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On the behaviour of the magnetorotational instability when the Rayleigh criterion is violated

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ABSTRACT

An elementary but revealing analysis of the dispersion relation of the magnetorotational instability (MRI) in the Rayleigh-unstable regime is described. The defining properties of the MRI – its maximum growth rate and the direction of the associated eigenvector displacement – remain unchanged as the Rayleigh discriminant passes from positive to negative values. At sufficiently negative discriminant values, however, the spectrum of unstable modes becomes dominated by zero wavenumber disturbances, and the problem loses its local character. These results may be relevant to understanding the level of turbulent fluid stress near the innermost stable circular orbit (ISCO) in an accretion disc around a black hole, since the Rayleigh discriminant changes sign at this location. Our conclusions are consistent with numerical simulations that find finite stress at the ISCO, some dependence of the stress magnitude with scale height, and a plunging region much closer to the event horizon than the ISCO radius.

Key words: accretion, accretion discs – black hole physics – instabilities – MHD.

1 INTRODUCTION

Determining the mass and spin of a black hole now falls, at least in principle, within the realm of observation. In practice, this requires reliable modelling of the accreting gas in the disc that generally surrounds the black hole. The behaviour of the gas in the vicinity of the innermost stable circular orbit (ISCO) of a Kerr black hole is very important. In the classical α disc model developed by Novikov & Thorne (1973), for example, the ISCO separates the outer region in the disc in which circular orbits were stable, from the interior region in which circular orbits were unstable (by the Rayleigh criterion), with the consequence that the unstable gas plunged into the hole. The ‘plunging region’, it was argued, can neither support nor maintain the greatly enhanced viscous-like stress that characterizes the gas dynamics of the outer region. Therefore, Novikov & Thorne adopted the boundary condition that the stress must vanish at the ISCO. This boundary condition dominates the local gas dynamics, forcing the surface density to drop rapidly to zero at the ISCO. The disc emissivity is then strongly dominated by the ISCO-neighbourhood gas, and it is possible to use this property to determine the location of the ISCO and hence the intrinsic properties of the black hole itself.

The Novikov–Thorne model was developed in an era in which there was profound ignorance of the nature of the enhanced torques

needed for accretion discs to operate. Such torques were (and still are) thought to be turbulent in origin, and as with most analytic models of turbulence, the Novikov–Thorne approach was heuristic. In fact, the model was not even really self-consistent. If turbulent in origin, the stress presumably had to arise from some kind of an instability, yet the only region in the disc that was demonstrably unstable (i.e. the gas interior to the ISCO) was precisely the region in which the turbulent stress supposedly vanished! If pressed on this point, early investigators would probably have argued that the turbulent stress was actually a consequence of a non-linear instability, whereas inside the ISCO the gas was by definition linearly unstable. The non-linear instability would be vigorous enough to produce a greatly enhanced turbulent stress, but gentle enough not to completely disrupt the circular orbits. The linear Rayleigh instability that took hold inside the ISCO would be, by way of contrast, catastrophic.

This scenario has survived, more or less intact to the present day, almost 40 years on. One major development that has occurred in the ensuing decades is that there is now good reason to believe that the origin of accretion disc turbulence, if not its ultimate resolution, is reasonably well understood. If any subthermal magnetic field is present in the disc, the magnetorotational instability (MRI) will destabilize circular orbits, and the outcome of all numerical simulations indicates that the result will be a level of enhanced non-linear turbulent stress that leaves the Keplerian disc structure intact but allows vigorous accretion to occur.

Much can be learned from the linear MRI dispersion relation alone. The most unstable displacements, for example, produce

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strong correlated growth in the radial and azimuthal magnetic field components, a result that endures into the non-linear turbulent regime. However, with only a handful of exceptions (e.g. Abramowicz, Brandenburg & Lasota 1996; Hawley, Balbus & Winters 1999), detailed analyses of the MRI dispersion relation have understandably focused on Keplerian and galactic discs, in which case the angular momentum gradient is strongly positive. Virtually nothing has been done by way of analysis for the regime that is relevant to the gas interior to the ISCO of a black hole accretion disc.

