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Abstract. We study the local repulsion between critical points of a stationary isotropic smooth planar Gaussian field. We show that the critical points can experience a soft repulsion which is maximal in the case of the random planar wave model, or a soft attraction of arbitrary high order. If the type of critical points is specified (extremum, saddle point), the points experience a hard local repulsion, that we quantify with the precise magnitude of the second factorial moment of the number of points in a small ball.

Key words: Gaussian random fields; Stationary random fields; Critical points; Kac-Rice formula; repulsive point process.

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1. Introduction

The main topic of this paper is a local analysis of the critical points of a smooth stationary planar Gaussian field. The study of critical points, their number as well as their positions, are important issues in various application areas such as sea waves modeling [CG13] , astronomy [LW04,ATW07,Lin72] or neuroimaging [NH03, TW07,WMNE96,WTTL04]. In these situations, practitioners are particularly interested in the detection of peaks of the random field under study or in high level asymptotics of maximal points [CS17,TW07,WMNE96]. At the opposite of these Extremes Theory results, some situations require the topological study of excursion sets over moderate levels [AT09,CX16] or the location study of critical points (not only extremal ones) [Mui20].

Repulsive point processes have known a surge of interest in the recent years, they are useful in a number of applications, such as sampling for quasi Monte-Carlo methods [BH20], data mining, training set selection in machine learning, or numerical integration, see for instance [KT12], or as coresets for subsampling large datasets [TBA19]. Critical points of Gaussian fields could be an alternative to determinantal point processes, which are commonly used for their repulsion properties despite the difficult issue of their synthesis [DGL17]. Several definitions exist to characterize the repulsion properties of a stationary point process. We will use the following informal definition of local repulsion: A stationary random set of points $\mathcal{X} \subset \mathbb{R}^2$ is *locally repulsive at the second order* if, denoting by $\mathcal{N}_\rho$ its number of points in a ball centred in 0 with radius ρ, we have

$$\mathsf{R}_{\mathcal{N}} := \lim_{\rho \to 0} \frac{\mathbb{E}(\mathcal{N}_\rho^{(2)})}{\mathbb{E}(\mathcal{N}_\rho)^2} < 1 \tag{1}$$

where for an integer $n, n^{(2)} = n(n-1)$ is the second order factorial power. This definition is motivated by the heuristic computation where we consider $x_1 \neq x_2$ randomly sampled in $\mathcal{N} \cap B_1$ and

$$\mathbb{E}(\mathcal{N}_\rho) = \mathbb{P}(x_1 \in B_\rho) + \text{remainder}$$
$$\mathbb{E}(\mathcal{N}_\rho^{(2)}) = \mathbb{P}(x_2 \in B_\rho\, , x_1 \in B_\rho) + \text{remainder},$$

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where the remainder terms are hopefully negligible when ρ is small. In other words, a point process is locally repulsive if the probability to find a point in a small ball diminishes if we know that there is already a point in this ball. The constant $\mathsf{R}_{\mathcal{N}}$ is called the *(second order) local repulsion factor*, it is a dimensionless parameter that is invariant under rescaling or rotation of the process $\mathcal{N}$. It equals 1 if $\mathcal{N}$ is a homogenous Poisson process, which is universally considered non-interacting. We say that the point process is *weakly locally repulsive* (resp. *attractive*) if $\mathsf{R}_{\mathcal{N}} \in (0,1)$ (resp. $(1,\infty)$), and strongly repulsive if $\mathsf{R}_{\mathcal{N}} = 0$.

We study the repulsion properties of the stationary process $\mathcal{X}_c$ formed by critical points of a planar stationary isotropic Gaussian field ψ. We show that, depending on the covariance function of the field, they form a weakly locally repulsive or a weakly locally attractive point process, and that the minimal repulsion factor is $\mathsf{R}_{\mathcal{X}_c} = \frac{1}{8\sqrt{3}}$, reached when ψ is a Gaussian random wave model, which hence yield the most locally repulsive process of critical points. There is on the other hand no maximal value for the limit. We also show that the subprocess formed by the local maxima of the field is strongly repulsive, as well as the subprocess formed by the saddle points, and give the precise magnitude of the ratio decay in the left hand member of (1).

