



**HAL**  
open science

# Mathematical issues in a fully constrained formulation of the Einstein equations

Isabel Cordero-Carrión, José-María Ibáñez, Éricourgoulhon, José Luis Jaramillo, Jérôme Novak

► **To cite this version:**

Isabel Cordero-Carrión, José-María Ibáñez, Éricourgoulhon, José Luis Jaramillo, Jérôme Novak. Mathematical issues in a fully constrained formulation of the Einstein equations. *Physical Review D*, 2008, 77, pp.84007. 10.1103/PhysRevD.77.084007. hal-03733441

**HAL Id: hal-03733441**

**<https://hal.science/hal-03733441v1>**

Submitted on 29 Jul 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**Mathematical issues in a fully constrained formulation of the Einstein equations**Isabel Cordero-Carrión,<sup>1,\*</sup> José María Ibáñez,<sup>1,†</sup> Ericourgoulhon,<sup>2,‡</sup> José Luis Jaramillo,<sup>2,3,§</sup> and Jérôme Novak<sup>2,||</sup><sup>1</sup>*Departamento de Astronomía y Astrofísica, Universidad de Valencia, Valencia, Spain*<sup>2</sup>*Laboratoire Univers et Théories (LUTH), Observatoire de Paris, CNRS, Université Paris Diderot, Place Jules Janssen, 92190 Meudon, France*<sup>3</sup>*Instituto de Astrofísica de Andalucía, CSIC, Apartado Postal 3004, Granada 18080, Spain*

(Received 28 January 2008; published 7 April 2008)

Bonazzola,ourgoulhon,Grandclément, and Novak [Phys. Rev. D **70**, 104007 (2004)] proposed a new formulation for 3 + 1 numerical relativity. Einstein equations result, according to that formalism, in a coupled elliptic-hyperbolic system. We have carried out a preliminary analysis of the mathematical structure of that system, in particular, focusing on the equations governing the evolution for the deviation of a conformal metric from a flat fiducial one. The choice of a Dirac gauge for the spatial coordinates guarantees the mathematical characterization of that system as a (strongly) hyperbolic system of conservation laws. In the presence of boundaries, this characterization also depends on the boundary conditions for the shift vector in the elliptic subsystem. This interplay between the hyperbolic and elliptic parts of the complete evolution system is used to assess the prescription of inner boundary conditions for the hyperbolic part when using an excision approach to black hole space-time evolutions.

DOI: [10.1103/PhysRevD.77.084007](https://doi.org/10.1103/PhysRevD.77.084007)

PACS numbers: 04.25.D-, 02.30.Jr, 02.60.Lj, 04.20.Ex

**I. A FULLY-CONSTRAINED EVOLUTION SCHEME**

A second-order, fully constrained, evolution formalism for the Einstein equations has been proposed in Ref. [1]. This evolution scheme, which will be referred to in the following as the fully constrained formulation (FCF), is based on a conformal 3 + 1 formulation of general relativity and makes use of an elliptic condition for the choice of spatial coordinates, a *generalized Dirac gauge*, and a maximal condition for the slicing. The enforcement of the constraints along the evolution, together with the elliptic nature of the employed gauge conditions, translates the FCF formalism into a mixed elliptic-hyperbolic partial differential equation (PDE) system, consisting in five quasilinear elliptic equations coupled with a tensorial second order in time and in space evolution equation for the conformal metric. In this article, we aim at gaining insight on some mathematical issues associated with this PDE system and, in particular, assessing the hyperbolicity of the tensorial evolution part. A good understanding of the mathematical structure of the system will be crucial in the context of full 3D numerical relativity simulations, since the choice of state-of-the-art numerical tools will be adapted to the specific structures of the whole system governing the evolution of matter fields in a dynamical space-time: spectral methods for the elliptic subsystem [2], and modern high-resolution shock-capturing techniques for the hyperbolic part [3,4]. The implementation of the scheme in [1] will naturally extend previous works—following the conformal flatness condition (CFC) approach of

Isenberg-Wilson-Mathews [5,6]—devoted to the study of some relevant astrophysical sources of gravitational radiation [7–10].

**A. Gauge reduction, PDE evolution systems, and well-posedness**

The gauge character of general relativity (GR) strongly conditions any attempt of finding a solution by solving a PDE problem. In its standard formulation through the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (1)$$

solutions are given in terms of space-time geometries ( $\mathcal{M}, g_{\mu\nu}$ ), i.e. classes of Lorentzian metrics  $g_{\mu\nu}$  equivalent under diffeomorphisms of  $\mathcal{M}$ , rather than by specific 4-metrics in some particular coordinate system. As a consequence of this, any attempt to cast (1) as a standard PDE system necessarily must go through a *gauge reduction* process. This fixing of the gauge involves four different (differential) systems: (i) the *reduced* system, whose solution provides the metric in a given coordinate system, (ii) the *constraint system*, consequence of the gauge character of the theory and that characterizes the solution manifold, (iii) the *gauge system*, which fixes the coordinate chart and permits one to write the reduced system as a standard PDE problem, and (iv) the *subsidiary system*, guaranteeing the overall consistency along the evolution and, in particular, between the reduced and gauge systems. The mathematical consistency of the evolution formalism involves two aspects. First, one must assess the analytic well-posedness of the PDE system that is actually solved during the evolution, which we will refer to in the following as the *evolution PDE system* that includes the *reduced system* but possibly other additional PDEs. Second, one

\*Isabel.Cordero@uv.es

†Jose.M.Ibanez@uv.es

‡eric.gourgoulhon@obspm.fr

§jarama@iaa.es

||Jerome.Novak@obspm.fr

must guarantee the fulfillment of the *subsidiary system* during the evolution.

As in other evolution formalisms based on the initial value problem for the Einstein equation [11], the constrained system in the FCF scheme follows from the Gauss-Codazzi-Ricci conditions

$$\begin{aligned} {}^{(3)}R - K_{ij}K^{ij} + K^2 &= 16\pi\rho, \\ D_j(K^{ij} - \gamma^{ij}K) &= 8\pi J^i, \end{aligned} \quad (2)$$

i.e. the Hamiltonian and momentum constraints in the 3 + 1 formulation ( $\rho$  is the energy density and  $J^i$  the current vector) which are elliptic in nature. The currently most successful numerical evolution formalisms are *free* schemes in which the constraint system (2) is not enforced during the evolution. This is the case of certain generalized harmonic formalisms [12,13] and the 3 + 1 BSSN (from Baumgarte, Shapiro, Shibata, and Nakamura; see Refs. [14,15]) used in recent binary black hole breakthroughs [16–19] and in fully 3D evolution of binary neutron stars (see e.g. [20]). In these free schemes, the corresponding evolution PDE system is formed by the respective reduced systems together with some additional evolution equations to fix the harmonic gauge sources, in the case of generalized harmonic schemes, or the lapse function and shift vector, in the BSSN case. No elliptic equation is solved during the evolution, and standard hyperbolic techniques can, in principle, be used to assess the well-posedness of the evolution system (cf., in this sense, [21] for the case of the BSSN system). In contrast, the FCF discussed here actually incorporates the constraints to the evolution PDE system. Moreover, the use of the above-mentioned elliptic gauge conditions adds additional elliptic equations during the evolution. The resulting FCF scheme presents some interesting properties as compared with free evolution schemes. Apart from the absence of constraint violations (an issue under control in current BSSN and generalized harmonic formulations), we can highlight the following features (cf. [1] for a more complete discussion): first, the FCF naturally generalizes (as commented above) the successful scheme employed in the CFC approximation to general relativity; second, it permits one to read the gravitational waveforms directly from the metric components; third, the scheme can be straightforwardly adapted to the extraction of gravitational radiation at null infinity by making use of hyperboloidal 3-slices implemented by means of a *constant mean curvature* elliptic gauge condition; and fourth, it provides a well-suited framework for the formulation of realistic (approximate) prescriptions in the construction of quasistationary astrophysically configurations [22]. However, the well-posedness analysis of such a mixed elliptic-hyperbolic system can be a formidable problem, since part of the dynamics related to the characteristic fields in the hyperbolic part is encoded in fields obtained only once the elliptic part is solved. Even though analyses of such systems exist in the GR literature (see e.g.

