.

Consider a maximal abelian subalgebra \mathfrak{t} of \mathfrak{h} . Its complexification $\mathfrak{t}^{\mathbb{C}}$ gives a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, for which we consider the root system $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}}$. Since $\mathrm{ad}(\mathfrak{t}^{\mathbb{C}})$ preserves $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$, $\mathfrak{g}_{\alpha}^{\mathbb{C}}$ must lie either in $\mathfrak{h}^{\mathbb{C}}$ or in $\mathfrak{m}^{\mathbb{C}}$. If $\mathfrak{g}_{\alpha}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ (resp. $\mathfrak{g}_{\alpha}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$) we say that the root α is **compact** (resp. **non-compact**) and denote the set of such roots by Δ_C (resp. Δ_Q). We choose an ordering of the roots in such a way that \mathfrak{m}^+ (resp. \mathfrak{m}^-) is spanned by the root vectors corresponding to the non-compact positive (resp. negative) roots. We use the superscript + (resp. -) to denote the positive (resp. negative) roots from a set of roots: Δ^+ , Δ_C^+ , Δ_Q^+ (resp. Δ^- , Δ_C^- , Δ_Q^-). Then,

$$\mathfrak{m}^{\pm} = \sum_{lpha \in \Delta_Q^{\pm}} \mathfrak{g}_{lpha}^{\mathbb{C}}.$$

We denote by $\langle \cdot, \cdot \rangle$ an invariant form on $\mathfrak{g}^{\mathbb{C}}$, a constant multiple of the Killing form (most often, the Killing form itself). For each root $\alpha \in \Delta$, let $H_{\alpha} \in i\mathfrak{t}$ be the dual of α , i.e.,

$$\alpha(Y) = \langle Y, H_{\alpha} \rangle$$
 for $Y \in i\mathfrak{t}$.

Define, as usual, $h_{\alpha} = \frac{2H_{\alpha}}{\langle H_{\alpha}, H_{\alpha} \rangle} \in i\mathfrak{t}$, and $e_{\alpha} \in \mathfrak{g}_{\alpha}^{\mathbb{C}}$ such that $[e_{\alpha}, e_{-\alpha}] = h_{\alpha}$ and $\tau e_{\alpha} = -e_{-\alpha}$, where τ is the involution of $\mathfrak{g}^{\mathbb{C}}$ fixing its compact real form $\mathfrak{h} + i\mathfrak{m}$. We define a real basis of \mathfrak{m} by taking for each $\alpha \in \Delta_Q^+$ the basis $(x_{\alpha} = e_{\alpha} + e_{-\alpha}, y_{\alpha} = i(e_{\alpha} - e_{-\alpha}))$ of $\mathfrak{g}_{\alpha}^{\mathbb{C}} \oplus \mathfrak{g}_{-\alpha}^{\mathbb{C}}$.

Two roots $\alpha, \beta \in \Delta$ are said to be strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root (equivalently $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\pm \beta}] = \{0\}$). A **system of strongly orthogonal roots** is a maximal set of strongly orthogonal positive non-compact roots. It has a number of elements equal to the rank r = rk(G/H) of the symmetric space G/H, i.e., the maximal dimension of a flat, totally geodesic submanifold of G/H. Moreover, for two strongly

orthogonal roots $\gamma \neq \gamma'$ we have

(1)
$$[e_{\pm\gamma}, e_{\pm\gamma'}] = 0, \quad [e_{\pm\gamma}, h_{\gamma'}] = 0.$$

For a strongly orthogonal system of roots Γ , consider

$$x_{\Gamma} = \sum_{\gamma \in \Gamma} x_{\gamma}, \quad y_{\Gamma} = \sum_{\gamma \in \Gamma} y_{\gamma}, \quad e_{\Gamma} = \sum_{\gamma \in \Gamma} e_{\gamma}, \quad c = \exp\left(\frac{\pi}{4}iy_{\Gamma}\right) \in U \subset G^{\mathbb{C}},$$

where $G^{\mathbb{C}}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and U is its compact real form (with Lie algebra $\mathfrak{h} \oplus i\mathfrak{m}$). We define the **Cayley transform** as the action of the element c on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ by $Ad(c): \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.

The Cayley transform $\mathrm{Ad}(c)$ satisfies $\mathrm{Ad}(c^8) = \mathrm{Id}$, $\mathrm{Ad}(c) \circ \theta = \theta \circ \mathrm{Ad}(c^{-1})$ for the Cartan involution θ , and consequently $\mathrm{Ad}(c^4)$ preserves \mathfrak{h} and \mathfrak{m} , even though $\mathrm{Ad}(c)$ does not preserve \mathfrak{g} . Since $(\mathrm{Ad}(c^4))^2 = \mathrm{Id}$, either $\mathrm{Ad}(c^4) = \mathrm{Id}$ (then the Hermitian symmetric space is said of **tube type**), or (for **non-tube type**) we can decompose \mathfrak{h} and \mathfrak{m} into ± 1 -eigenspaces for $\mathrm{Ad}(c^4)$:

$$\mathfrak{m} = \mathfrak{m}_T + \mathfrak{m}_2$$
 $\mathfrak{h} = \widetilde{\mathfrak{h}}_T + \mathfrak{q}_2.$

We define $\tilde{\mathfrak{g}}_T = \tilde{\mathfrak{h}}_T + \mathfrak{m}_T$, which is a Lie algebra as $\tilde{\mathfrak{h}}_T$ acts on \mathfrak{m}_T . Since $\tilde{\mathfrak{h}}_T$ may have a non-trivial ideal, we define $\mathfrak{h}_T = [\mathfrak{m}_T, \mathfrak{m}_T]$ and $\mathfrak{g}_T = \mathfrak{h}_T + \mathfrak{m}_T$ to get the Cartan decomposition $\mathfrak{g}_T = \mathfrak{h}_T + \mathfrak{m}_T$, associated to an irreducible Hermitian symmetric space. The subalgebras $\tilde{\mathfrak{g}}_T$ and $\tilde{\mathfrak{h}}_T$ are then the normalizers $\tilde{\mathfrak{g}}_T = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_T)$ and $\tilde{\mathfrak{h}}_T = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{h}_T)$. We also use the notation $\mathfrak{m}_T^{\pm} = \mathfrak{m}_T^{\mathbb{C}} \cap \mathfrak{m}^{\pm}$, $\mathfrak{m}_2^{\pm} = \mathfrak{m}_2^{\mathbb{C}} \cap \mathfrak{m}^{\pm}$.

We denote by G_T and H_T the (connected) subgroups of G with Lie algebras \mathfrak{g}_T and \mathfrak{h}_T . The group H_T is a maximal compact subgroup of G_T . The subgroups of G with Lie algebras $\tilde{\mathfrak{g}}_T$ and $\tilde{\mathfrak{h}}_T$ are $N_G(\mathfrak{g}_T)_0$ and $N_H(\mathfrak{h}_T)_0$. The Cartan decomposition is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_T) = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{h}_T) + \mathfrak{m}_T$ and the maximal compact subgroup of $N_G(\mathfrak{g}_T)_0$ is $N_H(\mathfrak{h}_T)_0$, whose complexification is $N_H \in (\mathfrak{h}_T^{\mathbb{C}})_0$.

Since $\operatorname{Ad}(c^4) = \operatorname{Id}$ on \mathfrak{g}_T , the Hermitian symmetric space G_T/H_T is now of tube type (the maximal 'subtube' of G/H; this is G/H if it was already of tube type). Moreover, $\operatorname{Ad}(c^2)$ commutes with θ , so it preserves \mathfrak{h}_T , and we get a decomposition into ± 1 -eigenspaces for $\operatorname{Ad}(c^2)$,

$$\mathfrak{h}_T = \mathfrak{h}' + i\mathfrak{m}'.$$

Related to this decomposition, there are two groups that will play a fundamental role in our work: the isotropy group H' of ie_{Γ} in H_T , whose Lie algebra is \mathfrak{h}' , and the non-compact dual of H_T , which we define as follows.

Definition 2.1. The subgroup $H_T^* \subset H_T^{\mathbb{C}}$ integrating the subalgebra $\mathfrak{h}' + \mathfrak{m}' \subset \mathfrak{h}_T^{\mathbb{C}}$ is called the non-compact dual of H_T . When G/H is of tube type, we say that H^* is the **non-compact dual of** H.

The group H' is thus the maximal compact subgroup of H_T^* and H_T^*/H' is a symmetric space of non-compact type (the non-compact dual of the Shilov boundary of G_T/H_T , which is given by H_T/H'). If G/H is of tube type the symmetric bounded domain corresponding to G/H is biholomorphic to the 'tube' $V+i\Omega \subset V^{\mathbb{C}}$, where $V=\mathrm{Ad}(c)\mathfrak{g}\cap \mathfrak{m}^+$ and $\Omega=H^*/H'$ is a symmetric cone in V. We refer to Table 1 for the list of tube

and non-tube type Hermitian groups (up to covering, and quotient by a subgroup of the centre).

The following lemma will be important for our Cayley correspondence.

Lemma 2.2. The maps $ad(e_{\Gamma}): \mathfrak{m}_{T}^{-} \to \mathfrak{m}'^{\mathbb{C}}$ and $ad(e_{\Gamma}): \mathfrak{m}'^{\mathbb{C}} \to \mathfrak{m}_{T}^{+}$ are $Ad(H'^{\mathbb{C}})$ -equivariant isomorphisms.

Proof. This lemma is well-known, but it is useful to give a short proof. The map is clearly $\mathrm{Ad}(H'^{\mathbb{C}})$ -equivariant, so there remains to prove the isomorphism statement. We can restrict ourselves to the tube type and forget the index T. Therefore $\mathrm{Ad}(c^4) = 1$ and there is a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_{-1}$ into ± 1 eigenspaces for $\mathrm{Ad}(c^2)$. As $\mathrm{Ad}(c^2)J = -J$, the spaces $\mathfrak{m}_{\pm 1}$ are exchanged by J. Since $\mathrm{Ad}(c)y_{\Gamma} = y_{\Gamma}$, there are maps

$$\mathfrak{m}_{-1} \xrightarrow{\operatorname{ad} y_{\Gamma}} i\mathfrak{m}' \xrightarrow{\operatorname{ad} y_{\Gamma}} \mathfrak{m}_{-1}.$$

Since $\ker \operatorname{ad} y_{\Gamma} \subset \ker(\operatorname{Ad}(c) - 1)$, both maps are injective, and therefore bijective. On the other hand $\operatorname{ad}(y_{\Gamma})|_{\mathfrak{m}_1} = 0$ because $\operatorname{Ad}(c^2) = \exp(i\frac{\pi}{2}\operatorname{ad}y_{\Gamma}) = 1$ on \mathfrak{m}_1 , but the eigenvalues of $\operatorname{ad}(y_{\Gamma})$ take only the values $\{\pm 2, \pm 1, 0\}$.

Now complexify this picture: $\mathfrak{m}_{-1}^{\mathbb{C}}$ projects bijectively to \mathfrak{m}_{-} , so we obtain an isomorphism $\mathfrak{m}_{-1}^{\mathbb{C}} \simeq \mathfrak{m}^{-} \to \mathfrak{m}'^{\mathbb{C}}$ given for $X \in \mathfrak{m}_{-1}^{\mathbb{C}}$ (so $JX \in \mathfrak{m}_{1}^{\mathbb{C}}$) by

$$X + iJX \longrightarrow \operatorname{ad}(y_{\Gamma})X = \operatorname{ad}(y_{\Gamma})(X + iJX) = \operatorname{ad}(ie_{\Gamma})(X + iJX).$$

The first isomorphism follows, the second one is similar.

2.2. Restricted root theory. Given a system $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal roots, let $\mathfrak{t}^- = \sum_{\Gamma} \mathbb{R}ih_{\gamma} \subset \mathfrak{t}$, denote by $\pi : (\mathfrak{t}^{\mathbb{C}})^* \to (i\mathfrak{t}^-)^*$ the restriction to $i\mathfrak{t}^-$. We will identify γ_i with $\pi(\gamma_i)$. The restricted root theorem says that the image by π of the root system Δ is

$$\pi(\Delta) \cup \{0\} = \begin{cases} \{\frac{\pm \gamma_i \pm \gamma_j}{2}, 1 \le i, j \le r\} & \text{in the tube type case,} \\ \{\frac{\pm \gamma_i \pm \gamma_j}{2}, 1 \le i, j \le r\} \cup \{\pm \frac{\gamma_i}{2}, 1 \le i \le r\} & \text{otherwise.} \end{cases}$$

Moreover, all the roots γ_i have the same length, therefore we shall note

$$\langle \gamma, \gamma \rangle := \langle \gamma_i, \gamma_i \rangle$$
 for any i .

We divide the roots according to their projection: the compact positive roots can project to $0, -\frac{1}{2}\gamma_i$ or $\frac{1}{2}(\gamma_j - \gamma_i)$ for j > i, so we set for j > i

$$C_0 = \Delta_C^+ \cap \pi^{-1}(0), \qquad C_i = \Delta_C^+ \cap \pi^{-1}(-\frac{1}{2}\gamma_i), \qquad C_{ij} = \pi^{-1}(\frac{1}{2}(\gamma_j - \gamma_i)).$$

Similarly the positive non-compact roots subdivide into the subsets (again j > i)

$$\Gamma$$
, $Q_i = \Delta_Q^+ \cap \pi^{-1}(\frac{1}{2}\gamma_i), \qquad Q_{ij} = \pi^{-1}(\frac{1}{2}(\gamma_j + \gamma_i)).$

Actually, the following translations are bijections:

(2)
$$C_i \xrightarrow{+\gamma_i} Q_i, \qquad C_{ij} \xrightarrow{+\gamma_i} Q_{ij}.$$

The projected roots appear with certain multiplicities: $\pm \gamma_j$ $(1 \le j \le r)$ with multiplicity 1, $\pm \frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k$ $(j \ne k)$ with multiplicity a, and possibly the roots $\pm \frac{1}{2}\gamma_j$ with even multiplicity b. (In the tube case, b = 0).

A relevant number in what follows is the dual Coxeter number N, an invariant of an irreducible root system, defined in general by considering the dual of the highest root θ expressed in the dual base of simple roots $\{\alpha_i\}$, $\theta^{\vee} = \sum a_i^{\vee} \alpha_i^{\vee}$: we then have $N = 1 + \sum a_i^{\vee}$. In our hermitian case, one has

(3)
$$N = a(r-1) + b + 2 = \frac{1}{r} (\dim_{\mathbb{C}} \mathfrak{m}^+ + \dim_{\mathbb{C}} \mathfrak{m}_T^+),$$

where $\dim_{\mathbb{C}} \mathfrak{m}_{T}^{+} = \frac{r(r-1)}{2}a + r$ and $\dim_{\mathbb{C}} \mathfrak{m}^{+} = \dim_{\mathbb{C}} \mathfrak{m}_{T}^{+} + rb$. Finally:

Lemma 2.3. For the Killing form, one has the equality $N = \frac{1}{\langle \gamma, \gamma \rangle}$.

