

Square-free pairs $n^2 + n + 1$, $n^2 + n + 2$

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Abstract

In this paper we prove that there exist infinitely many square-free pairs of the form $n^2 + n + 1$, $n^2 + n + 2$. We also establish an asymptotic formula for the number of such square-free pairs when n does not exceed given sufficiently large positive number. A key point in our proof is the application of bijective correspondence between the number of representations of number by binary quadratic form and the incongruent solutions of quadratic congruence.

Keywords: Square-free numbers, Asymptotic formula, Bijective function.

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1 Notations

Let X be a sufficiently large positive number. The letter ε denotes an arbitrary small positive number, not the same in all appearances. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t . Further $\mu(n)$ is Möbius' function and $\tau(n)$ denotes the number of positive divisors of n . Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. The relation $f(x) \ll g(x)$ means that $f(x) = \mathcal{O}(g(x))$. Moreover (m, n) is the greatest common divisor of m and n . The letter p always denotes a prime number. We write $e(t) = \exp(2\pi it)$ and $\psi(t) = \{t\} - 1/2$. For $x, y \in \mathbb{R}$ we write $x \equiv y(1)$ when $x - y \in \mathbb{Z}$. For any n and q such that $(n, q) = 1$ we denote by \bar{n}_q the inverse of n modulo q . By $G(q, m, n)$ we shall denote the Gauss sum

$$G(q, m, n) = \sum_{x=1}^q e\left(\frac{mx^2 + nx}{q}\right). \quad (1)$$

By $K(r, h)$ we shall denote the incomplete Kloosterman sum

$$K(r, h) = \sum_{\substack{\alpha \leq x < \beta \\ (x, r) = 1}} e\left(\frac{h\bar{x}_{|r|}}{r}\right), \quad (2)$$

where

$$h, r \in \mathbb{Z}, \quad hr \neq 0, \quad 0 < \beta - \alpha \leq 2|r|.$$

We also define

$$\lambda(q_1, q_2) = \sum_{\substack{1 \leq n \leq q_1 q_2 \\ n^2 + n + 1 \equiv 0 \pmod{q_1} \\ n^2 + n + 2 \equiv 0 \pmod{q_2}}} 1, \quad (3)$$

$$\Gamma(X) = \sum_{1 \leq n \leq X} \mu^2(n^2 + n + 1) \mu^2(n^2 + n + 2). \quad (4)$$

2 Introduction and statement of the result

We say that an integer $n \in \mathbb{N}$ is square-free if for any prime $p \mid n$, one has $p^2 \nmid n$. Information on the distribution of square-free numbers was given in 1885 by Gegenbauer [10]. He proved the following asymptotic formula

$$\sum_{n \leq X} \mu^2(n) = \frac{6}{\pi^2} X + \mathcal{O}\left(X^{\frac{1}{2}}\right). \quad (5)$$

Gegenbauer's argument is very simple, but despite the passage of 138 years the exponent $1/2$ appearing above has never been improved. The reminder term in (5) depends on the real part of the zeroes of the Riemann zeta function $\zeta(s)$. For this reason any reduction in the exponent $1/2$ would require the Riemann Hypothesis to be true. In 1932 Carlitz [5] showed that there exist infinitely many pairs of consecutive square-free numbers. More precisely he proved the asymptotic formula

$$\sum_{n \leq X} \mu^2(n) \mu^2(n+1) = \prod_p \left(1 - \frac{2}{p^2}\right) X + \mathcal{O}\left(X^{\frac{2}{3}+\varepsilon}\right). \quad (6)$$

Afterwards the reminder term of (6) was sharpened several times. Another interesting problem we know in number theory is square-free and consecutive square-free numbers of a special form. In this connection we first mention the theorems of Cao and Zhai [3], [4], concerning the study of square-free numbers of the form $[n^c]$. In 2020 the author [6] showed that there exist infinitely many consecutive square-free numbers of the form $x^2 + y^2 + 1$, $x^2 + y^2 + 2$. More precisely we proved that the asymptotic formula

$$\sum_{1 \leq x, y \leq H} \mu^2(x^2 + y^2 + 1) \mu^2(x^2 + y^2 + 2) = \prod_p \left(1 - \frac{\tilde{\lambda}(p^2, 1) + \tilde{\lambda}(1, p^2)}{p^4}\right) H^2 + \mathcal{O}\left(H^{\frac{8}{5}+\varepsilon}\right) \quad (7)$$

holds. Here

$$\tilde{\lambda}(q_1, q_2) = \sum_{\substack{1 \leq x, y \leq q_1 q_2 \\ x^2 + y^2 + 1 \equiv 0 \pmod{q_1} \\ x^2 + y^2 + 2 \equiv 0 \pmod{q_2}}} 1.$$

Subsequently the articles of B. Chen [1], Chen and Wang [2], the author [7], Fan and Zhai [9], Jing and Liu [13] and Zhou and Ding [16] continue the study of the representation of infinitely many square-free numbers by polynomials. Recently [8] the author showed that there exist infinitely many square-free numbers of the form $n^2 + n + 1$. More precisely we established the asymptotic formula

$$\sum_{1 \leq n \leq X} \mu^2(n^2 + n + 1) = cX + \mathcal{O}\left(X^{\frac{4}{5} + \varepsilon}\right),$$

where

$$c = \prod_p \left(1 - \frac{\lambda(p^2)}{p^2}\right)$$

and

$$\lambda(q) = \sum_{\substack{1 \leq n \leq q \\ n^2 + n + 1 \equiv 0 \pmod{q}}} 1.$$

Motivated by these investigations we prove the following theorem.

Theorem 1. *For the sum $\Gamma(X)$ defined by (4) the asymptotic formula*

$$\Gamma(X) = \sigma X + \mathcal{O}\left(X^{\frac{8}{9} + \varepsilon}\right) \tag{8}$$

holds. Here

$$\sigma = \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^2}\right). \tag{9}$$

From Theorem 1 it follows that there exist infinitely many consecutive square-free numbers of the form $n^2 + n + 1$, $n^2 + n + 2$, where n runs over naturals.

3 Lemmas

Lemma 1. *Let $(q_1, q_2) = 1$. Then for the Gauss sum denoted by (1) we have*

$$G(q_1 q_2, m_1 q_2 + m_2 q_1, n) = G(q_1, m_1 q_2^2, n) G(q_2, m_2 q_1^2, n).$$

Proof. See ([11], Theorem 5.1). □

Lemma 2. *For any $M \geq 2$, we have*

$$\psi(t) = - \sum_{1 \leq |m| \leq M} \frac{e(mt)}{2\pi im} + \mathcal{O}(f_M(t)),$$

where $f_M(t)$ is a positive function of t which is infinitely many times differentiable and periodic with period 1. It can be expanded into the Fourier series

$$f_M(t) = \sum_{m=-\infty}^{+\infty} b_M(m)e(mt),$$

with coefficients $b_M(m)$ such that

$$b_M(m) \ll \frac{\log M}{M} \quad \text{for all } m$$

and

$$\sum_{|m| > M^{1+\varepsilon}} |b_M(m)| \ll M^{-A}.$$

Here $A > 0$ is arbitrarily large and the constant in the \ll - symbol depends on A and ε .