We are motivated, therefore, to study the behaviour of linear disturbances in a weakly magnetized, Rayleigh-unstable gas. In particular, is there evidence in the dispersion relation for a profound change in the most unstable eigenvectors when the angular momentum gradient changes sign? We will show that there is no profound behavioural change evident in linear theory to distinguish gas interior and exterior to the ISCO, certainly nothing to warrant the notion of a discontinuous changes in the disc properties. The maximum growth rate remains the Oort A-value, the maximally unstable displacement eigenvector is associated with local wavenumbers and generates strongly correlated radial and azimuthal fields, and the one unstable branch of the dispersion relation (there is always only one such branch, with or without violation of the Rayleigh criterion) changes smoothly as the ISCO is passed. If the turbulent stress vanishes along the lines of Novikov & Thorne models, it is not evident from any changes in the dispersion relation of a weakly magnetized rotating gas. The stress could, in principle, vanish if the disc thickness within the ISCO became too thin to support MRI modes, but it is not obvious why this should be expected to occur.

The calculation is quite elementary, and the purpose of this Letter is simply to draw attention to certain dynamical behaviour that deserves to be more widely known among investigators modelling black hole discs. In Section 2, we present an analysis of the dispersion relation in question, and calculate the maximum growth rate and most unstable eigenvectors under very general circumstances. In Section 3, we discuss some of the astrophysical consequences of our results.

2 ANALYSIS

The dispersion relation for incompressible disturbances in a weakly magnetized rotating gas is by now well known (e.g. Balbus & Hawley 1998). We consider displacements in the plane of the disc. The cylindrical radius is denoted R , the azimuthal angle ϕ . The wavenumber must then have only a vertical (z) component, denoted k . The corresponding angular frequency is ω . Linear disturbances with a dependence on space and time of the plane wave form $\exp(ikz - i\omega t)$ satisfy the dispersion relation

$$\omega^4 - (\kappa^2 + 2k^2 v_A^2) \omega^2 + k^2 v_A^2 (k^2 v_A^2 + d\Omega^2/d \ln R) = 0. \quad (1)$$

Here, the angular rotation rate of the disc Ω depends only upon R , and κ is the disc epicyclic frequency. It is given by

$$\kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d \ln R}. \quad (2)$$

(In older fluid texts, the term ‘Rayleigh discriminant’ is used for κ^2 .) Do not be misled by the fact that this is a squared quantity; the focus here is on the case $\kappa^2 < 0$. As usual, v_A is an Alfvén velocity. If B is the z component of the magnetic field (the only one that enters into the dispersion relation), then

$$v_A^2 = \frac{B^2}{4\pi\rho}, \quad (3)$$

where ρ is the gas density. Since we will regard $k^2 v_A^2$ as a variable, we will henceforth denote it simply as x , always a positive quantity.

The solution to the dispersion relation (1) is

$$\omega^2 = \frac{\kappa^2}{2} + x \pm \sqrt{\frac{\kappa^4}{4} + 4\Omega^2 x}. \quad (4)$$

We distinguish three cases, $\kappa^2 > 0$, $\kappa^2 = 0$ and $\kappa^2 < 0$. (When numbers are used, the unit of frequency is Ω .)

(i) $\kappa^2 > 0$. This, of course, is the standard case. Taking the plus (+) branch, for $x \ll 1$,

$$\omega^2 \simeq \kappa^2 + (1 + 4\Omega^2/\kappa^2)x + \dots, \quad (5)$$

while for $x \gg 1$,

$$\omega^2 \simeq x. \quad (6)$$

These are clearly the epicyclic and Alfvén limits. For the minus (–) branch, the $x \ll 1$ limit is

$$\omega^2 \simeq \frac{x}{\kappa^2} \frac{d\Omega^2}{d \ln R}, \quad (7)$$

the MRI mode. The $x \gg 1$ limit is the same as the + branch, $\omega^2 \simeq x$. A typical case is plotted in Fig. 1.

(ii) $\kappa^2 = 0$. This corresponds to the neighbourhood of the ISCO. For $x \ll 1$, the + and – branches are given by

$$\omega^2 \simeq \pm 2\Omega\sqrt{x}, \quad (8)$$

while the $x \gg 1$ limit remains, as above, $\omega^2 \simeq x$. Fig. 2 shows a typical case.