Proof. First observe the following useful formula: for $Y \in \mathfrak{h}^{\mathbb{C}}$, one has

(4)
$$\langle J, Y \rangle = \operatorname{Tr}_{\mathfrak{m}^{\mathbb{C}}}(\operatorname{ad}(Y) \operatorname{ad}(J)) = 2i \operatorname{Tr}_{\mathfrak{m}^{+}}(\operatorname{ad} Y).$$

Therefore, for $\gamma \in \Gamma$,

$$\langle J, H_{\gamma} \rangle = 2i \operatorname{Tr}_{\mathfrak{m}^+}(\operatorname{ad} H_{\gamma}) = 2i \sum_{\alpha \in \Delta_O^+} \langle \alpha, \gamma \rangle.$$

Now we know all the orthogonal projections of roots in Δ_Q^+ on Γ : we obtain each γ_i once, $\frac{1}{2}(\gamma_i + \gamma_j)$ with multiplicity a and $\frac{1}{2}\gamma_i$ with multiplicity b. Therefore

$$\langle J, H_{\gamma} \rangle = i \langle \gamma, \gamma \rangle ((r-1)a + b + 2) = iN \langle \gamma, \gamma \rangle.$$

On the other hand, $\langle J, H_{\gamma} \rangle = \gamma(J) = i$, so the result follows.

2.3. The Toledo character. We introduce the Toledo character associated to a simple Lie algebra \mathfrak{g} of Hermitian type as the character on the Lie algebra $\mathfrak{h}^{\mathbb{C}}$ given as follows.

Definition 2.4. The **Toledo character** $\chi_T : \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ is defined, for $Y \in \mathfrak{h}^{\mathbb{C}}$, by

$$\chi_T(Y) = \langle -iJ, Y \rangle \langle \gamma, \gamma \rangle.$$

This is independent of the choice of invariant form on \mathfrak{g} . If one chooses the Killing form, then equivalently, from Lemma 2.3,

$$\chi_T(Y) = \frac{1}{N} \langle -iJ, Y \rangle.$$

Since J is in the center, χ_T vanishes on $[\mathfrak{h}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}}]$, hence determines a character.

We study now when the Toledo character lifts to a character of the group $H^{\mathbb{C}}$. Note that this depends on the choice of the pair (G, H) defining the same symmetric space. Let $Z_0^{\mathbb{C}}$ denote the identity component of $Z(H^{\mathbb{C}})$.

Proposition 2.5. Define o_J to be the order of $e^{2\pi J}$ and $\ell = |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$. For $q \in \mathbb{Q}$, the character $q\chi_T$ lifts to $H^{\mathbb{C}}$ if and only if q is an integral multiple of

(5)
$$q_T = \frac{\ell N}{o_J \dim \mathfrak{m}}.$$

Proof. Observe that $\chi_T(J) = \frac{1}{N} \langle -iJ, J \rangle = \frac{\dim \mathfrak{m}}{N}$. From the isomorphism $\mathbb{C}^* \simeq Z_0^{\mathbb{C}}$ given by $e^{\lambda} \mapsto e^{-i\lambda o_J J}$, it follows that $q\chi_T$ lifts to $Z_0^{\mathbb{C}}$ if and only if $q^{\frac{o_J \dim \mathfrak{m}}{N}} \in \mathbb{Z}$.

Now let $D = Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]$. Then $H^{\mathbb{C}} = [H^{\mathbb{C}}, H^{\mathbb{C}}] \times_D Z_0^{\mathbb{C}}$ and every character of $H^{\mathbb{C}}$ comes from a character of $[H^{\mathbb{C}}, H^{\mathbb{C}}] \times Z_0^{\mathbb{C}}$ (and therefore of $Z_0^{\mathbb{C}}$) containing D in its kernel. Since D is a finite subgroup of $Z_0^{\mathbb{C}} \simeq \mathbb{C}^*$, it is cyclic, so it must be generated by $e^{2\pi\frac{\sigma_J}{\ell}J}$. The proposition follows.

The value of q_T in the standard examples is given in Table 2. Note in particular that $q_T = \frac{1}{2}$ for all classical groups except SO*, for which $q_T = 1$. So for all classical groups the Toledo character lifts to $H^{\mathbb{C}}$. For the adjoint group, $o_J = 1$ so $q_T = \frac{\ell N}{\dim \mathfrak{m}}$. In the tube case $N = \frac{\dim \mathfrak{m}}{r}$ so this gives $q_T = \frac{\ell}{r}$. The values of q_T in the non-tube case are given in Table 3.

It follows from (4) that the lifted character $\tilde{\chi}_T$ can be interpreted as

(6)
$$\widetilde{\chi}_T(h) = \det \operatorname{Ad}_{\mathfrak{m}^+}(h)^{\frac{2}{N}}.$$

Of course, one has to take the power q_T if only $q_T \chi_T$ is liftable.

Finally, the following lemma will prove later that the Toledo invariants defined from the two points of view of Higgs bundles and representations coincide.

Lemma 2.6. The Toledo character χ_T defines a G-invariant form on G/H by

$$\omega(Y, Z) = i\chi_T([Y, Z]), \text{ for } Y, Z \in \mathfrak{m}.$$

This form is the Kähler form of the unique G-invariant metric on G/H with minimum holomorphic sectional curvature -1.

Proof. Every invariant p-form ω on a symmetric space is closed [25, p. 198], so the formula defines a closed homogeneous 2-form on G/H. The associated metric reads

$$g(Y,Z) = \omega(Y,JZ) = \langle J, [Y,[J,Z]] \rangle \langle \gamma, \gamma \rangle = \langle Y,Z \rangle \langle \gamma, \gamma \rangle$$

which is a positive metric since \mathfrak{m} is non-compact. So the formula indeed defines a Kähler metric on G/H.

Using the curvature $R(X,Y) = \mathrm{ad}_{\mathfrak{m}}[X,Y]$, we calculate the holomorphic sectional curvature for $X \in \mathfrak{m}$ such that g(X,X) = 1: we have

$$\kappa(X) = g(R(X,JX)X,JX) = -\omega(R(X,JX)X,X) = -i\chi_T([[[X,JX],X],X]).$$

For $X = x_{\gamma}$ we have $g(x_{\gamma}, x_{\gamma}) = \omega(x_{\gamma}, y_{\gamma}) = i\chi_T(-2ih_{\gamma}) = 2\chi_T(h_{\gamma})$ and

$$[[[x_{\gamma}, y_{\gamma}], x_{\gamma}], x_{\gamma}] = -8h_{\gamma}.$$

In general, it is sufficient to do the calculations for $X = \sum_{i=1}^{r} \lambda_i x_{\gamma_i}$, so

$$\kappa(X) = -\frac{\sum_{1}^{r} \lambda_i^4}{(\sum_{1}^{r} \lambda_i^2)^2} \frac{2}{\chi_T(h_\gamma)}.$$

But $\chi_T(h_\gamma) = \langle -iJ, h_\gamma \rangle \langle \gamma, \gamma \rangle = -2i\gamma(J) = 2$, so the result follows.

2.4. **Determinant and rank.** We define now a determinant polynomial, det, on \mathfrak{m}_T^+ , whose degree equals the rank of the symmetric space. This determinant is a familiar object in Jordan algebra theory [16], but it can be introduced in an elementary way as follows [35, Lemma 2.3]: it is the unique H'_0 -invariant polynomial on \mathfrak{m}_T^+ which restricts on $\mathfrak{a}^+ = \bigoplus_{1}^r \mathbb{C} e_{\gamma_i}$ to

$$\det \sum_{1}^{r} \lambda_{i} e_{\gamma_{i}} = \prod_{1}^{r} \lambda_{i}.$$

Here H'_0 is the identity component of H'. The existence comes from the Chevalley theorem on invariant polynomials, since the Weyl group acts exactly by all permutations on the (e_{γ_i}) (see again [35]).

The main useful property for us is the following equivariance:

Lemma 2.7. Let G be of tube type. For $h \in H^{\mathbb{C}}$ and $x \in \mathfrak{m}^+$ we have

(7)
$$\det(\operatorname{Ad}(h)x) = \tilde{\chi}_T(h)\det(x),$$

where $\tilde{\chi}_T$ is the lifting of χ_T to $H^{\mathbb{C}}$.

Note that we implicitly assumed here that the lifting $\tilde{\chi}_T$ exists, otherwise the same identity remains true after taking power q_T . Again note that the lemma is basically known in Jordan algebra theory, see [16, Chapter VIII] where the equivariance is established under the action of H^* , the non-compact dual real form of H in $H^{\mathbb{C}}$. The extension to $H^{\mathbb{C}}$ just requires the existence of the character $\tilde{\chi}_T$ on $H^{\mathbb{C}}$: this is not automatic because H^* is non-compact, but of course was proved in the previous section.

Proof. First, observe that if γ_i is a strongly orthogonal root, then $[h_{\gamma_i}, e_{\gamma_j}] = 2\delta_{ij}e_{\gamma_j}$ and $\chi_T(h_{\gamma_i}) = 2$, so the requested equivariance property is true for the torus generated by the h_{γ_i} .

Now observe that $\bigoplus_{1}^{r} \mathbb{R} h_{\gamma_{i}} \subset \mathfrak{m}'$ is a flat for the symmetric space H/H' (we are in the tube type case). Therefore any element X of \mathfrak{m}' is conjugate under H'_{0} to an element of $\bigoplus_{1}^{r} \mathbb{R} h_{\gamma_{i}}$, and it follows that (7) holds for all $\{\exp X, X \in \mathfrak{m}'\}$. Now $H'_{0} \exp \mathfrak{m}'$ is at least dense in H (recall H is connected, since G is connected), so (7) is true for all $h \in H$. Of course, the property extends immediately to the complex group $H^{\mathbb{C}}$.

We define a notion of rank on \mathfrak{m}^+ . Choose $\mathfrak{a}^+ = \bigoplus_{1}^r \mathbb{C} e_{\gamma_i}$. Any element of \mathfrak{m}^+ is conjugate under $H^{\mathbb{C}}$ to an element of \mathfrak{a}^+ .

Definition 2.8. Let $x \in \mathfrak{m}^+$, and $y = \sum \lambda_i e_{\gamma_i} \in \mathfrak{a}^+$ be conjugate to x under $H^{\mathbb{C}}$. Then we say that x has rank r' if y has exactly r' non-zero coefficients.

This is well defined because the Weyl group acts only by permutations on \mathfrak{a}^+ . Also, in the tube case, one can give a more intrinsic interpretation using the determinant: polarize the determinant to get an r-linear map C on \mathfrak{m}^+ such that $C(x,\ldots,x)=\det(x)$; then the rank of x is the maximal integer r' such that the (r-r')-form $C(x,\ldots,x,\cdot,\ldots,\cdot)$ is not identically zero, which is clearly an invariant notion.

Remark 2.9. In the case of SU(p,q) the rank on \mathfrak{m}^+ specializes to the notion of rank for a rectangular matrix $q \times p$. For $Sp(2n,\mathbb{R})$, the rank on \mathfrak{m}^+ is the rank for an element of $S^2(\mathbb{C}^n)$ seen as an endomorphism.

The following proposition plays an important role in what follows.

Proposition 2.10. Let $1 \leq r' \leq r$. The group $H^{\mathbb{C}}$ acts transitively on the set of elements of rank r' in \mathfrak{m}^+ . In particular, the set of regular (that is maximal rank) elements in \mathfrak{m}^+ is $H^{\mathbb{C}}/H'^{\mathbb{C}}$.

Proof. It is sufficient to prove this on elements of \mathfrak{a}^+ . Each h_{γ_i} generates a one parameter subgroup acting only on e_{γ_i} and fixing e_{γ_j} for $j \neq i$. And the Weyl group acts transitively on the basis (e_{γ_i}) , so the result follows.

3. Higgs bundles

In this section G is a real reductive Lie group, not necessarily of Hermitian type and not necessarily connected, and X is a compact Riemann surface of genus g. We fix a maximal compact subgroup H of G. The Lie algebra \mathfrak{g} of G is equipped with an involution θ that gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H. We fix a metric B in \mathfrak{g} with respect to which the Cartan decomposition is orthogonal. This metric is positive definite on \mathfrak{m} and negative definite on \mathfrak{h} . We have $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{h}$. From the isotropy representation $H \to \operatorname{Aut}(\mathfrak{m})$, we obtain the representation $\operatorname{Ad}: H^{\mathbb{C}} \to \operatorname{Aut}(\mathfrak{m}^{\mathbb{C}})$. When G is semisimple we take B to be the Killing form. In this case B and a choice of a maximal compact subgroup H determine a Cartan decomposition (see [31] for details).

3.1. Basic definitions. A G-Higgs bundle on X consists of a holomorphic principal $H^{\mathbb{C}}$ -bundle E together with a holomorphic section $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the associated vector bundle with fibre $\mathfrak{m}^{\mathbb{C}}$ via the complexified isotropy representation, and K is the canonical line bundle of X.

If G is compact, H = G and $\mathfrak{m} = 0$. A G-Higgs bundle is hence simply a holomorphic principal $G^{\mathbb{C}}$ -bundle. If $G = H^{\mathbb{C}}$, where now H is a compact Lie group, H is a maximal compact subgroup of G, and $\mathfrak{m} = i\mathfrak{h}$. In this case, a G-Higgs bundle is a principal $H^{\mathbb{C}}$ -bundle together with a section $\varphi \in H^0(X, E(\mathfrak{h}^{\mathbb{C}}) \otimes K) = H^0(X, E(\mathfrak{g}) \otimes K)$, where $E(\mathfrak{g})$ is the adjoint bundle. This is the original definition for complex Lie groups given by Hitchin in [28].

Let G' be a reductive subgroup of G. A maximal compact subgroup of G' is given by $H' = H \cap G'$ and we can take a compatible Cartan decomposition, in the sense that $\mathfrak{h}' \subset \mathfrak{h}$ and $\mathfrak{m}' \subset \mathfrak{m}$. Moreover, the isotropy representation of H' is the restriction of the isotropy representation Ad of H. We say that the structure group of a G-Higgs bundle (E, φ) reduces to G' when there is a reduction of the structure group of the underlying $H^{\mathbb{C}}$ -bundle to $H'^{\mathbb{C}}$, given by a subbundle E_{σ} , and the Higgs field $\varphi \in H^{0}(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ belongs to $H^{0}(X, E_{\sigma}(\mathfrak{m}'^{\mathbb{C}}) \otimes K)$.