Proof. See ([15], Theorem 1). □

Lemma 3. *Let $A, B \in \mathbb{Z} \setminus \{0\}$ and $(A, B) = 1$. Then*

$$\frac{\overline{A}_{|B|}}{B} + \frac{\overline{B}_{|A|}}{A} \equiv \frac{1}{AB} \pmod{1}.$$

Proof. The statement in the lemma is equivalent to

$$A\overline{A}_{|B|} + B\overline{B}_{|A|} \equiv 1 \pmod{|AB|}. \tag{10}$$

Bearing in mind (10) and $(A, B) = 1$ we derive the system

$$\begin{cases} A\overline{A}_{|B|} + B\overline{B}_{|A|} \equiv 1 \pmod{|A|} \\ A\overline{A}_{|B|} + B\overline{B}_{|A|} \equiv 1 \pmod{|B|} \end{cases}$$

which is equivalent to the system

$$\begin{cases} B\overline{B}_{|A|} \equiv 1 \pmod{|A|} \\ A\overline{A}_{|B|} \equiv 1 \pmod{|B|} \end{cases}.$$

Since the latter system is satisfied this proves the lemma. □

Lemma 4. For the sum denoted by (2) we have the estimate

$$K(r, h) \ll |r|^{\frac{1}{2}+\varepsilon} (r, h)^{\frac{1}{2}}$$

Proof. Follows easily from A. Weil's estimate for the Kloosterman sum. See ([12], Ch. 11, Corollary 11.12). \square

Lemma 5. The function $\lambda(q_1, q_2)$ defined by (3) is multiplicative, i.e. if

$$(q_1 q_2, q_3 q_4) = (q_1, q_2) = (q_3, q_4) = 1 \quad (11)$$

then

$$\lambda(q_1 q_2, q_3 q_4) = \lambda(q_1, q_3) \lambda(q_2, q_4).$$

Proof. On the one hand (1), (3), (11) and Lemma 1 imply

$$\begin{aligned} & \lambda(q_1 q_2, q_3 q_4) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{1 \leq n \leq q_1 q_2 q_3 q_4} \sum_{1 \leq h_1 \leq q_1 q_2} e\left(\frac{h_1(n^2 + n + 1)}{q_1 q_2}\right) \sum_{1 \leq h_2 \leq q_3 q_4} e\left(\frac{h_2(n^2 + n + 2)}{q_3 q_4}\right) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{1 \leq h_1 \leq q_1 q_2} e\left(\frac{h_1}{q_1 q_2}\right) \sum_{1 \leq h_2 \leq q_3 q_4} e\left(\frac{2h_2}{q_3 q_4}\right) \\ &\quad \times G(q_1 q_2 q_3 q_4, h_1 q_3 q_4 + h_2 q_1 q_2, h_1 q_3 q_4 + h_2 q_1 q_2) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{1 \leq h_1 \leq q_1 q_2} e\left(\frac{h_1}{q_1 q_2}\right) \sum_{1 \leq h_2 \leq q_3 q_4} e\left(\frac{2h_2}{q_3 q_4}\right) G(q_1 q_2, h_1 q_3^2 q_4^2, h_1 q_3 q_4 + h_2 q_1 q_2) \\ &\quad \times G(q_3 q_4, h_2 q_1^2 q_2^2, h_1 q_3 q_4 + h_2 q_1 q_2) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\ &\quad \times G(q_1 q_2, h_1 q_2 q_3^2 q_4^2 + h_2 q_1 q_3^2 q_4^2, h_1 q_2 q_3 q_4 + h_2 q_1 q_3 q_4) \\ &\quad \times G(q_3 q_4, h_3 q_1^2 q_2^2 q_4 + h_4 q_1^2 q_2^2 q_3, h_3 q_1 q_2 q_4 + h_4 q_1 q_2 q_3) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\ &\quad \times G(q_1, h_1 q_2^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4 + h_2 q_1 q_3 q_4) G(q_2, h_2 q_1^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4 + h_2 q_1 q_3 q_4) \\ &\quad \times G(q_3, h_3 q_1^2 q_2^2 q_4^2, h_3 q_1 q_2 q_4 + h_4 q_1 q_2 q_3) G(q_4, h_4 q_1^2 q_2^2 q_3^2, h_3 q_1 q_2 q_4 + h_4 q_1 q_2 q_3) \\ &= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) G(q_1, h_1 q_2^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4) G(q_2, h_2 q_1^2 q_3^2 q_4^2, h_2 q_1 q_3 q_4) \\ &\quad \times \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) G(q_3, h_3 q_1^2 q_2^2 q_4^2, h_3 q_1 q_2 q_4) G(q_4, h_4 q_1^2 q_2^2 q_3^2, h_4 q_1 q_2 q_3). \quad (12) \end{aligned}$$

On the other hand (1), (3) and Lemma 1 yield

$$\begin{aligned}
& \lambda(q_1, q_3)\lambda(q_2, q_4) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{1 \leq n \leq q_1 q_3} \sum_{1 \leq h_1 \leq q_1} e\left(\frac{h_1(n^2 + n + 1)}{q_1}\right) \sum_{1 \leq h_3 \leq q_3} e\left(\frac{h_3(n^2 + n + 2)}{q_3}\right) \\
&\times \sum_{1 \leq m \leq q_2 q_4} \sum_{1 \leq h_2 \leq q_2} e\left(\frac{h_2(m^2 + m + 1)}{q_2}\right) \sum_{1 \leq h_4 \leq q_4} e\left(\frac{h_4(m^2 + m + 2)}{q_4}\right) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\
&\times G(q_1 q_3, h_1 q_3 + h_3 q_1, h_1 q_3 + h_3 q_1) G(q_2 q_4, h_2 q_4 + h_4 q_2, h_2 q_4 + h_4 q_2) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\
&\times G(q_1, h_1 q_3^2, h_1 q_3 + h_3 q_1) G(q_3, h_3 q_1^2, h_1 q_3 + h_3 q_1) \\
&\times G(q_2, h_2 q_4^2, h_2 q_4 + h_4 q_2) G(q_4, h_4 q_2^2, h_2 q_4 + h_4 q_2) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) G(q_1, h_1 q_3^2, h_1 q_3) G(q_2, h_2 q_4^2, h_2 q_4) \\
&\times \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) G(q_3, h_3 q_1^2, h_3 q_1) G(q_4, h_4 q_2^2, h_4 q_2). \tag{13}
\end{aligned}$$

Using the substitution $n \rightarrow \overline{(q_2 q_4)}_{q_1 q_3} n$ we get

$$\begin{aligned}
G(q_1, h_1 q_2^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4) &= \sum_{n=1}^{q_1} e\left(\frac{h_1 q_2^2 q_3^2 q_4^2 n^2 + h_1 q_2 q_3 q_4 n}{q_1}\right) \\
&= \sum_{n=1}^{q_1} e\left(\frac{h_1 q_3^2 n^2 + h_1 q_3 n}{q_1}\right) = G(q_1, h_1 q_3^2, h_1 q_3). \tag{14}
\end{aligned}$$

Arguing in a similar way, we obtain

$$G(q_2, h_2 q_1^2 q_3^2 q_4^2, h_2 q_1 q_3 q_4) = G(q_2, h_2 q_4^2, h_2 q_4), \tag{15}$$

$$G(q_3, h_3 q_1^2 q_2^2 q_4^2, h_3 q_1 q_2 q_4) = G(q_3, h_3 q_1^2, h_3 q_1), \tag{16}$$

$$G(q_4, h_4 q_1^2 q_2^2 q_3^2, h_4 q_1 q_2 q_3) = G(q_4, h_4 q_2^2, h_4 q_2). \tag{17}$$

Summarizing (12) – (17) we complete the proof of the lemma. \square

The following lemma is the main weapon of the theorem.