Refs. [23–25] and particularly Ref. [26]) they deal with free evolution systems, in which the elliptic part follows only from the gauge conditions. The well-posedness analysis of the complete elliptic-hyperbolic system in the FCF scheme, which in addition includes the constraints, is beyond the scope of this work and we will mainly focus on the hyperbolicity analysis of the tensorial evolution equation. Before referring to the additional issues related to the subsidiary system, we must provide some details about the FCF formalism.

## B. Brief review of the FCF scheme

Following Ref. [1], we consider a standard 3 + 1 decomposition of an asymptotically flat space-time ( $\mathcal{M}$ ,  $g_{\mu\nu}$ ) in terms of a foliation by spacelike hypersurfaces ( $\Sigma_t$ ). We denote the unit timelike normal vector to the spacelike slice  $\Sigma_t$  by  $n^\mu$ , the spatial 3-metric by  $\gamma_{\mu\nu}$ , i.e.  $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ , and adopt the following sign convention for the extrinsic curvature:  $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$ . The evolution vector  $t^\mu \equiv (\partial_t)^\mu$  is decomposed in terms of the lapse function  $N$  and the shift vector  $\beta^\mu$ , as  $t^\mu = Nn^\mu + \beta^\mu$ .

Under this 3 + 1 decomposition, the Einstein equation (1) splits into the 3 + 1 constraints in (2) and a set of evolution equations for the extrinsic curvature that, together with the kinematical relation defining the extrinsic curvature, constitute the 3 + 1 evolution equations

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta)\gamma_{ij} &= -2NK_{ij}(\partial_t - \mathcal{L}_\beta)K_{ij} \\ &= -D_i D_j N + N\{{}^{(3)}R_{ij} + KK_{ij} - 2K_i^k K_{kj} \\ &\quad + 4\pi[(S - E)\gamma_{ij} - 2S_{ij}]\}. \end{aligned} \quad (3)$$

This is a first order in time and second order in space evolution system for  $(\gamma_{ij}, K^{ij})$ .

The first specific element in the FCF scheme is the introduction of a *time independent* fiducial flat metric  $f_{ij}$ , which satisfies  $\mathcal{L}_t f_{ij} = \partial_t f_{ij} = 0$ . This rigid structure is chosen to coincide with  $\gamma_{ij}$  at spatial infinity, capturing its asymptotic Euclidean character, and permits one to work with tensor quantities rather than with tensor densities. We will denote by  $\mathcal{D}_i$  the Levi-Civita connection associated with  $f_{ij}$ .

(a) *Conformal decomposition.*—As a step forward in the reduction process to the PDE system in the present FCF, we perform a conformal decomposition of the 3 + 1 fields:

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}, \quad K^{ij} = \Psi^4 \tilde{A}^{ij} + \frac{1}{3}K\gamma^{ij}, \quad (4)$$

where  $K = \gamma^{ij}K_{ij}$ , the representative  $\tilde{\gamma}_{ij}$  of the conformal class of the 3-metric is chosen to satisfy the unimodular condition  $\det(\tilde{\gamma}_{ij}) = \det(f_{ij})$ , and the traceless part  $\tilde{A}^{ij}$  of the extrinsic curvature is decomposed as

$$\tilde{A}^{ij} = \frac{1}{2N} \left( \tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij} + \partial_t \tilde{\gamma}^{ij} \right), \quad (5)$$

with  $\tilde{D}_i$  the Levi-Civita connection associated with  $\tilde{\gamma}_{ij}$ . Finally, in the following we will denote by  $h^{ij}$  the deviation of the conformal metric from the flat fiducial metric, i.e.

$$h^{ij} := \tilde{\gamma}^{ij} - f^{ij}. \quad (6)$$

Using these conformal decompositions of  $\gamma_{ij}$  and  $K^{ij}$ , the 3 + 1 constraints (2) and evolution system (3) can be expressed in terms of the basic variables  $h^{ij}$ ,  $\Psi$ ,  $N$ ,  $\beta^i$ ,  $K$ . Before giving more explicit expressions, let us remove the gauge freedom.

- (b) *Gauge system.*—Following the prescriptions in [1], namely, maximal slicing and the so-called generalized Dirac gauge, we choose

$$K = 0, \quad H^i := \mathcal{D}_k \tilde{\gamma}^{ki} = 0, \quad (7)$$

These gauge conditions fix the coordinates, even in the initial slice, up to boundary terms (see e.g. Secs. 9.3. and 9.4. in [27]). These two relations define the gauge system in the FCF scheme. Since the gauge system is meant to hold at all times, the following conditions must also be satisfied:

$$\dot{K} = 0, \quad \partial_t (\mathcal{D}_k \tilde{\gamma}^{ki}) = 0. \quad (8)$$

The FCF scheme actually enforces the first of these conditions,  $\dot{K} = 0$ , during the evolution. Taking the trace in the second equation in (3), and using the Hamiltonian constraint that is also enforced during the evolution (see below), an elliptic equation for the lapse follows,

$$\tilde{D}_k \tilde{D}^k N + 2\tilde{D}_k \ln \Psi \tilde{D}^k N = S_N[N, \Psi, \beta^i, \tilde{\gamma}_{ij}]. \quad (9)$$

- (c) *Main or reduced system.*—In the FCF scheme in Ref. [1] the reduced system is a second order in time and second order in space evolution system for the deviation tensor  $h^{ij}$ . This is obtained by (i) combining equations in (3) into a single second order in time equation; (ii) inserting into it the conformal decompositions (4) and (5), and (iii) imposing the gauges (7). The resulting expression is formally written as (see next section for a detailed account)

$$\begin{aligned} \frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4} \tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta \mathcal{L}_\beta h^{ij} \\ = S_h^{ij}, \end{aligned} \quad (10)$$

where the source  $S_h^{ij}$  does not contain second derivatives of  $h^{ij}$ . Use of the Dirac gauge results in the wavelike form of this equation, since it eliminates

certain second derivatives of the type  $\mathcal{D}^i \mathcal{D}_k h^{kj}$  coming from the expression of the Ricci tensor.

- (d) *Constrained system.* The Hamiltonian constraint in (2) can be written as an elliptic equation for the conformal factor  $\Psi$ :

$$\tilde{D}_k \tilde{D}^k \Psi - \frac{3\tilde{R}}{8} \Psi = S_\Psi[\Psi, N, \beta^i, \tilde{\gamma}_{ij}]. \quad (11)$$

Again  $S_\Psi[\Psi, N, \beta^i, \tilde{\gamma}_{ij}]$  represents a nonlinear source. The momentum constraint poses a more subtle issue. In Ref. [1] an elliptic equation for the shift vector is deduced using *both* the momentum constraint and the preservation in time of the Dirac gauge [second relation in (8)]:

$$\begin{aligned} \tilde{D}_k \tilde{D}^k \beta^i + \frac{1}{3} \tilde{D}^i \tilde{D}_k \beta^k + \tilde{R}^i_k \beta^k \\ = S_\beta^i[\Psi, N, \beta^i, \tilde{\gamma}_{ij}]. \end{aligned} \quad (12)$$

An equation for the shift could be derived from the momentum constraint alone, but the coupling to the tensorial equation (10) would become more complicated due to the presence of a mixed time-space second-order derivative of  $h^{ij}$ . This term is eliminated by the use of a Dirac, or a similar, gauge.

Alternatively, an elliptic equation for the shift can be drawn from the preservation of the Dirac gauge alone, renouncing, therefore, to the fully constrained character of the scheme—e.g. this is the strategy in Ref. [26], but using a spatial harmonic gauge condition instead of the Dirac one. At the end of the day, the choice (12) in the FCF scheme provides an elliptic equation for the shift that enforces the momentum constraint, *as long as the Dirac gauge is satisfied*.

- (e) *FCF evolution PDE system.*—The mixed elliptic-hyperbolic PDE system that evolves some initial data given on an Cauchy slice is formed by (a) Eqs. (9), (11), and (12), the elliptic part, and (b) Eq. (10), the wavelike tensorial equation. As we have pointed out, we will not consider here the well-posedness analysis of the whole system. To give an idea of the involved difficulties, we note that the elliptic part is very similar to the extended conformal thin sandwich (XCTS) [28,29] employed in the construction of initial data, though here it is solved all along the evolution. Even the restriction to the elliptic subsystem represents a very hard problem, as it is illustrated by the lack of the existence results for the XCTS system and the preliminary numerical [30] (see also [31]) and analytical [32,33] results pointing towards a generic non-uniqueness of the elliptic system. For these reasons, we will focus on the study of the hyperbolicity of the tensorial evolution equation (10), understanding this as a necessary condition for the overall well-posedness.