3.2. Stability of G-Higgs bundles. There is a notion of stability for G-Higgs bundles (see [18]). To explain this we consider the parabolic subgroups of $H^{\mathbb{C}}$ defined for $s \in i\mathfrak{h}$ as

$$P_s = \{g \in H^{\mathbb{C}} : e^{ts}ge^{-ts} \text{ is bounded as } t \to \infty\}.$$

When $H^{\mathbb{C}}$ is connected every parabolic subgroup is conjugate to one of the form P_s for some $s \in i\mathfrak{h}$, but this is not the case necessarily when $H^{\mathbb{C}}$ is non-connected. A Levi

subgroup of P_s is given by $L_s = \{g \in H^{\mathbb{C}} : \operatorname{Ad}(g)(s) = s\}$, and any Levi subgroup is given by pL_sp^{-1} for $p \in P_s$. Their Lie algebras are given by

$$\mathfrak{p}_s = \{ Y \in \mathfrak{h}^{\mathbb{C}} : \operatorname{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty \},$$

 $\mathfrak{l}_s = \{ Y \in \mathfrak{h}^{\mathbb{C}} : \operatorname{ad}(Y)(s) = [Y, s] = 0 \}.$

We consider the subspaces

$$\mathfrak{m}_s = \{ Y \in \mathfrak{m}^{\mathbb{C}} : \operatorname{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty \}$$

 $\mathfrak{m}_s^0 = \{ Y \in \mathfrak{m}^{\mathbb{C}} : \operatorname{Ad}(e^{ts})Y = Y \text{ for every } t \}.$

One has that \mathfrak{m}_s is invariant under the action of P_s and \mathfrak{m}_s^0 is invariant under the action of L_s . They are described in terms of root vectors by the following lemma.

Lemma 3.1. Given $s \in i\mathfrak{h}$, we have that

$$\mathfrak{p}_s = \langle Y \in \mathfrak{h}^{\mathbb{C}} : \operatorname{ad}(s)Y = \lambda_Y Y \text{ for } \lambda_Y \leq 0 \rangle, \qquad \mathfrak{l}_s = \langle Y \in \mathfrak{h}^{\mathbb{C}} : \operatorname{ad}(s)Y = 0 \rangle,$$
$$\mathfrak{m}_s = \langle Y \in \mathfrak{m}^{\mathbb{C}} : \operatorname{ad}(s)Y = \lambda_Y Y \text{ for } \lambda_Y \leq 0 \rangle, \qquad \mathfrak{m}_s^0 = \langle Y \in \mathfrak{m}^{\mathbb{C}} : \operatorname{ad}(s)Y = 0 \rangle.$$

Proof. We consider the endomorphism $\operatorname{ad}(s)$ and take $\{Y_{\delta}\}_{{\delta}\in D\subset\mathbb{C}}\subset\mathfrak{m}^{\mathbb{C}}$, a basis of eigenvectors such that $\operatorname{ad}(s)Y_{\delta}=\delta Y_{\delta}$. We have that

$$\operatorname{Ad}(e^{ts})Y_{\delta} = e^{\operatorname{ad}(ts_{\chi})}Y_{\delta} = \sum_{j=0}^{\infty} \frac{(\operatorname{ad}(ts))^{j}(Y_{\delta})}{j!} = \left(\sum_{j=0}^{\infty} \frac{(t\lambda)^{j}}{j!}\right)Y_{\delta} = e^{t\lambda}Y_{\delta}.$$

Therefore, Y_{δ} belongs to \mathfrak{m}_s (resp. \mathfrak{m}_s^0) if and only if $\lambda \leq 0$ (resp. $\lambda = 0$). By linearity, we obtain the result.

Remark 3.2. The subalgebra \mathfrak{m}_s is the non-compact part of the parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ defined by $s \in i\mathfrak{h}$. Define $\widetilde{\mathfrak{p}}_s = \{Y \in \mathfrak{g}^{\mathbb{C}} \mid \operatorname{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\}$. We have that $\mathfrak{p}_s = \widetilde{\mathfrak{p}}_s \cap \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}_s = \widetilde{\mathfrak{p}}_s \cap \mathfrak{m}^{\mathbb{C}}$. Analogously, define

$$\tilde{\mathfrak{l}}_s = \{ Y \in \mathfrak{g}^{\mathbb{C}} \mid \operatorname{Ad}(e^{ts})Y = Y \text{ for every } t \}.$$

Then $\mathfrak{l}_s = \widetilde{\mathfrak{l}}_s \cap \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}_s^0 = \widetilde{\mathfrak{l}}_s \cap \mathfrak{m}^{\mathbb{C}}$.

An element $s \in i\mathfrak{h}$ defines a character χ_s of \mathfrak{p}_s since $\langle s, [\mathfrak{p}_s, \mathfrak{p}_s] \rangle = 0$. Conversely, by the isomorphism $(\mathfrak{p}_s/[\mathfrak{p}_s, \mathfrak{p}_s])^* \cong \mathfrak{z}_{L_s}^*$, where \mathfrak{z}_{L_s} is the centre of the Levi subalgebra \mathfrak{l}_s , a character χ of \mathfrak{p}_s is given by an element in $\mathfrak{z}_{L_s}^*$, which gives, via the invariant metric, an element of $s_\chi \in \mathfrak{z}_{L_s} \subset i\mathfrak{h}$. When $\mathfrak{p}_s \subset \mathfrak{p}_{s_\chi}$, we say that χ is an antidominant character of \mathfrak{p} . When $\mathfrak{p}_s = \mathfrak{p}_{s_\chi}$ we say that χ is a strictly antidominant character. Note that for $s \in i\mathfrak{h}$, χ_s is a strictly antidominant character of \mathfrak{p}_s .

Remark 3.3. An approach based on root theory can be found in [18]. There, the antidominant characters are also described in terms of fundamental weights.

Let now (E, φ) be a G-Higgs bundle over X, and let $s \in i\mathfrak{h}$. Let P_s be defined as above. For $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ a reduction of the structure group of E from $H^{\mathbb{C}}$ to P_s , we define the degree relative to σ and s, or equivalently to σ and χ_s , as follows. When

a real multiple $\mu\chi_s$ of the character exponentiates to a character $\tilde{\chi}_s$ of P_s , we compute the degree as

$$\deg(E)(\sigma, s) = \frac{1}{\mu} \deg(E_{\sigma}(\tilde{\chi}_s)).$$

This condition is not always satisfied, but one shows ([18, Sec. 4.6]) that the antidominant character can be expressed as a linear combination of characters of the centre and fundamental weights, $\chi_s = \sum_j z_j \mu_j + \sum_k n_k \lambda_k$. [18, Lemma 2.4] states that there exists an integer multiple m of the characters of the centre and the fundamental weights exponentiating to the group, so we can define the degree as

$$\deg(E)(\sigma, s) = \frac{1}{m} \left(\sum_{j} z_{j} \deg(E_{\sigma}(\widetilde{m\mu_{j}})) + \sum_{k} n_{k} \deg(E_{\sigma}(\widetilde{m\lambda_{k}})) \right).$$

This value is independent of the expression of χ_s as sum of characters and the integer m.

There is also a definition of the degree in terms of the curvature of connections using Chern–Weil theory. This definition is more natural when considering gauge-theoretic equations as we do below. For this, define $H_s = H \cap L_s$ and $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{l}_s$. Then H_s is a maximal compact subgroup of L_s , so the inclusions $H_s \subset L_s$ is a homotopy equivalence. Since the inclusion $L_s \subset P_s$ is also a homotopy equivalence, given a reduction σ of the structure group of E to P_s one can further restrict the structure group of E to H_s in a unique way up to homotopy. Denote by E'_{σ} the resulting H_s principal bundle. Consider now a connection E'_{σ} and let $E'_{\sigma} \in \Omega^2(X, E'_{\sigma}(\mathfrak{h}_s))$ be its curvature. Then E'_{σ} is a 2-form on E'_{σ} with values in E'_{σ} and

(8)
$$\deg(E)(\sigma, s) := \frac{i}{2\pi} \int_X \chi_s(F_A).$$

We define the subalgebra \mathfrak{h}_{ad} as follows. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h}, \mathfrak{h}]$, where \mathfrak{z} is the centre of \mathfrak{h} , and the isotropy representation ad $= ad : \mathfrak{h} \to \operatorname{End}(\mathfrak{m})$. Let $\mathfrak{z}' = \ker(ad_{\mathfrak{z}})$ and take \mathfrak{z}'' such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. Define the subalgebra $\mathfrak{h}_{ad} := \mathfrak{z}'' + [\mathfrak{h}, \mathfrak{h}]$. The subindex ad denotes that we have taken away the part of the centre \mathfrak{z} acting trivially via the isotropy representation ad.

Remark 3.4. For groups of Hermitian type, $\mathfrak{z}'=0$ since an element both in \mathfrak{z} and $\ker(\mathrm{ad})$ belongs to the centre of \mathfrak{g} , which is zero, as \mathfrak{g} is semisimple. Hence $\mathfrak{h}_{\mathrm{ad}}=\mathfrak{h}$.

With $L_s \mathfrak{m}_s$ and \mathfrak{m}_s^0 defined as above. We have the following.

Definition 3.5. Let $\alpha \in i\mathfrak{z} \subset \mathfrak{z}^{\mathbb{C}}$. We say that a G-Higgs bundle (E,φ) is:

 α -semistable if for any $s \in i\mathfrak{h}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_{\sigma}(\mathfrak{m}_s) \otimes K)$, we have that $\deg(E)(\sigma, s) - \chi_s(\alpha) \geq 0$.

 α -stable if for any $s \in i\mathfrak{h}_{ad}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_{\sigma}(\mathfrak{m}_s) \otimes K)$, we have that $\deg(E)(\sigma, s) - \chi_s(\alpha) > 0$.

 α -polystable if it is α -semistable and for any $s \in \mathfrak{ih}_{ad}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_{\sigma}(\mathfrak{m}_s) \otimes K)$ and $\deg(E)(\sigma, s) - \chi_s(\alpha) = 0$, there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma(E_{\sigma}(P_s/L_s))$ to a Levi subgroup L_s such that $\varphi \in H^0(X, E_{\sigma_L}(\mathfrak{m}_s^0) \otimes K) \subset H^0(X, E_{\sigma}(\mathfrak{m}_s) \otimes K)$.

- Remark 3.6. We may define a real group $G_{L_s} = (L_s \cap H) \exp(\mathfrak{m}_s^0 \cap \mathfrak{m})$ with maximal compact subgroup the compact real form $L_s \cap H$ of the complex group L_s and $\mathfrak{m}_s^0 \cap \mathfrak{m}$ as isotropy representation. Thus, an α polystable G-Higgs bundle reduces to an α -polystable G_{L_s} -Higgs bundle since φ belongs $H^0(X, E_{\sigma_L}(\mathfrak{m}_s^0) \otimes K)$.
- **Remark 3.7.** We can replace K in the definition of G-Higgs bundle by any holomorphic line bundle L on X. More precisely, a L-twisted G-Higgs bundle (E, φ) consists of a principal $H^{\mathbb{C}}$ -bundle E, and a holomorphic section $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$. We reserve the name G-Higgs bundle for the K-twisted case. The stability criteria are as in Definition 3.5, replacing K by L.
- Remark 3.8. For G semisimple, the notion of α -stability with $\alpha \neq 0$ only makes sense for groups of Hermitian type, since α belongs to the centre of \mathfrak{h} , which is not zero if and only if the centre of a maximal compact subgroup H is non-discrete, i.e., if G is of Hermitian type. For reductive groups which are not of Hermitian type, α -stability makes sense, but there is only one value of α for which the condition is not void. This value is fixed by the topology of the principal bundle (see [21] for details).
- **Remark 3.9.** When $H^{\mathbb{C}}$ is connected, as mentioned above, every parabolic subgroup of $H^{\mathbb{C}}$ is conjugate to one of the form P_s for $s \in i\mathfrak{h}$. In this situation, we can formulate the stability conditions in Definition 3.5 in terms of any parabolic subgroup $P \subset H^{\mathbb{C}}$, replacing s by s_{χ} , for any antidominant character χ of $\mathfrak{h}^{\mathbb{C}}$.

Two G-Higgs bundles (E, φ) and (E', φ') are isomorphic if there is an isomorphism $f: E \to E'$ such that $\varphi' = f^*\varphi$, where f^* is the map $E(\mathfrak{m}^{\mathbb{C}}) \otimes K \to E'(\mathfrak{m}^{\mathbb{C}}) \otimes K$ induced by f.

The moduli space of α -polystable G-Higgs bundles $\mathcal{M}^{\alpha}(G)$ is defined as the set of isomorphism classes of α -polystable G-Higgs bundles on X. When $\alpha = 0$ we simply denote $\mathcal{M}(G) := \mathcal{M}^0(G)$.

Remark 3.10. Similarly, we can define the moduli space of α -polystable L-twisted G-Higgs bundles which will be denoted by $\mathcal{M}_L^{\alpha}(G)$.

These moduli spaces have the structure of a complex analytic varieties, as one can see by the standard slice method, which gives local models via the so-called Kuranishi map (see, e.g., [34]). When G is algebraic and under fairly general conditions, the moduli spaces $\mathcal{M}^{\alpha}(G)$ can be constructed by geometric invariant theory and hence are complex algebraic varieties. The work of Schmitt [43, 44] deals with the construction of the moduli space of L-twisted G-Higgs bundles for G a real reductive Lie group. This construction generalizes the constructions of the moduli space of G-Higgs bundles done by Ramanathan [41] when G is compact, and by Simpson [46, 47] when G is a complex reductive algebraic (see also [40] for $G = GL(n, \mathbb{C})$).

The notion of stability emerges from the study of the Hitchin equations. The equivalence between the existence of solutions to these equations and the α -polystability of Higgs bundles is known as the **Hitchin-Kobayashi correspondence**, which we state below.

Theorem 3.11. Let (E, φ) be a G-Higgs bundle over a Riemann surface X with volume form ω . Then (E, φ) is α -polystable if and only if there exists a reduction h of the

structure group of E from $H^{\mathbb{C}}$ to H, that is a smooth section of $E(H^{\mathbb{C}}/H)$, such that

(9)
$$F_h - [\varphi, \tau_h(\varphi)] = -i\alpha\omega$$

where $\tau_h: \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h and the conjugation of 1-forms, and F_h is the curvature of the unique H-connection compatible with the holomorphic structure of E (the Chern connection).