Lemma 6. *Let $n \geq 5$. There exists a bijective function from the solution set of the equation*

$$x^2 + xy + 2y^2 = n, \quad (x, y) = 1, \quad x \in \mathbb{N}, \quad y \in \mathbb{Z} \setminus \{0\} \quad (18)$$

to the incongruent solutions modulo n of the congruence

$$z^2 + z + 2 \equiv 0 (n). \quad (19)$$

Proof. Let us denote by F the set of ordered pairs (x, y) satisfying (18) and by E the set of solutions of the congruence (19). Every residue class modulo n with representatives satisfying (19) will be considered as one solution of (19).

Let $(x, y) \in F$. By (18) we have that $(n, y) = 1$. Then there exists a unique residue class z modulo n such that

$$zy \equiv x (n). \quad (20)$$

For this class we write

$$(z^2 + z + 2)y^2 \equiv (zy)^2 + (zy)y + 2y^2 \equiv x^2 + xy + 2y^2 \equiv 0 (n).$$

The last congruence and $(n, y) = 1$ yield $z^2 + z + 2 \equiv 0 (n)$ that is $z \in E$. We define the map

$$\beta : F \rightarrow E \quad (21)$$

that associates to each pair $(x, y) \in F$ the residue class $z = x\bar{y}_n$ satisfying (20).

We will first prove that the map (21) is a injection. Let $(x, y), (x', y') \in F$ that is

$$\left| \begin{array}{l} x^2 + xy + 2y^2 = n \\ x'^2 + x'y' + 2y'^2 = n \end{array} \right. , \quad (22)$$

$$(x, y) = (x', y') = 1 \quad (23)$$

and

$$(x, y) \neq (x', y'). \quad (24)$$

Assume that

$$\beta(x, y) = \beta(x', y'). \quad (25)$$

Hence there exists $z \in E$ such that

$$\left| \begin{array}{l} zy \equiv x (n) \\ zy' \equiv x' (n) \end{array} \right. . \quad (26)$$

The system (26) yields

$$xy' - x'y \equiv 0 (n). \quad (27)$$

Since the discriminants of the quadratic equations in (22) must be nonnegative and $n \geq 5$ we derive

$$\begin{cases} 0 < x, x' \leq \sqrt{\frac{8n}{7}} \\ 0 < |y|, |y'| \leq \sqrt{\frac{4n}{7}} \end{cases} . \quad (28)$$

We first consider the case

$$yy' > 0. \quad (29)$$

By (28) it follows

$$\begin{cases} 0 < |xy'| < \frac{4\sqrt{2n}}{7} \\ 0 < |x'y| < \frac{4\sqrt{2n}}{7} \end{cases}$$

and bearing in mind (29) we obtain

$$-n < xy' - x'y < n. \quad (30)$$

Now (27) and (30) lead to

$$xy' - x'y = 0$$

which together with (23) gives us

$$x = x', \quad y = y'. \quad (31)$$

From (24) and (31) we get a contradiction.

Next we consider the case

$$yy' < 0. \quad (32)$$

By (22), (28) and (32) we deduce

$$\begin{cases} 0 < x \leq \sqrt{\frac{8n}{7}} \\ 0 < |y| \leq \sqrt{\frac{4n}{7}} \\ 0 < x' < \sqrt{n} \\ 0 < |y'| < \sqrt{\frac{n}{2}} \end{cases} \quad \text{or} \quad \begin{cases} 0 < x < \sqrt{n} \\ 0 < |y| < \sqrt{\frac{n}{2}} \\ 0 < x' \leq \sqrt{\frac{8n}{7}} \\ 0 < |y'| \leq \sqrt{\frac{4n}{7}} \end{cases} \quad (33)$$

and therefore

$$-2n < xy' - x'y < 2n. \quad (34)$$

Now (27), (32) and (34) imply

$$xy' - x'y = n \quad (35)$$

or

$$xy' - x'y = -n. \quad (36)$$

After multiplying the congruence (19) by yy' and using (26) we deduce

$$z^2yy' + zyy' + 2yy' \equiv 0 \pmod{n}$$

thus

$$xx' + xy' + 2yy' \equiv 0 \pmod{n} \quad (37)$$

and

$$xx' + x'y + 2yy' \equiv 0 \pmod{n}. \quad (38)$$

On the other hand (33) yields

$$\begin{cases} 0 < xx' < \frac{2\sqrt{2}n}{\sqrt{7}} \\ 0 < |2yy'| < \frac{2\sqrt{2}n}{\sqrt{7}} \\ 0 < |xy'| < \frac{2n}{\sqrt{7}} \\ 0 < |x'y| < \frac{2n}{\sqrt{7}} \end{cases}$$

which together with (32) gives us

$$|xx' + xy' + 2yy'| \leq |xx' + 2yy'| + |xy'| < \frac{2\sqrt{2} + 2}{\sqrt{7}}n < 2n \quad (39)$$

and

$$|xx' + x'y + 2yy'| \leq |xx' + 2yy'| + |x'y| < \frac{2\sqrt{2} + 2}{\sqrt{7}}n < 2n. \quad (40)$$

Let (35) be true. Now (37) and (39) lead to three possibilities

$$xx' + xy' + 2yy' = 0 \quad (41)$$

or

$$xx' + xy' + 2yy' = n \quad (42)$$

or

$$xx' + xy' + 2yy' = -n. \quad (43)$$

On the one hand (23) and (41) imply $x = \pm y'$ and therefore

$$x' + y' = \pm 2y.$$

From the last equation and (22) we derive

$$xy \pm xx' = 2y^2$$

which contradicts (23). On the other hand (35) and (42) give us

$$xx' + x'y + 2yy' = 0 \quad (44)$$

which together with (23) yields $x' = \pm y$ and therefore

$$x + y = \pm 2y'.$$

From the last equation and (22) we derive

$$x'y' \pm xx' = 2y'^2$$

which contradicts (23). Finally (35) and (43) lead to

$$xx' + x'y + 2yy' = -2n$$

which contradicts (40).

Let (36) be true. Now (38) and (40) lead to three possibilities

$$xx' + x'y + 2yy' = 0$$

or

$$xx' + x'y + 2yy' = n \quad (45)$$

or

$$xx' + x'y + 2yy' = -n. \quad (46)$$

The first equation coincides with (44). The equation (45) due to (36) coincides with (41). The equation (46) due to (36) implies

$$xx' + xy' + 2yy' = -2n$$

which contradicts (39). The resulting contradictions show that the assumption (25) is not true. This proves the injectivity of β .

It remains to show that the map (21) is a surjection. We note that according to (20) the solutions (x, y) and $(-x, -y)$ of $x^2 + xy + 2y^2 = n$ will produce the same z . That is why if it happens that (x, y) is such a solution of $x^2 + xy + 2y^2 = n$ that $x < 0$ then we replace (x, y) with $(-x, -y)$. In the proof, we will omit this consideration.

Let $z \in E$. According to Dirichlet's approximation theorem there exist integers a and q such that

$$\left| \frac{z}{n} - \frac{a}{q} \right| < \frac{1}{q\sqrt{n}}, \quad 1 \leq q \leq \sqrt{n}, \quad (a, q) = 1. \quad (47)$$

Put

$$r = zq - an. \quad (48)$$

Therefore

$$r^2 + rq + 2q^2 = z^2q^2 - 2zqan + a^2n^2 + (zq - an)q + 2q^2 \equiv (z^2 + z + 2)q^2 (n). \quad (49)$$

Now (19) and (49) imply

$$r^2 + rq + 2q^2 \equiv 0 (n). \quad (50)$$

By (47) and (48) we get

$$|r| < \sqrt{n}. \quad (51)$$

Using (47) and (51) we deduce

$$0 < r^2 + rq + 2q^2 < 4n. \quad (52)$$

From (50) and (52) it follows that $r^2 + rq + 2q^2 = n$, $r^2 + rq + 2q^2 = 2n$ or $r^2 + rq + 2q^2 = 3n$. Consider all cases.

