

# Square-free pairs $n^2 + n + 1$ , $n^2 + n + 2$

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## Abstract

In this paper we prove by asymptotic formula that there exist infinitely many square-free pairs of the form  $n^2 + n + 1$ ,  $n^2 + n + 2$ . A key point in our proof is the establishment of bijective correspondence between the number of representations of number by binary quadratic form and the incongruent solutions of quadratic congruence.

**Keywords:** Square-free numbers, Asymptotic formula, Bijective function.

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## 1 Notations

Let  $X$  be a sufficiently large positive number. The letter  $\varepsilon$  denotes an arbitrary small positive number, not the same in all appearances. As usual  $[t]$  and  $\{t\}$  denote the integer part, respectively, the fractional part of  $t$ . Further  $\mu(n)$  is Möbius' function and  $\tau(n)$  denotes the number of positive divisors of  $n$ . Instead of  $m \equiv n \pmod{k}$  we write for simplicity  $m \equiv n(k)$ . Moreover  $(m, n)$  is the greatest common divisor of  $m$  and  $n$ . The letter  $p$  will always denote prime number. We write  $e(t) = \exp(2\pi it)$  and  $\psi(t) = \{t\} - 1/2$ . For  $x, y \in \mathbb{R}$  we write  $x \equiv y(1)$  when  $x - y \in \mathbb{Z}$ . For any  $n$  and  $q$  such that  $(n, q) = 1$  we denote by  $\bar{n}_q$  the inverse of  $n$  modulo  $q$ . By  $G(q, m, n)$  we shall denote the Gauss sums

$$G(q, m, n) = \sum_{x=1}^q e\left(\frac{mx^2 + nx}{q}\right). \quad (1)$$

By  $K(r, h)$  we shall denote the incomplete Kloosterman sum

$$K(r, h) = \sum_{\substack{\alpha \leq x < \beta \\ (x, r) = 1}} e\left(\frac{h\bar{x}_{|r|}}{r}\right), \quad (2)$$

where

$$h, r \in \mathbb{Z}, \quad hr \neq 0, \quad 0 < \beta - \alpha \leq 2|r|.$$

We also define

$$\lambda(q_1, q_2) = \sum_{\substack{1 \leq n \leq q_1 q_2 \\ n^2 + n + 1 \equiv 0 \pmod{q_1} \\ n^2 + n + 2 \equiv 0 \pmod{q_2}}} 1, \quad (3)$$

$$\Gamma(X) = \sum_{1 \leq n \leq X} \mu^2(n^2 + n + 1) \mu^2(n^2 + n + 2), \quad (4)$$

## 2 Introduction and statement of the result

We say that an integer  $n \in \mathbb{N}$  is square-free if for any prime  $p \mid n$ , one has  $p^2 \nmid n$ . Information on the distribution of square-free numbers was given in 1885 by Gegenbauer [6]. He proved the following asymptotic formula

$$\sum_{n \leq X} \mu^2(n) = \frac{6}{\pi^2} X + \mathcal{O}\left(X^{\frac{1}{2}}\right). \quad (5)$$

Gegenbauer's argument is very simple, but despite the passage of 137 years the exponent  $1/2$  appearing above has never been improved. Any reduction in the exponent  $1/2$  would appear to require a quasi Riemann Hypothesis.

Let  $k$  and  $n$  be integers and  $k \geq 2$ . We say that  $n$  is  $k$ -free if there is no prime  $p$  such that  $p^k \mid n$ . Consider the irreducible polynomial  $f(x) \in \mathbb{Z}[x]$  of degree  $d$ . Assume that for every prime  $p$  there is at least one integer  $n_p$  for which  $p^k \nmid f(n_p)$ . It is conjectured that the set  $f(\mathbb{Z}) = \{f(n), n \in \mathbb{Z}\}$  contains infinitely many  $k$ -free values. A lot of articles are devoted to problems of this type. We point out the papers [1], [4], [8], [9], [10], [11], [15].

Another interesting problem we know in number theory is consecutive square-free numbers. In 1932 Carlitz [2] showed that there exist infinitely many pairs of consecutive square-free numbers. More precisely he proved the asymptotic formula

$$\sum_{n \leq X} \mu^2(n) \mu^2(n+1) = \prod_p \left(1 - \frac{2}{p^2}\right) X + \mathcal{O}(X^{\theta+\varepsilon}), \quad (6)$$

where  $\theta = 2/3$ . Afterwards the remainder term of (6) was improved by Mirsky [13] and Heath-Brown [7]. The best result up to now belongs to Reuss [14] with  $\theta = (26 + \sqrt{433})/81$ . Recently [3] the author showed that there exist infinitely many square-free numbers of

the form  $n^2 + n + 1$ . More precisely we established the asymptotic formula

$$\sum_{1 \leq n \leq X} \mu^2(n^2 + n + 1) = cX + \mathcal{O}\left(X^{\frac{4}{5} + \varepsilon}\right),$$

where

$$c = \prod_p \left(1 - \frac{\lambda(p^2)}{p^2}\right)$$

and

$$\lambda(q) = \sum_{\substack{1 \leq n \leq q \\ n^2 + n + 1 \equiv 0 \pmod{q}}} 1.$$

Motivated by these investigations we prove the following theorem.

**Theorem 1.** *For the sum  $\Gamma(X)$  defined by (4) the asymptotic formula*

$$\Gamma(X) = \sigma X + \mathcal{O}\left(X^{\frac{8}{9} + \varepsilon}\right) \tag{7}$$

*holds. Here*

$$\sigma = \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^2}\right). \tag{8}$$

From Theorem 1 it follows that there exist infinitely many consecutive square-free numbers of the form  $n^2 + n + 1$ ,  $n^2 + n + 2$ , where  $n$  runs over naturals.

### 3 Lemmas

**Lemma 1.** *Let  $(q_1, q_2) = 1$ . Then for the Gauss sum denoted by (1) we have*

$$G(q_1 q_2, m_1 q_2 + m_2 q_1, n) = G(q_1, m_1 q_2^2, n) G(q_2, m_2 q_1^2, n).$$

*Proof.* See [5]. □

**Lemma 2.** *For any  $M \geq 2$ , we have*

$$\psi(t) = - \sum_{1 \leq |m| \leq M} \frac{e(mt)}{2\pi i m} + \mathcal{O}(f_M(t)),$$

*where  $f_M(t)$  is a positive function of  $t$  which is infinitely many times differentiable and periodic with period 1. It can be expanded into the Fourier series*

$$f_M(t) = \sum_{m=-\infty}^{+\infty} b_M(m) e(mt),$$

with coefficients  $b_M(m)$  such that

$$b_M(m) \ll \frac{\log M}{M} \quad \text{for all } m$$

and

$$\sum_{|m| > M^{1+\varepsilon}} |b_M(m)| \ll M^{-A}.$$

Here  $A > 0$  is arbitrarily large and the constant in the  $\ll$ -symbol depends on  $A$  and  $\varepsilon$ .

*Proof.* See ([16], Theorem 1). □

The next lemma we need is well-known.