(iii) $\kappa^2 < 0$. This applies to the disc region inside the ISCO. For the + branch, the $x \ll 1$ limit is

$$\omega^2 \simeq x(1 + 4\Omega^2/|\kappa^2|), \quad (9)$$

while the – branch in this limit is

$$\omega^2 \simeq -|\kappa^2| + x(1 - 4\Omega^2/|\kappa^2|). \quad (10)$$

Note that the slope at $x = 0$ changes from negative to positive as $|\kappa^2|$ passes from less than to larger than $4\Omega^2$.

On both branches, the $x \gg 1$ limit is, as always, $\omega^2 \simeq x$. Figs 3 and 4 show this case for the subcases $|\kappa^2| < 4\Omega^2$ and $|\kappa^2| > 4\Omega^2$, respectively.

Regarded as a function of x , $-\omega^2$ is maximized when the wavenumber satisfies $d\omega^2/dx = 0$. This is a straightforward calculation and yields

$$x_{\max} = -\frac{\kappa^2 + 4\Omega^2}{16\Omega^2} \frac{d\Omega^2}{d \ln R}. \quad (11)$$

When $\kappa^2 < 0$, it is necessary that $\kappa^2 + 4\Omega^2 > 0$ for the existence of x_{\max} . This is the very same requirement that ω^2 be a decreasing function of x when $x \ll 1$. When this requirement is not fulfilled, $\omega^2(x)$ is everywhere an increasing function of x (on either the + or – branch), and has no extremum. When this requirement is fulfilled, the value for x_{\max} is identical to that found for the standard $\kappa^2 > 0$ MRI. The associated maximum growth rate is given by

$$|\omega_{\max}| = -\frac{1}{2} \frac{d\Omega}{d \ln R} \quad (12)$$

and is also identical to the standard $\kappa^2 > 0$ MRI. This is a rather remarkable result. The onset of the Rayleigh instability does not change the character of the instability. Both the wavenumber of maximum growth and the maximum growth rate itself are identical to those found by Balbus & Hawley (1992) for the standard MRI. It is only when the magnitude of κ^2 exceeds $4\Omega^2$ that the existence

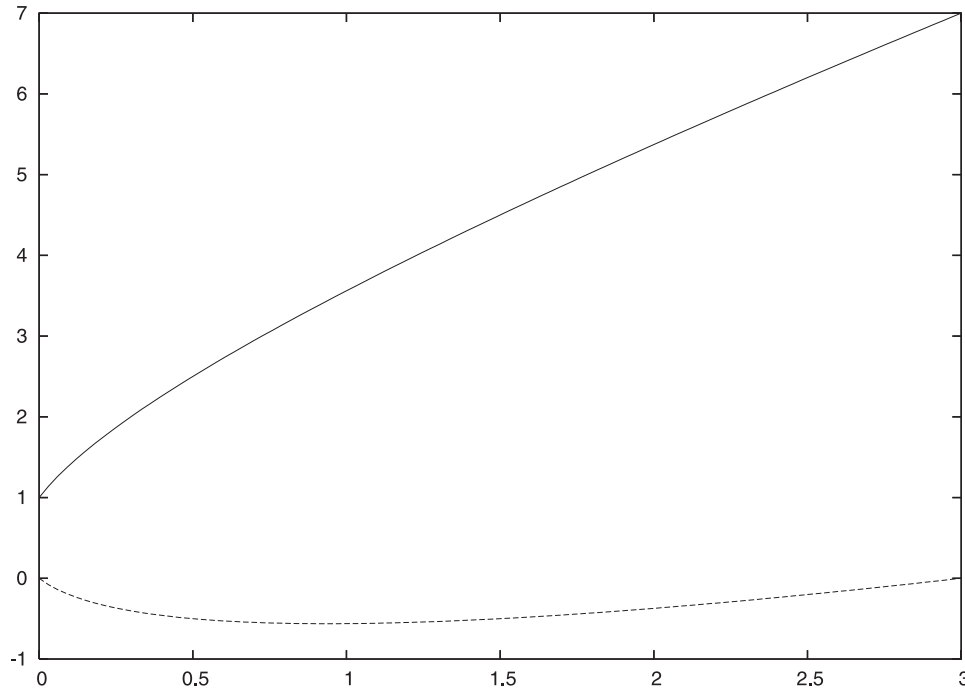


Figure 1. ω^2 versus $(kv_A)^2$ from equation (4) for the case $\kappa^2 = 1$. Frequency unit is Ω . The upper branch corresponds to the + sign, the lower branch to the – sign. The lower curve is the standard MRI.