Let us quote two recent articles that are concerned with a very similar question. The first one, which has been a source of inspiration, is [BCW19]. In this paper, Belyaev, Cammarota and Wigman study the repulsion of the critical points for a particular Gaussian field, the *Berry's Planar Random Wave Model*, whose spectral measure is uniformly spread on a circle centred in 0. They obtain the exact repulsion ratio for critical points and upper bounds for the repulsivity for specific types of critical points (saddle, extrema). Azais and Delmas [AD22] have studied the attraction or repulsion of critical points for general stationary Gaussian fields in any dimension. Using a different computation method, they get an upper bound for the second factorial moment which is compatible with the order of magnitude that we obtain. Their method is borrowed from techniques in random matrix theory, as suggested by Fyodorov [Fyo04]. Namely, an explicit expression for the joint density of GOE eigenvalues is exploited.

In order to quantify the repulsion of the critical points, we compute the second factorial moment using the Rice or Kac-Rice formulas (see [AT09] or [AW09] for details), as the vast majority of works concerned with counting the zeros or critical points of a random field. We get the asymptotics as the ball radius tends to 0 by performing a fine asymptotic analysis on the conditional expectations that are involved in the Kac-Rice formulas.

The paper is organized as follows: In Section 2, we present the Gaussian fields, which are the probabilistic object of our study, and the basic tools we will use for their study. In Section 3, we derive the Kac-Rice formula, in a context adapted to our framework. The purpose of section 4 is to compute the expectation of the number of critical points and also the number of extrema, minima, maxima and saddle (see Proposition 3). In Section 5, we study the second factorial moment and discuss the repulsion properties of the critical points.

2. Assumptions and tools

The main actors of this article are centered random Gaussian functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ whose law is invariant under translations, and whose realisations are smooth. Formally it means that for $x_1, \dots, x_n \in \mathbb{R}^2$, $(\psi(x_1), \dots, \psi(x_n))$ is a centered Gaussian vector which law is invariant under translation of the x_i's (and rotations if isotropy is further assumed), and that the sample paths $\{\psi(x); x \in \mathbb{R}^2\}$ are a.s. of class $\mathcal{C}^2$ (or more). See [AT09] for a rigourous and detailed exposition of Gaussian fields. Such a field is characterised by its reduced covariance function Γ

$$\mathbb{E}[\psi(z)\psi(w)] := \Gamma(z-w)$$

for some $\Gamma : \mathbb{R}^2 \to \mathbb{R}$, and if the field is furthermore assumed to be isotropic (i.e. its law is invariant under rotations)

$$\Gamma(z-w) = \sigma(|z-w|^2) \tag{2}$$

for some $\sigma : \mathbb{R}_+ \to \mathbb{R}$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^2$.

We denote by $\nabla\psi(z)$ the gradient of ψ at $z \in \mathbb{R}^2$, by $H_\psi(z)$ the Hessian matrix evaluated at z, when these quantities are well defined. For a smooth random field ψ, the set of critical points is denoted by

$$\mathcal{X}_c = \mathcal{X}_c(\psi) := \{x \in \mathbb{R}^2 : \nabla\psi(c) = 0\},$$

and the number of critical points in a small disc B_ρ of radius $\rho > 0$ is defined by

$$\mathcal{N}_\rho^c(\psi) := \#\mathcal{X}_c \cap B_\rho.$$

When there is no ambiguity about the random field ψ, we simply write $\mathcal{N}_\rho^c$ instead of $\mathcal{N}_\rho^c(\psi)$. Similarly, we denote by resp. $\mathcal{N}_\rho^s(\psi), \mathcal{N}_\rho^e(\psi), \mathcal{N}_\rho^{max}(\psi), \mathcal{N}_\rho^{min}(\psi)$ the number of resp. saddles, extrema, maxima and minima, critical points characterised by the signs of the Hessian eigenvalues.

As will be explained at Section 5, to perform a second order local analysis of the repulsion of ψ's critical points, we must assume fourth order differentiation of ψ, and for technical reasons we further assume that the fourth order derivative is α-Hölder for some $\alpha > 0$, we call this property $\mathcal{C}^{4+\alpha}$ regularity. It is implied by σ being of class $\mathcal{C}^{8+\beta}$ for some $\beta > 2\alpha$, see Proposition 1 below. In this case, the Hölder constant is a random variable with Gaussian tail (see below).

Assumption 2.1. *Assume that ψ is a non-constant stationary Gaussian field on $\mathbb{R}^2$ and its reduced covariance Γ is of class $\mathcal{C}^{8+\beta}$ for some $\beta > 0$.*

This assumption implies the $\mathcal{C}^{4+\alpha}$ regularity of ψ by applying the proposition below to ψ's 4th order derivatives.