**Lemma 3.** *Let  $A, B \in \mathbb{Z} \setminus \{0\}$  and  $(A, B) = 1$ . Then*

$$\frac{\overline{A}_{|B|}}{B} + \frac{\overline{B}_{|A|}}{A} \equiv \frac{1}{AB} \pmod{1}.$$

**Lemma 4.** *For the sum denoted by (2) the estimate*

$$K(r, h) \ll |r|^{\frac{1}{2}+\varepsilon} (r, h)^{\frac{1}{2}}$$

*holds.*

*Proof.* Follows easily from A. Weil's estimate for the Kloosterman sum. See ([12], Ch. 11, Corollary 11.12). □

**Lemma 5.** *The function  $\lambda(q_1, q_2)$  defined by (3) is multiplicative, i.e. if*

$$(q_1 q_2, q_3 q_4) = (q_1, q_2) = (q_3, q_4) = 1 \tag{9}$$

*then*

$$\lambda(q_1 q_2, q_3 q_4) = \lambda(q_1, q_3) \lambda(q_2, q_4).$$

*Proof.* On the one hand (1), (3), (9) and Lemma 1 imply

$$\begin{aligned}
& \lambda(q_1q_2, q_3q_4) = \\
&= \frac{1}{q_1q_2q_3q_4} \sum_{1 \leq n \leq q_1q_2q_3q_4} \sum_{1 \leq h_1 \leq q_1q_2} e\left(\frac{h_1(n^2 + n + 1)}{q_1q_2}\right) \sum_{1 \leq h_2 \leq q_3q_4} e\left(\frac{h_2(n^2 + n + 2)}{q_3q_4}\right) \\
&= \frac{1}{q_1q_2q_3q_4} \sum_{1 \leq h_1 \leq q_1q_2} e\left(\frac{h_1}{q_1q_2}\right) \sum_{1 \leq h_2 \leq q_3q_4} e\left(\frac{2h_2}{q_3q_4}\right) \\
&\quad \times G(q_1q_2q_3q_4, h_1q_3q_4 + h_2q_1q_2, h_1q_3q_4 + h_2q_1q_2) \\
&= \frac{1}{q_1q_2q_3q_4} \sum_{1 \leq h_1 \leq q_1q_2} e\left(\frac{h_1}{q_1q_2}\right) \sum_{1 \leq h_2 \leq q_3q_4} e\left(\frac{2h_2}{q_3q_4}\right) G(q_1q_2, h_1q_3^2q_4^2, h_1q_3q_4 + h_2q_1q_2) \\
&\quad \times G(q_3q_4, h_2q_1^2q_2^2, h_1q_3q_4 + h_2q_1q_2) \\
&= \frac{1}{q_1q_2q_3q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1q_2 + h_2q_1}{q_1q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3q_4 + h_4q_3)}{q_3q_4}\right) \\
&\quad \times G(q_1q_2, h_1q_2q_3^2q_4^2 + h_2q_1q_3^2q_4^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4) \\
&\quad \times G(q_3q_4, h_3q_4q_1^2q_2^2 + h_4q_3q_1^2q_2^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4) \\
&= \frac{1}{q_1q_2q_3q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1q_2 + h_2q_1}{q_1q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3q_4 + h_4q_3)}{q_3q_4}\right) \\
&\quad \times G(q_1, h_1q_2^2q_3^2q_4^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4) \\
&\quad \times G(q_2, h_2q_1^2q_3^2q_4^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4) \\
&\quad \times G(q_3, h_3q_1^2q_2^2q_4^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4) \\
&\quad \times G(q_4, h_4q_1^2q_2^2q_3^2, h_1q_2q_3q_4 + q_1h_2q_3q_4 + q_1q_2h_3q_4 + q_1q_2q_3h_4). \tag{10}
\end{aligned}$$

On the other hand (1), (3) and Lemma 1 yield

$$\begin{aligned}
& \lambda(q_1, q_3)\lambda(q_2, q_4) = \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{1 \leq n \leq q_1 q_3} \sum_{1 \leq h_1 \leq q_1} e\left(\frac{h_1(n^2 + n + 1)}{q_1}\right) \sum_{1 \leq h_3 \leq q_3} e\left(\frac{h_3(n^2 + n + 2)}{q_3}\right) \\
&\times \sum_{1 \leq m \leq q_2 q_4} \sum_{1 \leq h_2 \leq q_2} e\left(\frac{h_2(m^2 + m + 1)}{q_2}\right) \sum_{1 \leq h_4 \leq q_4} e\left(\frac{h_4(m^2 + m + 2)}{q_4}\right) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\
&\times G(q_1 q_3, h_1 q_3 + h_3 q_1, h_1 q_3 + h_3 q_1) G(q_2 q_4, h_2 q_4 + h_4 q_2, h_2 q_4 + h_4 q_2) \\
&= \frac{1}{q_1 q_2 q_3 q_4} \sum_{\substack{1 \leq h_1 \leq q_1 \\ 1 \leq h_2 \leq q_2}} e\left(\frac{h_1 q_2 + h_2 q_1}{q_1 q_2}\right) \sum_{\substack{1 \leq h_3 \leq q_3 \\ 1 \leq h_4 \leq q_4}} e\left(\frac{2(h_3 q_4 + h_4 q_3)}{q_3 q_4}\right) \\
&\times G(q_1, h_1 q_3^2, h_1 q_3 + h_3 q_1) G(q_3, h_3 q_1^2, h_1 q_3 + h_3 q_1) \\
&\times G(q_2, h_2 q_4^2, h_2 q_4 + h_4 q_2) G(q_4, h_4 q_2^2, h_2 q_4 + h_4 q_2). \tag{11}
\end{aligned}$$

Using the substitution  $n \rightarrow \overline{(q_2 q_4)}_{q_1 q_3} n$  we get

$$\begin{aligned}
& G(q_1, h_1 q_2^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4 + q_1 h_2 q_3 q_4 + q_1 q_2 h_3 q_4 + q_1 q_2 q_3 h_4) \\
&= \sum_{n=1}^{q_1} e\left(\frac{h_1 q_2^2 q_3^2 q_4^2 n^2 + (h_1 q_2 q_3 q_4 + q_1 h_2 q_3 q_4 + q_1 q_2 h_3 q_4 + q_1 q_2 q_3 h_4)n}{q_1}\right) \\
&= \sum_{n=1}^{q_1} e\left(\frac{h_1 q_3^2 n^2 + (h_1 q_3 + h_3 q_1)n}{q_1}\right) \\
&= G(q_1, h_1 q_3^2, h_1 q_3 + h_3 q_1) \tag{12}
\end{aligned}$$

Arguing in a similar way, we obtain

$$G(q_2, h_2 q_1^2 q_3^2 q_4^2, h_1 q_2 q_3 q_4 + q_1 h_2 q_3 q_4 + q_1 q_2 h_3 q_4 + q_1 q_2 q_3 h_4) = G(q_2, h_2 q_4^2, h_2 q_4 + h_4 q_2), \tag{13}$$

$$G(q_3, h_3 q_1^2 q_2^2 q_4^2, h_1 q_2 q_3 q_4 + q_1 h_2 q_3 q_4 + q_1 q_2 h_3 q_4 + q_1 q_2 q_3 h_4) = G(q_3, h_3 q_1^2, h_1 q_3 + h_3 q_1), \tag{14}$$

$$G(q_4, h_4 q_1^2 q_2^2 q_3^2, h_1 q_2 q_3 q_4 + q_1 h_2 q_3 q_4 + q_1 q_2 h_3 q_4 + q_1 q_2 q_3 h_4) = G(q_4, h_4 q_2^2, h_2 q_4 + h_4 q_2). \tag{15}$$

Summarizing (11) – (15) we complete the proof of the lemma.  $\square$

The following lemma is the main weapon of the theorem.