This theorem was proved by Hitchin in the case of $SL(2, \mathbb{C})$, by Simpson when G is complex, and in [7, 18] for a general reductive real Lie group G

Remark 3.12. There is a theorem similar to Theorem 3.11 for L-twisted G-Higgs bundles (see Remark 3.7 and [18]) for an arbitrary line bundle L. If (E, φ) is such a pair, one fixes a Hermitian metric h_L on L, and looks for a reduction of structure group h of E from $H^{\mathbb{C}}$ to H satisfying

(10)
$$F_h - [\varphi, \tau_h(\varphi)]\omega = -i\alpha\omega,$$

where now $\tau_h: \Omega^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) \to \Omega^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h and the metric h_L .

3.3. Higgs bundles and representations. Fix a base point $x \in X$. A representation of $\pi_1(X, x)$ in G is a homomorphism $\pi_1(X, x) \longrightarrow G$. After fixing a presentation of $\pi_1(X, x)$, the set of all such homomorphisms, $\operatorname{Hom}(\pi_1(X, x), G)$, can be identified with the subset of G^{2g} consisting of 2g-tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^g [A_i, B_i] = 1$. This shows that $\operatorname{Hom}(\pi_1(X, x), G)$ is an algebraic variety.

The group G acts on $\text{Hom}(\pi_1(X,x),G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1},$$

where $g \in G$, $\rho \in \text{Hom}(\pi_1(X, x), G)$ and $\gamma \in \pi_1(X, x)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X, x), G)$ consisting of reductive representations, the orbit space is Hausdorff. We recall that a **reductive representation** is one whose composition with the adjoint representation in \mathfrak{g} decomposes as a direct sum of irreducible representations. This is equivalent to the condition that the Zariski closure of the image of $\pi_1(X, x)$ in G is a reductive group. Define the **moduli space of representations** of $\pi_1(X, x)$ in G to be the orbit space

$$\mathcal{R}(G) = \operatorname{Hom}^+(\pi_1(X, x), G)/G.$$

This is a real algebraic variety. For another point $x' \in X$, the fundamental groups $\pi_1(X,x)$ and $\pi_1(X,x')$ are identified by an isomorphism unique up to an inner automorphism. Consequently, $\mathcal{R}(G)$ is independent of the choice of the base point x.

Given a representation $\rho \colon \pi_1(X,x) \longrightarrow G$, there is an associated flat principal G-bundle on X, defined as

$$E_{\rho} = \widetilde{X} \times_{\rho} G,$$

where $\widetilde{X} \longrightarrow X$ is the universal cover and $\pi_1(X, x)$ acts on G via ρ . This gives in fact an identification between the set of equivalence classes of representations $\operatorname{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat principal G-bundles, which in turn is parametrized by the (non-abelian) cohomology set $H^1(X, G)$. We have the following.

Theorem 3.13. Let G be a semisimple real Lie group. Then there is a homeomorphism $\mathcal{R}(G) \cong \mathcal{M}(G)$.

The proof of Theorem 3.13 is the combination of Theorem 3.11 and the following theorem of Corlette [12], also proved by Donaldson [13] when $G = SL(2, \mathbb{C})$.

Theorem 3.14. Let ρ be a representation of $\pi_1(X)$ in G with corresponding flat G-bundle E_{ρ} . Let $E_{\rho}(G/H)$ be the associated G/H-bundle. Then the existence of a harmonic section of $E_{\rho}(G/H)$ is equivalent to the reductiveness of ρ .

Remark 3.15. Theorem 3.13 can be extended to reductive groups replacing $\pi_1(X)$ by its universal central extension.

4. HERMITIAN GROUPS, TOLEDO INVARIANT AND MILNOR-WOOD INEQUALITY

In this section we will assume that G is a connected, non-compact real simple Lie group of Hermitian type with finite centre (see Section 2.1 for definition), and X is a compact Riemann surface of genus g. We fix a maximal compact subgroup $H \subset G$, with Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Let (E,φ) be a G-Higgs bundle over X. The decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$ gives a vector bundle decomposition $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^+) \oplus E(\mathfrak{m}^-)$ and hence the Higgs field has two components:

$$\varphi = (\varphi^+, \varphi^-) \in H^0(X, E(\mathfrak{m}^+) \otimes K) \oplus H^0(X, E(\mathfrak{m}^-) \otimes K) = H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K).$$

When the group G is a classical group, or more generally when H is a classical group, it is useful to take the standard representation of $H^{\mathbb{C}}$ to describe a G-Higgs bundle in terms of associated vector bundles. This is the approach taken in [3, 4, 6, 19].

4.1. **Toledo invariant.** Let (E, φ) be a G-Higgs bundle. Consider the Toledo character χ_T defined in Section 2.3. Maybe up to an integer multiple, χ_T lifts to a character $\tilde{\chi}_T$ of $H^{\mathbb{C}}$. Let $E(\tilde{\chi}_T)$ be the line bundle associated to E via the character $\tilde{\chi}_T$.

Definition 4.1. We define the **Toledo invariant** τ of (E, φ) by

$$\tau = \tau(E) := \deg(E(\tilde{\chi}_T)).$$

If $\tilde{\chi}_T$ is not defined, but only $\tilde{\chi}_T^q$, one must replace the definition by $\frac{1}{q} \deg E(\tilde{\chi}_T^q)$.

We denote by $\mathcal{M}_{\tau}^{\alpha}(G)$ the subspace of $\mathcal{M}^{\alpha}(G)$ corresponding to G-Higgs bundles whose Toledo invariant equals τ . For $\alpha = 0$ we simplify our notation setting $\mathcal{M}_{\tau}(G) := \mathcal{M}_{\tau}^{0}(G)$

The following proposition relates our Toledo invariant to the usual Toledo invariant of a representation, first defined in [48].

Proposition 4.2. Let $\rho: \pi_1(X) \to G$ be reductive and let (E, φ) be the corresponding polystable G-Higgs bundle given by Theorem 3.13. Let $f: \tilde{X} \to G/H$ be the corresponding harmonic metric. Then

$$\tau(E) = \frac{1}{2\pi} \int_X f^* \omega,$$

where ω is the Kähler form of the symmetric metric on G/H with minimal holomorphic sectional curvature -1, computed in Lemma 2.6.

In particular, $\tau(E)$ is the Toledo invariant of ρ .

Proof. The harmonic metric f defines a solution h to the Hitchin equations (9), that is a reduction of structure group of E to H. Write $\Phi = \varphi - \tau_h(\varphi) = \Phi_x dx + \Phi_y dy$ in local coordinates, where τ_h is defined in Theorem 3.11. Then $[\varphi, \tau_h(\varphi)] = [\Phi_x, \Phi_y] dx \wedge dy$. From the Hitchin equation $F_h = [\varphi, \tau_h(\varphi)]$ and Lemma 2.6,

$$\frac{i}{2\pi}\chi_T(F_h) = \frac{i}{2\pi}\chi_T([\Phi_x, \Phi_y])dx \wedge dy = \frac{1}{2\pi}\omega(\Phi_x, \Phi_y)dx \wedge dy.$$

Since $\Phi = df$, the expected formula follows by integration.

The Toledo invariant is related to the topological class of the bundle E defined as an element of $\pi_1(H)$. To explain this, assume that $H^{\mathbb{C}}$ is connected. The topological classification of $H^{\mathbb{C}}$ -bundles E on X is given by a characteristic class $c(E) \in \pi_1(H^{\mathbb{C}})$ as follows. From the exact sequence

$$1 \to \pi_1(H^{\mathbb{C}}) \to \widetilde{H^{\mathbb{C}}} \to H^{\mathbb{C}} \to 1$$

we obtain a long exact sequence in cohomology and, in particular, the connecting homomorphism

(11)
$$H^{1}(X, \underline{H}^{\mathbb{C}}) \xrightarrow{c} H^{2}(X, \pi_{1}(H^{\mathbb{C}})),$$

where $\underline{H}^{\mathbb{C}}$ is the sheaf of local holomorphic functions in X with values in $H^{\mathbb{C}}$. The cohomology set $H^1(X,\underline{H}^{\mathbb{C}})$ (not necessarily a group since $H^{\mathbb{C}}$ is in general not abelian) parametrizes isomorphism classes of principal $H^{\mathbb{C}}$ -bundles over X. On the other hand, since $\dim_{\mathbb{R}} X = 2$, by the universal coefficient theorem and the fact that the fundamental group of a Lie group is abelian, $H^2(X, \pi_1(H^{\mathbb{C}}))$ is isomorphic to $\pi_1(H^{\mathbb{C}})$. Moreover, $\pi_1(H^{\mathbb{C}}) \cong \pi_1(H) \cong \pi_1(G)$ since H is a deformation retract for both $H^{\mathbb{C}}$ and G. This map thus associates a topological invariant in $\pi_1(H)$ to any G-Higgs bundle on X.

By the relation between the fundamental group and the centre of a Lie group, the topological class in $\pi_1(H)$ is of special interest when H has a non-discrete centre, i.e., when G is of Hermitian type. In this case, $\pi_1(H)$ is isomorphic to \mathbb{Z} plus possibly a torsion group (among the classical groups, $SO_0(2, n)$ is the only one with torsion). Very often (see for example [4]), the Toledo invariant of a G-Higgs bundle (E, φ) is defined as the projection of c(E) defined by (11) on the torsion-free part, \mathbb{Z} . The general relation is the following.

Proposition 4.3. Let (E,φ) be a G-Higgs bundle, and $d \in \mathbb{Z}$ the projection on the torsion-free part of the class c(E) defined by (11). Then d is related to the Toledo invariant by

$$\tau = \frac{d}{q_T}.$$

Proof. The character $\tilde{\chi}_T: H^{\mathbb{C}} \to \mathbb{C}^*$ induces a morphism

$$(\tilde{\chi}_T)_*: \pi_1(H^{\mathbb{C}}) \longrightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z},$$

defined by $(\tilde{\chi}_T)_*(\gamma) = \tau(E)$, where E is the $H^{\mathbb{C}}$ -bundle corresponding to $\gamma \in \pi_1(H^{\mathbb{C}})$ under the bijection of $H^1(X, H^{\mathbb{C}})$ with $\pi_1(H^{\mathbb{C}})$. This map must be a multiple of the projection $\pi_1(H^{\mathbb{C}}) \to \mathbb{Z}$ on the torsion-free part.

We compute this multiple by considering the image by $\tilde{\chi}_T$ of a loop generating $\pi_1(H^{\mathbb{C}})$. Recall that $H^{\mathbb{C}} = [H^{\mathbb{C}}, H^{\mathbb{C}}] \times_D Z_0^{\mathbb{C}}$, where $Z_0^{\mathbb{C}}$ is the identity component of the centre of $H^{\mathbb{C}}$ and $D = [H^{\mathbb{C}}, H^{\mathbb{C}}] \cap Z_0^{\mathbb{C}}$. We have seen in the proof of Proposition 2.5 that the torsion-free part of $\pi_1(H^{\mathbb{C}})$ is generated by the loop $\exp(2\pi \frac{o_J}{\ell} J \theta)$, and hence, since $(\chi_T)_*(\frac{o_J}{\ell} J) = \frac{o_J}{\ell} \frac{\dim \mathfrak{m}}{N} i$, the result follows.

4.2. **Milnor-Wood inequality.** In this section we give a bound of the Toledo invariant for an α -semistable G-Higgs bundle $(E, \varphi^+, \varphi^-)$ involving the ranks of φ^+ and φ^- , leading to the familiar Milnor-Wood inequality.

Definition 2.8 gives the ranks of φ^+ and φ^- at a point $x \in X$. The space of elements of \mathfrak{m}^{\pm} with rank at most ρ is an algebraic subvariety of \mathfrak{m}^{\pm} , so the ranks of φ^+ and φ^- are the same at all points of X except a finite number of points where it it smaller. We therefore have a well defined notion of rank of φ^+ and φ^- :

Definition 4.4. The generic value on X of the rank of φ^+ is called the rank of φ^+ and denoted $\operatorname{rk} \varphi^+$. Analogously we define the rank $\operatorname{rk} \varphi^-$ of φ^- .

The main result of this section is the following.

Theorem 4.5. Let $\alpha \in i\mathfrak{z}$ such that $\alpha = i\lambda J$ for $\lambda \in \mathbb{R}$. Let $(E, \varphi^+, \varphi^-)$ be an α -semistable G-Higgs bundle. Then, the Toledo invariant of E satisfies:

$$-\operatorname{rk}(\varphi^+)(2g-2) - \left(\frac{\dim\mathfrak{m}}{N} - \operatorname{rk}(\varphi^+)\right)\lambda \leq \tau \leq \operatorname{rk}(\varphi^-)(2g-2) - \left(\frac{\dim\mathfrak{m}}{N} - \operatorname{rk}(\varphi^-)\right)\lambda,$$

where N is the dual Coxeter number. In the tube case, this simplifies to:

$$-\operatorname{rk}(\varphi^+)(2g-2) - (r - \operatorname{rk}(\varphi^+))\lambda \le \tau \le \operatorname{rk}(\varphi^-)(2g-2) - (r - \operatorname{rk}(\varphi^-))\lambda.$$

The rest of this subsection is devoted to the proof of the theorem. We will focus on φ^+ to prove the left inequality, the right one is a consequence of the same proof for φ^- .

We will apply the semistability hypothesis to a carefully defined parabolic reduction of E. Recall that a nilpotent endomorphism U in a vector space has a unique decreasing filtration $W = (W_k)$, such that $U(W_k) \subset W_{k+2}$ and U^k induces an isomorphism on the graded spaces: $W_{-k}/W_{-k+1} \to W_k/W_{k+1}$. If one completes U into a representation (S, U, V) of $\mathfrak{sl}(2, \mathbb{R})$, then W_k is the sum of the eigenspaces of S for the eigenvalues $\lambda > k$.

Now consider a point $x \in X$. To begin with, fix a trivialization of E, so that we can consider φ_x^+ as an element of \mathfrak{m}^+ . Then $U = \operatorname{ad} \varphi_x^+$ acts on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$, so from the associated W filtration, we can define

$$\mathfrak{p}_x = W_0 \cap \mathfrak{h}^{\mathbb{C}}.$$

This W filtration is very simple, because $U(\mathfrak{m}^+) \subset \mathfrak{h}^{\mathbb{C}}$, $U(\mathfrak{h}^{\mathbb{C}}) \subset \mathfrak{m}^- \subset \ker U$, so actually

$$\mathfrak{p}_x = \ker U|_{\mathfrak{h}^{\mathbb{C}}} + \operatorname{im} U|_{\mathfrak{m}^-}.$$

The element φ_x^+ can be completed in a \mathfrak{sl}_2 -triple (h, φ_x^+, v) such that $h \in \mathfrak{h}^{\mathbb{C}}$ (and $v \in \mathfrak{m}^-$). Then a \mathfrak{sl}_2 -triple associated to U is $(\operatorname{ad} h, U, \operatorname{ad} v)$. From the definition of W_0 using the eigenspaces of $\operatorname{ad} h$, it follows now that \mathfrak{p}_x is a parabolic subalgebra of $\mathfrak{h}^{\mathbb{C}}$.