(f) *Subsidiary system.*—The resolution of the PDE evolution system only guarantees the consistency between the reduced and gauge systems as far as the slicing condition is regarded, since Eq. (9) for the lapse is indeed enforced. This is, in principle, not the case for the Dirac gauge. More dramatically, if the Dirac gauge is actually not satisfied, the FCF scheme is not really fully constrained, since in that situation Eq. (12) no longer enforces the momentum constraint. Control of the evolution of the Dirac gauge is therefore crucial in the scheme. A wavelike equation for  $\mathcal{D}_k h^{ki}$  can be obtained by taking the divergence of the tensorial equation (10). The vanishing of  $\mathcal{D}_k h^{ki}$  in the evolution would then follow from the initial conditions  $\mathcal{D}_k h^{ki} = 0$  and  $\partial_t(\mathcal{D}_k h^{ki} = 0) = 0$  imposed in the construction of the initial data, and the satisfaction of Eq. (91) in Ref. [1] for  $\hat{\beta}^i$ . The latter can be considered as the *subsidiary system* in the FCF scheme.

### C. Specific objectives and organization

Though the wave character of Eq. (10) essentially guarantees its hyperbolicity, we aim here at developing a more detailed analysis. This is motivated by the need of controlling the characteristics in initial boundary problems and also when trying to make use of first-order techniques employed in matter evolutions. Our main specific goal in this article is the development of a hyperbolicity analysis of a *first-order* version of the evolution part in the FCF formalism, where  $N$ ,  $\Psi$ , and  $\beta^i$  are considered as fixed parameters. In particular, we aim at obtaining explicit expressions for the characteristic fields and speeds. As pointed out above, this point represents a fundamental ingredient in the study of the appropriate boundary conditions if boundaries are present in the integration domain. This constitutes only a preliminary study of the well-posedness of the evolution system since no stability analy-

sis whatsoever will be considered. Certainly further analysis is required. However, in the absence of a full treatment and being ultimately motivated by practical numerical implementation needs, the level of rigor and completeness in this article is adapted to the achievement of limited but concrete results.

On behalf of self-consistency, and in spite of the lack of a fully rigorous treatment of the FCF subsidiary system, we also aim at discussing certain (numerical) algorithms devised to guarantee the fulfillment of the Dirac gauge along the evolution. Though this is not a substitute for a formal proof, it provides, on the one hand, support for the coherence among the reduced, gauge and constrained systems. On the other hand, and more importantly from a practical point of view, the implementation of the FCF scheme is then guaranteed to be fully constrained, even in numerical implementations where errors can occur even if analytic well-posedness has been established.

The article is organized as follows. Section II presents a first-order formulation of the FCF scheme, more concretely, of its reduced system. In Sec. III the characteristic structure of the reduced system is analyzed, with a brief application to inner boundaries in excised black hole space-time evolutions. Section IV discusses the possibility of writing the first-order reduced FCF system as a system of conservation laws, by making explicit use of the Dirac gauge. In Sec. V two different manners of enforcing the Dirac gauge in the evolution are introduced, providing key support for overall consistency and guaranteeing the fully constrained character of the scheme. Finally Sec. VI concludes with a discussion of the results.

## II. FIRST-ORDER REDUCTION OF THE REDUCED SYSTEM IN THE FCF

Equations governing the evolution of  $h^{ij}$  in the FCF are

$$\begin{aligned}
\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\psi^4} \tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} - 2 \mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta \mathcal{L}_\beta h^{ij} &= \mathcal{L}_{\hat{\beta}} h^{ij} + \frac{4}{3} \mathcal{D}_k \beta^k \left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) h^{ij} - \frac{N}{\psi^6} \mathcal{D}_k Q (\mathcal{D}^i h^{jk} + \mathcal{D}^j h^{ik} - \mathcal{D}^k h^{ij}) \\
&+ \left[ \left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \ln N \right] \left[ \left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) h^{ij} - \frac{2}{3} \mathcal{D}_k \beta^k h^{ij} + (L\beta)^{ij} \right] \\
&+ \frac{2}{3} \left[ \left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \mathcal{D}_k \beta^k - \frac{2}{3} (\mathcal{D}_k \beta^k)^2 \right] h^{ij} - \left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) (L\beta)^{ij} \\
&+ \frac{2}{3} \mathcal{D}_k \beta^k (L\beta)^{ij} + 2N\psi^{-4} Z^{ij} \\
&+ (2N)^2 \left[ \tilde{\gamma}_{kl} A^{ik} A^{jl} - 4\pi \left( \psi^4 S^{ij} - \frac{1}{3} S \tilde{\gamma}^{ij} \right) \right] \\
&- 2N\psi^{-6} \left[ \tilde{\gamma}^{ik} \tilde{\gamma}^{jl} \mathcal{D}_k \mathcal{D}_l Q + \frac{1}{2} (h^{ik} \mathcal{D}_l h^{lj} + h^{jk} \mathcal{D}_k h^{il} - h^{kl} \mathcal{D}_k h^{ij}) \right. \\
&\left. \times \mathcal{D}_l Q - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l Q \right], \tag{13}
\end{aligned}$$



where  $S^{ij}$  and  $S$  are, respectively, the spatial components of the stress tensor  $S_{\alpha\beta} := \gamma_\alpha^\mu \gamma_\beta^\nu T_{\mu\nu}$ , associated with the matter energy-momentum tensor  $T_{\mu\nu}$ , and its trace.  $(L\beta)^{ij}$  is the conformal Killing operator associated with the flat metric  $f_{ij}$  acting on the vector field  $\beta^i$ :

$$(L\beta)^{ij} := \mathcal{D}^i \beta^j + \mathcal{D}^j \beta^i - \frac{2}{3} \mathcal{D}_k \beta^k f^{ij}, \quad (14)$$

and the auxiliary quantities  $Q$  and  $Z^{ij}$  are

$$Q := N\psi^2, \quad (15)$$

$$\begin{aligned} Z^{ij} = & N[\tilde{R}_*^{ij} + 8\psi^{-2}(\tilde{\gamma}^{ik} \mathcal{D}_k \psi)(\tilde{\gamma}^{jl} \mathcal{D}_l \psi)] \\ & + 4\psi^{-1}(\tilde{\gamma}^{ik} \mathcal{D}_k \psi)(\tilde{\gamma}^{jl} \mathcal{D}_l N) \\ & + 4\psi^{-1}(\tilde{\gamma}^{jk} \mathcal{D}_k \psi)(\tilde{\gamma}^{il} \mathcal{D}_l N) \\ & - \frac{1}{3} N[\tilde{R}_* + 8\psi^{-2} \mathcal{D}_k \psi(\tilde{\gamma}^{kl} \mathcal{D}_l \psi)] \tilde{\gamma}^{ij} \\ & - \frac{8}{3} \psi^{-1} \mathcal{D}_k \psi(\tilde{\gamma}^{kl} \mathcal{D}_k N) \tilde{\gamma}^{ij}. \end{aligned} \quad (16)$$

The symmetric tensor  $\tilde{R}_*^{ij}$  is defined by

$$\begin{aligned} \tilde{R}_*^{ij} := & \frac{1}{2} [-\mathcal{D}_l h^{ik} \mathcal{D}_k h^{jl} - \tilde{\gamma}_{kl} \tilde{\gamma}^{mn} \mathcal{D}_m h^{ik} \mathcal{D}_n h^{jl} \\ & + \tilde{\gamma}^{nl} \mathcal{D}_k h^{mn} (\tilde{\gamma}^{ik} \mathcal{D}_m h^{jl} + \tilde{\gamma}^{jk} \mathcal{D}_m h^{il})] \\ & + \frac{1}{4} \tilde{\gamma}^{ik} \tilde{\gamma}^{jl} \mathcal{D}_k h^{mn} \mathcal{D}_l \tilde{\gamma}_{mn}, \end{aligned} \quad (17)$$

and the scalar  $\tilde{R}_*$  is

$$\tilde{R}_* := \frac{1}{4} \tilde{\gamma}^{kl} \mathcal{D}_k h^{mn} \mathcal{D}_l \tilde{\gamma}_{mn} - \frac{1}{2} \tilde{\gamma}^{kl} \mathcal{D}_k h^{mn} \mathcal{D}_n \tilde{\gamma}_{mn}. \quad (18)$$