**Lemma 6.** *Let  $n \geq 5$ . There exists a bijective function from the solution set of the equation*

$$x^2 + xy + 2y^2 = n, \quad (x, y) = 1, \quad x, y \in \mathbb{Z} \setminus \{0\} \quad (16)$$

*to the incongruent solutions modulo  $n$  of the congruence*

$$z^2 + z + 2 \equiv 0 (n). \quad (17)$$

*Proof.* Let us denote by  $F$  the set of ordered pairs  $(x, y)$  satisfying (16) and by  $E$  the set of solutions of the congruence (17). Every residue class modulo  $n$  with representatives satisfying (17) will be considered as one solution of (17).

Let  $(x, y) \in F$ . By (16) we have that  $(n, y) = 1$ . Then there exists a unique residue class  $z$  modulo  $n$  such that

$$zy \equiv x (n). \quad (18)$$

For this class we write

$$(z^2 + z + 2)y^2 \equiv (zy)^2 + (zy)y + 2y^2 \equiv x^2 + xy + 2y^2 \equiv 0 (n).$$

The last congruence and  $(n, y) = 1$  yield  $z^2 + z + 2 \equiv 0 (n)$  that is  $z \in E$ . We define the map

$$\beta : F \rightarrow E \quad (19)$$

that associates to each pair  $(x, y) \in F$  the residue class  $z = x\bar{y}_n$  satisfying (18).

We will first prove that the map (19) is an injection. Let  $(x, y), (x', y') \in F$  that is

$$\begin{cases} x^2 + xy + 2y^2 = n \\ x'^2 + x'y' + 2y'^2 = n \end{cases}, \quad (20)$$

$$(x, y) = (x', y') = 1 \quad (21)$$

and

$$(x, y) \neq (x', y'). \quad (22)$$

Assume that

$$\beta(x, y) = \beta(x', y'). \quad (23)$$

Hence there exists  $z \in E$  such that

$$\begin{cases} zy \equiv x (n) \\ zy' \equiv x' (n) \end{cases}. \quad (24)$$

The system (24) yields

$$xy' - x'y \equiv 0 (n). \quad (25)$$

Since the discriminants of the quadratic equations in (20) must be nonnegative and  $n \geq 5$  we derive

$$\begin{cases} 0 < |x|, |x'| \leq \sqrt{\frac{8n}{7}} \\ 0 < |y|, |y'| \leq \sqrt{\frac{4n}{7}} \end{cases}. \quad (26)$$

We first consider the case

$$xx'yy' > 0. \quad (27)$$

By (26) it follows

$$\begin{cases} 0 < |xy'| < \frac{4\sqrt{2n}}{7} \\ 0 < |x'y| < \frac{4\sqrt{2n}}{7} \end{cases}$$

and bearing in mind (27) we obtain

$$-n < xy' - x'y < n. \quad (28)$$

Now (25) and (28) lead to

$$xy' - x'y = 0$$

which together with (21) gives us

$$x = x', \quad y = y'. \quad (29)$$

From (22) and (29) we get a contradiction.

Next we consider the case

$$xx'yy' < 0. \quad (30)$$

By (20), (26) and (30) we deduce

$$\begin{cases} 0 < |x| \leq \sqrt{\frac{8n}{7}} \\ 0 < |y| \leq \sqrt{\frac{4n}{7}} \\ 0 < |x'| < \sqrt{n} \\ 0 < |y'| < \sqrt{\frac{n}{2}} \end{cases} \quad \text{or} \quad \begin{cases} 0 < |x| < \sqrt{n} \\ 0 < |y| < \sqrt{\frac{n}{2}} \\ 0 < |x'| \leq \sqrt{\frac{8n}{7}} \\ 0 < |y'| \leq \sqrt{\frac{4n}{7}} \end{cases} \quad (31)$$

and therefore

$$-2n < xy' - x'y < 2n. \quad (32)$$

Now (25) and (32) imply

$$xy' - x'y = n \quad (33)$$

or

$$xy' - x'y = -n. \quad (34)$$



After multiplying the congruence (17) by  $yy'$  and using (24) we deduce

$$z^2yy' + zyy' + 2yy' \equiv 0 \pmod{n}$$

thus

$$xx' + xy' + 2yy' \equiv 0 \pmod{n} \quad (35)$$

and

$$xx' + x'y + 2yy' \equiv 0 \pmod{n}. \quad (36)$$

On the other hand (31) yields

$$\begin{cases} 0 < |xx'| < \frac{2\sqrt{2}n}{\sqrt{7}} \\ 0 < |2yy'| < \frac{2\sqrt{2}n}{\sqrt{7}} \\ 0 < |xy'| < \frac{2n}{\sqrt{7}} \\ 0 < |x'y| < \frac{2n}{\sqrt{7}} \end{cases}$$

which together with (30) gives us

$$|xx' + xy' + 2yy'| \leq |xx' + 2yy'| + |xy'| < \frac{2\sqrt{2} + 2}{\sqrt{7}}n < 2n \quad (37)$$

and

$$|xx' + x'y + 2yy'| \leq |xx' + 2yy'| + |x'y| < \frac{2\sqrt{2} + 2}{\sqrt{7}}n < 2n. \quad (38)$$

Let (33) be true. Now (35) and (37) lead to three possibilities

$$xx' + xy' + 2yy' = 0 \quad (39)$$

or

$$xx' + xy' + 2yy' = n \quad (40)$$

or

$$xx' + xy' + 2yy' = -n. \quad (41)$$

On the one hand (21) and (39) imply  $x = \pm y'$  and therefore

$$x' + y' = \pm 2y.$$

From the last equation and (20) we derive

$$xy \pm xx' = 2y^2$$

which contradicts (21). On the other hand (33) and (40) give us

$$xx' + x'y + 2yy' = 0 \quad (42)$$

which together with (21) yields  $x' = \pm y$  and therefore

$$x + y = \pm 2y'.$$

From the last equation and (20) we derive

$$x'y' \pm xx' = 2y'^2$$

which contradicts (21). Finally (33) and (41) lead to

$$xx' + x'y + 2yy' = -2n$$

which contradicts (38).

Let (34) be true. Now (36) and (38) lead to three possibilities

$$xx' + x'y + 2yy' = 0$$

or

$$xx' + x'y + 2yy' = n \quad (43)$$

or

$$xx' + x'y + 2yy' = -n. \quad (44)$$

The first equation coincides with (42). The equation (43) due to (34) coincides with (39). The equation (44) due to (34) implies

$$xx' + xy' + 2yy' = -2n$$

which contradicts (37). The resulting contradictions show that the assumption (23) is not true. This proves the injectivity of  $\beta$ .

It remains to show that the map (19) is a surjection. Let  $z \in E$ . According to Dirichlet's approximation theorem there exist integers  $a$  and  $q$  such that

$$\left| \frac{z}{n} - \frac{a}{q} \right| < \frac{1}{q\sqrt{n}}, \quad 1 \leq q \leq \sqrt{n}, \quad (a, q) = 1. \quad (45)$$

Put

$$r = zq - an. \quad (46)$$

Therefore

$$r^2 + rq + 2q^2 = z^2q^2 - 2zqan + a^2n^2 + (zq - an)q + 2q^2 \equiv (z^2 + z + 2)q^2 (n). \quad (47)$$

Now (17) and (47) imply

$$r^2 + rq + 2q^2 \equiv 0 (n). \quad (48)$$

By (45) and (46) we get

$$|r| < \sqrt{n}. \quad (49)$$

Using (45) and (49) we deduce

$$0 < r^2 + rq + 2q^2 < 4n. \quad (50)$$

From (48) and (50) it follows that  $r^2 + rq + 2q^2 = n$ ,  $r^2 + rq + 2q^2 = 2n$  or  $r^2 + rq + 2q^2 = 3n$ . Consider all cases.