Case 1

$$r^2 + rq + 2q^2 = n. \quad (53)$$

From (48) and (53) we obtain

$$n = (zq - an)^2 + (zq - an)q + 2q^2 = (z^2 + z + 2)q^2 - ran - zqan - qan$$

and thus

$$ra + 1 = kq, \quad (54)$$

where

$$k = \frac{z^2 + z + 2}{n}q - az - a. \quad (55)$$

By (19) and (55) it follows that $k \in \mathbb{Z}$ and bearing in mind (54) we get

$$(r, q) = 1. \quad (56)$$

On the other hand (53), (56) and $n \geq 5$ give us $r \neq 0$. Put

$$x = r, \quad y = q. \quad (57)$$

From (53), (56) and (57) it follows that $(x, y) \in F$. Also (48) and (57) yield (20). Therefore $\beta(x, y) = z$.

Case 2

$$r^2 + rq + 2q^2 = 2n. \quad (58)$$

From (48) and (58) it follows that

$$2n = (zq - an)^2 + (zq - an)q + 2q^2 = (z^2 + z + 2)q^2 - ran - zqan - qan$$

and thus

$$ra + 2 = kq, \quad (59)$$

where k is denoted by (55). From (59) we deduce

$$(r, q) \leq 2. \quad (60)$$

Now (58), (60) and $n \geq 5$ lead to $r \neq 0$.

Case 2.1

$$r = 2r_0, \quad q = 2q_0 + 1. \quad (61)$$

By (58) and (61) we get

$$q^2 + qr_0 + 2r_0^2 = n$$

which is equivalent to

$$(q + r_0)^2 - r_0(q + r_0) + 2r_0^2 = n. \quad (62)$$

Put

$$x = q + r_0, \quad y = -r_0. \quad (63)$$

Now (60) and (61) imply

$$(q + r_0, r_0) = 1. \quad (64)$$

From (62), (63) and (64) we obtain that $(x, y) \in F$. Further (48) and (63) give us

$$2(zy - x) = -(z^2 + z + 2)q + zan + an. \quad (65)$$

Bearing in mind (19) and (65) we deduce

$$2(zy - x) \equiv 0(n). \quad (66)$$

Case 2.1.1

$$r_0 = 2r_1. \quad (67)$$

Now (61), (62) and (67) assure us that n is odd. Hence (66) yields (20). Consequently $\beta(x, y) = z$.

Case 2.1.2

$$r_0 = 2r_1 + 1. \quad (68)$$

From (61), (62), (63) and (68) it follows that n and x are even. Now (61) leads to

$$zq = r + an \equiv 0 \pmod{2}$$

that is z is even. Thus

$$zy - x \text{ is even.} \quad (69)$$

Case 2.1.2a

$$n = 2n_0, \quad n_0 \text{ is odd.} \quad (70)$$

Now (66), (69) and (70) imply (20). Therefore $\beta(x, y) = z$.

Case 2.1.2b

$$n = 2^{l+1}n_0, \quad l \geq 1, \quad n_0 \text{ is odd.} \quad (71)$$

By (19) and (71) we obtain

$$z^2 + z + 2 \equiv 0 \pmod{2^{l+1}}. \quad (72)$$

Assume that

$$zy - x = 2^l h, \quad h \text{ is odd.} \quad (73)$$

Now (18), (71), (72) and (73) give us consistently

$$\begin{aligned} (yz)^2 + (yz)y + 2y^2 &\equiv 0 \pmod{2^{l+1}}, \\ (2^l h + x)^2 + (2^l h + x)y + 2y^2 &\equiv 0 \pmod{2^{l+1}}, \\ 2^{2l} h^2 + 2^{l+1} hx + x^2 + xy + 2y^2 + 2^l hy &\equiv 0 \pmod{2^{l+1}}, \\ 2^l hy &\equiv 0 \pmod{2^{l+1}}. \end{aligned} \quad (74)$$

From (63), (68) and (74) we get a contradiction. Consequently

$$zy - x \equiv 0 \pmod{2^{l+1}}. \quad (75)$$

Now (66), (71) and (75) yield (20). Thus $\beta(x, y) = z$.

Case 2.2

$$r = 2r_0, \quad q = 2q_0. \quad (76)$$

Bearing in mind (60) and (76) we get

$$(r_0, q_0) = 1. \quad (77)$$

By (58) and (76) we find

$$4q_0^2 + 2q_0r_0 + 2r_0^2 = n. \quad (78)$$

Case 2.2.1

$$r_0 = 2r_1, \quad q_0 = 2q_1 + 1. \quad (79)$$

We write equation (78) in the form

$$q^2 - q(q_0 + r_0) + 2(q_0 + r_0)^2 = n. \quad (80)$$

Put

$$x = -q, \quad y = q_0 + r_0. \quad (81)$$

Now (76), (77) and (79) lead to

$$(-q, r_0 + q_0) = 1. \quad (82)$$

Taking into account (80) – (82) we conclude that $(x, y) \in F$. Further (76) – (79) assure us that $4 \mid n$ and $4 \mid r$. Therefore

$$zq = r + an \equiv 0 \pmod{4}. \quad (83)$$

From (76), (79) and (83) it follows that

$$z = 2z_0. \quad (84)$$

Using (48), (76), (81) and (84) we write

$$2(z_0y - x) = -(z_0^2 + z_0 + 2)2q_0 - 2nz_0a. \quad (85)$$

By (19) and (85) we obtain (20). Consequently $\beta(x, y) = z$.

Case 2.2.2

$$r_0 = 2r_1 + 1, \quad q_0 = 2q_1 + 1. \quad (86)$$

We write equation (78) in the form

$$(2q_0 + r_0)^2 - r_0(2q_0 + r_0) + 2r_0^2 = n. \quad (87)$$

Put

$$x = 2q_0 + r_0, \quad y = -r_0. \quad (88)$$

Now (77) and (86) imply

$$(2q_0 + r_0, -r_0) = 1. \quad (89)$$

From (87) – (89) we get $(x, y) \in F$. Further (78) and (86) give us

$$n \equiv 0 \pmod{4}. \quad (90)$$

If we assume that (84) is true then (48), (76), (90) and (84) yield

$$2r_0 = zq - an = 4z_0q_0 - an \equiv 0 \pmod{4}$$

which contradicts (86). This means that z is odd, that is

$$z + 1 = 2z_0. \quad (91)$$

Bearing in mind (48), (76) and (88) we deduce

$$2(zy - x) = -(z^2 + z + 2)2q_0 + an(z + 1). \quad (92)$$

From (19), (91) and (92) we establish (20). Therefore $\beta(x, y) = z$.

Case 2.2.3

$$r_0 = 2r_1 + 1, \quad q_0 = 2q_1. \quad (93)$$

When z is even then Case 2.2.3 coincides with Case 2.2.1. When z is odd then Case 2.2.3 coincides with Case 2.2.2.