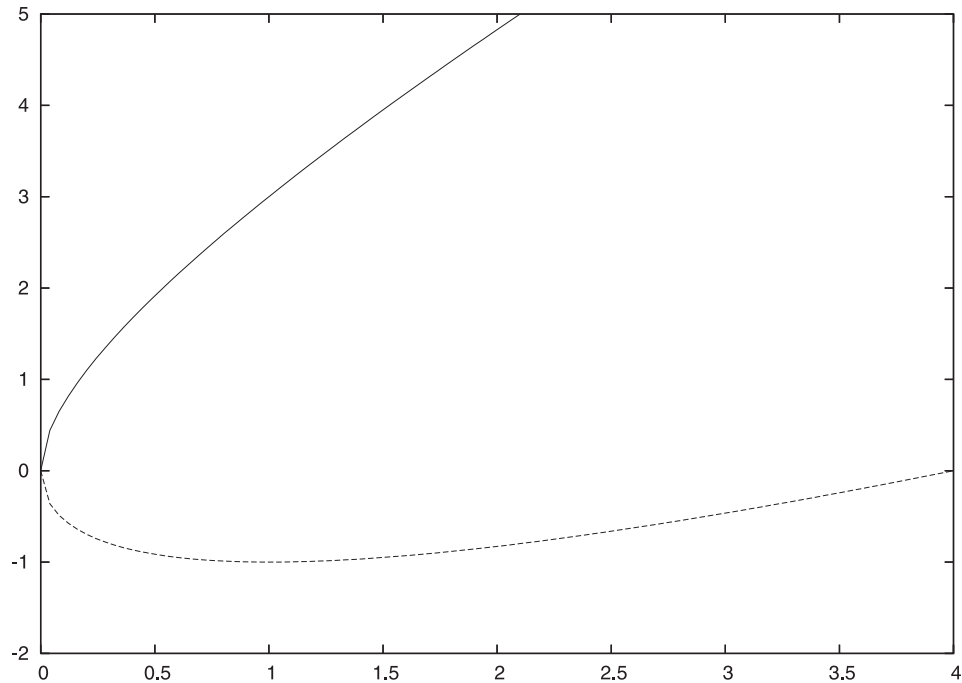


Figure 2. ω^2 versus $(kv_A)^2$ from equation (4) for the case $\kappa^2 = 0$. Frequency unit is Ω . The upper branch corresponds to the + sign, the lower branch to the – sign. The lower curve is very similar in its properties to the standard MRI.

of a local maximum in the growth rate at a particular value of x disappears.

Finally, we may ask what the eigenvector of the most rapidly growing mode looks like, and what is the Maxwell–Reynolds stress associated with it. To this end, we adopt equation (2.3d) from Balbus & Hawley (1991). Instead of the angular frequency ω , it is more convenient to use the growth rate $\sigma \equiv -i\omega$. In terms of σ , the Eulerian perturbed azimuthal velocity δv_ϕ is related to the Eulerian

perturbed radial velocity δv_R by

$$\delta v_\phi = -\frac{\delta v_R}{\sigma} \left(\frac{\kappa^2}{2\Omega} + \frac{x}{\sigma^2} \frac{d\Omega}{d \ln R} \right) \left(1 + \frac{x}{\sigma^2} \right)^{-1}. \quad (13)$$

Using equations (11) and (12) in (13), after some expansion and simplification we find the result