Let us write Eq. (13) as a first-order system, by introducing the following auxiliary variables:

$$u^{ij} := \frac{\partial h^{ij}}{\partial t}, \quad (19)$$

$$w_k^{ij} := \mathcal{D}_k h^{ij}. \quad (20)$$

With these new variables the system for  $h^{ij}$  can be cast into the form

$$\begin{aligned} \frac{\partial u^{ij}}{\partial t} - \frac{N^2}{\psi^4} \tilde{\gamma}^{kl} \mathcal{D}_k w_l^{ij} - 2\beta^k \mathcal{D}_k u^{ij} + \beta^k \beta^l \mathcal{D}_k w_l^{ij} \\ = \phi^{ij}(\beta^k, N, \psi, \partial_\mu \beta^k, \partial_\mu N, \partial_\mu \psi, h^{ij}, u^{ij}, w_k^{ij}), \end{aligned} \quad (21)$$

where  $\phi^{ij}$  are source terms which do not contain partial derivatives of  $u^{ij}$  or  $w_k^{ij}$ . From definition (20) we obtain

$$\frac{\partial w_k^{ij}}{\partial t} = \mathcal{D}_k u^{ij}, \quad (22)$$

where we have taken into account that  $\partial_t f^{ij} = 0$ . In terms of the above new auxiliary variables, the system of Eqs. (19), (21), and (22) can be written as

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + \mathbf{A}^l \mathcal{D}_l \bar{\mathbf{v}} = \mathbf{g}(\beta^k, N, \psi, \partial_\mu \beta^k, \partial_\mu N, \partial_\mu \psi, h^{ij}, u^{ij}, w_k^{ij}), \quad (23)$$

where the vector  $\bar{\mathbf{v}}$  is

$$\bar{\mathbf{v}} = \begin{pmatrix} (h^{ij}) \\ (u^{ij}) \\ (w_k^{ij}) \end{pmatrix}, \quad (24)$$

and the source  $\mathbf{g}$  is

$$\mathbf{g}(\beta^k, N, \psi, \partial_\mu \beta^k, \partial_\mu N, \partial_\mu \psi, h^{ij}, u^{ij}, w_k^{ij}) = \begin{pmatrix} (u^{ij}) \\ (\phi^{ij}) \\ (0) \end{pmatrix}. \quad (25)$$

In these equations,  $\bar{\mathbf{v}}$  and  $\mathbf{g}$  are vectors of dimension 30, as it results from the symmetry properties of  $h^{ij}$ ,  $u^{ij}$ , and  $w_k^{ij}$ . Let us note that, besides the above symmetry properties, the following algebraic constraints have to be satisfied: (i)  $\det \tilde{\gamma}_{ij} = \det f_{ij}$ ; and (ii)  $w_i^{ij} = 0$ , which is equivalent to Dirac's gauge. In order to write the matrices of the system in a simple way, the following auxiliary quantities are defined:

$$q^{ij} := \beta^i \beta^j - N^2 \psi^{-4} \tilde{\gamma}^{ij}, \quad (26)$$

$$Q^i := (q^{1i} \quad q^{2i} \quad q^{3i}), \quad (27)$$

$$-\delta^i := \begin{pmatrix} -\delta_1^i \\ -\delta_2^i \\ -\delta_3^i \end{pmatrix}. \quad (28)$$

Then, the explicit form of the matrices  $\mathbf{A}^l$  is

$$\mathbf{A}^l = \left( \begin{array}{c|c|c|c|c} \hline 0_{6 \times 6} & & & & 0_{6 \times 24} \\ \hline & & & & Q^l & 0 \\ & & & & \ddots & \\ & & & & 0 & Q^l \\ \hline & -\delta^l & & & 0_{3 \times 5} & \\ \hline 0_{24 \times 6} & & & & \hline & -\delta^l & & & 0_{3 \times 4} & \\ \hline & & & & \hline & & -\delta^l & & 0_{3 \times 3} & \\ \hline & & & & \hline & & & -\delta^l & 0_{3 \times 2} & \\ \hline & & & & \hline & & & & -\delta^l & 0_3 & \\ \hline & & & & & 0_3 & -\delta^l & \\ \hline & & & & & & & -\delta^l \\ \hline \end{array} \right) \quad (29)$$

### III. CHARACTERISTIC STRUCTURE OF THE REDUCED SYSTEM

Let us present here a preliminary analysis of the mathematical structure of system (23).

First, we give the explicit expressions of the characteristic speeds in terms of the functions  $\psi$ ,  $N$ ,  $\beta^i$ , and  $\tilde{\gamma}^{ij}$ .

*Lemma 1.*—Let us consider the evolution vector  $\partial_t$ , whose components are  $\xi^\alpha = (1, 0, 0, 0)$ , and a generic spacelike covector of components  $\zeta_\alpha = (0, \zeta_i)$  orthogonal to the evolution vector. The associated eigenvalue problem (see, e.g., Ref. [34]),

$$[\mathbf{A}^l \zeta_l - \lambda \mathbf{I}] \mathbf{X}_\lambda = 0, \quad (30)$$

where  $\lambda$  denotes the eigenvalue and  $\mathbf{X}_\lambda$  the corresponding eigenvector, has the following solution:

$$\begin{aligned} \lambda_0 &= 0, \\ \lambda_\pm^{(\zeta)} &= -\beta^\mu \zeta_\mu \pm \frac{N}{\psi^2} (\tilde{\gamma}^{\mu\nu} \zeta_\mu \zeta_\nu)^{1/2} \\ &= -\beta^\mu \zeta_\mu \pm N (\zeta^\mu \zeta_\mu)^{1/2}, \end{aligned} \quad (31)$$

where  $\lambda_0$  has multiplicity 18, and each  $\lambda_\pm^{(\zeta)}$  has multiplicity 6.

Imposing Dirac's gauge in (7) indeed guarantees the real character of the eigenvalues corresponding to matrices  $\mathbf{A}^i$ , and therefore the hyperbolicity of the evolution system. Even though this is not a prerogative of the Dirac gauge, other prescriptions for  $H^i$  in condition (7) lead to a more

complicated structure of the resulting sources. As mentioned after Eq. (12), a more important point is the fact that other choices of  $H^i$  will generally introduce time derivatives of  $h^{ij}$  in the elliptic subsystem, complicating further the complete PDE system. Of course, if no gauge is imposed at all, one can check that the  $\mathbf{A}^l$  matrices admit complex eigenvalues. This reflects the property that Einstein equations by themselves do not have a definite type, without the specification of a gauge. We conclude that when imposing Dirac's gauge the eigenvalues of the linear combination  $\mathbf{A}^l \zeta_l$  are real.

*Lemma 2.*—Dirac's gauge is a sufficient condition for the hyperbolicity of system (23).

In the above eigenvalue problem, the first 6 eigenvectors, with 0 eigenvalue and associated with the  $h^{ij}$  components of  $\bar{\mathbf{v}}$  in (24), completely decouple from the other eigenvectors. Therefore, the rest of the eigenvectors can be studied independently. For the sake of clarity in the notation, let us define some auxiliary quantities before writing the matrix of (right) eigenvectors:

$$\mathbf{C}_1 := \begin{pmatrix} -\zeta_i q^{i2} & -\zeta_i q^{i3} \\ \zeta_i q^{i1} & 0 \\ 0 & \zeta_i q^{i1} \end{pmatrix}, \quad \mathbf{C}_2 := \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}. \quad (32)$$

The matrix of (right) eigenvectors,  $\mathbf{R}^{(\zeta)}$ , associated with the eigenvalue problem described in the above Lemma 1 is

$$\mathbf{R}^{(\zeta)} = \left( \begin{array}{c|c|c|c} \hline I_6 & & & 0_{6 \times 24} \\ \hline & 0_{6 \times 12} & -\lambda_+^{(\zeta)} I_6 & -\lambda_-^{(\zeta)} I_6 \\ \hline & \mathbf{C}_1 & 0 & \mathbf{C}_2 & 0 & \mathbf{C}_2 & 0 \\ \hline 0_{24 \times 6} & & & \mathbf{C}_1 & & \mathbf{C}_2 & \\ \hline & & & \ddots & & \ddots & \\ \hline & & & 0 & \mathbf{C}_1 & 0 & \mathbf{C}_2 & 0 & \mathbf{C}_2 \end{array} \right) \quad (33)$$

If the determinant of this matrix vanishes, the set of eigenvalues is not complete. This happens in the following cases:

*Case 1:*  $\lambda_+^{(\xi)} = \lambda_-^{(\xi)}$ . Since

$$\lambda_+^{(\xi)} = \lambda_-^{(\xi)} \Rightarrow N^2 \psi^{-4} \zeta_i \zeta_j \tilde{\gamma}^{ij} = N^2 \zeta_i \zeta^i = 0, \quad (34)$$

and  $\zeta_i \zeta^i$  does not vanish ( $\zeta^i$  is a spatial vector different from zero), noncompleteness only occurs if the lapse  $N$  vanishes.