Case 2.3

$$r = 2r_0 + 1, \quad q = 2q_0 + 1. \quad (94)$$

By (58) and (94) we obtain

$$q^2 - q(q_0 + r_0 + 1) + 2(q_0 + r_0 + 1)^2 = n. \quad (95)$$

Put

$$x = -q, \quad y = q_0 + r_0 + 1. \quad (96)$$

Now (60) and (94) assure us that

$$(-q, q_0 + r_0 + 1) = 1. \quad (97)$$

From (95) – (97) we conclude that $(x, y) \in F$. Further (48) and (96) lead to

$$2(zy - x) = (z^2 + z + 2)q - zan. \quad (98)$$

Using (19) and (98) we establish that (66) holds.

Case 2.3.1 The numbers r_0 and q_0 are of different parity. By (95) and (96) it follows that n is odd. Hence (66) implies (20). Thus $\beta(x, y) = z$.

Case 2.3.2 The numbers r_0 and q_0 are of the same parity.

From (95) and (96) it follows that y is odd and n is even. Now (48) and (94) give us that zq is odd. Therefore z is odd. Consequently $zy - x$ is even. It remains to be seen that when n has the shape (70) then Case 2.3.2 coincides with Case 2.1.2a and when n has the form (71) then Case 2.3.2 coincides with Case 2.1.2b. Hence $\beta(x, y) = z$.

Case 3

$$r^2 + rq + 2q^2 = 3n. \quad (99)$$

Direct verifications will prove that Case 3 is impossible.

Case 3.1

$$r = 3r_0, \quad q = 3q_0. \quad (100)$$

Now (99) and (100) yield

$$3(r_0^2 + r_0q_0 + 2q_0^2) = n,$$

i.e. $3 \mid n$ that contradicts (19) because the congruences

$$z^2 + z + 2 \equiv 0 \pmod{3}$$

has no solution.

Case 3.2

$$r = 3r_0, \quad q = 3q_0 + 1. \quad (101)$$

By (99) and (101) we get

$$3(3r_0^2 + 3r_0q_0 + r_0 + 6q_0^2 + 4q_0) + 2 \equiv 0 \pmod{3}$$

which is impossible.

Case 3.3

$$r = 3r_0, \quad q = 3q_0 + 2. \quad (102)$$

Using (99) and (102) we deduce

$$3(3r_0^2 + 3r_0q_0 + 2r_0 + 6q_0^2 + 8q_0) + 8 \equiv 0 \pmod{3}$$

which is a contradiction.

Case 3.4

$$r = 3r_0 + 1, \quad q = 3q_0. \quad (103)$$

Now (99) and (103) give us

$$3(3r_0^2 + 3r_0q_0 + 2r_0 + 6q_0^2 + q_0) + 1 \equiv 0 \pmod{3}$$

which is impossible.

Case 3.5

$$r = 3r_0 + 1, \quad q = 3q_0 + 1. \quad (104)$$

From (99) and (104) it follows

$$3(3r_0^2 + 3r_0q_0 + 3r_0 + 6q_0^2 + 5q_0) + 4 \equiv 0 \pmod{3}$$

which is a contradiction.

Case 3.6

$$r = 3r_0 + 1, \quad q = 3q_0 + 2. \quad (105)$$

Now (99) and (105) lead to

$$3(3r_0^2 + 3r_0q_0 + 4r_0 + 6q_0^2 + 9q_0) + 11 \equiv 0 \pmod{3}$$

which is impossible.

Case 3.7

$$r = 3r_0 + 2, \quad q = 3q_0. \quad (106)$$

By (99) and (106) we obtain

$$3(3r_0^2 + 3r_0q_0 + 4r_0 + 3q_0^2 + 2q_0) + 4 \equiv 0 \pmod{3}$$

which is a contradiction.

Case 3.8

$$r = 3r_0 + 2, \quad q = 3q_0 + 1. \quad (107)$$

Now (99) and (107) assure us that

$$3(3r_0^2 + 3r_0q_0 + 5r_0 + 6q_0^2 + 6q_0) + 8 \equiv 0 \pmod{3}$$

which is impossible.

Case 3.9

$$r = 3r_0 + 2, \quad q = 3q_0 + 2. \quad (108)$$

From (99) and (108) we write

$$3(3r_0^2 + 3r_0q_0 + 6r_0 + 6q_0^2 + 10q_0) + 16 \equiv 0 \pmod{3}$$

which is a contradiction. This proves that β is a surjective map.

For the benefit of the readers, we will give a second shorter proof of the surjectivity of β . Note that this proof is valid only when the quadratic form has class number one. We denote

$$f(x, y) = x^2 + xy + 2y^2. \quad (109)$$

For each root z of the congruence (19) we define the set

$$\Lambda_z = \{(x, y) \in \mathbb{Z}^2 \mid x - zy \equiv 0 \pmod{n}\}. \quad (110)$$

We choose a basis

$$(a_1, a_2) = (n, 0), \quad (b_1, b_2) = (z, 1) \quad (111)$$

of Λ_z . Clearly, the vectors (111) are linearly independent elements of Λ_z . Further, (110) shows that the lattice determinant of Λ_z is divisible by n , therefore $(a_1, a_2), (b_1, b_2)$ form a basis of this lattice. Bearing in mind (19), (109) and (110) we obtain

$$f(a_1X + b_1Y, a_2X + b_2Y) = ng(X, Y), \quad (112)$$

where

$$g(X, Y) = nX^2 + (2z + 1)XY + \frac{z^2 + z + 2}{n}Y^2. \quad (113)$$

A direct discriminant calculation shows that the discriminant of f and g is -7 . From the fact that there exists a unique class of binary quadratic forms with discriminant equal to -7 (see [14], Theorem 198), it follows that g is equivalent to f , that is

$$g(X, Y) = f(aX + bY, cX + dY), \quad (114)$$

where a, b, c, d are integers with $ad - bc = \pm 1$. Consider a, b, c, d to be fixed. Now let (X_0, Y_0) be the unique solution of the system

$$\begin{cases} aX + bY = 1 \\ cX + dY = 0 \end{cases}. \quad (115)$$

We construct a solution (x, y) of (18) of the form

$$(x, y) = X_0(a_1, a_2) + Y_0(b_1, b_2) = (nX_0 + zY_0, Y_0). \quad (116)$$

If $nX_0 + zY_0 < 0$, then replace (X_0, Y_0) by $(-X_0, -Y_0)$ and note that (X_0, Y_0) is the unique solution of the system

$$\begin{cases} aX + bY = -1 \\ cX + dY = 0 \end{cases}. \quad (117)$$

Since $ad - bc = \pm 1$ then $X_0, Y_0 \in \mathbb{Z}$. On the one hand $aX_0 + bY_0 = \pm 1$ leads to

$$(X_0, Y_0) = 1. \quad (118)$$

On the other hand (109), (113), (114), (115) and (117) give us

$$nX_0^2 + (2z + 1)X_0Y_0 + \frac{z^2 + z + 2}{n}Y_0^2 = g(X_0, Y_0) = f(\pm 1, 0) = 1 \quad (119)$$

and consequently

$$(n, Y_0) = 1. \quad (120)$$

Now (116), (118) and (120) yield

$$(x, y) = 1. \quad (121)$$

By (112), (116) and (119) we derive

$$f(x, y) = ng(X_0, Y_0) = n. \quad (122)$$

Finally (109), (110), (121) and (122) imply that β is a surjection.