$$\delta v_\phi = \delta v_R, \quad (14)$$

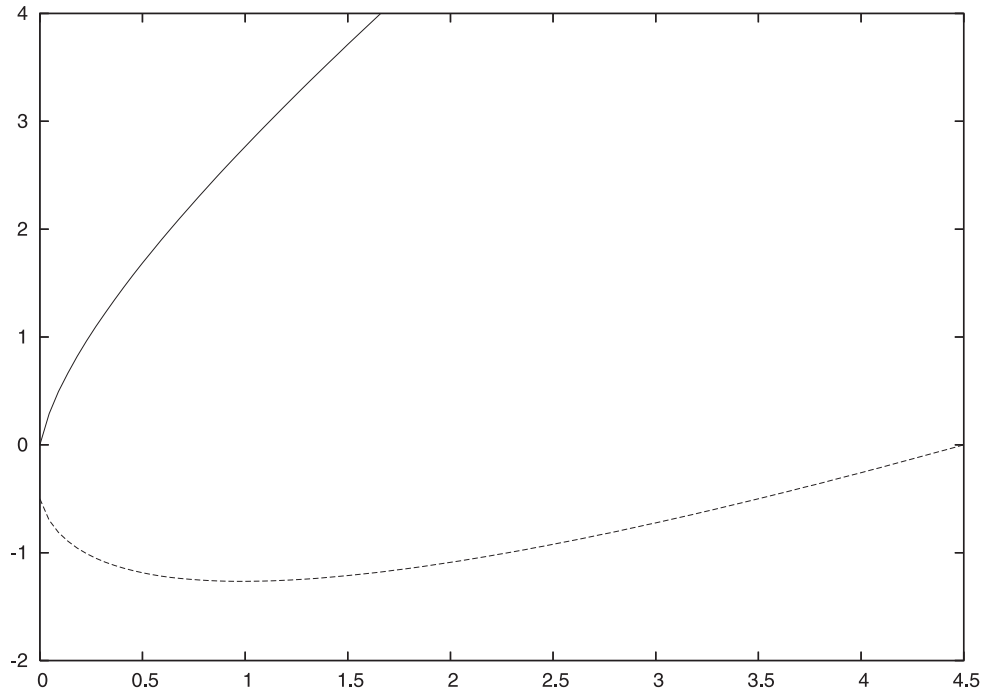


Figure 3. ω^2 versus $(kv_A)^2$ from equation (4) for the case $\kappa^2 = -0.5$. Frequency unit is Ω . The upper branch corresponds to the $+$ sign, the lower branch to the $-$ sign. Once again, the lower curve is very similar in its properties to the standard MRI, but with a finite growth rate at $x = 0$.

which is identical to the standard MRI eigenvector. Obviously, the Reynolds stress $\delta v_\phi \delta v_R$ is finite and positive. To follow the path of a fluid element, we require the Lagrangian perturbed velocity. If ξ is the Lagrangian displacement vector of a fluid element, then the perturbed Lagrangian velocity for the eigenvector of maximum growth is

$$\begin{aligned} \dot{\xi}_\phi &= \delta v_\phi + \xi_R \frac{d\Omega}{d \ln R} = \delta v_\phi + \frac{\delta v_R}{\sigma} \frac{d\Omega}{d \ln R} \\ &= \delta v_R \left(1 + \frac{2d\Omega/d \ln R}{|d\Omega/d \ln R|} \right) = -\delta v_R. \end{aligned} \quad (15)$$

Once again, this Lagrangian displacement is exactly the same as that found in the standard MRI (Balbus & Hawley 1992).

3 DISCUSSION

It is clear from the analysis of Section 2 that the onset of Rayleigh instability does not mark the ISCO as a very special location. In particular, the ISCO is not, in fact, the ‘innermost stable circular orbit’. The bulk of the disc is unstable, which is of course why it is turbulent. The one new feature of a WKB analysis within the ISCO is that $k = 0$ modes become unstable, a hint of the beginning of the breakdown of local physics. (This is a pattern similar to that seen in convectively unstable discs, in which the longest wavelength modes are dominated by hydrodynamics, whereas the local modes remain magnetohydrodynamic in character; Balbus & Hawley 2002; Narayan et al. 2002.) The fastest growing modes are still locally triggered upon entering the region interior to the ISCO, however. Were it not for the slowly growing $k = 0$ modes, the radius 0.9 (in ISCO units) would be indistinguishable in its formal stability properties from the radius 1.1.

If there is a location that marks the onset of qualitatively different behaviour and merits its own moniker, it is the radius at which $\kappa^2 = -4\Omega^2$. As Fig. 4 makes clear, inside this location, the

most rapidly growing eigenmodes are associated with global wavelengths, and it makes no sense at all to think of the flow stability as a local phenomenon. However, this location is very near to the event horizon itself.