Proposition 1. Let φ be a stationary Gaussian field $\mathbb{R}^2 \to \mathbb{R}$, with reduced covariance function $\gamma : \mathbb{R}^2 \to \mathbb{R}$. Then if for some $C, \beta > 0$, for $\delta > 0$ sufficiently small

$$|\gamma(x) - \gamma(0)| \leqslant C|x|^\beta, |x| \leqslant \delta,$$

then for $0 < \varepsilon < \beta/2$ there is a random variable U_ε with Gaussian tail such that for all $x, y \in B_1$,

$$|\varphi(x) - \varphi(y)| \leqslant U_\varepsilon |x-y|^{\beta/2-\varepsilon}.$$

Proof. It follows from the classical result from Landau and Shepp [AT09, (2.1.4)] that for a centred Gaussian field f a.s. bounded on a Euclidean compact T, there is $c > 0$ such that for large enough u,

$$\mathbb{P}(\sup_{t\in T} |f(t)| \geqslant u) \leqslant 2\exp(-cu^2).$$

We wish to apply this result to $T = B_1 \times B_1$ and

$$f(x,y) = |x-y|^{-\alpha}(\varphi(x) - \varphi(y)), (x,y) \in T.$$

Let $\alpha = \beta/2 - \varepsilon$. The fact that f is bounded is the consequence of the fact that φ's path are locally α-Holder for $\alpha < \beta/2$, see for instance [Pot09, Corollary 4.8]. □

Definition 1. Say that some random variables X, Y satisfy $X = O_{\mathbb{P}(Y)}$ if $X \leqslant UY$ where U is a random variable with a Gaussian tail, i.e.

$$\mathbb{P}(|U| > t) \leqslant c\exp(-c't^2), t \geqslant 0$$

for some $c < \infty, c' > 0$.

Proposition 1 hence implies that if a stationary field ψ's reduced covariance Γ is of class $\mathcal{C}^{k+\eta}$, then

$$\partial^k \psi(t+h) = \partial^k \psi(t) + \mathcal{O}_{\mathbb{P}}(h^{\eta/2}), t \in \mathbb{R}^d.$$

2.1. **Dependency structure.** Stationarity conveys strong constraints on the dependence structure between the field's partial derivatives at a given point. Let us recall formula [AT09, (5.5.4)-(5.5.5)]: if Γ is $\mathcal{C}^{k+\eta}$ differentiable for some $k \in \mathbb{N}, \eta > 0$, for natural integers $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta \leqslant k, \gamma + \delta \leqslant k$,

$$\mathbb{E}\left(\partial_1^\alpha \partial_2^\beta \psi(t) \cdot \partial_1^\gamma \partial_2^\delta \psi(s)\right) = \frac{\partial}{\partial_{t_1}^\alpha} \frac{\partial}{\partial_{t_2}^\beta} \frac{\partial}{\partial_{s_1}^\gamma} \frac{\partial}{\partial_{s_2}^\delta} \Gamma(t-s), s, t \in \mathbb{R}^2.$$

In particular if $s = t$ we have the spectral representation

$$\mathbb{E}\left(\partial_1^\alpha \partial_2^\beta \psi(t) \cdot \partial_1^\gamma \partial_2^\delta \psi(t)\right) = (-1)^{\gamma+\delta} \frac{\partial^{\alpha+\beta+\gamma+\delta}\Gamma(0)}{\partial t_1^\alpha \partial t_2^\beta \partial t_1^\gamma \partial t_2^\delta} = m_{\alpha+\gamma,\beta+\delta} \text{ where } m_{a,b} := (-1)^a \imath^{a+b} \int_{\mathbb{R}^2} \lambda_1^a \lambda_2^b F(d\lambda), t \in \mathbb{R}^2 \tag{3}$$

where the symmetric spectral measure F is uniquely defined by

$$\Gamma(t) = \int_{\mathbb{R}^2} \exp(-\imath \lambda \cdot t) F(d\lambda), t \in \mathbb{R}^2. \tag{4}$$

Let us state important consequences of (3), and in particular of the fact that, due to the symmetry of F, the integral vanishes if a or b is an odd number. For this reason, $(-1)^a = (-1)^b$ when the integral does not vanish, and $m_{a,b}$ is symmetric in a and b.