This completes the proof of the lemma. \square

4 Proof of the theorem

Using (4) and the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

we get

$$\Gamma(X) = \sum_{\substack{d_1, d_2 \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{1 \leq n \leq X \\ n^2+n+1 \equiv 0 \pmod{d_1^2} \\ n^2+n+2 \equiv 0 \pmod{d_2^2}}} 1 = \Gamma_1(X) + \Gamma_2(X), \quad (123)$$

where

$$\Gamma_1(X) = \sum_{\substack{d_1 d_2 \leq z \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \quad (124)$$

$$\Gamma_2(X) = \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \quad (125)$$

$$\Sigma(X, d_1^2, d_2^2) = \sum_{\substack{1 \leq n \leq X \\ n^2+n+1 \equiv 0 \pmod{d_1^2} \\ n^2+n+2 \equiv 0 \pmod{d_2^2}}} 1, \quad (126)$$

$$\sqrt{X} \leq z < X, \quad (127)$$

where z is to be chosen later.

4.1 Estimation of $\Gamma_1(\mathbf{X})$

Assume that $q_1 = d_1^2$, $q_2 = d_2^2$, where d_1 and d_2 are square-free, $(q_1, q_2) = 1$ and $d_1 d_2 \leq z$. Define

$$\Omega(X, q_1, q_2, n) = \sum_{\substack{m \leq X \\ m \equiv n \pmod{q_1 q_2}}} 1. \quad (128)$$

Obviously

$$\Omega(X, q_1, q_2, n) = \frac{X}{q_1 q_2} + \mathcal{O}(1). \quad (129)$$

By (126) and (128) we obtain upon partitioning the sum (126) into residue classes modulo $q_1 q_2$

$$\Sigma(X, q_1, q_2) = \sum_{\substack{1 \leq n \leq q_1 q_2 \\ n^2 + n + 1 \equiv 0 \pmod{q_1} \\ n^2 + n + 2 \equiv 0 \pmod{q_2}}} \Omega(X, q_1, q_2, n). \quad (130)$$

From (3), (130) and (129) we get

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(\lambda(q_1, q_2)). \quad (131)$$

Taking into account (3) and that the number of solutions of the congruence

$$n^2 + n + 1 \equiv a \pmod{q_1}$$

is less than or equal to $\tau(q_1)$ we deduce

$$\lambda(q_1, q_2) \ll \lambda(q_1, 1) \ll \tau(q_1) \ll \tau(q_1 q_2). \quad (132)$$

Now (131), (132) and the inequalities

$$\tau(q_1 q_2) \ll (q_1 q_2)^\varepsilon \ll X^\varepsilon$$

imply

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(X^\varepsilon). \quad (133)$$

Using (124), (127) and (133) we obtain

$$\begin{aligned} \Gamma_1(X) &= X \sum_{\substack{d_1 d_2 \leq z \\ (d_1, d_2) = 1}} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(z X^\varepsilon) \\ &= \sigma X - X \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2) = 1}} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(z X^\varepsilon), \end{aligned} \quad (134)$$

where

$$\sigma = \sum_{\substack{d_1, d_2=1 \\ (d_1, d_2)=1}}^{\infty} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2 d_2^2}. \quad (135)$$

By (132) it follows

$$\sum_{\substack{d_1 d_2 > z \\ (d_1, d_2)=1}} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} \ll \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2)=1}} \frac{(d_1 d_2)^\varepsilon}{(d_1 d_2)^2} \ll \sum_{n > z} \frac{\tau(n)}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}. \quad (136)$$

It remains to see that the product (9) and the sum (135) coincide. From Lemma 5 and $(d_1, d_2) = 1$ we have

$$\lambda(d_1^2, d_2^2) = \lambda(d_1^2, 1)\lambda(1, d_2^2). \quad (137)$$

Now (135) and (137) yield

$$\sigma = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2), \quad (138)$$

where

$$f_{d_1}(d_2) = \begin{cases} 1 & \text{if } (d_1, d_2) = 1, \\ 0 & \text{if } (d_1, d_2) > 1. \end{cases}$$

It is easy to see that the function

$$\frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is multiplicative with respect to d_2 and the series

$$\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is absolutely convergent. Using the Euler product we write

$$\begin{aligned} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2) &= \prod_{p \nmid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \\ &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}. \end{aligned} \quad (139)$$

From (138) and (139) we get

$$\begin{aligned} \sigma &= \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}. \end{aligned} \quad (140)$$

It is easy to see that the function

$$\frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}$$

is multiplicative with respect to d_1 and the series

$$\sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}$$

is absolutely convergent. Using again the Euler product from (3) and (140) we deduce

$$\begin{aligned} \sigma &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_p \left(1 - \frac{\lambda(p^2, 1)}{p^2} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}\right) \\ &= \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^2}\right). \end{aligned} \quad (141)$$

Summarizing (127), (134), (136) and (141) we establish

$$\Gamma_1(X) = \sigma X + \mathcal{O}(zX^\epsilon), \quad (142)$$

where σ is given by the product (9).

4.2 Estimation of $\Gamma_2(\mathbf{X})$

Bearing in mind (125), (126) and splitting the range of d_1 and d_2 into dyadic subintervals of the form $D_1 \leq d_1 < 2D_1$, $D_2 \leq d_2 < 2D_2$ we obtain

$$\Gamma_2(X) \ll (\log X)^2 \sum_{n \leq X} \sum_{\substack{D_1 \leq d_1 < 2D_1 \\ n^2 + n + 1 \equiv 0 \pmod{d_1^2}}} \sum_{\substack{D_2 \leq d_2 < 2D_2 \\ n^2 + n + 2 \equiv 0 \pmod{d_2^2}}} 1, \quad (143)$$

where

$$\frac{1}{2} \leq D_1, D_2 \leq \sqrt{X^2 + X + 2}, \quad D_1 D_2 > \frac{z}{4}. \quad (144)$$

On the one hand (143) implies

$$\Gamma_2(X) \ll X^\epsilon \Sigma_1, \quad (145)$$

where

$$\Sigma_1 = \sum_{n \leq X} \sum_{\substack{D_1 \leq d_1 < 2D_1 \\ n^2 + n + 1 \equiv 0 \pmod{d_1^2}}} 1. \quad (146)$$

On the other hand (143) gives us

$$\Gamma_2(X) \ll X^\varepsilon \Sigma_2, \quad (147)$$

where

$$\Sigma_2 = \sum_{n \leq X} \sum_{\substack{D_2 \leq d_2 < 2D_2 \\ n^2 + n + 2 \equiv 0 \pmod{d_2^2}}} 1. \quad (148)$$

Estimation of Σ_1

Arguing as in [?] we get

$$\Sigma_1 \ll X^{1+\varepsilon} D_1^{-\frac{1}{4}}. \quad (149)$$

Estimation of Σ_2

Define

$$\mathcal{N}(d) = \{n \in \mathbb{N} : 1 \leq n \leq d, \quad n^2 + n + 2 \equiv 0 \pmod{d}\}, \quad (150)$$

$$\mathcal{N}'(d) = \{n \in \mathbb{N} : 1 \leq n \leq d^2, \quad n^2 + n + 2 \equiv 0 \pmod{d^2}\}. \quad (151)$$

From (148) and (151) we write

$$\begin{aligned} \Sigma_2 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \sum_{\substack{m \leq X \\ m \equiv n \pmod{d_2^2}}} 1 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \left(\left[\frac{X-n}{d_2^2} \right] - \left[\frac{-n}{d_2^2} \right] \right) \\ &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \left(\frac{X}{d_2^2} + \psi\left(\frac{-n}{d_2^2}\right) - \psi\left(\frac{X-n}{d_2^2}\right) \right) \\ &\ll X^{1+\varepsilon} D_2^{-1} + |\Sigma'_2| + |\Sigma''_2|, \end{aligned} \quad (152)$$

where

$$\Sigma'_2 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \psi\left(\frac{-n}{d_2^2}\right), \quad (153)$$

$$\Sigma''_2 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \psi\left(\frac{X-n}{d_2^2}\right). \quad (154)$$