*Case 2:*  $\zeta_i \zeta_j q^{ij} = 0$ . From the definition of  $q^{ij}$ , it follows that

$$\zeta_i \zeta_j (\beta^i \beta^j - N^2 \psi^{-4} \tilde{\gamma}^{ij}) = 0 \Leftrightarrow (\zeta_i \beta^i)^2 = N^2 (\zeta_i \zeta^i). \quad (35)$$

One can see that the previous equality depends only on the direction of the vector  $\zeta^i$  (i.e.  $\zeta^i \zeta_i = 1$ ). From now until the end of the study of the different cases, the vector  $\zeta^i$  will be considered to be unitary. So (35) leads to

$$\zeta_i \zeta_j (\beta^i \beta^j - N^2 \psi^{-4} \tilde{\gamma}^{ij}) = 0 \Leftrightarrow (\zeta_i \beta^i)^2 = N^2. \quad (36)$$

Decomposing  $\beta^i$  into components parallel and normal to  $\zeta^i$ , we write  $\beta^i = (\beta^{\parallel}) \zeta^i + (\beta^{\perp})^i$ , where  $(\beta^{\parallel}) = \zeta_i \beta^i$  and  $\zeta_i (\beta^{\perp})^i = 0$ . From (36), we conclude

$$\zeta_i \zeta_j q^{ij} = 0 \Leftrightarrow (\beta^{\parallel})^2 = N^2. \quad (37)$$

Note that this case is independent of the choice of  $\zeta^i$ , since it corresponds to  $(\beta^{\parallel})^i (\beta^{\parallel})_i$ , i.e.  $|\zeta^i \beta_i|^2$ . Therefore, non-completeness occurs if  $|\beta^{\parallel}| = N$ .

*Case 3:*  $\zeta_i q^{ij} = 0$ ,  $\forall j = 1, 2, 3$ . This is a stronger case than the previous one. Again, from the definition of  $q^{ij}$ , we have

$$\zeta_i (\beta^i \beta^j - N^2 \psi^{-4} \tilde{\gamma}^{ij}) = 0 \Leftrightarrow (\zeta_i \beta^i) \beta^j = N^2 \zeta^j. \quad (38)$$

From this, and the decomposition  $\beta^i = (\beta^{\parallel}) \zeta^i + (\beta^{\perp})^i$ , it follows that

$$\zeta_i q^{ij} = 0 \Leftrightarrow (\beta^{\perp})^i = 0 \quad \text{and} \quad (\beta^{\parallel})^2 = N^2. \quad (39)$$

This is just a stronger version of the second case above.

As a consequence of the above analysis we can set up the following lemma.

*Lemma 3.*—The (right) eigenvectors associated with the matrix  $\mathbf{A}^l \zeta_l$  define a complete system iff (i) the lapse  $N$  does not vanish, and (ii) the projection of the evolution

vector onto the plane spanned by  $n^\mu$  and  $\zeta^\mu$ , i.e.  $(t^{\parallel})^\mu = N n^\mu + b_\beta \zeta^\mu$ , is non-null, i.e.  $(\beta^{\parallel})^2 \neq N^2$ .

In the eigenvalue problem (30),  $\zeta^i$  stands for an arbitrary spatial vector. In particular, we can always choose  $\zeta^i = \beta^i$ . In that case, the degeneracy condition in cases 2 and 3 above reduces to  $\beta^i \beta_i = N^2$ . This happens if the vector  $t^\mu$  becomes null. Moreover, if the vector  $t^\mu$  is spacelike then we are in case 2, since then there exists a vector  $\zeta^i$  (in fact, a cone obtained by the rotation of the nonvanishing  $\beta^i$  by an appropriate angle) such that the projection of  $\beta^i$  onto that  $\zeta^i$ , referred to as  $(\beta^{\parallel})^i$ , satisfies  $(\beta^{\parallel})^i (\beta^{\parallel})_i = (\zeta_i \beta^i)^2 = N^2$ . We conclude as follows.

*Proposition 1.*—The system (23) is strongly hyperbolic if  $t^\mu$  is timelike, i.e. if  $N \neq 0$  and  $N^2 - \beta^i \beta_i > 0$ .

In some particular cases, degeneracy in the eigenvalues can occur. In particular, it could happen that one of the eigenvalues  $\lambda_+$  or  $\lambda_-$  coincides with  $\lambda_0$ . These degeneracies can appear where

$$\lambda_+^{(\xi)} \lambda_-^{(\xi)} = 0 \Leftrightarrow (\beta^\mu \zeta_\mu)^2 = N^2 (\zeta^\mu \zeta_\mu). \quad (40)$$

Again, one can consider  $\zeta^i$  to be unitary. Hence, either  $\lambda_+$  or  $\lambda_-$  vanishes when  $(\beta^{\parallel})^2 = N^2$ . As seen in (36), in this case the system of eigenvectors is incomplete.

Another relevant property is the following.

*Proposition 2.*—All the characteristic fields associated with the eigenvalue problem (30) are linearly degenerate; i.e., they satisfy the following condition:

$$\mathbf{D} \lambda_p(\bar{v}) \cdot r_p(\bar{v}) = 0, \quad (41)$$

where  $r_p$  is the eigenvector associated to the eigenvalue  $\lambda_p$ , and the operator  $\mathbf{D}$  is defined in the space of the variables of the system.

This shows the good behavior of the Dirac gauge since, in the language from fluid dynamics, it means that no shocks can be propagated along these curves, in particular, gauge shocks. Hence, if there were discontinuities, they would have to be contact discontinuities.

Regarding the characteristics speeds  $\lambda_\pm^{(\xi)}$ , we have the following.

*Corollary 1.*—The nonzero eigenvalues associated with  $\zeta^i$  correspond to the coordinate velocity of light.

This feature, which is an expected result, can be shown by considering a unitary  $\zeta^i$  and a curve whose spatial part points in the  $\zeta^i$  direction:  $\frac{dx^i}{dt} = |\frac{dx^i}{dt}| \zeta^i$ . Using the 3 + 1 expression of the metric, the vanishing of the line element of the curve, where the component of  $\beta^i$  in the  $\zeta^i$  direction is considered, is imposed. It follows, using the expression for  $\lambda^{(\xi)}$  in (31), that  $\lambda^{(\xi)} = |\frac{dx}{dt}|$ .

### Application to inner boundary conditions

The explicit expressions (31) for the characteristic speeds are especially useful in the assessment of the



boundary conditions to be imposed on a given border. We illustrate this by considering inner boundaries in the context of excised black hole space-times. Before doing so, let us underline that the FCF can be employed in combination with any of the standard techniques dealing with the black hole singularity in numerical evolutions of black hole space-times, namely, *excision*, *punctures*, or *stuffed black holes*. However, the excision technique is favored if (the elliptic subsystem of) the FCF is implemented by means of spectral methods. Focusing on the excision approach, let us denote by  $\mathcal{S}_t$  the inner sphere employed as an inner boundary at a given spacelike slice  $\Sigma_t$ , and by  $\mathcal{H}$  the world-tube hypersurface generated along the evolution by *piling up* the different  $\mathcal{S}_t$ . A natural expectation is that no inner boundary conditions should be prescribed for radiation fields on inner superluminal (growing) inner boundaries. This would avoid the need to incorporate boundary conditions in the well-posedness analysis of the associated initial boundary value problem. For this reason, spacelike inner hypersurfaces  $\mathcal{H}$  are good candidates for inner boundary conditions. However, this general idea must be assessed in the context of every specific evolution scheme. In our particular case, we must check that characteristic speeds (31) are outgoing (with respect to the integration domain). The tangent vector  $h^\mu$  to  $\mathcal{H}$ , which is normal to each  $\mathcal{S}_t$  and transports  $\mathcal{S}_t$  into  $\mathcal{S}_{t+\delta t}$ , can be written as

$$h^\mu = Nn^\mu + h_s s^\mu, \quad (42)$$

where  $s^\mu$  is the normal vector to  $\mathcal{S}_t$ , lying on  $\Sigma_t$  and pointing toward spatial infinity. Then, since the norm of  $h^\mu$  is given by  $h^\mu h_\mu = -N^2 + h_s^2$ , it follows that  $\mathcal{H}$  is spacelike as long as  $b > N$ . Choosing a coordinate system adapted to  $\mathcal{H}$ , i.e. where all the spheres  $\mathcal{S}_t$  stay at the same coordinate position—say  $r = \text{const} = r_o$ —it follows that  $h_s = \beta^i s_i \equiv \beta^\perp$ . In this case,  $\mathcal{H}$  is spacelike as long as  $\beta^\perp > N$ . Evaluating expression (31) for  $\zeta^i = s^i$ , it follows that

$$\lambda_\pm^{(s)} = -\beta^\perp \pm N. \quad (43)$$

From this we get the following.