Remark 1. For all $t \in \mathbb{R}^2, \psi(t)$ and $\partial_j \psi(t)$ are independent for $j = 1, 2$, hence $\partial_1 \psi$ and $\partial_2 \psi$ are independent, and furthermore for any two natural integers k, l which difference is odd, any partial derivatives of orders k and l

$$\partial_{i_1,\dots,i_k} \psi(0) \text{ and } \partial_{j_1,\dots,j_l} \psi(0) \text{ are independent.}$$

Non-independence and technical difficulties will mainly emerge from dependence between even degrees of differentiation of the field, such as $\psi(t)$ and $\partial_{11}\psi(t)$, or $\partial_{11}\psi(t)$ and $\partial_{22}\psi(t)$, or between the values of the field at different locations, say $\psi(s)$ and $\psi(t), s \neq t$. A case we must discard is that of constant ψ, i.e. $\psi(t) = U$ for some Gaussian variable U, and this is what we call a *trivial* Gaussian field.

Also, Cauchy-Schwarz inequality yields that for $\alpha, \beta, \gamma, \delta \in \mathbb{N}$

$$|m_{\alpha+\gamma,\beta+\delta}|^2 \leqslant m_{2\alpha,2\beta} m_{2\gamma,2\delta},$$

and there is equality only if $\lambda_1^\alpha \lambda_2^\beta$ is proportionnal to $\lambda_1^\gamma \lambda_2^\delta$ dF-a.s. In the isotropic case (i.e. F is invariant under spatial rotations), unless $\alpha = \gamma, \beta = \delta$ it can only happen if dF is the Dirac mass in 0, i.e.

$$m_{\alpha+\gamma,\beta+\delta}^2 < m_{2\alpha,2\beta} m_{2\gamma,2\delta}, \alpha \neq \gamma \text{ or } \beta \neq \delta \text{ if } \psi \text{ is non-trivial isotropic.} \tag{5}$$

Proposition 2. Let ψ be an isotropic Gaussian field $\mathbb{R}^2 \to \mathbb{R}$ that satisfies Assumption 2.1 with covariance under the form (2). We indicate the first derivatives of σ at point $0 \in \mathbb{R}$ by $\sigma'(0) = \eta_0$, $\sigma''(0) = \mu_0$, $\sigma^{(3)}(0) = \nu_0$, $\sigma^4(0) = \upsilon$. Then

$$\begin{aligned}
\mathrm{Var}(\partial_i \psi(0)) &= -2\eta_0 = m_{2,0} > 0, \\
\mathrm{Var}(\partial_{12} \psi(0)) &= 2^2 \mu_0 = m_{2,2} > 0, \\
\mathrm{Var}(\partial_{ii} \psi(0)) &= 3 \cdot 2^2 \mu_0 = m_{4,0} > 0, i = 1, 2, \\
\mathrm{Var}(\partial_{iii} \psi(0)) &= -15 \cdot 2^3 \nu_0 = m_{6,0} > 0, i = 1, 2.
\end{aligned} \tag{6}$$

The two last equalities illustrate the fact that isotropy and polar change of coordinates yield other relations between the $m_{a,b}$ of the form

$$m_{a,b} = \alpha_{a,b} m_{a+b,0}$$

where the coefficients $\alpha_{a,b}$ don't depend on F.

Example 1. Let J_0 be the Bessel function of the first order

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ix\cos(\theta)} d\theta, x \in \mathbb{R}.$$

For $k > 0$ let ψ be the *Gaussian random wave* with parameter k, i.e. the isotropic stationary Gaussian field with reduced covariance function

$$\Gamma(z) = J_0(k|z|).$$

As is apparent from (4), this is the centered Gaussian field whose spectral measure is the uniform law on the centred circle with radius k. It is important as it is the unique (in law) stationary Gaussian field for which

$$\partial_{11}\psi + \partial_{22}\psi + k^2\psi = 0 \text{ a.s.}$$

up to a multiplicative constant. See for instance [BCW19, MRV, NPR19] and references therein for recent works about diverse aspects of planar random wave models. As proved at Section 6.2, it is the only non-trivial stationary isotropic stationary field satisfying a non-trivial linear partial differential equation of order three or less. As critical points are not modified by adding a constant, we also consider *shifted Gaussian random waves* (SGRW), of the form $\tau U + \sigma\psi$, where $\tau \geqslant 0, \sigma > 0, \psi$ is a GRW and U is an independent centered standard Gaussian variable. The spectral measure of a SGRW is the sum of a uniform measure on a circle of $\mathbb{R}^2$ centred in 0 and a finite mass in $\{0\}$.

3. The Kac-Rice formula

The Kac-Rice formula gives a description of the factorial moments of the zeros of a random field. Let us give a formula adapted to counting the critical points of a certain type. The following result can be proved by combining the proofs of Theorems 6.3 and 6.4 from [AW09], see also [AD22, Appendix A].