Arguing as in [?] we deduce

$$\Sigma'_2 \ll X^\varepsilon D_2^{-1}. \quad (155)$$

Further we consider the sum Σ''_2 defined by (154). Let $D_2 \leq X^{\frac{1}{2}}$. The trivial estimation leads to

$$\Sigma''_2 \ll \sum_{D_2 \leq d_2 < 2D_2} d_2^\varepsilon \ll X^{\frac{1}{2}+\varepsilon}. \quad (156)$$

Let

$$D_2 > X^{\frac{1}{2}}. \quad (157)$$

We notice that all summands in the sum (148) for which $7 \mid d_2$ are equal to zero because the congruences

$$n^2 + n + 2 \equiv 0 \pmod{49} \quad (158)$$

has no solution. That's why in the estimation of (148) we will consider that $7 \nmid d_2$.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial with integral coefficients and r_1, \dots, r_k be all solutions of the congruence

$$f(x) \equiv 0 \pmod{p^{l-1}}. \quad (159)$$

From the theory of the congruences we know that when $p \nmid f'(r_i)$ for $i = 1, \dots, k$ then the number of solutions of the congruence

$$f(x) \equiv 0 \pmod{p^l}. \quad (160)$$

is also equal to k , that is, congruences (159) and (160) have an equal number of solutions. Taking into account the above considerations, we derive that the congruences

$$n^2 + n + 2 \equiv 0 \pmod{d_2^2} \quad (161)$$

and

$$n^2 + n + 2 \equiv 0 \pmod{d_2} \quad (162)$$

will have an equal number of solutions if we show that for arbitrary prime divisor p of d_2 and arbitrary solution r of (162) we have that

$$p \nmid 2r + 1.$$

We assume the opposite. Hence

$$r = \frac{ph - 1}{2}, \quad (163)$$

where $h \in \mathbb{Z}$. Now (163) and

$$r^2 + r + 2 \equiv 0 \pmod{p}$$

yield

$$p^2 h^2 + 7 \equiv 0 \pmod{4p}$$

which means $p = 7$. But we have already excluded the case when 7 is a prime divisor of d_2 . Therefore congruences (161) and (162) have an equal number of solutions. We note

that the sum over n in (148) does not contain terms with $n = \frac{d_2^2}{2}$ and $n = d_2^2$. Moreover for any n satisfying the congruence (161) and such that $1 \leq n < \frac{d_2^2}{2}$ the number $d_2^2 - n - 1$ satisfies the same congruence. The same is true for the congruence (162). We also note that if $n = d_2^2 - n - 1$ then $d_2^2 \mid 7$ which is impossible and if $n = d_2 - n - 1$ then $d_2 = 7$ which we excluded as an possibility. Using this facts and notations (150), (151) we denote

$$k = \#\mathcal{N}(d_2) = \#\mathcal{N}'(d_2), \quad (164)$$

$$n_1, \dots, n_k \in \mathcal{N}(d_2), \quad n'_1, \dots, n'_k \in \mathcal{N}'(d_2). \quad (165)$$

Now (150), (151), (157), (164), (165) and $d_2 \geq D_2$ give us

$$\begin{aligned} \sum_{n \in \mathcal{N}'(d_2)} \psi \left(\frac{X-n}{d_2^2} \right) &= \sum_{n \in \mathcal{N}'(d_2)} \left(\frac{X-n}{d_2^2} - \frac{1}{2} \right) \\ &= \sum_{n \in \mathcal{N}'(d_2)} \left(\frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{n'_1 + \dots + n'_{k/2} + (d_2^2 - n'_1 - 1) + \dots + (d_2^2 - n'_{k/2} - 1)}{d_2^2} \\ &= \sum_{n \in \mathcal{N}(d_2)} \left(\frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{k(d_2^2 - 1)}{2d_2^2} \\ &= \sum_{n \in \mathcal{N}(d_2)} \left(\frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{n_1 + \dots + n_{k/2} + (d_2 - n_1 - 1) + \dots + (d_2 - n_{k/2} - 1)}{d_2} \left(1 + \frac{1}{d_2} \right) \\ &= \sum_{n \in \mathcal{N}(d_2)} \left(\frac{X}{d_2^2} - \frac{1}{2} \right) - \left(1 + \frac{1}{d_2} \right) \sum_{n \in \mathcal{N}(d_2)} \frac{n}{d_2} \\ &= \sum_{n \in \mathcal{N}(d_2)} \left(\frac{X}{d_2^2} - \frac{\sqrt{X}}{d_2} - \frac{n}{d_2^2} \right) + \sum_{n \in \mathcal{N}(d_2)} \left(\frac{\sqrt{X} - n}{d_2} - \frac{1}{2} \right) \\ &= \sum_{n \in \mathcal{N}(d_2)} \left(\frac{X}{d_2^2} - \frac{\sqrt{X}}{d_2} - \frac{n}{d_2^2} \right) + \sum_{n \in \mathcal{N}(d_2)} \psi \left(\frac{\sqrt{X} - n}{d_2} \right). \end{aligned} \quad (166)$$

From (154), (157) and (166) it follows

$$\Sigma_2'' \ll X^{\frac{1}{2}+\varepsilon} + |\Sigma_3|, \quad (167)$$

where

$$\Sigma_3 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \psi \left(\frac{\sqrt{X} - n}{d_2} \right). \quad (168)$$

Using (168) and Lemma 2 with

$$M = X^{\frac{1}{2}} \quad (169)$$

we obtain

$$\begin{aligned}\Sigma_3 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \left(- \sum_{1 \leq |m| \leq M} \frac{e\left(m \left(\frac{\sqrt{X}-n}{d_2}\right)\right)}{2\pi i m} + \mathcal{O}\left(f_M\left(\frac{\sqrt{X}-n}{d_2}\right)\right) \right) \\ &= \Sigma_4 + \Sigma_5,\end{aligned}\tag{170}$$

where

$$\Sigma_4 = \sum_{1 \leq |m| \leq M} \frac{\Theta_m}{2\pi i m},\tag{171}$$

$$\Theta_m = \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{\sqrt{X}m}{d_2}\right) \sum_{n \in \mathcal{N}(d_2)} e\left(-\frac{nm}{d_2}\right),\tag{172}$$

$$\Sigma_5 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} f_M\left(\frac{\sqrt{X}-n}{d_2}\right).\tag{173}$$

By (172), (173) and Lemma 2 we derive

$$\begin{aligned}\Sigma_5 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \sum_{m=-\infty}^{+\infty} b_M(m) e\left(\frac{\sqrt{X}-n}{d_2} m\right) = \sum_{m=-\infty}^{+\infty} b_M(m) \Theta_m \\ &\ll \frac{\log M}{M} |\Theta_0| + \frac{\log M}{M} \sum_{1 \leq |m| \leq M^{1+\varepsilon}} |\Theta_m| + \sum_{|m| > M^{1+\varepsilon}} |b_M(m)| |\Theta_m| \\ &\ll \frac{\log M}{M} D_2^{1+\varepsilon} + \frac{\log M}{M} \sum_{1 \leq m \leq M^{1+\varepsilon}} |\Theta_m| + D_2^{1+\varepsilon} \sum_{|m| > M^{1+\varepsilon}} |b_M(m)| \\ &\ll \frac{\log M}{M} D_2^{1+\varepsilon} + \frac{\log M}{M} \sum_{1 \leq m \leq M^{1+\varepsilon}} |\Theta_m|.\end{aligned}\tag{174}$$