Numerical simulations designed to measure the stress near the ISCO have been underway for several years now (Krolik, Hawley & Hirose 2005). The calculation is difficult, and as a consequence, the interpretation of the results is controversial. If we somewhat boldly extrapolate from the linear analysis presented here, the following predictions seem natural.

No feature in the flow should be identifiable with the ISCO. The reasons for this have been discussed above. Simply put, the MRI stability properties and the associated stress generation change seamlessly as one passes the radius marking the onset of Rayleigh instability.

There should be no particular sensitivity to H/R beyond a minimum value. As long as a reasonably broad range of unstable vertical wavenumbers fit inside the disc, the $R\phi$ component of the stress tensor will vigorously develop.

Below a certain threshold, there should be a sensitivity to H/R . If H/R allows a very limited range of unstable wavenumbers, the level of saturated $R\phi$ stress could be affected: mixing to smaller wavenumbers effectively kills the growth. Obviously, at small enough H/R , the MRI is completely suppressed.

There should be a plunging region only very near the horizon. This is associated (in the pseudo-Newtonian formalism) with the radius at which $\kappa^2 = -4\Omega^2$, at which point the zero wavenumber disturbances dominate the instability – a true Rayleigh regime.

Magnetic field geometry should influence the level of the stress. This is not a very startling conclusion: this feature has been noted in MRI simulations since the very early days of the subject.

The behaviour described above is consistent with the findings of the studies of Reynolds & Fabian (2008) and Noble, Krolik & Hawley (2010). These are both global disc simulations. Studies

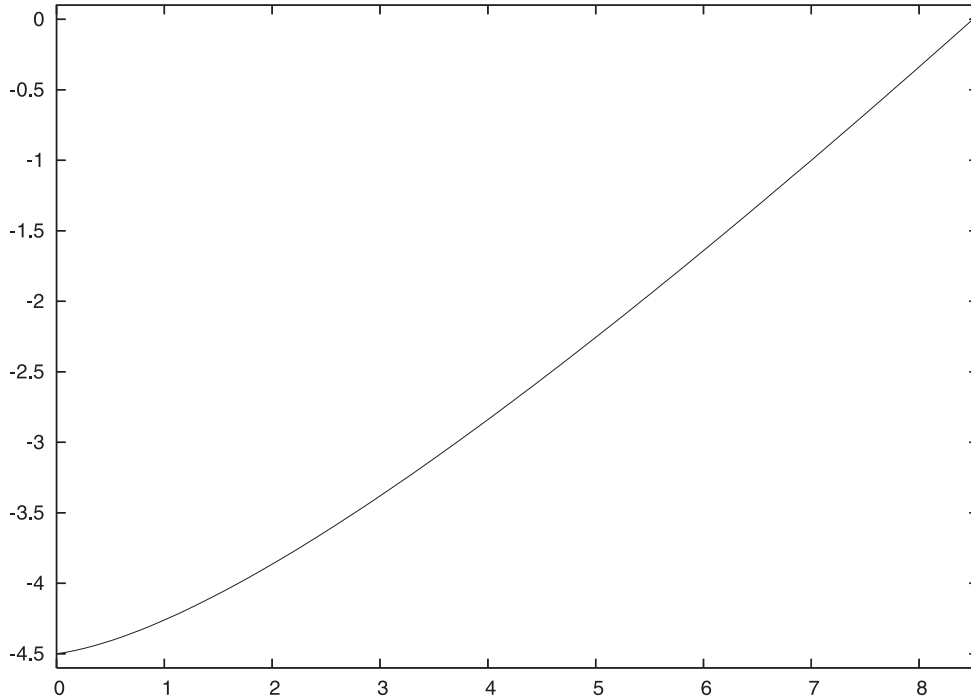


Figure 4. ω^2 versus $(kv_A)^2$ from equation (4) for the case $\kappa^2 = -4.5$. Frequency unit is Ω . Only the unstable ‘minus’ branch is shown. The behaviour is now significantly different, since the maximum growth rate is no longer at a finite value of x , but at $x = 0$. The instability is likely to be dominated by global modes.

of local shearing boxes for which $\kappa^2 < 0$ would be a revealing numerical laboratory, since there is greater control of the external parameters in such systems. On the other hand, the evolution of the unstable $k = 0$ mode may be pathological in shearing box geometry, so that care must be taken with its interpretation. We will report on the findings of such a study in a future publication.

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