One can make things more explicit: choose a system of strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_r\}$ as in Section 2, suppose $\operatorname{rk} \varphi_x^+ = r'$, then φ_x^+ is conjugate to the element

$$(13) u = e_{\gamma_1} + \dots + e_{\gamma_{n'}} \in \mathfrak{m}^+.$$

Then we choose

$$(14) v = e_{-\gamma_1} + \dots + e_{-\gamma_{r'}}$$

$$(15) h = h_{\gamma_1} + \dots + h_{\gamma_{r'}}$$

so that we obtain the description

$$\mathfrak{p} = \mathfrak{t}^{\mathbb{C}} \oplus \underset{\alpha \in \Delta_C, \alpha(h) \geq 0}{\oplus} \mathfrak{g}_{\alpha}.$$

Denote by P the corresponding parabolic subgroup of $H^{\mathbb{C}}$.

All the previous description was just at the point x. But since the construction is canonical (indeed, the W filtration depends only on φ_x^+), we obtain a reduction of the structure group of E to P on the open set where the rank of φ_x^+ is $r' = \operatorname{rk} \varphi^+$. The description (12) with $U = \operatorname{ad} \varphi^+$ shows that the reduction extends over the singular points (since the kernel or the image of a morphism between holomorphic bundles do). Therefore φ^+ defines a global reduction σ to P of the $H^{\mathbb{C}}$ -bundle E.

To apply the semistability criterion, the second ingredient is an antidominant character of \mathfrak{p} . We consider

$$\chi_{r'} = \gamma_1 + \dots + \gamma_{r'}$$

to define a character of \mathfrak{p} by

$$\chi = \chi_T - \chi_{r'}.$$

Let $s_{\chi} \in \mathfrak{t}^{\mathbb{C}}$ be dual to χ via the invariant product. Then:

Lemma 4.6. For $\alpha \in \Delta$ one has

$$\frac{1}{\langle \gamma, \gamma \rangle} \alpha(s_{\chi}) = \begin{cases} 1 - \frac{1}{2}\alpha(h) & \text{if } \alpha \in \Delta_{Q}^{+}, \\ -\frac{1}{2}\alpha(h) & \text{if } \alpha \in \Delta_{C}, \\ -1 - \frac{1}{2}\alpha(h) & \text{if } \alpha \in \Delta_{Q}^{-}. \end{cases}$$

In particular, s_{χ} defines a strictly antidominant character of \mathfrak{p} , such that $\mathfrak{m}^- \subset \mathfrak{m}_{s_{\chi}}$ and $u \in \mathfrak{m}^0_{s_{\chi}}$.

Proof. Recall that the dual of γ_i is $s_{\gamma_i} = \frac{\langle \gamma_i, \gamma_i \rangle}{2} h_{\gamma_i}$, so the dual of χ is

$$s_{\chi} = s_{\chi_T} - \frac{\langle \gamma_i, \gamma_i \rangle}{2} h = \langle \gamma_i, \gamma_i \rangle \left(-iJ - \frac{h}{2} \right).$$

The formula in the lemma follows immediately. In particular, s_{χ} defines a strictly antidominant character for \mathfrak{p} , since on $\mathfrak{h}^{\mathbb{C}}$ it coincides (up to a positive constant) with -h. Using [h,u]=2u and $u\in\mathfrak{m}^+$, we deduce that $[s_{\chi},u]=0$ so u is actually in the Levi part $\mathfrak{m}_s^0\subset\mathfrak{p}$. Moreover, since all the eigenvalues of ad h are in $\{-2,\ldots,2\}$, we have $-1-\frac{1}{2}\alpha(h)\leq 0$ for all α , and hence $\mathfrak{m}^-\subset\mathfrak{m}_{s_{\chi}}$.

The last algebraic ingredient needed for the proof of the theorem is the remark that the choice of r' strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_{r'}\}$ defines a 'subtube'. This will also give a useful interpretation of χ in terms of the subtube. Indeed, we can define, following the notations of Section 2.2, (we will give a more intrinsic construction later)

$$C_{r'} = \bigcup_{1 \le i, j \le r'} C_{ij}, \quad Q_{r'} = \bigcup_{1 \le i, j \le r'} Q_{ij},$$

and

$$\mathfrak{h}_{r'}^{\mathbb{C}} = \langle h_{\gamma}, \gamma \in Q_{r'} \rangle \oplus \bigoplus_{\alpha \in C_{r'}} \mathfrak{g}_{\alpha}, \quad \mathfrak{m}_{r'}^{\pm} = \bigoplus_{\alpha \in Q_{r'}} \mathfrak{g}_{\alpha}.$$

Intersecting with \mathfrak{g} we obtain well-defined real forms $\mathfrak{h}_{r'}$ and $\mathfrak{m}_{r'}$.

This construction is illustrated in the case of SU(p,q) by the matrix

$$\begin{pmatrix}
0 & & & & \\
& \mathfrak{h}_{r'}^{\mathbb{C}} & \mathfrak{m}_{r'}^{+} & & \\
\hline
& \mathfrak{m}_{r'}^{-} & \mathfrak{h}_{r'}^{\mathbb{C}} & & & \\
& & & & & 0
\end{pmatrix}.$$

In general, $\mathfrak{g}_{r'} = \mathfrak{h}_{r'} \oplus \mathfrak{m}_{r'}$ has all the properties of a symmetric pair of tube type of rank r', having $\{\gamma_1, \ldots, \gamma_{r'}\}$ as a set of strongly orthogonal roots. The character $\chi_{r'}$ defined in (16) clearly identifies to the Toledo character of this subtube. Let $H_{r'} \subset H$ and $G_{r'} \subset G$ the corresponding subgroups. All the algebraic study of Section 2 applies: in particular we have a determinant $\det_{r'}$ on $\mathfrak{m}_{r'}^+$, a Toledo character $\chi_{r'}$ lifting to a Toledo character $\tilde{\chi}_{r'}$ of $H_{r'}^{\mathbb{C}}$, so that for $h \in H_{r'}^{\mathbb{C}}$ and $m \in \mathfrak{m}_{r'}^+$, we have (maybe taking some power if the lifting does not exist)

(18)
$$\det_{r'}(\operatorname{Ad}(h)m) = \tilde{\chi}_{r'}(h) \det_{r'}(m).$$

Of course, coming back to our bundle E, we must see that this construction is intrinsic so that it defines bundles over the surface X. It is easily checked that $\mathfrak{m}_{r'}^+$ coincides with the step W_2 of the W filtration of ad φ^+ (which equals the eigenspace of ad h for the eigenvalue 2 because this is the highest eigenvalue). Moreover, the Levi factor L of P acts on $\mathfrak{m}_{r'}^+$ by Ad, so we can can consider

$$L' = \{ g \in L : \operatorname{Ad}(g) |_{\mathfrak{m}_{r'}^+} = 1 \},$$

and we obtain a faithful action of

$$\tilde{L} = L/L'$$

on $\mathfrak{m}_{r'}^+$; the group \tilde{L} identifies to $H_{r'}^{\mathbb{C}}$ (so it is more intrinsically defined as a quotient rather than a subgroup).

All this depends canonically on φ^+ , so produces a pair

$$(E_{H_{n'}^{\mathbb{C}}}, \varphi^+)$$

of a holomorphic $H_{r'}^{\mathbb{C}}$ principal bundle $E_{H_{r'}^{\mathbb{C}}}$ and a section $\varphi^+ \in H^0(X, E_{H_{r'}^{\mathbb{C}}}(\mathfrak{m}_{r'}^+) \otimes K)$. Because $\operatorname{rk} \varphi^+ = r' = \operatorname{rk} G_{r'}/H_{r'}$, the determinant $\det_{r'}$ produces a non-zero section

$$\det_{r'} \varphi^+ \in H^0(X, E_{H^{\mathbb{C}}_{-l'}}(\tilde{\chi}_{r'}) \otimes K^{r'}),$$

which implies

(19)
$$\deg E_{H^{\mathbb{C}}}(\tilde{\chi}_{r'}) + r'(2g - 2) \ge 0.$$

We have now all the ingredients needed for the proof of Theorem 4.5.

of Theorem 4.5. We constructed from φ^+ a reduction σ of E from $H^{\mathbb{C}}$ to $P = P_{s_{\chi}}$. Call E_{σ} the reduced P-bundle. From Lemma 4.6, we have $\varphi \in H^0(X, E_{\sigma}(\mathfrak{m}_{s_{\chi}}) \otimes K)$. The semistability condition gives

(20)
$$\deg(E)(\sigma, s_{\chi}) - \langle \alpha, s_{\chi} \rangle \ge 0.$$

The character χ lifts to a character of P given by

$$\tilde{\chi} = \tilde{\chi}_T \tilde{\chi}_{r'}^{-1}$$
.

(Again, it may be necessary to take some power). Therefore

$$\deg(E)(\sigma, s) = \deg(E_P(\tilde{\chi}_T)) - \deg(E_P(\tilde{\chi}_{r'})),$$

$$= \tau - \deg(E_P(\tilde{\chi}_{r'}))$$

$$\leq \tau + r'(2g - 2)$$

where at the last line, we have used (19). So (20) becomes

(21)
$$\tau \ge -r'(2g-2) + \langle \alpha, s_{\chi} \rangle.$$

From (4) we obtain

$$\langle \alpha, s_{\chi} \rangle = (\chi_T - \chi_{r'})(\lambda i J) = -\lambda \left(\frac{\dim \mathfrak{m}}{N} - r'\right).$$

As an immediate corollary of Theorem 4.5 we have the following.

Proposition 4.7. Let $\alpha = i\lambda J$. If (E, β, γ) is an α -semistable G-Higgs bundle with Toledo invariant τ , we have that

$$|\tau| \le \begin{cases} (2g - 2 + \lambda) \operatorname{rk}(G/H) - \frac{\dim \mathfrak{m}}{N} \lambda & \text{if } \lambda > -(2g - 2), \\ -\frac{\dim \mathfrak{m}}{N} \lambda & \text{if } \lambda \le -(2g - 2). \end{cases}$$

The case $\alpha = 0$ is of special interest because of the relation of the moduli space of polystable G-Higgs bundles over a Riemann surface X with the moduli space of representations of $\pi_1(X)$ into G as explained in Section 3.3. In this situation, we will simply talk about **stability** of a G-Higgs bundle, meaning 0-stability, and analogously

for **semistability** and **polystability**. When $\alpha = 0$, we have that $\lambda = 0$ in Theorem 4.5, and obtain the following.

Theorem 4.8. Let $(E, \varphi^+, \varphi^-)$ be a semistable G-Higgs bundle. Then the Toledo invariant τ satisfies

$$-\operatorname{rk}(\varphi^+)(2g-2) \le \tau \le \operatorname{rk}(\varphi^-)(2g-2).$$

In particular, we obtain the familiar Milnor-Wood inequality:

$$|\tau| \le \operatorname{rk}(G/H)(2g-2),$$

and the equality holds if and only if φ^+ (resp. φ^-) is regular at each point in the case $\tau < 0$ (resp. $\tau > 0$).

Proof. There only remains to prove the last statement on the equality case, but this is an immediate consequence of the proof of Theorem 4.5: for simplicity, restrict to the case $\tau < 0$, then equality in the Milnor–Wood inequality implies equality in (19), which implies that $\det_r \varphi^+$ does not vanish. The converse is immediate.

Theorem 4.8 was proved on a case by case basis for the classical groups [27, 24, 3, 4, 6, 19]. In these references, the bound given is for the integer $d \in \pi_1(H) \cong \pi_1(H^{\mathbb{C}}) \cong \mathbb{Z}$ associated naturally to the $H^{\mathbb{C}}$ -bundle E. This differs from the Toledo invariant by a rational multiple. From Table 2 and Proposition 4.3 combined with Theorem 4.8 we obtain the Milnor-Wood inequalities given in [4] for the classical Hermitian groups. Our intrinsic general approach covers of course the exceptional groups and quotients and covers of classical groups that have not been studied previously.

A polystable G-Higgs bundle (E, φ) is, by Theorem 3.13, in correspondence with a reductive representation $\rho: \pi_1(X) \to G$, and from Proposition 4.2 the Toledo invariant of (E, φ) coincides with the Toledo invariant of a representation of the fundamental group in G. In the context of representations the inequality $|\tau| \leq \text{rk}(G/H)(2g-2)$, goes back to Milnor [39], who studies the case $G = \text{PSL}(2, \mathbb{R})$, and was proved in various cases in [49, 15, 14, 11], and in general in [9]. We should point out that the Higgs bundle

approach gives the Milnor–Wood inequality for an arbitrary representation, as the other approaches do, since such a representation can always be deformed to a reductive one.

Remark 4.9. Lemma 2.6 and Proposition 4.2 indeed provide the way to translate the Milnor-Wood inequality for Higgs bundles into the inequality for representations, for any group of Hermitian type with finite centre, in a classification-independent way. Translating the Milnor-Wood inequality for representations into that for Higgs bundles is the aim of [30], where it is done for the groups SU(p,q) and $Sp(2n,\mathbb{R})$, or a matrix Lie group admitting a so-called admissible representation.

4.3. Involutions and the Toledo invariant. Let $G^{\mathbb{C}}$ be a complex Lie group and let $\operatorname{Aut}(G^{\mathbb{C}})$ be the group of its holomorphic automorphisms. Let $\operatorname{Int}(G^{\mathbb{C}}) \subset \operatorname{Aut}(G^{\mathbb{C}})$ be the subgroup of inner automorphisms. The group of outer automorphisms of $G^{\mathbb{C}}$ is defined as

$$\operatorname{Out}(G^{\mathbb{C}}) := \operatorname{Aut}(G^{\mathbb{C}}) / \operatorname{Int}(G^{\mathbb{C}}).$$

We thus have a sequence

$$(22) 1 \longrightarrow \operatorname{Int}(G^{\mathbb{C}}) \longrightarrow \operatorname{Aut}(G^{\mathbb{C}}) \longrightarrow \operatorname{Out}(G^{\mathbb{C}}) \longrightarrow 1.$$

A real Lie subgroup G of the underlying real Lie group to $G^{\mathbb{C}}$ is a **real form** of $G^{\mathbb{C}}$ if it is the fixed point set of a conjugation (i.e. an anti-holomorphic involution) σ of $G^{\mathbb{C}}$.