*Corollary 2.*—For a coordinate system adapted to a spacelike inner world tube  $\mathcal{H}$ , where  $\beta^\perp > N$ , no ingoing radiative modes flow into the integration domain  $\Sigma_t$  at the excision surface.

Under these conditions no inner boundary conditions whatsoever must be prescribed for the hyperbolic part. Of course, it is not obvious how to choose dynamically an inner boundary  $\mathcal{H}$  that is guaranteed to be spacelike during the evolution. A proposal in this line has been presented in [35] in the context of the dynamical trapping

horizon framework (see e.g. Ref. [36]). Quasilocal approaches to black hole horizons aim at modeling the boundary of a black hole region as world tubes of apparent horizons ( $\mathcal{S}_t$ ). Dynamical horizons provide a geometric prescription for  $\mathcal{H}$  that is guaranteed to be spacelike, as long as the black hole is dynamical, and remain inside the event horizon, if cosmic censorship holds. The corresponding geometric dynamical horizon characterization is enforced as an inner boundary condition on the elliptic part of the FCF, in particular, on the shift equation (12). This shows the key interplay between elliptic and hyperbolic modes in the coupled fully constrained PDE evolution system. Note however that, according to Proposition 1, the hyperbolic evolution system ceases to be strongly hyperbolic. In fact, the evolution vector  $t^\mu$ , tangent to  $\mathcal{H}$  in the adapted coordinate system, becomes spacelike in a finite region. This can be bypassed by adopting a coordinate system in which the coordinate radii of the  $\mathcal{S}_t$  slices grow in time:  $r = r(t) \neq 0$ , where  $r(t)$  is appropriately *chosen*. In this case,  $h_s = \beta^\perp$  holds no longer, and this relation is rather substituted by  $\beta^\perp = h_s - [r(t) - r_o]$ . This condition is again under *control* through the appropriate boundary condition on the elliptic equation for  $\beta^i$ . Note that in this case the characteristics are still *outgoing* from the integration domain though, in this case with a coordinate growing excision sphere, this feature is no longer characterized by the negativity of the characteristics speeds  $\lambda_\pm^{(s)}$ . The outgoing character is guaranteed by the characterization of  $\lambda_\pm^{(s)}$  as the coordinate velocity of light in Corollary 1, together with the spacelike character of  $\mathcal{H}$ .

#### IV. DIRAC GAUGE AND SYSTEM OF CONSERVATION LAWS

A hyperbolic system of conservation laws, without sources, is

$$\partial_t \mathbf{u} + \mathbf{D}_i \mathbf{f}^i(\mathbf{u}) = 0. \quad (44)$$

In this system we can identify the set of unknowns, i.e., the vector of conserved quantities  $\mathbf{u}$ , and their corresponding fluxes  $\mathbf{f}(\mathbf{u})$ .

The choice of Dirac's gauge allows us to find the following set of  $l$  vector fluxes  $\mathbf{f}^l$  ( $l = 1, 2, 3$ ), of dimension 30:

$$\mathbf{f}^l := \begin{pmatrix} (0_6) \\ (-2u^{ij}\beta^l + w_k^{ij}[\beta^k\beta^l - N^2\psi^{-4}\tilde{\gamma}^{kl}]) \\ (-u^{ij}\delta_k^l) \end{pmatrix} \quad (45)$$

in terms of which system (23) can be rewritten as a *hyperbolic system of conservation laws (with sources)*. The Jacobian matrices associated to the fluxes  $\mathbf{f}^l$ ,  $(\mathbf{A}^*)^l$  are



Thorne [39], with the same notations:  $T^{L_0, \ell m}$ ,  $T^{E_1, \ell m}$ ,  $T^{B_1, \ell m}$ ,  $T^{E_2, \ell m}$ ,  $T^{B_2, \ell m}$ ,  $T^{T_0, \ell m}$ . If we note the coefficients of  $h^{ij}$  in this basis  $(c^{L_0, \ell m}, c^{E_1, \ell m}, c^{B_1, \ell m}, c^{E_2, \ell m}, c^{B_2, \ell m}, c^{T_0, \ell m})$ , we can define for any rank 2 symmetric tensor the following six scalar fields:

$$\begin{aligned} L_0 &:= \sum_{\ell, m} c^{L_0, \ell m} Y_{\ell m} = h^{rr}, & \eta &:= \sum_{\ell \geq 1, m} c^{E_1, \ell m} Y_{\ell m}, \\ \mu &:= \sum_{\ell \geq 1, m} c^{B_1, \ell m} Y_{\ell m}, & \mathcal{W} &:= \sum_{\ell \geq 2, m} c^{E_2, \ell m} Y_{\ell m}, \\ \mathcal{X} &:= \sum_{\ell \geq 2, m} c^{B_2, \ell m} Y_{\ell m}, & T_0 &:= \sum_{\ell, m} c^{T_0, \ell m} Y_{\ell m}, \end{aligned} \quad (52)$$

where  $Y_{\ell m}(\theta, \varphi)$  are the scalar spherical harmonics, which are eigenfunctions of the angular Laplace operator  $\Delta_{\theta\varphi} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$ . Note that there is a one-to-one relation between the six components of  $h^{ij}$  and these six scalar fields. The trace condition (51) simply turns into  $T_0 + h^{rr} = 0$ ; therefore we shall replace  $T_0$  with  $-h^{rr}$  in all forthcoming expressions. The divergence-free conditions (50) turn into

$$\frac{\partial h^{rr}}{\partial r} + \frac{3h^{rr}}{r} + \frac{1}{r} \Delta_{\theta\varphi} \eta = 0, \quad (53)$$

$$\frac{\partial \eta}{\partial r} + \frac{3\eta}{r} + (\Delta_{\theta\varphi} + 2) \frac{\mathcal{W}}{r} - \frac{h^{rr}}{2r} = 0, \quad (54)$$

$$\frac{\partial \mu}{\partial r} + \frac{3\mu}{r} + (\Delta_{\theta\varphi} + 2) \frac{\mathcal{X}}{r} = 0, \quad (55)$$

where all the angular derivatives are expressed in terms of  $\Delta_{\theta\varphi}$ , introduced in Eq. (48).

A first way to solve the system (49)–(51) has been described in Ref. [1] and uses evolution equations for  $h^{rr}$  and  $\mu$ , from which other scalar fields are deduced through the gauge equations (53)–(55) as solutions of the angular Laplace operator, with radial derivatives as sources. However, this method has the great disadvantage of requiring the computation of two radial derivatives to get  $h^{ij}$ , when the source  $S^{ij}$  already contains second-order radial derivatives of  $h^{ij}$ . This fourth-order derivation introduces a great amount of numerical noise, which has been observed to rapidly spoil the numerical integration. An alternative way is to evolve two other scalar fields and then to integrate (or solve PDEs coming from) the Dirac gauge condition to obtain the others. Unfortunately, this is not possible using only the six scalar fields (52), but one can devise the following procedure in a similar spirit.