Theorem 3.1. *Let ψ isotropic satisfying Assumption 2.1. Let $k \in \{1, 2\}$, B_1, B_2 some open subsets of $\mathbb{R}^d$,*

$$\mathcal{N}_\rho^{B_1} = \{t \in B(0,\rho) : \nabla\psi(t) = 0, H_\psi(t) \in B_1\}.$$

Then for ρ sufficiently small

$$\mathbb{E}[\mathcal{N}_\rho^{B_1}] = \int_{\mathcal{B}_\rho} \mathrm{K}_1^{B_1}(t)\,\mathrm{d}t, \tag{7}$$

$$\mathbb{E}[\mathcal{N}_\rho^{B_1}(\mathcal{N}_\rho^{B_2} - 1)] = \int_{\mathcal{B}_\rho^2} \mathrm{K}_2^{B_1,B_2}(t_1,t_2)\,\mathrm{d}t, \tag{8}$$

where we have the k-point correlation function :

$$\mathrm{K}_1^{B_1}(t) = \phi_{\nabla\psi(t)}(0)\ \mathbb{E}\left[|\det H_\psi(t)|\ \ \mathbf{1}_{B_1}(H_\psi(t)) \,\middle|\, \nabla\psi(t) = 0\right],$$

$$\mathrm{K}_2^{B_1,B_2}(t_1,t_2) = \phi_{(\nabla\psi(t_1),\nabla\psi(t_2))}(0,0)\ \mathbb{E}\left[\prod_{i=1}^{2} |\det H_\psi(t_i)|\ \ \mathbf{1}_{B_i}(H_\psi(t_i)) \,\middle|\, \nabla\psi(t_1) = \nabla\psi(t_2) = 0\right],$$

where ϕ_V is the density probability function of a Gaussian vector V.

We are specifically interested in a finite class of sets B_i, namely

$$\begin{aligned} B_c =& \mathcal{M}_d(\mathbb{R}) \text{ the class of } d\times d \text{ square matrices},\\ B_{ext} =& \overset{-1}{\det}((0,\infty))\\ B_s =& \overset{-1}{\det}((-\infty,0)),\\ B_{min} =& \{H \text{ definite positive}\},\\ B_{max} =& \{H \text{ definite negative}\}. \end{aligned}$$

In this case, the exponent in $\mathcal{N}$ or K_i is replaced by the subscript of B, e.g.

$$\mathcal{N}_\rho^c = \mathcal{N}_\rho^{B_c}, K_2^{s,s} = K_2^{B_s,B_s}, etc...$$

Proof. With $Z(t) = \nabla\psi(t), Y^t = H_\psi(t), g(t,H) = \mathbf{1}_{\{\det(H)\in B\}}$, we have

$$\mathcal{N}_\rho^B = \sum_{t:Z(t)=0} g(t,Y^t).$$

Let us show that hypothesis (iii') of [AW09, Th.6.3] is satisfied, that is for $t,s\in\mathcal{B}_\rho$, the law of $(\nabla\psi(t),\nabla\psi(s))$ is non-degenerated for ρ small enough. Let us expand

$$\mathbb{E}(\partial_i\psi(s)\partial_j\psi(t)) = \frac{\partial^2}{\partial_{s_i}\partial_{t_j}}\sigma(|s-t|^2) = \begin{cases} -2\sigma'(|s-t|^2) - 4(s_i-t_i)^2\sigma''(|s-t|^2) \text{ if } i=j\\ -4(s_i-t_i)(s_j-t_j)\sigma''(|s-t|^2) \text{ if } i\neq j. \end{cases}$$

By isotropy it suffices to evaluate it in $t=(r,0), s=(-r,0)$ for $r\geqslant 0$. Let us write the 4×4 covariance matrix in function of $\eta_r = \sigma'(4r^2), \mu_r = \sigma''(4r^2)$

$$\Sigma = -2\begin{pmatrix} \eta_0 I & \eta_r I + 2\mu_r A_r\\ \eta_r I + 2\mu_r A_r & \eta_0 I \end{pmatrix} \tag{9}$$

where

$$A_r = \begin{pmatrix} 4r^2 & 0\\ 0 & 0 \end{pmatrix}. \tag{10}$$

Hence the block determinant is

$$\begin{aligned} 16\det(\eta_0^2 I - (\eta_r I + 2\mu_r A_r)^2) &= 16\det((\eta_0^2-\eta_r^2)I - 4\mu_r\eta_r A_r - 4\mu_r^2 A_r^2)\\ &= 16(\eta_0^2-\eta_r^2)((\eta_0^2-\eta_r^2) - 16\mu_r\eta_r r^2 - 64\mu_r^2 r^4). \end{aligned}$$