Assume now that $G^{\mathbb{C}}$ is semisimple. Then there is a **compact real form**, i.e. a maximal compact subgroup of $G^{\mathbb{C}}$, defined by a conjugation σ_c . Let $\text{Conj}(G^{\mathbb{C}})$ be the set of conjugations of $G^{\mathbb{C}}$. We define the following equivalence relation in $\text{Conj}(G^{\mathbb{C}})$:

$$\sigma \sim \sigma'$$
 if there is $\alpha \in \text{Int}(G^{\mathbb{C}})$ such that $\sigma' = \alpha \sigma \alpha^{-1}$.

We can define a similar relation \sim in the set $\operatorname{Aut}_2(G^{\mathbb{C}})$ of automorphisms of $G^{\mathbb{C}}$ of order 2. Cartan [10] shows that there is a bijection

$$\operatorname{Conj}(G^{\mathbb{C}})/\sim \longleftrightarrow \operatorname{Aut}_2(G^{\mathbb{C}})/\sim.$$

More concretely, given a compact conjugation σ_c , in each class in $\operatorname{Conj}(G^{\mathbb{C}})/\sim$ one can find a representative σ commuting with σ_c so that $\theta := \sigma \sigma_c$ is an element of $\operatorname{Aut}_2(G^{\mathbb{C}})$, and similarly if we start with a class in $\operatorname{Aut}_2(G^{\mathbb{C}})/\sim$. The natural map $\operatorname{Aut}_2(G^{\mathbb{C}})\longrightarrow\operatorname{Out}_2(G^{\mathbb{C}})$, where $\operatorname{Out}_2(G^{\mathbb{C}})$ are the elements of order 2 in $\operatorname{Out}(G^{\mathbb{C}})$, is surjective since the extension (22) splits in this situation (see [45]), and descends to give a map $\operatorname{Aut}_2(G^{\mathbb{C}})/\sim\longrightarrow\operatorname{Out}_2(G^{\mathbb{C}})$. This combined with the bijection with conjugations defines a map

$$c: \operatorname{Conj}(G^{\mathbb{C}})/\sim \longrightarrow \operatorname{Out}_2(G^{\mathbb{C}}).$$

Clearly the image of the compact conjugation σ_c under this map is the trivial element in $\operatorname{Out}_2(G^{\mathbb{C}})$.

Since $G^{\mathbb{C}}$ is semisimple we also have the **split real form**, defined by a conjugation σ_s which can be chosen to commute with the compact conjugation σ_c .

Consider now a real form G of Hermitian type of $G^{\mathbb{C}}$, defined by a conjugation σ . It is well-known that this real form is inner equivalent to the compact one, that is $c(\sigma) = 1$. Starting with σ we can choose conjugations σ_c and σ_s commuting with σ and commuting between themselves. This is easy to see if σ_s is also inner equivalent to σ_c , otherwise it requires a little argument [1]. The maps $\theta := \sigma \sigma_c$ and $\psi := \sigma_s \sigma$ are holomorphic involutions of $G^{\mathbb{C}}$, i.e. elements in $\operatorname{Aut}_2(G^{\mathbb{C}})$ which commute. The group $H = (G^{\mathbb{C}})^{\sigma_c} \cap (G^{\mathbb{C}})^{\sigma}$ is a maximal compact subgroup of G whose complexification $H^{\mathbb{C}}$ coincides with $(G^{\mathbb{C}})^{\theta}$. The following is straightforward.

Proposition 4.10. (1) The involution $\psi: G^{\mathbb{C}} \to G^{\mathbb{C}}$ leaves invariant $G, H^{\mathbb{C}}$ and H.

- (2) The differential of ψ preserves the Cartan decomposition, sends $J \in \mathfrak{z}(\mathfrak{h})$ to -J and exchanges \mathfrak{m}^+ and \mathfrak{m}^- .
- (3) The involution of $\pi_1(H)$ induced by ψ sends d to -d under the isomorphism of the free part of $\pi_1(H)$ with \mathbb{Z} .
- (4) The involution of the symmetric space G/H induced by ψ is an antiholomorphic isometry.

If now $(E, \varphi^+, \varphi^-)$ is a G-Higgs bundle from (1) in Proposition 4.10 we can define the $H^{\mathbb{C}}$ -principal bundle

$$\psi(E) := E \times_{\psi} H^{\mathbb{C}},$$

and From (2) in Proposition 4.10 we have isomorphisms

$$\psi: E(\mathfrak{m}^{\pm}) \to \psi(E)(\mathfrak{m}^{\mp})$$

which can be used to define the G-Higgs bundle $(\psi(E), \psi(\varphi^-), \psi(\varphi^+))$ (here we are abusing notation using ψ also for the induced bundle isomorphisms).

The following is a consequence of Proposition 4.10.

Proposition 4.11. Let G be a real form of Hermitian type of a complex semisimple Lie group, and let ψ be defined as above. Then the map

$$(E, \varphi^+, \varphi^-) \mapsto (\psi(E), \psi(\varphi^-), \psi(\varphi^+))$$

defines an isomorphism between $\mathcal{M}^{\alpha}_{\tau}(G)$ to $\mathcal{M}^{-\alpha}_{-\tau}(G)$. In particular it defines an involution of $\mathcal{M}^0_0(G)$ (simply denoted $\mathcal{M}_0(G)$).

A simple Lie group G of Hermitian type need not have a complexification as we are assuming. For instance, any non-trivial finite covering G of $\operatorname{Sp}(2n,\mathbb{R})$ does not sit in any group with Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$, since $\operatorname{Sp}(2n,\mathbb{C})$ is simply connected and we have $\operatorname{Sp}(2n,\mathbb{R}) \subset \operatorname{Sp}(2n,\mathbb{C})$. However, the symmetric space M := G/H has always antiholomorphic involutive automorphisms. These have been classified in [32, 33, 38]. From this, one can obtain involutions of the adjoint group $\operatorname{Ad}(G) := G/Z(G)$ that satisfy the properties in Proposition 4.10. To explain this, let us assume that G is of adjoint type, i.e. Z(G) = 1, otherwise we consider $\operatorname{Ad}(G)$. In this situation, G is the connected component of the identity of $\operatorname{Isom}(M)$, the group of isometries of M = G/H, and consists of course of holomorphic isometries. We are now interested in studying conjugations of M, that is anti-holomorphic isometries of M. Let us denote by $\operatorname{Conj}(M)$ the set of all such conjugations. The group G act on $\operatorname{Conj}(M)$ by sending $\sigma \in \operatorname{Conj}(M)$ to $g\sigma g^{-1}$ for any $g \in G$. Also if $o \in M$ corresponds to the coset H, we consider $\operatorname{Conj}_o(M) := \{\sigma \in \operatorname{Conj}(M) : \sigma(o) = o\}$. In this case H acts on $\operatorname{Conj}_o(M)$ by conjugation. A key result in [32, 33, 38] is the following.

Proposition 4.12. The sets Conj(M)/G and $Conj_o(M)/H$ are finite and the natural map $Conj_o(M)/H \to Conj(M)/G$ is a bijection.

Moreover, [32, 33, 38] give an explicit classification for the irreducible symmetric spaces. In all cases the fixed points of any conjugation are connected symmetric subspaces $G'/H' \subset G/H$, whose real dimension equals the complex dimension of G/H.

If G is a real form of a complex group $G^{\mathbb{C}}$ the involution of M = G/H induced by ψ in Proposition 4.10 defines a class in $\operatorname{Conj}_o(M)/H$. It is perhaps plausible that every class in $\operatorname{Conj}_o(M)/H$ is defined in a similar way to the one induced by ψ , by replacing the split conjugation of $G^{\mathbb{C}}$ by another conjugation of $G^{\mathbb{C}}$ commuting with the compact conjugation of $G^{\mathbb{C}}$ and the conjugation defining G.

Now, fix $\sigma \in \operatorname{Conj}_o(M)$. Since $\sigma \in \operatorname{Isom}(M)$ and G is the connected component of the identity of $\operatorname{Isom}(M)$ we can define an automorphism ψ_{σ} of G given by the rule $g \mapsto \sigma g \sigma^{-1}$ for $g \in G$. One can check that this automorphism satisfies all the properties

in Proposition 4.10, besides the fact that, since G may not have a complexification, ψ_{σ} is only defined on G, and one has the following.

Corollary 4.13. Let G be a Hermitian Lie group such that Z(G) = 1. Let $\sigma \in \operatorname{Conj}_o(M)$ and let ψ_{σ} be the involution of G defined as above. Then the map

$$(E, \varphi^+, \varphi^-) \mapsto (\psi_{\sigma}(E), \psi_{\sigma}(\varphi^-), \psi_{\sigma}(\varphi^+))$$

defines an isomorphism between $\mathcal{M}^{\alpha}_{\tau}(G)$ to $\mathcal{M}^{-\alpha}_{-\tau}(G)$. In particular it defines an involution of $\mathcal{M}_0(G)$. Moreover the involution on the moduli space only depends on the class of σ in $\operatorname{Conj}_o(M)/H$.

It is clear that ψ_{σ} defines an isometry of the symmetric space $H^{\mathbb{C}}/H$ and hence if h is a solution to the Hitchin equations for (E, φ) , i.e. a reduction of structure group of E to H satisfying (9) then $\psi_{\sigma}(h)$ is a solution to the Hitchin equations for $(\psi_{\sigma}(E), \psi_{\sigma}(\varphi))$. As a consequence of this, the involution of $\mathcal{M}^{\alpha}(G)$ is in fact an isometry for the natural Kähler metric defined on $\mathcal{M}^{\alpha}(G)$ by solving the Hitchin equations (see [18]).

If a Hermitian group G has no complexification and is not of adjoint type we can consider the adjoint group $\operatorname{Ad}(G)$ and apply to it the previous results. It is not clear whether the automorphisms of $\operatorname{Ad}(G)$ obtained from conjugations of M = G/H can be lifted to G to deduce a result similar to Proposition 4.13. However, if we consider the map $\mathcal{M}^{\alpha}(G) \to \mathcal{M}^{\alpha}(\operatorname{Ad}(G))$ and now consider the involution $\mathcal{M}^{\alpha}(\operatorname{Ad}(G)) \to \mathcal{M}^{\alpha}(\operatorname{Ad}(G))$ defined by a ψ_{σ} as above, it is very plausible that this can be lifted in a compatible way to an involution of $\mathcal{M}^{\alpha}(G)$. For this, we note that the class in $H^{2}(X, Z(G))$ of the $H^{\mathbb{C}}/Z(G)$ -bundle E' associated to an $H^{\mathbb{C}}$ -bundle E is trivial, and the same is true for $\psi_{\sigma}(E')$.

5. HERMITIAN GROUPS OF TUBE TYPE AND CAYLEY CORRESPONDENCE

In this section we assume that G is a connected, non-compact, real simple Hermitian Lie group of tube type with finite centre (see Section 2.1 for definition) with a fixed maximal compact subgroup $H \subset G$, and X is a compact Riemann surface. We consider the stability parameter α to be 0, and, as above, we refer to 0-stability of a G-Higgs bundle over X simply as stability (analogously for semistability and polystability). In this case the Milnor-Wood inequality for the Toledo invariant of a G-Higgs bundle is given by Theorem 4.8. We define a polystable Higgs bundle (E,φ) to be **maximal** if its Toledo invariant τ attains one of the bounds of the inequality i.e., $\tau = \pm r(2g-2)$, where r = rk(G/H). We denote $\tau_{\text{max}} = \text{rk}(G/H)(2g-2)$.

Let H^* be the non-compact dual of H as defined in Definition 2.1. In this section we establish a bijective correspondence between maximal G-Higgs bundles over X and K^2 -twisted H^* -Higgs bundles over X, as defined in Remark 3.7, where K^2 is the square of the canonical line bundle.

Suppose that (E,φ) is a polystable maximal G-Higgs bundle, and choose for example $\tau=-r(2g-2)$. By Theorem 4.8, the field φ^+ has rank r at each point. Let $Z_0^{\mathbb{C}}\simeq\mathbb{C}^*$ be the connected component of the identity of the center of $H^{\mathbb{C}}$. There is an exact sequence

$$(23) 1 \longrightarrow Z_0^{\mathbb{C}} \longrightarrow H^{\mathbb{C}} \longrightarrow H^{\mathbb{C}}/Z_0^{\mathbb{C}} \longrightarrow 1,$$

so there is an action of $Z_0^{\mathbb{C}}$ -bundles on $H^{\mathbb{C}}$ -bundles that we will denote by \otimes : in this way, if κ is a line bundle over X, we can define $E \otimes \kappa$ (here we are identifying the line bundle κ with its corresponding \mathbb{C}^* -bundle).

Lemma 5.1. If κ is an o_J -root of K, where o_J is the order of $e^{2\pi J}$, then φ^+ defines a reduction of the $H^{\mathbb{C}}$ -bundle $E \otimes \kappa$ to the group $H'^{\mathbb{C}}$.

Proof. Recall that an infinitesimal generator of $Z_0^{\mathbb{C}}$ is $-io_J J$, so

(24)
$$(E \otimes \kappa)(\mathfrak{m}^+) = E(\mathfrak{m}^+) \otimes \kappa^{o_J} = E(\mathfrak{m}^+) \otimes K.$$

Therefore, φ^+ is a section of $(E \otimes \kappa)(\mathfrak{m}^+)$, and since it has rank r at each point, its stabilizer at each point is isomorphic to $H'^{\mathbb{C}}$. Therefore φ^+ defines a reduction of the structure of $E \otimes \kappa$ to $H'^{\mathbb{C}}$.

Of course, such κ exists only if o_J divides 2g-2. We will now suppose that κ exists and is fixed. Denote by E' the reduction of $E \otimes \kappa$ to $H'^{\mathbb{C}}$. As we have seen, $\varphi^+ \in H^0(X, E'(\mathfrak{m}^+))$, and similarly $\varphi^- \in H^0(X, E'(\mathfrak{m}^-) \otimes K^2)$. From Lemma 2.2, we have an isomorphism

(25)
$$\operatorname{ad} \varphi^{+}: E'(\mathfrak{m}^{-}) \longrightarrow E'(\mathfrak{m}'^{\mathbb{C}}),$$

so that we can define a Higgs field

$$\varphi' = [\varphi^+, \varphi^-] \in H^0(X, E'(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2).$$

The data (E', φ') is a K^2 -twisted H^* -Higgs bundle.