Any rank 2 symmetric tensor  $T^{ij}$  can be split into two pieces:

$$T^{ij} = (\hat{L}V)^{ij} + \tilde{T}^{ij} \equiv \mathcal{D}^i V^j + \mathcal{D}^j V^i + \tilde{T}^{ij}, \quad (56)$$

with  $\mathcal{D}_j \tilde{T}^{ij} = 0$ . For a given  $T^{ij}$  the divergence of Eq. (56) allows for the determination of the vector  $V^i$  through the elliptic PDE

$$\mathcal{D}^k \mathcal{D}_k V^i + \mathcal{D}^i \mathcal{D}_j V^j = \mathcal{D}_j T^{ij}, \quad (57)$$

where  $V^i$  is fixed up to isometries of  $f_{ij}$ , which are set by the choice of boundary conditions. If we now return to the case  $T^{ij} = h^{ij}$  and consider only the asymptotically flat spatial metric defined on  $\mathbb{R}^3$ —no holes—the Dirac gauge condition (50) is equivalent to having  $V^i = 0$ , since there are no Euclidean symmetries vanishing at infinity. If one similarly seeks three scalar fields  $(A, B, C)$  such that

$$A = B = C = 0 \Leftrightarrow \tilde{T}^{ij} = 0, \quad (58)$$

one can check that a solution is

$$A = \frac{\partial \mathcal{X}}{\partial r} - \frac{\mu}{r}, \quad (59)$$

$$B = \frac{\partial \mathcal{W}}{\partial r} - \frac{\Delta_{\theta\varphi} \mathcal{W}}{2r} - \frac{\eta}{r} - \frac{h^{rr}}{4r}, \quad (60)$$

$$C = \frac{\partial h^{rr}}{\partial r} + \frac{3h^{rr}}{r} + 2\Delta_{\theta\varphi} \left( \frac{\partial \mathcal{W}}{\partial r} + \frac{\mathcal{W}}{r} \right). \quad (61)$$

In the present case where the trace (or the determinant) is given,  $B$  and  $C$  are actually coupled and it is sufficient to consider

$$\tilde{B} = \sum_{\ell, m} \tilde{B}^{\ell m} Y_{\ell m}, \quad \text{with}$$

$$\begin{aligned} \tilde{B}^{\ell m} &= (\ell + 2) \left( \frac{\partial \mathcal{W}}{\partial r} + \ell \frac{\mathcal{W}}{r} \right) - \frac{2\eta}{r} \\ &\quad - \frac{1}{2(\ell + 1)} \left( \frac{\partial h^{rr}}{\partial r} + (\ell + 4) \frac{h^{rr}}{r} \right), \end{aligned} \quad (62)$$

to recover  $B$  and  $C$  using the trace. A nice property of  $A$  and  $\tilde{B}$  is that, when expressed in terms of these potentials related to  $h^{ij}$ , the tensor Poisson equation, with  $F^{ij}$  being a symmetric tensor representing a source,

$$\Delta h^{ij} = F^{ij}, \quad (63)$$

has a rather simple form. Namely, if we define  $F^A$  and  $F^{\tilde{B}}$  as the scalar potentials similar to  $A$  and  $\tilde{B}$ , but deduced from  $F^{ij}$ , a consequence of Eq. (63) is

$$\Delta A = F^A, \quad \tilde{\Delta} \tilde{B} = F^{\tilde{B}}, \quad (64)$$

with

$$\begin{aligned} \tilde{\Delta} &:= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \tilde{\Delta}_{\theta\varphi} \quad \text{and} \\ \tilde{\Delta}_{\theta\varphi} Y_{\ell m} &:= -\ell(\ell - 1) Y_{\ell m}. \end{aligned} \quad (65)$$

Obviously, a very similar property holds for the wave equation (49). Therefore, a way of solving numerically the constrained system of Eqs. (49)–(51), by making use of the potentials  $A$  and  $B$ , is the following. With the source  $S^{ij}$  and  $h^{ij}$  known at the initial hypersurface, it is possible to deduce the potentials  $S^A$  and  $S^{\tilde{B}}$  of the source and thus

to advance the potentials  $A$  and  $\tilde{B}$  of  $h^{ij}$  to the next time step through the evolution equations

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)A = \mathcal{S}^A, \quad \left(\frac{\partial^2}{\partial t^2} - \tilde{\Delta}\right)\tilde{B} = \mathcal{S}^{\tilde{B}}. \quad (66)$$

Then the six scalar fields (52) can be computed by solving the PDE system formed by the following five elliptic equations: the definitions of  $A$  and  $\tilde{B}$ , i.e. Eqs. (59) and (62), together with the Dirac gauge conditions (53)–(55) plus the trace-free condition (51)—used to get  $T_0$ . All the components of  $h^{ij}$  can be finally recovered by taking angular derivatives of the scalar fields defined in Eqs. (52). With this algorithm, only two scalar potentials,  $A$  and  $\tilde{B}$ , are evolved in time. The whole tensor is deduced from these potentials and the gauge and trace conditions. Note that, when decomposing all the scalar fields onto a spherical harmonics function basis, the elliptic system of five PDEs described above reduces to a system of coupled ordinary differential equations in the radial coordinate  $r$ .

With either of these approaches (the one described here or that presented in Ref. [1]) it is possible to evolve two scalar potentials using hyperbolic wavelike operators and recover the symmetric tensor  $h^{ij}$  through an elliptic system of PDEs obtained from the gauge conditions. A numerical implementation of these techniques being beyond the scope of the present article, here we have only exhibited both algorithms in order to show that it is, in principle, possible to build up the whole conformal metric from the gauge conditions, while being consistent with the evolution equations. This might inversely be linked toward the property of the Dirac gauge system being preserved by the 3 + 1 evolution system. Future numerical developments in these directions shall certainly bring better insight into the problem.

## VI. DISCUSSION

All evolution formalisms for the resolution of Einstein equations as an initial value boundary problem exploit the intrinsic hyperbolicity of Eq. (1), although the associated evolution systems are not necessarily hyperbolic from the PDE theory point of view [40]. In the present case of the FCF formalism [1], Einstein equations result in a coupled elliptic-hyperbolic PDE system. The hyperbolic part of the evolution system consists of the *reduced system*, governing the evolution of the gravitational degrees of freedom, whereas the elliptic part is formed by the *constrained system* and part of the *gauge system* (maximal slicing equation). In fact, in the context of the algorithms presented in Sec. V, the elliptic Dirac system, Eqs. (53)–(55), can actually be seen as a part of the PDE evolution system. In summary, the evolution PDE system is formed by the reduced, constraint, and gauge systems, whereas the fulfillment of the subsidiary system, represented by Eq. (91) in Ref. [1] for  $\hat{\beta}^i$ , can be used as a control test of the scheme along the evolution. We have carried out a first analysis of

the mathematical structure of the PDE evolution system, paying particular attention to the equations (10) governing the evolution for the deviation  $h^{ij}$  of the conformal metric from the flat fiducial one  $f_{ij}$ , i.e.  $h^{ij} = \tilde{\gamma}^{ij} - f^{ij}$ . Dirac's gauge plays an important role in getting a well-defined hyperbolic structure. This elliptic gauge is close in spirit and properties to other gauges employed in the literature, like the *spatial harmonic* gauge in [26], the *minimal distortion* introduced by York & Smarr, the *new minimal distortion* gauge introduced by Jantzen & York, or the numerically motivated *pseudominimal distortion* gauge by Nakamura, *approximate minimal distortion* by Shibata, or the *Gamma freezing* (cf. Secs. 9.3. and 9.4 in Ref. [27] for a review of them). In particular, all of them can be written as elliptic equations on the shift vector  $\beta^i$ . The Dirac gauge fixes spatial coordinates in the evolution (including on the initial data, as the *spatial harmonic gauge* does) up to boundary conditions. For boundary conditions (enforced when solving the elliptic PDE for  $\beta^i$ ) such that the evolution vector is timelike, the Dirac gauge provides a sufficient condition for the strong hyperbolicity of Eq. (10). Moreover, using this gauge it is possible to derive a *flux vector* in terms of which the first-order system of equations, equivalent to (10), has the structure of a hyperbolic system of conservation laws (with sources). Likewise, the analysis of the characteristics sheds light on the prescription of inner boundary conditions on a spacelike inner cylinder, when employing an excision approach to black hole evolutions. More generally, maximal and Dirac gauges can be relaxed to admit more general gauges, while preserving the hyperbolic properties of the system but possibly complicating the structure of the sources.