**Case 1**

$$r^2 + rq + 2q^2 = n. \quad (51)$$

From (46) and (51) we obtain

$$n = (zq - an)^2 + (zq - an)q + 2q^2 = (z^2 + z + 2)q^2 - ran - zqan - qan$$

and thus

$$ra + 1 = kq, \quad (52)$$

where

$$k = \frac{z^2 + z + 2}{n}q - az - a. \quad (53)$$

By (17) and (53) it follows that  $k \in \mathbb{Z}$  and bearing in mind (52) we get

$$(r, q) = 1. \quad (54)$$

On the other hand (51), (54) and  $n \geq 5$  give us  $r \neq 0$ . Put

$$x = r, \quad y = q. \quad (55)$$

From (51), (54) and (55) it follows that  $(x, y) \in F$ . Also (46) and (55) yield (18). Therefore  $\beta(x, y) = z$ .

**Case 2**

$$r^2 + rq + 2q^2 = 2n. \quad (56)$$

From (46) and (56) it follows that

$$2n = (zq - an)^2 + (zq - an)q + 2q^2 = (z^2 + z + 2)q^2 - ran - zqan - qan$$

and thus

$$ra + 2 = kq, \tag{57}$$

where  $k$  is denoted by (53). From (57) we deduce

$$(r, q) \leq 2. \tag{58}$$

Now (56), (58) and  $n \geq 5$  lead to  $r \neq 0$ .

**Case 2.1**

$$r = 2r_0, \quad q = 2q_0 + 1. \tag{59}$$

By (56) and (59) we get

$$q^2 + qr_0 + 2r_0^2 = n$$

which is equivalent to

$$(q + r_0)^2 - r_0(q + r_0) + 2r_0^2 = n. \tag{60}$$

Put

$$x = q + r_0, \quad y = -r_0. \tag{61}$$

Now (58) and (59) imply

$$(q + r_0, r_0) = 1. \tag{62}$$

From (60), (61) and (62) we obtain that  $(x, y) \in F$ . Further (46) and (61) give us

$$2(zy - x) = -(z^2 + z + 2)q + zan + an. \tag{63}$$

Bearing in mind (17) and (63) we deduce

$$2(zy - x) \equiv 0(n). \tag{64}$$

**Case 2.1.1**

$$r_0 = 2r_1. \tag{65}$$

Now (59), (60) and (65) assure us that  $n$  is odd. Hence (64) yields (18). Consequently  $\beta(x, y) = z$ .

**Case 2.1.2**

$$r_0 = 2r_1 + 1. \tag{66}$$

From (59), (60), (61) and (66) it follows that  $n$  and  $x$  are even. Now (59) leads to

$$zq = r + an \equiv 0 \pmod{2}$$

that is  $z$  is even. Thus

$$zy - x \text{ is even.} \tag{67}$$

**Case 2.1.2a**

$$n = 2n_0, \quad n_0 \text{ is odd.} \tag{68}$$

Now (64), (67) and (68) imply (18). Therefore  $\beta(x, y) = z$ .

**Case 2.1.2b**

$$n = 2^{l+1}n_0, \quad l \geq 1, \quad n_0 \text{ is odd.} \tag{69}$$

By (17) and (69) we obtain

$$z^2 + z + 2 \equiv 0 \pmod{2^{l+1}}. \tag{70}$$

Assume that

$$zy - x = 2^l h, \quad h \text{ is odd.} \tag{71}$$

Now (16), (69), (70) and (71) give us consistently

$$\begin{aligned} (yz)^2 + (yz)y + 2y^2 &\equiv 0 \pmod{2^{l+1}}, \\ (2^l h + x)^2 + (2^l h + x)y + 2y^2 &\equiv 0 \pmod{2^{l+1}}, \\ 2^{2l} h^2 + 2^{l+1} hx + x^2 + xy + 2y^2 + 2^l hy &\equiv 0 \pmod{2^{l+1}}, \\ 2^l hy &\equiv 0 \pmod{2^{l+1}}. \end{aligned} \tag{72}$$

From (61), (66) and (72) we get a contradiction. Consequently

$$zy - x \equiv 0 \pmod{2^{l+1}}. \tag{73}$$

Now (64), (69) and (73) yield (18). Thus  $\beta(x, y) = z$ .

**Case 2.2**

$$r = 2r_0, \quad q = 2q_0. \tag{74}$$

Bearing in mind (58) and (74) we get

$$(r_0, q_0) = 1. \tag{75}$$

By (56) and (74) we find

$$4q_0^2 + 2q_0r_0 + 2r_0^2 = n. \tag{76}$$

**Case 2.2.1**

$$r_0 = 2r_1, \quad q_0 = 2q_1 + 1. \quad (77)$$

We write equation (76) in the form

$$q^2 - q(q_0 + r_0) + 2(q_0 + r_0)^2 = n. \quad (78)$$

Put

$$x = -q, \quad y = q_0 + r_0. \quad (79)$$

Now (74), (75) and (77) lead to

$$(-q, r_0 + q_0) = 1. \quad (80)$$

Taking into account (78) – (80) we conclude that  $(x, y) \in F$ . Further (74) – (77) assure us that  $4 \mid n$  and  $4 \mid r$ . Therefore

$$zq = r + an \equiv 0 \pmod{4}. \quad (81)$$

From (74), (77) and (81) it follows that

$$z = 2z_0. \quad (82)$$

Using (46), (74), (79) and (82) we write

$$2(z_0 y - x) = -(z_0^2 + z_0 + 2)2q_0 - 2nz_0 a. \quad (83)$$

By (17) and (83) we obtain (18). Consequently  $\beta(x, y) = z$ .

**Case 2.2.2**

$$r_0 = 2r_1 + 1, \quad q_0 = 2q_1 + 1. \quad (84)$$

We write equation (76) in the form

$$(2q_0 + r_0)^2 - r_0(2q_0 + r_0) + 2r_0^2 = n. \quad (85)$$

Put

$$x = 2q_0 + r_0, \quad y = -r_0. \quad (86)$$

Now (75) and (84) imply

$$(2q_0 + r_0, -r_0) = 1. \quad (87)$$

From (85) – (87) we get  $(x, y) \in F$ . Further (76) and (84) give us

$$n \equiv 0 \pmod{4}. \quad (88)$$

If we assume that (82) is true then (46), (74), (88) and (82) yield

$$2r_0 = zq - an = 4z_0q_0 - an \equiv 0 \pmod{4}$$

which contradicts (84). This means that  $z$  is odd, that is

$$z + 1 = 2z_0. \tag{89}$$

Bearing in mind (46), (74) and (86) we deduce

$$2(z y - x) = -(z^2 + z + 2)2q_0 + an(z + 1). \tag{90}$$

From (17), (89) and (90) we establish (18). Therefore  $\beta(x, y) = z$ .

**Case 2.2.3**

$$r_0 = 2r_1 + 1, \quad q_0 = 2q_1. \tag{91}$$

When  $z$  is even then Case 2.2.3 coincides with Case 2.2.1. When  $z$  is odd then Case 2.2.3 coincides with Case 2.2.2.

**Case 2.3**

$$r = 2r_0 + 1, \quad q = 2q_0 + 1. \tag{92}$$

By (56) and (92) we obtain

$$q^2 - q(q_0 + r_0 + 1) + 2(q_0 + r_0 + 1)^2 = n. \tag{93}$$

Put

$$x = -q, \quad y = q_0 + r_0 + 1. \tag{94}$$

Now (58) and (92) assure us that

$$(-q, q_0 + r_0 + 1) = 1. \tag{95}$$

From (93) – (95) we conclude that  $(x, y) \in F$ . Further (46) and (94) lead to

$$2(z y - x) = (z^2 + z + 2)q - zan. \tag{96}$$

Using (17) and (96) we establish that (64) holds.