Conversely, from a K^2 -twisted H^* -Higgs bundle (E', φ') we can reconstruct (E, φ) in the following way. The bundle is $E = E' \otimes \kappa^{-1}$. Observe that for the $H'^{\mathbb{C}}$ -bundle E' we have canonical section $e_{\Gamma} \in H^0(X, E'(\mathfrak{m}^+))$ corresponding to the element $e_{\Gamma} \in \mathfrak{m}^+$ fixed by H', which becomes by (24) a section $\varphi^+ \in H^0(X, E(\mathfrak{m}^+) \otimes K)$. Finally, φ^- is reconstructed from (25) as $(\operatorname{ad} \varphi^+)^{-1}(\varphi')$. Therefore, κ being fixed, we obtain a complete correspondence between maximal G-Higgs bundles and K^2 -twisted H^* -Higgs bundles. We refer to (E', φ') as the **Cayley partner** of (E, φ) . The main result of this section consists in showing that this correspondence preserves stability.

Theorem 5.2 (Cayley correspondence). Let G be a connected non-compact real simple Hermitian Lie group of tube type with finite centre. Let H be a maximal compact subgroup of G and H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element in \mathfrak{z} (the centre of \mathfrak{h}) defining the almost complex structure on \mathfrak{m} . If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides (2g-2), then there is an isomorphism of complex algebraic varieties

(26)
$$\mathcal{M}_{\max}(G) \cong \mathcal{M}_{K^2}(H^*)$$

given by $(E, \varphi) \mapsto (E', \varphi')$ as above.

Remark 5.3. The condition $o_J|(2g-2)$ is always satisfied for a group of adjoint type, since in this case $o_J = 1$. Table 2 shows that the o_J divides (2g-2) for the classical and exceptional groups. This may not happen for coverings of these groups, where o_J may be bigger.

The rest of this section is devoted to the proof of Theorem 5.2. We begin by the easy direction.

Lemma 5.4. Let (E, φ) be a maximal G-Higgs bundle, and let (E', φ') be the corresponding K^2 -twisted H^* -Higgs bundle. Suppose (E, φ) is (poly, semi)stable, then (E', φ') is (poly, semi)stable.

Proof. Suppose that we have a reduction $(E'_{P'_s}, \sigma')$ of the structure group of E' to the parabolic P'_s defined by a $s \in i\mathfrak{h}'$, such that φ' takes values in $E'_{P'_s}(\mathfrak{m}'_s) \otimes K^2$. Since $\mathfrak{h}' \subset \mathfrak{h}$, the element s defines a parabolic subgroup $P \subset H^{\mathbb{C}}$ as well, and using the map $H'^{\mathbb{C}}/P'_s \to H^{\mathbb{C}}/P_s$ we obtain from σ' a reduction σ of the structure group of E to P_s , resulting in a P_s -bundle E_P . Since $s \in i\mathfrak{h}'$ stabilizes e_Γ , we have $e_\Gamma \in \mathfrak{m}_s^{+,0}$, which translates into the fact that φ^+ takes values in $E_{P_s}(\mathfrak{m}_s^{+,0}) \otimes K$. For the same reason, the map $\mathrm{ad}(e_\Gamma)$ from Lemma 2.2 sends \mathfrak{m}_s^- to $\mathfrak{m}_s'^{\mathbb{C}}$, so we get that φ^- takes values in $E(\mathfrak{m}_s^-) \otimes K$.

Therefore, from the reduction $(E'_{P'_s}, \sigma')$ we constructed a reduction (E_{P_s}, σ) such that φ takes values in $E_{P_s}(\mathfrak{m}_s^{\mathbb{C}} \otimes K)$. By polystability of (E, φ) , one has $\deg(E)(\sigma, s) \geq 0$. Now, since $s \in i\mathfrak{h}'$ and $J \in i\mathfrak{m}'$ we have that $\langle s, iJ \rangle = 0$ and hence in the computation of $\deg E(\sigma, s)$ given by (8) there is no contribution coming from the twisting $E' = E \otimes \kappa$, so that

(27)
$$\deg(E)(\sigma, s) = \deg(E')(\sigma', s).$$

It follows that the (poly, semi)stability of (E, φ) implies the (poly, semi)stability of (E', φ') . For polystability, one must just check additionally that in the equality case, reduction for the Levi subgroup $L_s \subset P_s$ to a $E_{L_s} \subset E$ implies reduction for the Levi subgroup $L'_s \subset P'_s$, but it is sufficient to take $E_{L'_s} = (E_{L_s} \otimes \kappa) \cap E'$.

The other direction is more difficult. We begin by the following construction: given a G-Higgs bundle (E,φ) , we can associate to it a K^2 -twisted $H^{\mathbb{C}}$ -Higgs bundle given by $(E, [\varphi^+, \varphi^-])$, where $[\varphi^+, \varphi^-] \in H^0(X, E(\mathfrak{h}^{\mathbb{C}}) \otimes K^2)$ is defined using the Lie bracket on $\mathfrak{h}^{\mathbb{C}}$ combined with the tensor product of K with itself. The strategy to prove the Theorem will be to show that if (E,φ) is maximal and (E',φ') is its Cayley partner, then the polystability of (E',φ') implies the α -polystability of $(E, [\varphi^+,\varphi^-])$, where α is determined by the topology of E. In turn, to prove this, we will use the correspondence between polystability of the Higgs bundles involved with solutions to the corresponding Hitchin equations given by Theorem 3.11 (see Remark 3.12 in relation to the K^2 -twisting). To complete the proof we will show that the α -polystability of $(E, [\varphi^+, \varphi^-])$, implies the polystability of (E, φ) . We prove these various steps separately, so that the Theorem is a consequence of Lemmas 5.5 and 5.7. Again we restrict to the case where φ^+ is regular.

Lemma 5.5. Let (E, φ) be a maximal G-Higgs bundle, and let (E', φ') be the associated K^2 -twisted H^* -Higgs bundle. If (E', φ') is polystable then the K^2 -twisted $H^{\mathbb{C}}$ -Higgs bundle $(E, [\varphi^+, \varphi^-])$ is -iJ-polystable, where $J \in \mathfrak{z}(\mathfrak{h})$ is the element defining the complex structure of \mathfrak{m} .

Proof. By Theorem 3.11 (see also Remark 3.12) the polystability of (E', φ') implies the existence of a metric on E', that is a smooth section h' of $E'(H'^{\mathbb{C}}/H')$, satisfying the equation

(28)
$$F_{h'} - [\varphi', \tau_{h'}(\varphi')]\omega = 0.$$

The choice of the fixed Kähler form ω on X is not really important here, and to simplify notations we will choose the Kähler form of the hyperbolic metric, so that $F_K = -i\omega$. We use ω to define metrics on all powers of K, especially that on K^2 used to define τ'_h (see Remark 3.12).

Out of the metric h' and the metric on κ we obtain a metric h on $E = E' \otimes \kappa^{-1}$; since in (23) the infinitesimal generator of $Z_0^{\mathbb{C}}$ is $o_J J$, we obtain

$$F_h = F_{h'} - \frac{1}{i} o_J F_L J = F_{h'} + i F_K J = F_{h'} + \omega J.$$

Using the Hermite-Einstein equation (28) and the identity

$$[\varphi', \tau_{h'}(\varphi')] = [\varphi', \tau_h(\varphi')] = [[\varphi^+, \varphi^-], \tau_h([\varphi^+, \varphi^-])],$$

we therefore obtain

$$F_h - [[\varphi^+, \varphi^-], \tau_h([\varphi^+, \varphi^-])] = \omega J.$$

The lemma follows.

To relate the stability of a G-Higgs bundle $(E, \varphi^+, \varphi^-)$ with the stability of the corresponding $H^{\mathbb{C}}$ -Higgs bundle $(E, [\varphi^+, \varphi^-])$ we need the following result from GIT.

Lemma 5.6. If $\varphi^+ \in \mathfrak{m}_{reg}^+$, and for some $s \in i\mathfrak{h}$ we have $\varphi^+ \in \mathfrak{m}_s^+$, then $\langle -iJ, s \rangle \geq 0$, and if equality holds then $\varphi^+ \in \mathfrak{m}_s^{+,0}$.

Proof. First, define $h = [e_{\Gamma}, \tau(e_{\Gamma})] = h_{\gamma_1} + \cdots + h_{\gamma_r}$. In the tube case, h is an element of the center of \mathfrak{h} , and therefore is a multiple of iJ. Since for any strongly orthogonal root one has $\gamma(h) = 2$ and $\gamma(J) = i$, it follows that

$$h = -2iJ.$$

Observe that the adjoint action of H on \mathfrak{m}^+ with its standard flat symplectic structure is Hamiltonian, with moment map μ satisfying $i\mu(\varphi^+) = [\varphi^+, \tau(\varphi^+)]$, where τ here is the compact conjugation defining H. So in particular, for $\varphi^+ = e_{\Gamma}$, we have

$$(29) i\mu(\varphi^+) = -2iJ.$$

In particular e_{Γ} , and therefore the whole $H^{\mathbb{C}}$ -orbit of e_{Γ} , is polystable for the problem (29). Therefore all regular elements of \mathfrak{m}^+ are polystable. We leave to the reader as an exercice in finite dimensional GIT theory that the polystability condition is exactly the conclusion of the lemma.

Lemma 5.7. Let (E, φ) be a G-Higgs bundle with Toledo invariant τ such that φ^+ is generically regular (in particular if (E, φ) is maximal). Let $(E, [\varphi^+, \varphi^-])$ be the associated $H^{\mathbb{C}}$ -Higgs bundle. If $(E, [\varphi^+, \varphi^-])$ is -iJ-semistable (resp. polystable, stable), then (E, φ) is semistable (resp. polystable, stable).

Proof. We must analyse the degree of a reduction σ of E to E_{P_s} for a parabolic subgroup P_s such that $\varphi \in H^0(X, E_{P_s}(\mathfrak{m}_s^{\mathbb{C}}) \otimes K)$. This means that both φ^{\pm} are sections of $E_{P_s}(\mathfrak{m}_s^{\mathbb{C}}) \otimes K$, so their bracket $[\varphi^+, \varphi^-]$ is a section of $E_{P_s}(\mathfrak{m}_s^{\mathbb{C}}) \otimes K^2$. So if $(E, [\varphi^+, \varphi^-])$ is -iJ-semistable we get

(30)
$$\det E(\sigma, s) \ge \langle -iJ, s \rangle.$$

Since φ^+ lies in $E_{P_s}(\mathfrak{m}_s^+) \otimes K$, we can apply Lemma 5.6 to deduce that $\langle -iJ, s \rangle \geq 0$. Therefore -iJ-semistability of $(E, [\varphi^+, \varphi^-])$ implies semistability of (E, φ) . Also the implication for stability follows immediately, since strict inequality in (30) implies $\det E(\sigma, s) > 0$.

For polystability, observe that the equality $\deg E(\sigma,s)=0$ implies $\langle iJ,s\rangle=0$ and the equality in (30). Therefore $(E,[\varphi^+,\varphi^-])$ reduces to a Levi subgroup $L_s\subset P_s$ and, again from Lemma 5.6, φ^+ takes values in $E_{L_s}(\mathfrak{m}_s^{+,0})\otimes K$). From the isomorphism (25), the condition that $[\varphi^+,\varphi^-]$ lies in $E_{L_s}(\mathfrak{m}_s^{-,0})\otimes K^2$ implies that φ^- lies in $E_{L_s}(\mathfrak{m}_s^{-,0})\otimes K$). Therefore (E,φ) also reduces to the Levi subgroup L_s .

This lemma finishes the proof of Theorem 5.2. See Table 4 for the case of irreducible tube-type Hermitian symmetric spaces G/H.

For the classical groups, Theorem 5.2 was proved in [27, 24, 3, 4, 6, 19], where it is sometimes referred to as the Cayley correspondence, inspired by the fact that the symmetric space G/H is realized as a tube domain via the Cayley transform described in Section 2.1.

This result is interpreted as a rigidity result for Higgs bundles since the structure group of the K^2 twisted H^* -Higgs bundles is smaller and reveals new invariants coming from the group H^* . For example, when $G = \operatorname{Sp}(2n, \mathbb{R})$, we have $H^* = \operatorname{GL}(n, \mathbb{R})$ with $H' = \operatorname{O}(n)$ as a maximal compact subgroup. To a $\operatorname{Sp}(2n, \mathbb{R})$ -Higgs bundle we can thus attach the first and second Stiefel-Whitney classes of the principal $\operatorname{O}(n, \mathbb{C})$ -bundle given by the corresponding $\operatorname{GL}(n, \mathbb{R})$ -Higgs pair via the Cayley correspondence. In general, similar invariants may come from the non-connectedness and non-simply connectedness of H^* .

In the previous case by case proofs many of the geometrical ingredients were identified but not explicitly used. Moreover, our result generalizes the work for the classical groups in two ways. First, by considering quotients and coverings of the classical groups, even though they may not be matrix groups. And second, by including the exceptional case, stated as follows.

Theorem 5.8. The Cayley correspondence defines an isomorphism of complex algebraic varieties

$$\mathcal{M}_{\mathrm{max}}(\mathrm{E}_7^{-25}) \cong \mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*)$$

A maximal compact subgroup of $H^* = E_6^{-26} \ltimes \mathbb{R}^*$ is given by $H' = F_4 \times \mathbb{Z}_2$. Since $H'^{\mathbb{C}}$ is non-connected, we consider the short exact sequence $1 \to H_0'^{\mathbb{C}} \to H'^{\mathbb{C}} \to \pi_0(H'^{\mathbb{C}}) \cong \mathbb{Z}_2 \to 1$ and the following homomorphism of its induced long exact sequence in cohomology,

$$H^1(X, H'^{\mathbb{C}}) \to H^1(X, \pi_0(H'^{\mathbb{C}})) \cong \mathbb{Z}_2^{2g}.$$

This map associates an invariant in \mathbb{Z}_2^{2g} to any K^2 -twisted H^* -Higgs bundle, and hence to any G-Higgs bundle. This implies that $\mathcal{M}_{K^2}(\mathbb{E}_6^{-26} \ltimes \mathbb{R}^*)$, and hence $\mathcal{M}_{\max}(\mathbb{E}_7^{-25})$, has at least 2^{2g} connected components.