This is equivalent to

$$16\cdot 8\eta_0(-\mu_0 r^2)(8\eta_0(-\mu_0 r^2) + \mathcal{O}_{\mathbb{P}}(r^2) - 16\mu_0\eta_0 r^2 + \mathcal{O}_{\mathbb{P}}(r^2)) \sim -128\eta_0\mu_0 r^2(-24\eta_0\mu_0 r^2) = 3\cdot 2^{10}\mu_0^2\eta_0^2 r^4, \tag{11}$$

where we have $\mu_0\eta_0\neq 0$ in virtue of (6). Hence the determinant is non zero for $r\neq 0$ sufficiently small. Then the modification of the proof of Theorem 6.3 following the proof of Theorem 6.4 of [AW09] yields the result, see Appendix A in [AD22].

It yields in particular

$$\phi_{(\nabla\psi(t_1),\nabla\psi(t_2))}(0,0) = \frac{1}{\sqrt{\det(\Sigma)}(2\pi)^2} = \frac{1}{2^7\pi^2\sqrt{3}|\mu_0\eta_0|r^2}(1+o_{r\to 0}(1)). \tag{12}$$

□

4. First order

In this section, we are interested in the computation of the expectation of the number of critical points in a Borel set $B \subset \mathbb{R}^2$.

Proposition 3. Let $\psi = \{\psi(z) : z \in \mathbb{R}^2\}$ be a non-trivial isotropic stationary Gaussian field $\mathbb{R}^2 \to \mathbb{R}$ which is a.s. of class $\mathcal{C}^2$ and let σ be defined by (2). Let

$$\lambda_c = \frac{4}{\sqrt{3}\pi} \frac{\sigma''(0)}{(-\sigma'(0))}.$$

In virtue of Proposition 2, $\lambda_c \in (0, \infty)$. Then, for every $\rho > 0$, we have

$$\mathbb{E}[\mathcal{N}_\rho^c] = \lambda_c |B_\rho|, \tag{13}$$

$$\mathbb{E}[\mathcal{N}_\rho^e] = \mathbb{E}[\mathcal{N}_\rho^s] = \frac{1}{2}\mathbb{E}[\mathcal{N}_\rho^c], \tag{14}$$

$$\mathbb{E}[\mathcal{N}_\rho^{min}] = \frac{1}{4}\mathbb{E}[\mathcal{N}_\rho^c]. \tag{15}$$

A sufficient condition for ψ being of class $\mathcal{C}^2$ is that σ is of classe $\mathcal{C}^{4+\beta}$ for some $\beta > 0$, see Proposition 1.

Remark 2. By stationarity, λ_c is the intensity of $\mathcal{X}_c(\psi)$, i.e. the mean number of critical points per unit volume.

Proof. According to Theorem 3.1, we must simply evaluate

$$\mathrm{K}_1(z) = \phi_{\nabla\psi(z)}(0,0)\ \ \mathbb{E}\big[\ |\det H_\psi(z)|\ \big|\ \nabla\psi(z) = 0\ \big]. \tag{16}$$

The stationarity of ψ implies that $K_1(z)$ is independent of z, see formula (8). So, we get

$$\mathbb{E}[\mathcal{N}_\rho^c] = |B_\rho| \mathrm{K}_1(0). \tag{17}$$

Using the matrix Σ with $r = 0$ in (9), we immediately obtain the probability density function of (two-dimensional vector) $\nabla\psi(z)$ evaluated at point $(0,0)$:

$$\phi_{\nabla\psi(z)}(0,0) = \frac{1}{2\pi}\frac{1}{\sqrt{4\eta_0^2}} = \frac{1}{4\pi|\eta_0|}, \tag{18}$$

where $\eta_0 = \sigma'(0)$. From this point until the end of the proof we will use the method of the article [BCW19]. Since the first and the second derivatives of $\psi(z)$ are independent at every fixed point $z \in \mathbb{R}^2$, then:

$$\mathbb{E}\big[|\det H_\psi(z)|\ \big|\nabla\psi(z) = 0\big] = \mathbb{E}\big[\ |\det H_\psi(z)|\ \big] = \mathbb{E}\big[\ |\det H_\psi(z)|\ \big] = \mathbb{E}\big[\ \big|\partial_{11}\psi(0)\partial_{22}\psi(0) - \partial_{12}^2\psi(0)\big|\ \big]. \tag{19}$$