Now (170), (171) and (174) imply

$$\Sigma_3 \ll X^\varepsilon \left(\frac{D_2}{M} + \sum_{1 \leq m \leq M^{1+\varepsilon}} \frac{|\Theta_m|}{m} \right).\tag{175}$$

Define

$$\mathcal{F}(d) = \{(u, v) : u^2 + uv + 2v^2 = d, (u, v) = 1, u \in \mathbb{N}, v \in \mathbb{Z} \setminus \{0\}\}.\tag{176}$$

According to Lemma 6 there exists a bijection

$$\beta : \mathcal{F}(d) \rightarrow \mathcal{N}(d)$$

from $\mathcal{F}(d)$ to $\mathcal{N}(d)$ defined by (150) that associates to each couple $(u, v) \in \mathcal{F}(d)$ the element $n \in \mathcal{N}(d)$ satisfying

$$nv \equiv u (d). \quad (177)$$

Now (177) yields

$$n_{u,v} \equiv u\bar{v}_d (d)$$

thus

$$\frac{n_{u,v}}{d} \equiv u \frac{\bar{v}_{u^2+uv+2v^2}}{u^2+uv+2v^2} (1). \quad (178)$$

From (178) and Lemma 3 it follows

$$\frac{n_{u,v}}{d} \equiv \frac{u}{v(u^2+uv+2v^2)} - \frac{\bar{u}_{|v|}}{v} (1), \quad (179)$$

$$\frac{n_{u,v}}{d} \equiv -\frac{u+2v}{u(u^2+uv+2v^2)} + \frac{\bar{v}_u}{u} (1). \quad (180)$$

Bearing in mind (172), (176), (179) and (180) we get

$$\begin{aligned} \Theta_m &= \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{(u,v) \in \mathcal{F}(d_2)} e\left(-\frac{n_{u,v}}{d_2}m\right) \\ &= \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{\substack{(u,v) \in \mathcal{F}(d_2) \\ 0 < u < |v|}} e\left(-\frac{mu}{v(u^2+uv+2v^2)} + \frac{m\bar{u}_{|v|}}{v}\right) \\ &+ \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{\substack{(u,v) \in \mathcal{F}(d_2) \\ 0 < |v| < u}} e\left(\frac{m(u+2v)}{u(u^2+uv+2v^2)} - \frac{m\bar{v}_u}{u}\right) \\ &= \sum_{\substack{D_2 \leq u^2+uv+2v^2 < 2D_2 \\ 0 < u < |v| \\ (u,v)=1}} e\left(\frac{m\sqrt{X}}{u^2+uv+2v^2} - \frac{mu}{v(u^2+uv+2v^2)} + \frac{m\bar{u}_{|v|}}{v}\right) \\ &+ \sum_{\substack{D_2 \leq u^2+uv+2v^2 < 2D_2 \\ 0 < |v| < u \\ (u,v)=1}} e\left(\frac{m\sqrt{X}}{u^2+uv+2v^2} + \frac{m(u+2v)}{u(u^2+uv+2v^2)} - \frac{m\bar{v}_u}{u}\right) \\ &= \Theta'_m + \Theta''_m, \end{aligned} \quad (181)$$

say. Consider Θ'_m . Let for any fixed $\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}$ the interval $[\eta_1(v), \eta_2(v)]$ is a solution with respect to u of the system

$$\begin{cases} u^2 + uv + 2v^2 < 2D_2 \\ u^2 + uv + 2v^2 \geq D_2 \\ 0 < u < |v| \end{cases} . \quad (182)$$

Denote

$$g(u) = e \left(\frac{m\sqrt{X}}{u^2 + uv + 2v^2} - \frac{mu}{v(u^2 + uv + 2v^2)} \right), \quad (183)$$

$$K_{v,m}(t) = \sum_{\substack{\eta_1(v) \leq u < t \\ (u,v)=1}} e \left(\frac{m\bar{u}_{|v|}}{v} \right). \quad (184)$$

By (181) – (184) and Abel's summation formula we deduce

$$\begin{aligned} \Theta'_m &= \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \sum_{\substack{\eta_1(v) \leq u < \eta_2(v) \\ (u,v)=1}} g(u) e \left(\frac{m\bar{u}_{|v|}}{v} \right) \\ &= \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left(g(\eta_2(v)) K_{v,m}(\eta_2(v)) - \int_{\eta_1(v)}^{\eta_2(v)} K_{v,m}(t) \left(\frac{d}{dt} g(t) \right) dt \right) \\ &\ll \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left(1 + \frac{m\sqrt{X}}{v^2} \right) \max_{\eta_1(v) \leq t \leq \eta_2(v)} |K_{v,m}(t)|. \end{aligned} \quad (185)$$

We are now in a good position to apply Lemma 4 because the sum defined by (184) is an incomplete Kloosterman sum. We have

$$K_{v,m}(t) \ll |v|^{\frac{1}{2}+\varepsilon} (v, m)^{\frac{1}{2}}. \quad (186)$$

From (185) and (186) we derive

$$\begin{aligned} \Theta'_m &\ll \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left(1 + \frac{m\sqrt{X}}{v^2} \right) |v|^{\frac{1}{2}+\varepsilon} (v, m)^{\frac{1}{2}} \\ &\ll X^\varepsilon \left(D_2^{\frac{1}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{3}{4}} \right) \sum_{0 < v < \sqrt{D_2}} (v, m)^{\frac{1}{2}}. \end{aligned} \quad (187)$$

On the other hand

$$\sum_{0 < v < \sqrt{D_2}} (v, m)^{\frac{1}{2}} \leq \sum_{l|m} l^{\frac{1}{2}} \sum_{\substack{v \leq \sqrt{D_2} \\ v \equiv 0 \pmod{l}}} 1 \ll D_2^{\frac{1}{2}} \sum_{l|m} l^{-\frac{1}{2}} \ll D_2^{\frac{1}{2}} \tau(m) \ll X^\varepsilon D_2^{\frac{1}{2}}. \quad (188)$$

The estimations (187) and (188) lead to

$$\Theta'_m \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}} \right). \quad (189)$$

Working in a similar way for Θ_m'' from (181) we deduce

$$\Theta_m'' \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}} \right). \quad (190)$$

Now (181), (189) and (190) yield

$$\Theta_m \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}} \right). \quad (191)$$

From (175) and (191) we write

$$\Sigma_3 \ll X^\varepsilon \left(D_2 M^{-1} + D_2^{\frac{3}{4}} + X^{\frac{1}{2}} M D_2^{-\frac{1}{4}} \right). \quad (192)$$

Using (144), (169) and (192) we obtain

$$\Sigma_3 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \quad (193)$$

Bearing in mind (144), (152), (155), (156), (167) and (193) we find

$$\Sigma_2 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \quad (194)$$

Estimation of $\Gamma_2(X)$

Now (144), (145), (147), (149) and (194) imply

$$\Gamma_2(X) \ll X^{1+\varepsilon} z^{-\frac{1}{8}}. \quad (195)$$

4.3 The end of the proof

Summarizing (123), (142), (195) and choosing $z = X^{\frac{8}{9}}$ we establish the asymptotic formula (8). This completes the proof of Theorem 1.

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