Having said this, it is clear that further analysis is necessary. First, particular attention should be paid to the source terms in Eq. (13). They can introduce, in the so-called stiff case, new characteristic time scales (relaxation times in the language of fluid dynamics) which may be much smaller than the CFL (Courant-Friedrichs-Lewy) numerical time step (see, e.g., [41–43]). In particular, authors in Ref. [41] have studied general hyperbolic systems with supercharacteristic relaxations, and they show in which conditions a source term can be damping or, on the contrary, enforces growth of instabilities. Looking, in our case, at the quantity  $R_*^{ij}$  [Eq. (17)], one can notice the presence of quadratic terms in the  $w_k^{ij}$ ; it suggests that huge spatial gradients of  $h^{jk}$  can introduce some degree of stiffness in the source terms. Second, nothing has been said about the possible outer boundary conditions to be prescribed when studying the initial boundary value problem with an outer timelike cylinder. Certainly, in this case the well-posedness analysis is more complicated. However, thanks to the enforcement of the constraint along the evolution, there is no need for devising specific constraint preserving boundary conditions, and Sommerfeld-



like conditions as in [44,45] can be straightforwardly employed. Third, nothing has been said about the elliptic part and its coupling with the hyperbolic subsystem. On the one hand, this coupling is crucial in the overall well-posedness of the problem, as clearly illustrated in the inner boundary conditions issue, where inner boundary conditions on the elliptic part determine the ingoing or outgoing nature of the characteristics in the hyperbolic part. On the other hand, the analysis of the elliptic system by itself represents an outstanding challenge. This is illustrated by the XCTS elliptic system [28,29] referred to in Sec. I B, very closely related to the FCF elliptic subsystem. We note that, in this case, no results on existence are available, and very little is known on uniqueness, where recent numerical [30,31] and analytical works [32,33] point toward the essential non-uniqueness of the system (related to a wrong sign in the differential operator of the maximal slicing equation). Fourth, nothing has been said about consequences on well-posedness of coupling matter equations to the gravitational degrees of freedom.

Although our analysis is far from being exhaustive, it has the advantage of giving some clues about which numerical strategies are the most convenient in order to solve Einstein equations in the FCF formalism. In this sense, we have attempted to obtain some limited but concrete results, rather than remain frozen by the “nonattainability” of complete and fully rigorous results.

## ACKNOWLEDGMENTS

I.C.-C. acknowledges support from the Spanish Ministerio de Educación y Ciencia (MEC) (No. AP2005-2857). J.L.J. acknowledges the support of Marie Curie Contract No. MERG-CT-2006-043501 in the 6th European Community Framework Program. E. G. and J. N. acknowledge support from ANR Grant No. 06-2-134423 MATH-GR. This work is supported by Grant No. AYA2004-08067-C03-01 from the MEC, and a France-Spain bilateral research grant (No. HF2005-0115), as well as the Hubert Curien exchange grant, from the French ministry for foreign affairs.

- 
- [1] S. Bonazzola, E.ourgoulhon, P. Grandclément, and J. Novak, *Phys. Rev. D* **70**, 104007 (2004).
  - [2] P. Grandclément and J. Novak, arXiv:0706.2286 [Living Rev. Relat. (to be published)].
  - [3] J.Ma. Martí and E. Müller, *Living Rev. Relativity* **2**, 3 (1999), <http://www.livingreviews.org/lrr-1999-3>.
  - [4] J.A. Font, *Living Rev. Relativity* **3**, 2 (2000), <http://www.livingreviews.org/lrr-2000-2>.
  - [5] J. Isenberg, arXiv:gr-qc/0702113.
  - [6] J.R. Wilson, G.J. Mathews, and P. Marronetti, *Phys. Rev. D* **54**, 1317 (1996).
  - [7] H. Dimmelmeier, J.A. Font, and E. Müller, *Astrophys. J.* **560**, L163 (2001).
  - [8] H. Dimmelmeier, J.A. Font, and E. Müller, *Astron. Astrophys.* **388**, 917 (2002).
  - [9] H. Dimmelmeier, J.A. Font, and E. Müller, *Astron. Astrophys.* **393**, 523 (2002).
  - [10] H. Dimmelmeier, J. Novak, J.A. Font, J.Ma. Ibáñez, and E. Müller, *Phys. Rev. D* **71**, 064023 (2005).
  - [11] H. Friedrich and A.D. Rendall, *Lect. Notes Phys.* **540**, 127 (2000).
  - [12] F. Pretorius, *Classical Quantum Gravity* **22**, 425 (2005).
  - [13] C. Gundlach, J.M. Martín-García, G. Calabrese, and I. Hinder, *Classical Quantum Gravity* **22**, 3767 (2005).
  - [14] M. Shibata and T. Nakamura, *Phys. Rev. D* **52**, 5428 (1995).
  - [15] T.W. Baumgarte and S.L. Shapiro, *Phys. Rev. D* **59**, 024007 (1998).
  - [16] F. Pretorius, *Phys. Rev. Lett.* **95**, 121101 (2005).
  - [17] M. Campanelli, C.O. Lousto, P. Marronetti, and Y. Zlochower, *Phys. Rev. Lett.* **96**, 111101 (2006).
  - [18] J.G. Baker, J. Centrella, D.I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev. Lett.* **96**, 111102 (2006).
  - [19] F. Pretorius, arXiv:0710.1338.
  - [20] M. Shibata and K. Uryū, *Phys. Rev. D* **61**, 064001 (2000).
  - [21] C. Gundlach and J.M. Martín-García, *Phys. Rev. D* **74**, 024016 (2006).
  - [22] K. Uryu, F. Limousin, J.L. Friedman, E.ourgoulhon, and M. Shibata, *Phys. Rev. Lett.* **97**, 171101 (2006).
  - [23] Y. Choquet-Bruhat and J.W. York, *C.R. Acad. Sci. Paris, Ser. I* **321**, No. 8, 1089 (1995).
  - [24] Y. Choquet-Bruhat and J.W. York, arXiv:gr-qc/9606001.
  - [25] Y. Choquet-Bruhat and J.W. York, arXiv:gr-qc/9601030.
  - [26] L. Andersson and V. Moncrief, *Ann. Inst. Henri Poincaré, A* **4**, 1 (2003).
  - [27] E.ourgoulhon, arXiv:gr-qc/0703035.
  - [28] H.P. Pfeiffer and J.W. York, *Phys. Rev. D* **67**, 044022 (2003).
  - [29] H.P. Pfeiffer and J. Hyperbol, *Diff. Equ.* **2**, 497 (2005).
  - [30] H.P. Pfeiffer and J.W. York, *Phys. Rev. Lett.* **95**, 091101 (2005).
  - [31] J.L. Jaramillo, M. Ansorg, and F. Limousin, *Phys. Rev. D* **75**, 024019 (2007).
  - [32] T.W. Baumgarte, N. O’Murchadha, and H.P. Pfeiffer, *Phys. Rev. D* **75**, 044009 (2007).
  - [33] D.M. Walsh, *Classical Quantum Gravity* **24**, 1911 (2007).
  - [34] A.M. Anile, *Relativistic Fluids and Magneto-Fluids* (Cambridge University Press, Cambridge, England, 1989).
  - [35] J.L. Jaramillo, E.ourgoulhon, I. Cordero-Carrión, and J.Ma. Ibáñez, *Phys. Rev. D* **77**, 047501 (2008).
  - [36] E.ourgoulhon and J.L. Jaramillo, *Phys. Rep.* **423**, 159 (2006).
  - [37] J. Mathews, *J. Soc. Ind. Appl. Math.* **10**, 768 (1962).



- [38] F.J. Zerilli, *J. Math. Phys. (N.Y.)* **11**, 2203 (1970).
- [39] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299 (1980).
- [40] H. Friedrich, *Ann. Phys. (Leipzig)* **15**, 84 (2006).
- [41] S. Jin and M. A. Katsoulakis, *SIAM J. Appl. Math.* **61**, 273 (2000).
- [42] J. A. Rossmann, D. S. Bale, and R. J. LeVeque, *J. Comput. Phys.* **199**, 631 (2004).
- [43] F. Miniati and P. Colella, *J. Comput. Phys.* **224**, 519 (2007).
- [44] J. Novak and S. Bonazzola, *J. Comput. Phys.* **197**, 186 (2004).
- [45] L. T. Buchman and O. C. A. Sarbach, *Classical Quantum Gravity* **23**, 6709 (2006).