To evaluate (19), we consider the transformation $W_1 = \partial_{11}\psi(0)$, $W_2 = \partial_{12}\psi(0)$, $W_3 = \partial_{11}\psi(0) + \partial_{22}\psi(0)$ and we write $\mathbb{E}\big[\ \big|\partial_{11}\psi(0)\partial_{22}\psi(0) - \partial_{12}^2\psi(0)\big|\ \big]$ in terms of a conditional expectation as follows:

$$\mathbb{E}\big[\ \big|\partial_{11}\psi(0)\partial_{22}\psi(0) - \partial_{12}^2\psi(0)\big|\ \big] = \mathbb{E}\big[\ \big|W_1W_3 - W_1^2 - W_2^2\big|\ \big] = \mathbb{E}\big[\ \mathbb{E}\big[\ |W_1W_3 - W_1^2 - W_2^2|\ \big|W_3\ \big]\big], \tag{20}$$

where $W = (W_1, W_2, W_3)$ is a centered Gaussian vector field with covariance matrix D.

Use Proposition 2 and Remark 1, we have $D = \begin{pmatrix} 12\mu_0 & 0 & 16\mu_0 \\ 0 & 4\mu_0 & 0 \\ 16\mu_0 & 0 & 32\mu_0 \end{pmatrix}$.

The conditional distribution of $(W_1, W_2)\big|W_3$ is Gaussian with covariance matrix $\Sigma_{(W_1,W_2)|W_3} = \begin{pmatrix} 4\mu & 0 \\ 0 & 4\mu \end{pmatrix}$

and expectation $\mathbb{E}[\ (W_1,W_2)\ |W_3=t] = \begin{pmatrix} \frac{t}{2} \\ 0 \end{pmatrix}$ for $t\in\mathbb{R}$.

The conditioned Gaussian vector $(W_1,W_2)|W_3=t$ is distributed as $(2\sqrt{\mu_0}Z_1+\frac{t}{2}, 2\sqrt{\mu_0}\ Z_2)$ where Z_1, Z_2 are two independent standard Gaussian random variables, hence we have

$$\begin{aligned}
\mathbb{E}\left[\ |W_1W_3 - W_1^2 - W_2^2|\ \middle|\ W_3=t\ \right] &= \mathbb{E}[\ |\ W_1t - W_1^2 - W_2^2\ |\ |W_3=t\ ]] \\
&= \mathbb{E}[\ |(2\sqrt{\mu_0}Z_1+t/2)t - (2\sqrt{\mu_0}Z_1+t/2)^2 - 4\mu_0\ Z_2^2|\ ] \\
&= \mathbb{E}[\ |-4\mu_0 Z_1^2 - 4\mu_0 Z_2^2 + \frac{t^2}{4}|\ ] \\
&= 4\mu_0\ \mathbb{E}\left[\ \left|-X+\frac{t^2}{16\mu_0}\right|\right],
\end{aligned}$$

where X is a $\chi-$square random variable with density $f_X(x)=\frac{1}{2}e^{-\frac{x}{2}}, x>0$.
So

$$\begin{aligned}
\mathbb{E}[\ |-X+\frac{t^2}{16\mu_0}|\ ] &= \frac{1}{2}\int_0^{\frac{t^2}{16\mu_0}} (\frac{t^2}{16\mu_0}-x)e^{-\frac{x}{2}}dx + \frac{1}{2}\int_{\frac{t^2}{16\mu_0}}^{+\infty}(-\frac{t^2}{16\mu_0}+x)e^{-\frac{x}{2}}dx \\
&= -2 + 4e^{-\frac{t^2}{32\mu_0}} + \frac{t^2}{16\mu_0},
\end{aligned}$$

then

$$\begin{aligned}
\mathbb{E}\left[\ |W_1W_3 - W_1^2 - W_2^2|\ \middle|\ W_3=t\ \right] &= 4\mu_0\ \frac{1}{8\sqrt{\pi\mu}}\int_{\mathbb{R}} \mathbb{E}\left[\ \left|-X+\frac{t^2}{16\mu_0}\right|\right]\ e^{-\frac{t^2}{64\mu_0}}dt \\
&= \frac{\sqrt{\mu_0}}{2\sqrt{\pi}}\int_{\mathbb{R}} e^{-\frac{t^2}{64\mu}}\left(-2+4e^{-\frac{t^2}{32\mu_0}}+\frac{t^2}{16\mu_0}\right)\ dt = \frac{16\mu_0}{\sqrt{3}}. \qquad (21)
\end{aligned}$$

By combining Equations (17), (18),(19) and (21), we obtain Formula (13).