**Case 2.3.1** The numbers  $r_0$  and  $q_0$  are of different parity. By (93) and (94) it follows that  $n$  is odd. Hence (64) implies (18). Thus  $\beta(x, y) = z$ .

**Case 2.3.2** The numbers  $r_0$  and  $q_0$  are of the same parity.

From (93) and (94) it follows that  $y$  is odd and  $n$  is even. Now (46) and (92) give us that  $zq$  is odd. Therefore  $z$  is odd. Consequently  $zy - x$  is even. It remains to be seen

that when  $n$  has the shape (68) then Case 2.3.2 coincides with Case 2.1.2a and when  $n$  has the form (69) then Case 2.3.2 coincides with Case 2.1.2b. Hence  $\beta(x, y) = z$ .

**Case 3**

$$r^2 + rq + 2q^2 = 3n. \quad (97)$$

Direct verifications will prove that Case 3 is impossible.

**Case 3.1**

$$r = 3r_0, \quad q = 3q_0. \quad (98)$$

Now (97) and (98) yield

$$3(r_0^2 + r_0q_0 + 2q_0^2) = n,$$

i.e.  $3 \mid n$  that contradicts (17) because the congruences

$$z^2 + z + 2 \equiv 0 \pmod{3}$$

has no solution.

**Case 3.2**

$$r = 3r_0, \quad q = 3q_0 + 1. \quad (99)$$

By (97) and (99) we get

$$3(3r_0^2 + 3r_0q_0 + r_0 + 6q_0^2 + 4q_0) + 2 \equiv 0 \pmod{3}$$

which is impossible.

**Case 3.3**

$$r = 3r_0, \quad q = 3q_0 + 2. \quad (100)$$

Using (97) and (100) we deduce

$$3(3r_0^2 + 3r_0q_0 + 2r_0 + 6q_0^2 + 8q_0) + 8 \equiv 0 \pmod{3}$$

which is a contradiction.

**Case 3.4**

$$r = 3r_0 + 1, \quad q = 3q_0. \quad (101)$$

Now (97) and (101) give us

$$3(3r_0^2 + 3r_0q_0 + 2r_0 + 6q_0^2 + q_0) + 1 \equiv 0 \pmod{3}$$

which is impossible.



**Case 3.5**

$$r = 3r_0 + 1, \quad q = 3q_0 + 1. \quad (102)$$

From (97) and (102) it follows

$$3(3r_0^2 + 3r_0q_0 + 3r_0 + 6q_0^2 + 5q_0) + 4 \equiv 0 \pmod{3}$$

which is a contradiction.

**Case 3.6**

$$r = 3r_0 + 1, \quad q = 3q_0 + 2. \quad (103)$$

Now (97) and (103) lead to

$$3(3r_0^2 + 3r_0q_0 + 4r_0 + 6q_0^2 + 9q_0) + 11 \equiv 0 \pmod{3}$$

which is impossible.

**Case 3.7**

$$r = 3r_0 + 2, \quad q = 3q_0. \quad (104)$$

By (97) and (104) we obtain

$$3(3r_0^2 + 3r_0q_0 + 4r_0 + 3q_0^2 + 2q_0) + 4 \equiv 0 \pmod{3}$$

which is a contradiction.

**Case 3.8**

$$r = 3r_0 + 2, \quad q = 3q_0 + 1. \quad (105)$$

Now (97) and (105) assure us that

$$3(3r_0^2 + 3r_0q_0 + 5r_0 + 6q_0^2 + 6q_0) + 8 \equiv 0 \pmod{3}$$

which is impossible.

**Case 3.9**

$$r = 3r_0 + 2, \quad q = 3q_0 + 2. \quad (106)$$

From (97) and (106) we write

$$3(3r_0^2 + 3r_0q_0 + 6r_0 + 6q_0^2 + 10q_0) + 16 \equiv 0 \pmod{3}$$

which is a contradiction.

This completes the proof of the lemma. □

## 4 Proof of the theorem

Using (4) and the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

we get

$$\Gamma(X) = \sum_{\substack{d_1, d_2 \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{1 \leq n \leq X \\ n^2+n+1 \equiv 0 \pmod{d_1^2} \\ n^2+n+2 \equiv 0 \pmod{d_2^2}}} 1 = \Gamma_1(X) + \Gamma_2(X), \quad (107)$$

where

$$\Gamma_1(X) = \sum_{\substack{d_1 d_2 \leq z \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \quad (108)$$

$$\Gamma_2(X) = \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \quad (109)$$

$$\Sigma(X, d_1^2, d_2^2) = \sum_{\substack{1 \leq n \leq X \\ n^2+n+1 \equiv 0 \pmod{d_1^2} \\ n^2+n+2 \equiv 0 \pmod{d_2^2}}} 1, \quad (110)$$

$$\sqrt{X} \leq z < X, \quad (111)$$

where  $z$  is to be chosen later.

### 4.1 Estimation of $\Gamma_1(\mathbf{X})$

Assume that  $q_1 = d_1^2$ ,  $q_2 = d_2^2$ , where  $d_1$  and  $d_2$  are square-free,  $(q_1, q_2) = 1$  and  $d_1 d_2 \leq z$ . Define

$$\Omega(X, q_1, q_2, n) = \sum_{\substack{m \leq X \\ m \equiv n \pmod{q_1 q_2}}} 1. \quad (112)$$

Obviously

$$\Omega(X, q_1, q_2, n) = \frac{X}{q_1 q_2} + \mathcal{O}(1). \quad (113)$$

By (110) and (112) we obtain upon partitioning the sum (110) into residue classes modulo  $q_1 q_2$

$$\Sigma(X, q_1, q_2) = \sum_{\substack{1 \leq n \leq q_1 q_2 \\ n^2+n+1 \equiv 0 \pmod{q_1} \\ n^2+n+2 \equiv 0 \pmod{q_2}}} \Omega(X, q_1, q_2, n). \quad (114)$$

From (3), (114) and (113) we get

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(\lambda(q_1, q_2)). \quad (115)$$

Taking into account (3) and that the number of solutions of the congruence

$$n^2 + n + 1 \equiv a(q_1)$$

is less than or equal to  $\tau(q_1)$  we deduce

$$\lambda(q_1, q_2) \ll \lambda(q_1, 1) \ll \tau(q_1) \ll \tau(q_1 q_2). \quad (116)$$

Now (115), (116) and the inequalities

$$\tau(q_1 q_2) \ll (q_1 q_2)^\varepsilon \ll X^\varepsilon$$

imply

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(X^\varepsilon). \quad (117)$$

Using (108), (111) and (117) we obtain

$$\begin{aligned} \Gamma_1(X) &= X \sum_{\substack{d_1 d_2 \leq z \\ (d_1, d_2) = 1}} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(z X^\varepsilon) \\ &= \sigma X - X \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2) = 1}} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(z X^\varepsilon), \end{aligned} \quad (118)$$

where

$$\sigma = \sum_{\substack{d_1, d_2 = 1 \\ (d_1, d_2) = 1}}^{\infty} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2}. \quad (119)$$