The Cayley correspondence can be adapted to L-twisted G-Higgs bundles as follows. First, a version of the inequality of Milnor-Wood for an L-twisted Higgs bundle gives a

different bound $|\tau| \leq \operatorname{rk}(G/H) \operatorname{deg} L$. Let $\mathcal{M}_{L,\max}(G)$ be the moduli space of L-twisted G-Higgs

Theorem 5.9. Let G be a connected non-compact real simple Hermitian Lie group of tube type with finite centre. Let H be a maximal compact subgroup of G and H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element in the centre of the Lie algebra \mathfrak{g} defining the almost complex structure on \mathfrak{m} . If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides $\deg L$, then there is an isomorphism of complex algebraic varieties

(31)
$$\mathcal{M}_{L,\max}(G) \cong \mathcal{M}_{L^2}(H^*).$$

6. HERMITIAN GROUPS OF NON-TUBE TYPE AND RIGIDITY

In this section we describe a rigidity phenomenon of maximal G-Higgs bundles for groups of non-tube type.

Theorem 6.1. Let G be a simple Hermitian group of non-tube type and let H be its maximal compact subgroup. Then, there are no stable G-Higgs bundles with maximal Toledo invariant. In fact, every polystable maximal G-Higgs bundle reduces to a stable $N_G(\mathfrak{g}_T)_0$ -Higgs bundle, where $N_G(\mathfrak{g}_T)_0$ is the identity component of the normalizer of \mathfrak{g}_T in G, and \mathfrak{g}_T is the maximal tube subalgebra of \mathfrak{g} .

Proof. Let $(E, \varphi^+, \varphi^-)$ be a polystable G-Higgs bundle. Suppose that the Toledo invariant τ is maximal. We assume that it is negative, $\tau = -r(2g-2)$ where $r = \operatorname{rk}(G/H)$, without loss of generality. Then, by Theorem 4.5, $\operatorname{rk}(\varphi^+) = \operatorname{rk}(G/H) = r$. We define $P_r \subset H^{\mathbb{C}}$, σ and χ as in the proof of Theorem 4.5. Since the Toledo invariant is maximal, $\deg(E)(\sigma,\chi) = 0$, and the polystability condition yields a reduction of E to a Levi subgroup $E \subset P_r$, and the condition $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$. By definition of $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$ are the rad $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$. By definition of $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$ is easily seen to be also $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$. Similarly $F \in H^0(X,E(\mathfrak{m}^0_{s_\chi})\otimes K)$.

From Section 2.1, $\tilde{\mathfrak{g}}_T = \tilde{\mathfrak{h}}_T + \mathfrak{m}_T$ is the Lie algebra of the group $N_G(\mathfrak{g}_T)_0$, so the previous observations can be summarized by saying that (E,φ) reduces to a $N_G(\mathfrak{g}_T)_0$ -Higgs bundle. Again from Section 2.1, note that the maximal compact subgroup of $N_G(\mathfrak{g}_T)_0$ is $N_H(\mathfrak{h}_T)_0$, with Lie algebra $\tilde{\mathfrak{h}}_T$.

Note that in the tube case, the argument of Theorem 6.1 does not work since the parabolic subgroup given by Theorem 4.5 is the whole group $H^{\mathbb{C}}$ and hence, there is no reduction of the structure group.

Theorem 6.1 was proved in [4, 6] for the classical groups. This general approach extends the result to quotients and coverings and to exceptional groups. For example, every maximal E_6^{-14} -Higgs bundle is strictly polystable and reduces to a stable $\mathrm{Spin}_0(2,8) \times_{\mathbb{Z}_4} \mathrm{U}(1)$ -Higgs bundle. From the point of view of representations similar results were proved in [48, 26, 9].

Continue to consider the maximal case studied in Theorem 6.1. In the group G_T of the subtube, we have the central subgroup $D' = Z(G_T) \cap H_T$, the adjoint group $G_T^{Ad} = G_T/D'$, and its maximal compact subgroup $H_T^{Ad} = H_T/D'$. As already seen in

the proof of Theorem 4.5, we have actually

$$H_T^{\mathrm{Ad}} = N_H(\mathfrak{h}_T)_0/L', \quad L' = N_H(\mathfrak{h}_T)_0 \cap \ker \operatorname{Ad}|_{\mathfrak{m}_T}.$$

Therefore a $N_G(\mathfrak{g}_T)_0$ -Higgs bundle gives a G_T^{Ad} -Higgs bundle at the quotient. From this we have the following.

Theorem 6.2. Under the hypotheses of Theorem 6.1, the maximal representations for G and for the subtube G_T are related by a fibration of complex algebraic varieties,

$$\mathcal{M}_0(L') \longrightarrow \mathcal{M}_{\max}(G) \longrightarrow \mathcal{M}_{\max}(G_T^{\mathrm{Ad}}).$$

Proof. From Theorem 6.1, we can replace maximal G-Higgs bundles by $N_G(\mathfrak{g}_T)_0$ -Higgs bundles. Then we have already seen the arrow $\mathcal{M}_{\max}(G) \longrightarrow \mathcal{M}_{\max}(G_T^{\mathrm{Ad}})$. Moreover we have the exact sequence $1 \to L' \to N_G(\mathfrak{g}_T)_0 \to G_T^{\mathrm{Ad}} \to 1$. But, since there is a direct sum decomposition $\tilde{\mathfrak{g}}_T = \mathfrak{g}_T \oplus \mathfrak{l}'$, actually

$$N_G(\mathfrak{g}_T)_0 = L' \times_{D'} G_T.$$

Therefore, starting from a maximal G_T^{Ad} -Higgs bundle, we can lift it into a G_T -Higgs bundle, the ambiguity being a line bundle M such that $M^{|D'|}=1$. Giving a L'-bundle then gives by product a $L'\times G_T^{\mathrm{Ad}}$ -Higgs bundle, and passing to the quotient by D' a $N_G(G_T)_0$ -Higgs bundle (and the ambiguity on the lift is killed by this quotient).

Observe that by maximality, the Toledo invariant of the G_T^{Ad} -Higgs bundle and the Toledo invariant of the G-bundle are the same, which implies that the Toledo invariant of the L'-bundle vanishes, hence the index '0'.

Finally observe that the direct sum $\tilde{\mathfrak{g}}_T = \mathfrak{g}_T \oplus \mathfrak{l}'$ implies that the Hermite-Einstein equation passes to the L' and G_T^{Ad} -bundles, hence polystability is preserved. The fibration is proved.

We give two applications of the previous theorem. For p < q one has

$$\mathcal{M}_0(\mathrm{U}_{q-p} \ltimes \mathbb{Z}_{2p}) \longrightarrow \mathcal{M}_{\mathrm{max}}(\mathrm{SU}(p,q)) \longrightarrow \mathcal{M}_{\mathrm{max}}(\mathrm{PU}(p,p)).$$

Since both the basis and the fibres are connected [2, 21, 3], this gives a simpler proof of the connectedness of $\mathcal{M}_{\text{max}}(SU(p,q))$ than in [4].

The second example is the above mentioned case of E₆: we have

$$\mathcal{M}_0(\mathrm{U}_1) \longrightarrow \mathcal{M}_{\mathrm{max}}(\mathrm{E}_6^{-14}) \longrightarrow \mathcal{M}_{\mathrm{max}}(\mathrm{PSO}_o(2,8)).$$

So the number of connected components of $\mathcal{M}_{\max}(E_6^{-14})$ equals that of $\mathcal{M}_{\max}(PSO_o(2,8))$. It is known [4] that the space $\mathcal{M}_{\max}(SO_0(2,8))$ has 2^{2g+1} components, but the space $\mathcal{M}_{\max}(PSO_o(2,8))$ may have other components.

APPENDIX A. TABLES

We use the following notation for Table 1:

- Δ_{10}^{\pm} are the half-spinor representations of the group Spin(10, \mathbb{C}). They are 16-dimensional.
- M and M^* are the irreducible 27-dimensional representations of E_6 , which are dual to each other.
- η^r is the representation $\eta^r: \mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto z^r$.

| G | Н | $H^{\mathbb{C}}$ | $\mathfrak{m}^\mathbb{C}=\mathfrak{m}^++\mathfrak{m}^-$ |
|------------------------------------|---|---|---|
| SU(p,q) | $S(U(p) \times U(q))$ | $S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$ | $\operatorname{Hom}(\mathbb{C}^q,\mathbb{C}^p) + \operatorname{Hom}(\mathbb{C}^p,\mathbb{C}^q)$ |
| $\operatorname{Sp}(2n,\mathbb{R})$ | U(n) | $\mathrm{GL}(n,\mathbb{C})$ | $S^2(\mathbb{C}^n) + S^2(\mathbb{C}^{n*})$ |
| $SO^*(2n)$ | $\mathrm{U}(n)$ | $\mathrm{GL}(n,\mathbb{C})$ | $\Lambda^2(\mathbb{C}^n) + \Lambda^2(\mathbb{C}^{n*})$ |
| $SO_0(2,n)$ | $SO(2) \times SO(n)$ | $SO(2,\mathbb{C}) \times SO(n,\mathbb{C})$ | $\operatorname{Hom}(\mathbb{C}^n,\mathbb{C}) + \operatorname{Hom}(\mathbb{C},\mathbb{C}^n)$ |
| E_6^{-14} | $\operatorname{Spin}(10) \times_{\mathbb{Z}_4} \operatorname{U}(1)$ | $\operatorname{Spin}(10,\mathbb{C}) \times_{\mathbb{Z}_4} \mathbb{C}^*$ | $\Delta_{10}^+ \otimes \eta^3 + \Delta_{10}^- \otimes \eta^{-3}$ |
| E_{7}^{-25} | $E_6^{-78} \times_{\mathbb{Z}_3} U(1)$ | $E_6 \times_{\mathbb{Z}_3} \mathbb{C}^*$ | $M\otimes \eta^2 + M^*\otimes \eta^{-2}$ |

Table 1. Irreducible Hermitian symmetric spaces ${\cal G}/{\cal H}$

| G | Н | N | $\dim \mathfrak{m}$ | ℓ | $o(e^{2\pi J})$ | q_T |
|------------------------------------|---|--------|---------------------|----------|-------------------------|----------|
| SU(p,q) | $S(U(p) \times U(q))$ | p+q | 2pq | lcm(p,q) | $\frac{p+q}{\gcd(p,q)}$ | 1/2 |
| $\operatorname{Sp}(2n,\mathbb{R})$ | U(n) | n+1 | n(n+1) | n | 2 | $1/_{2}$ |
| $SO^*(2n)$ | U(n) | 2(n-1) | n(n-1) | n | 2 | 1 |
| $SO_0(2,n)$ | $SO(2) \times SO(n)$ | n | 2n | 1 | 1 | 1/2 |
| E_6^{-14} | $\operatorname{Spin}(10) \times_{\mathbb{Z}_4} \operatorname{U}(1)$ | 12 | 32 | 4 | 3 | 1/2 |
| E_7^{-25} | $E_6^{-78} \times_{\mathbb{Z}_3} U(1)$ | 18 | 54 | 3 | 2 | 1/2 |

Table 2. Toledo character data for the classical and exceptional groups

| G | Н | N | $\dim \mathfrak{m}$ | ℓ | $o(e^{2\pi J})$ | q_T |
|--------------------------|---|--------|---------------------|----------|-----------------|-------------------------|
| PSU(p,q) | $\mathrm{PS}(\mathrm{U}(p) \times \mathrm{U}(q))$ | p+q | 2pq | gcd(p,q) | 1 | $\frac{p+q}{2lcm(p,q)}$ |
| $PSO^*(2n = 4m + 2)$ | U(n) | 2(n-1) | n(n-1) | n | 1 | 2 |
| E_6^{-14}/\mathbb{Z}_3 | $\mathrm{Spin}(10) \times_{\mathbb{Z}_4} \mathrm{U}(1)$ | 12 | 32 | 4 | 1 | 3/2 |

Table 3. Toledo character data for adjoint groups of non-tube type.

| G | Н | H^* | H' | $\check{S} = H/H'$ | \mathfrak{m}' | $\mathfrak{m}'^{\mathbb{C}}$ |
|------------------------------------|--|---|---|---|--------------------------------------|--|
| SU(n,n) | $S(U(n) \times U(n))$ | $\{A \in \operatorname{GL}(n,\mathbb{C}) \mid \det(A)^2 \in \mathbb{R}^+\}$ | $ \{A \in \mathrm{U}(n) \mid \det(A)^2 = 1 \} $ | U(n) | $\operatorname{Herm}(n,\mathbb{C})$ | $\mathrm{Mat}(n,\mathbb{C})$ |
| $\operatorname{Sp}(2n,\mathbb{R})$ | U(n) | $\mathrm{GL}(n,\mathbb{R})$ | O(n) | $\mathrm{U}(n)/\mathrm{O}(n)$ | $\operatorname{Sym}(n,\mathbb{R})$ | $\operatorname{Sym}(n,\mathbb{C})$ |
| $SO^*(2n)$ $n = 2m$ | U(n) | $\mathrm{U}^*(n)$ | $\operatorname{Sp}(n)$ | $U(n)/\mathrm{Sp}(n)$ | $\operatorname{Herm}(m,\mathbb{H})$ | $\mathrm{Skew}(n,\mathbb{C})$ |
| $SO_0(2,n)$ | $SO(2) \times SO(n)$ | $SO_0(1,1) \times SO(1,n-1)$ | O(n-1) | $\frac{\mathrm{U}(1)\times S^{n-1}}{\mathbb{Z}_2}$ | $\mathbb{R} \times \mathbb{R}^{n-1}$ | $\mathbb{C} \times \mathbb{C}^{n-1}$ |
| E_7^{-25} | $E_6^{-78} \times_{\mathbb{Z}_3} U(1)$ | $\mathrm{E}_6^{-26}\ltimes\mathbb{R}^*$ | $F_4 \times \mathbb{Z}_2$ | $\frac{\mathrm{E}_6^{-78}\cdot\mathrm{U}(1)}{\mathrm{F}_4}$ | $\operatorname{Herm}(3,\mathbb{O})$ | $\operatorname{Herm}(3,\mathbb{O})\otimes\mathbb{C}$ |

Table 4. Irreducible Hermitian symmetric spaces G/H of tube type

| G | Н | G_T | L' |
|----------------|---|------------------------------|----------------------------------|
| SU(p,q), p < q | $S(U(p) \times U(q))$ | SU(p,p) | $U(q-p) \ltimes \mathbb{Z}_{2p}$ |
| $SO^*(4m+2)$ | U(2m+1) | $SO^*(4m)$ | U(1) |
| E_6^{-14} | $\operatorname{Spin}(10) \times_{\mathbb{Z}_4} \operatorname{U}(1)$ | $\operatorname{Spin}_0(2,8)$ | U(1) |

Table 5. Irreducible Hermitian symmetric spaces G/H of non-tube type

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