By (116) it follows

$$\sum_{\substack{d_1 d_2 > z \\ (d_1, d_2) = 1}} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} \ll \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2) = 1}} \frac{(d_1 d_2)^\varepsilon}{(d_1 d_2)^2} \ll \sum_{n > z} \frac{\tau(n)}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}. \quad (120)$$

It remains to see that the product (8) and the sum (119) coincide. From Lemma 5 and  $(d_1, d_2) = 1$  we have

$$\lambda(d_1^2, d_2^2) = \lambda(d_1^2, 1) \lambda(1, d_2^2). \quad (121)$$

Now (119) and (121) yield

$$\sigma = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2), \quad (122)$$

where

$$f_{d_1}(d_2) = \begin{cases} 1 & \text{if } (d_1, d_2) = 1, \\ 0 & \text{if } (d_1, d_2) > 1. \end{cases}$$

It is easy to see that the function

$$\frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is multiplicative with respect to  $d_2$  and the series

$$\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is absolutely convergent. Using the Euler product we write

$$\begin{aligned} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2) &= \prod_{p \nmid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \\ &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}. \end{aligned} \quad (123)$$

From (122) and (123) we get

$$\begin{aligned} \sigma &= \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}. \end{aligned} \quad (124)$$

It is easy to see that the function

$$\frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}$$

is multiplicative with respect to  $d_1$  and the series

$$\sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}$$

is absolutely convergent. Using again the Euler product from (3) and (124) we deduce

$$\begin{aligned}\sigma &= \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_p \left(1 - \frac{\lambda(p^2, 1)}{p^2} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}\right) \\ &= \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^2}\right).\end{aligned}\tag{125}$$

Summarizing (111), (118), (120) and (125) we establish

$$\Gamma_1(X) = \sigma X + \mathcal{O}(zX^\varepsilon),\tag{126}$$

where  $\sigma$  is given by the product (8).

## 4.2 Estimation of $\Gamma_2(\mathbf{X})$

Bearing in mind (109), (110) and splitting the range of  $d_1$  and  $d_2$  into dyadic subintervals of the form  $D_1 \leq d_1 < 2D_1$ ,  $D_2 \leq d_2 < 2D_2$  we obtain

$$\Gamma_2(X) \ll (\log X)^2 \sum_{n \leq X} \sum_{\substack{D_1 \leq d_1 < 2D_1 \\ n^2 + n + 1 \equiv 0 \pmod{d_1^2}}} \sum_{\substack{D_2 \leq d_2 < 2D_2 \\ n^2 + n + 2 \equiv 0 \pmod{d_2^2}}} 1,\tag{127}$$

where

$$\frac{1}{2} \leq D_1, D_2 \leq \sqrt{X^2 + X + 2}, \quad D_1 D_2 > \frac{z}{4}.\tag{128}$$

On the one hand (127) implies

$$\Gamma_2(X) \ll X^\varepsilon \Sigma_1,\tag{129}$$

where

$$\Sigma_1 = \sum_{n \leq X} \sum_{\substack{D_1 \leq d_1 < 2D_1 \\ n^2 + n + 1 \equiv 0 \pmod{d_1^2}}} 1.\tag{130}$$

On the other hand (127) gives us

$$\Gamma_2(X) \ll X^\varepsilon \Sigma_2,\tag{131}$$

where

$$\Sigma_2 = \sum_{n \leq X} \sum_{\substack{D_2 \leq d_2 < 2D_2 \\ n^2 + n + 2 \equiv 0 \pmod{d_2^2}}} 1.\tag{132}$$

### Estimation of $\Sigma_1$

Arguing as in [3] we get

$$\Sigma_1 \ll X^{1+\varepsilon} D_1^{-\frac{1}{4}}. \quad (133)$$

### Estimation of $\Sigma_2$

Define

$$\mathcal{N}(d) = \{n \in \mathbb{N} : 1 \leq n \leq d, \quad n^2 + n + 2 \equiv 0 \pmod{d}\}, \quad (134)$$

$$\mathcal{N}'(d) = \{n \in \mathbb{N} : 1 \leq n \leq d^2, \quad n^2 + n + 2 \equiv 0 \pmod{d^2}\}. \quad (135)$$

From (132) and (135) we write

$$\begin{aligned} \Sigma_2 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \sum_{\substack{m \leq X \\ m \equiv n \pmod{d_2^2}}} 1 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \left( \left[ \frac{X-n}{d_2^2} \right] - \left[ \frac{-n}{d_2^2} \right] \right) \\ &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \left( \frac{X}{d_2^2} + \psi \left( \frac{-n}{d_2^2} \right) - \psi \left( \frac{X-n}{d_2^2} \right) \right) \\ &\ll X^{1+\varepsilon} D_2^{-1} + |\Sigma'_2| + |\Sigma''_2|, \end{aligned} \quad (136)$$

where

$$\Sigma'_2 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \psi \left( \frac{-n}{d_2^2} \right), \quad (137)$$

$$\Sigma''_2 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}'(d_2)} \psi \left( \frac{X-n}{d_2^2} \right). \quad (138)$$

Arguing as in [3] we deduce

$$\Sigma'_2 \ll X^\varepsilon D_2^{-1}. \quad (139)$$

Further we consider the sum  $\Sigma''_2$  defined by (138). Let  $D_2 \leq X^{\frac{1}{2}}$ . The trivial estimation leads to

$$\Sigma''_2 \ll \sum_{D_2 \leq d_2 < 2D_2} d_2^\varepsilon \ll X^{\frac{1}{2}+\varepsilon}. \quad (140)$$

Let

$$D_2 > X^{\frac{1}{2}}. \quad (141)$$

We notice that all summands in the sum (132) for which  $7 \mid d_2$  are equal to zero because the congruences

$$n^2 + n + 2 \equiv 0 \pmod{49} \quad (142)$$

has no solution. That's why in the estimation of (132) we will consider that  $7 \nmid d_2$ .

Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  be a polynomial with integral coefficients and  $r_1, \dots, r_k$  be all solutions of the congruence

$$f(x) \equiv 0 \pmod{p^{l-1}}. \quad (143)$$

From the theory of the congruences we know that when  $p \nmid f'(r_i)$  for  $i = 1, \dots, k$  then the number of solutions of the congruence

$$f(x) \equiv 0 \pmod{p^l}. \quad (144)$$

is also equal to  $k$ , that is, congruences (143) and (144) have an equal number of solutions. Taking into account the above considerations, we derive that the congruences

$$n^2 + n + 2 \equiv 0 \pmod{d_2^2} \quad (145)$$

and

$$n^2 + n + 2 \equiv 0 \pmod{d_2} \quad (146)$$

will have an equal number of solutions if we show that for arbitrary prime divisor  $p$  of  $d_2$  and arbitrary solution  $r$  of (146) we have that

$$p \nmid 2r + 1.$$

We assume the opposite. Hence

$$r = \frac{ph - 1}{2}, \quad (147)$$

where  $h \in \mathbb{Z}$ . Now (147) and

$$r^2 + r + 2 \equiv 0 \pmod{p}$$

yield

$$p^2 h^2 + 7 \equiv 0 \pmod{4p}$$

which means  $p = 7$ . But we have already excluded the case when 7 is a prime divisor of  $d_2$ . Therefore congruences (145) and (146) have an equal number of solutions. We note that the sum over  $n$  in (132) does not contain terms with  $n = \frac{d_2^2}{2}$  and  $n = d_2^2$ . Moreover for any  $n$  satisfying the congruence (145) and such that  $1 \leq n < \frac{d_2^2}{2}$  the number  $d_2^2 - n - 1$  satisfies the same congruence. The same is true for the congruence (146). We also note that if  $n = d_2^2 - n - 1$  then  $d_2^2 \mid 7$  which is impossible and if  $n = d_2 - n - 1$  then  $d_2 = 7$  which we excluded as an possibility. Using this facts and notations (134), (135) we denote

$$k = \#\mathcal{N}(d_2) = \#\mathcal{N}'(d_2), \quad (148)$$

$$n_1, \dots, n_k \in \mathcal{N}(d_2), \quad n'_1, \dots, n'_k \in \mathcal{N}'(d_2). \quad (149)$$

Now (134), (135), (141), (148), (149) and  $d_2 \geq D_2$  give us

$$\begin{aligned}
& \sum_{n \in \mathcal{N}'(d_2)} \psi \left( \frac{X-n}{d_2^2} \right) = \sum_{n \in \mathcal{N}'(d_2)} \left( \frac{X-n}{d_2^2} - \frac{1}{2} \right) \\
&= \sum_{n \in \mathcal{N}'(d_2)} \left( \frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{n'_1 + \dots + n'_{k/2} + (d_2^2 - n'_1 - 1) + \dots + (d_2^2 - n'_{k/2} - 1)}{d_2^2} \\
&= \sum_{n \in \mathcal{N}(d_2)} \left( \frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{k(d_2^2 - 1)}{2d_2^2} \\
&= \sum_{n \in \mathcal{N}(d_2)} \left( \frac{X}{d_2^2} - \frac{1}{2} \right) - \frac{n_1 + \dots + n_{k/2} + (d_2 - n_1 - 1) + \dots + (d_2 - n_{k/2} - 1)}{d_2} \left( 1 + \frac{1}{d_2} \right) \\
&= \sum_{n \in \mathcal{N}(d_2)} \left( \frac{X}{d_2^2} - \frac{1}{2} \right) - \left( 1 + \frac{1}{d_2} \right) \sum_{n \in \mathcal{N}(d_2)} \frac{n}{d_2} \\
&= \sum_{n \in \mathcal{N}(d_2)} \left( \frac{X}{d_2^2} - \frac{\sqrt{X}}{d_2} - \frac{n}{d_2^2} \right) + \sum_{n \in \mathcal{N}(d_2)} \left( \frac{\sqrt{X} - n}{d_2} - \frac{1}{2} \right) \\
&= \sum_{n \in \mathcal{N}(d_2)} \left( \frac{X}{d_2^2} - \frac{\sqrt{X}}{d_2} - \frac{n}{d_2^2} \right) + \sum_{n \in \mathcal{N}(d_2)} \psi \left( \frac{\sqrt{X} - n}{d_2} \right). \quad (150)
\end{aligned}$$

From (138), (141) and (150) it follows

$$\Sigma_2'' \ll X^{\frac{1}{2}+\varepsilon} + |\Sigma_3|, \quad (151)$$

where

$$\Sigma_3 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \psi \left( \frac{\sqrt{X} - n}{d_2} \right). \quad (152)$$

Using (152) and Lemma 2 with

$$M = X^{\frac{1}{2}} \quad (153)$$

we obtain

$$\begin{aligned}
\Sigma_3 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \left( - \sum_{1 \leq |m| \leq M} \frac{e \left( m \left( \frac{\sqrt{X} - n}{d_2} \right) \right)}{2\pi i m} + \mathcal{O} \left( f_M \left( \frac{\sqrt{X} - n}{d_2} \right) \right) \right) \\
&= \Sigma_4 + \Sigma_5, \quad (154)
\end{aligned}$$



where

$$\Sigma_4 = \sum_{1 \leq |m| \leq M} \frac{\Theta_m}{2\pi i m}, \quad (155)$$

$$\Theta_m = \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{\sqrt{X}m}{d_2}\right) \sum_{n \in \mathcal{N}(d_2)} e\left(-\frac{nm}{d_2}\right), \quad (156)$$

$$\Sigma_5 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} f_M\left(\frac{\sqrt{X} - n}{d_2}\right). \quad (157)$$

By (156), (157) and Lemma 2 we derive

$$\begin{aligned} \Sigma_5 &= \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}(d_2)} \sum_{m=-\infty}^{+\infty} b_M(m) e\left(\frac{\sqrt{X} - n}{d_2} m\right) = \sum_{m=-\infty}^{+\infty} b_M(m) \Theta_m \\ &\ll \frac{\log M}{M} |\Theta_0| + \frac{\log M}{M} \sum_{1 \leq |m| \leq M^{1+\varepsilon}} |\Theta_m| + \sum_{|m| > M^{1+\varepsilon}} |b_M(m)| |\Theta_m| \\ &\ll \frac{\log M}{M} D_2^{1+\varepsilon} + \frac{\log M}{M} \sum_{1 \leq m \leq M^{1+\varepsilon}} |\Theta_m| + D_2^{1+\varepsilon} \sum_{|m| > M^{1+\varepsilon}} |b_M(m)| \\ &\ll \frac{\log M}{M} D_2^{1+\varepsilon} + \frac{\log M}{M} \sum_{1 \leq m \leq M^{1+\varepsilon}} |\Theta_m|. \end{aligned} \quad (158)$$

Now (154), (155) and (158) imply

$$\Sigma_3 \ll X^\varepsilon \left( \frac{D_2}{M} + \sum_{1 \leq m \leq M^{1+\varepsilon}} \frac{|\Theta_m|}{m} \right). \quad (159)$$

Define

$$\mathcal{F}(d) = \{(u, v) : u^2 + uv + 2v^2 = d, (u, v) = 1, u, v \in \mathbb{Z} \setminus \{0\}\}. \quad (160)$$

According to Lemma 6 there exists a bijection

$$\beta : \mathcal{F}(d) \rightarrow \mathcal{N}(d)$$

from  $\mathcal{F}(d)$  to  $\mathcal{N}(d)$  defined by (134) that associates to each couple  $(u, v) \in \mathcal{F}(d)$  the element  $n \in \mathcal{N}(d)$  satisfying

$$nv \equiv u \pmod{d}. \quad (161)$$

Now (161) yields

$$n_{u,v} \equiv u \bar{v}_d \pmod{d}$$

thus

$$\frac{n_{u,v}}{d} \equiv u \frac{\bar{v}u^2+uv+2v^2}{u^2+uv+2v^2} \quad (1). \quad (162)$$

From (162) and Lemma 3 it follows

$$\frac{n_{u,v}}{d} \equiv \frac{u}{v(u^2+uv+2v^2)} - \frac{\bar{u}|v|}{v} \quad (1), \quad (163)$$

$$\frac{n_{u,v}}{d} \equiv -\frac{u+2v}{u(u^2+uv+2v^2)} + \frac{\bar{v}_u}{u} \quad (1). \quad (164)$$

Bearing in mind (156), (160), (163) and (164) we get

$$\begin{aligned} \Theta_m &= \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{(u,v) \in \mathcal{F}(d_2)} e\left(-\frac{n_{u,v}}{d_2}m\right) \\ &= \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{\substack{(u,v) \in \mathcal{F}(d_2) \\ 0 < |u| < |v|}} e\left(-\frac{mu}{v(u^2+uv+2v^2)} + \frac{m\bar{u}|v|}{v}\right) \\ &+ \sum_{D_2 \leq d_2 < 2D_2} e\left(\frac{m\sqrt{X}}{d_2}\right) \sum_{\substack{(u,v) \in \mathcal{F}(d_2) \\ 0 < |v| < |u|}} e\left(\frac{m(u+2v)}{u(u^2+uv+2v^2)} - \frac{m\bar{v}_u}{u}\right) \\ &= \sum_{\substack{D_2 \leq u^2+uv+2v^2 < 2D_2 \\ 0 < |u| < |v| \\ (u,v)=1}} e\left(\frac{m\sqrt{X}}{u^2+uv+2v^2} - \frac{mu}{v(u^2+uv+2v^2)} + \frac{m\bar{u}|v|}{v}\right) \\ &+ \sum_{\substack{D_2 \leq u^2+uv+2v^2 < 2D_2 \\ 0 < |v| < |u| \\ (u,v)=1}} e\left(\frac{m\sqrt{X}}{u^2+uv+2v^2} + \frac{m(u+2v)}{u(u^2+uv+2v^2)} - \frac{m\bar{v}_u}{u}\right) \\ &= \Theta'_m + \Theta''_m, \end{aligned} \quad (165)$$

say. Consider  $\Theta'_m$ . Let for any fixed  $\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}$  the interval  $[\eta_1(v), \eta_2(v)]$  is a solution with respect to  $u$  of the system

$$\begin{cases} u^2 + uv + 2v^2 < 2D_2 \\ u^2 + uv + 2v^2 \geq D_2 \\ 0 < |u| < |v| \end{cases} . \quad (166)$$

Denote

$$g(u) = e\left(\frac{m\sqrt{X}}{u^2+uv+2v^2} - \frac{mu}{v(u^2+uv+2v^2)}\right), \quad (167)$$

$$K_{v,m}(t) = \sum_{\substack{\eta_1(v) \leq u < t \\ (u,v)=1}} e\left(\frac{m\bar{u}|v|}{v}\right). \quad (168)$$

By (165) – (168) and Abel’s summation formula we deduce

$$\begin{aligned}
\Theta'_m &= \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \sum_{\substack{\eta_1(v) \leq u < \eta_2(v) \\ (u,v)=1}} g(u) e\left(\frac{m\bar{u}|v|}{v}\right) \\
&= \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left( g(\eta_2(v)) K_{v,m}(\eta_2(v)) - \int_{\eta_1(v)}^{\eta_2(v)} K_{v,m}(t) \left(\frac{d}{dt} g(t)\right) dt \right) \\
&\ll \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left(1 + \frac{m\sqrt{X}}{v^2}\right) \max_{\eta_1(v) \leq t \leq \eta_2(v)} |K_{v,m}(t)|. \tag{169}
\end{aligned}$$

We are now in a good position to apply Lemma 4 because the sum defined by (168) is incomplete Kloosterman sum. We have

$$K_{v,m}(t) \ll |v|^{\frac{1}{2}+\varepsilon} (v, m)^{\frac{1}{2}}. \tag{170}$$

From (169) and (170) we derive

$$\begin{aligned}
\Theta'_m &\ll \sum_{\sqrt{\frac{D_2}{4}} \leq |v| < \sqrt{D_2}} \left(1 + \frac{m\sqrt{X}}{v^2}\right) |v|^{\frac{1}{2}+\varepsilon} (v, m)^{\frac{1}{2}} \\
&\ll X^\varepsilon \left(D_2^{\frac{1}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{3}{4}}\right) \sum_{0 < v < \sqrt{D_2}} (v, m)^{\frac{1}{2}}. \tag{171}
\end{aligned}$$

On the other hand

$$\sum_{0 < v < \sqrt{D_2}} (v, m)^{\frac{1}{2}} \leq \sum_{l|m} l^{\frac{1}{2}} \sum_{\substack{v \leq \sqrt{D_2} \\ v \equiv 0 \pmod{l}}} 1 \ll D_2^{\frac{1}{2}} \sum_{l|m} l^{-\frac{1}{2}} \ll D_2^{\frac{1}{2}} \tau(m) \ll X^\varepsilon D_2^{\frac{1}{2}}. \tag{172}$$

The estimations (171) and (172) lead to

$$\Theta'_m \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}}\right). \tag{173}$$

Working in a similar way for  $\Theta''_m$  from (165) we deduce

$$\Theta''_m \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}}\right). \tag{174}$$

Now (165), (173) and (174) yield

$$\Theta_m \ll X^\varepsilon \left(D_2^{\frac{3}{4}} + mX^{\frac{1}{2}} D_2^{-\frac{1}{4}}\right). \tag{175}$$

From (159) and (175) we write

$$\Sigma_3 \ll X^\varepsilon \left( D_2 M^{-1} + D_2^{\frac{3}{4}} + X^{\frac{1}{2}} M D_2^{-\frac{1}{4}} \right). \quad (176)$$

Using (128), (153) and (176) we obtain

$$\Sigma_3 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \quad (177)$$

Bearing in mind (128), (136), (139), (140), (151) and (177) we find

$$\Sigma_2 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \quad (178)$$

### Estimation of $\Gamma_2(\mathbf{X})$

Now (128), (129), (131), (133) and (178) imply

$$\Gamma_2(X) \ll X^{1+\varepsilon} z^{-\frac{1}{8}}. \quad (179)$$

## 4.3 The end of the proof

Summarizing (107), (126), (179) and choosing  $z = X^{\frac{8}{9}}$  we establish the asymptotic formula (7).

This completes the proof of the theorem.

## References

- [1] T. D. Browning, *Power-free values of polynomials*, Arch. Math., **96**, (2011), 139 – 150.
- [2] L. Carlitz, *On a problem in additive arithmetic II*, Quart. J. Math., **3**, (1932), 273 – 290.
- [3] S. I. Dimitrov, *Square-free values of  $n^2 + n + 1$* , arXiv:2205.02488.
- [4] P. Erdős, *Arithmetical properties of polynomials*, J. London Math. Soc., **28**, (1953), 416 – 425.
- [5] T. Estermann, *A new application of the Hardy-Littlewood-Kloosterman method*, Proc. London Math. Soc., **12**, (3), (1962), 425 – 444.
- [6] L. Gegenbauer, *Asymptotische Gesetze der Zahlentheorie*, Denkschriften Akad. Wiss. Wien, **49**, (1885), 37 – 80.

- [7] D. R. Heath-Brown, *The Square-Sieve and Consecutive Square-Free Numbers*, Math. Ann., **266**, (1984), 251 – 259.
- [8] D. R. Heath-Brown, *Counting rational points on algebraic varieties*, Analytic number theory, 51 – 95, Lecture Notes in Math., 1891, Springer, Berlin, 2006.
- [9] D. R. Heath-Brown, *Square-free values of  $n^2 + 1$* , Acta Arith., **155**, (2012), 1 – 13.
- [10] D. R. Heath-Brown, *Power-free values of polynomials*, Quart. J. Math. **64**, (2013), 177 – 188.
- [11] C. Hooley, *On the power-free values of polynomials*, Mathematika, **14**, (1967), 21 – 26.
- [12] H. Iwaniec, E. Kowalski, *Analytic number theory*, Colloquium Publications, **53**, Am. Math. Soc., (2004).
- [13] L. Mirsky, *On the frequency of pairs of square-free numbers with a given difference*, Bull. Amer. Math. Soc., **55**, (1949), 936 – 939.
- [14] T. Reuss, T. Reuss, *The Determinant Method and Applications*, Thesis, University of Oxford, (2015).
- [15] T. Reuss, *Power-free values of polynomials*, Bull. London Math. Soc., **47**, (2015), 270 – 284.
- [16] D. I. Tolev, *On the exponential sum with squarefree numbers*, Bull. Lond. Math. Soc., **37**, 6, (2005), 827 – 834.

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