Now, we turn to the evaluation of the expected number of the extrema and saddle points. We have

$$\begin{aligned}
\mathcal{N}_\rho^e := \mathcal{N}_\rho^{(0,\infty)} &= \#\{x\in B_\rho : \nabla\psi(x)=0, \det H_\psi(z)>0\} \\
\mathcal{N}_\rho^s := \mathcal{N}_\rho^{(-\infty,0)} &= \#\{x\in B_\rho : \nabla\psi(x)=0, \det H_\psi(z)<0\}.
\end{aligned}$$

As previously, we apply the Kac-Rice formula from Section 3. We get:

$$\mathbb{E}[\mathcal{N}_\rho^e] = \int_{B_\rho} \mathrm{K}_1^e(z)dz \qquad \text{and} \qquad \mathbb{E}[\mathcal{N}_\rho^s] = \int_{B_\rho} \mathrm{K}_1^s(z)dz,$$

where

$$\begin{aligned}
\mathrm{K}_1^e(z) &= \phi_{\nabla\psi(z)}(0,0)\ \ \mathbb{E}\left[\ |\det H_\psi(z)|\ \mathbf{1}_{\{\det H_\psi(z)>0\}}\ \middle|\nabla\psi(z)=0\ \right], \\
\mathrm{K}_1^s(z) &= \phi_{\nabla\psi(z)}(0,0)\ \ \mathbb{E}\left[\ |\det H_\psi(z)|\ \mathbf{1}_{\{\det H_\psi(z)<0\}}\ \middle|\nabla\psi(z)=0\ \right].
\end{aligned}$$

Since the first and the second derivatives of $\psi(z)$ are independent at every fixed point $z\in\mathbb{R}^2$, we obtain

$$\mathbb{E}[\mathcal{N}_\rho^e] = \pi\rho^2\phi_{\nabla\psi(z)}(0,0)\ \ \mathbb{E}\left[\ |\det H_\psi(z)|\ \mathbf{1}_{\{\det H_\psi(z)>0\}}\right], \qquad (22)$$

$$\mathbb{E}[\mathcal{N}_\rho^s] = \pi\rho^2\phi_{\nabla\psi(z)}(0,0)\ \ \mathbb{E}\left[\ |\det H_\psi(z)|\ \mathbf{1}_{\{\det H_\psi(z)<0\}}\right]. \qquad (23)$$

Using the same argument as in the case of critical points, we write

$$\begin{aligned}\mathbb{E}\left[\ |\det H_\psi(z)|\,\mathbf{1}_{\{detH_\psi(z)>0\}}\ \right] &= \mathbb{E}\left[|\partial_{11}\psi(0)\partial_{22}\psi(0)-\partial_{12}^2\psi(0)|\ \mathbf{1}_{\{\partial_{11}\psi(0)\partial_{22}\psi(0)-\partial_{12}^2>0\}}\right]\\ &= 4\mu\ \frac{1}{8\sqrt{\pi\mu}}\int_{\mathbb{R}}\mathbb{E}\left[\ \left|-X+\frac{t^2}{16\mu_0}\right|\mathbf{1}_{\{-X+\frac{t^2}{16\mu_0}>0\}}\right]\ e^{-\frac{t^2}{64\mu_0}}\,dt\\ &= \frac{\sqrt{\mu_0}}{2\sqrt{\pi}}\int_{\mathbb{R}}\left(-2+2e^{-\frac{t^2}{32\mu_0}}+\frac{t^2}{16\mu_0}\right)\ e^{-\frac{t^2}{64\mu_0}}\,dt\\ &= \frac{8\mu_0}{\sqrt{3}},\end{aligned}\tag{24}$$

and

$$\begin{aligned}\mathbb{E}\left[\ |\det H_\psi(z)|\,\mathbf{1}_{\{detH_\psi(z)<0\}}\ \right] &= \mathbb{E}\left[|\partial_{11}\psi(0)\partial_{22}\psi(0)-\partial_{12}^2|\ \mathbf{1}_{\{\partial_{11}\psi(0)\partial_{22}\psi(0)-\partial_{12}^2<0\}}\right]\\ &= 4\mu\ \frac{1}{8\sqrt{\pi\mu_0}}\int_{\mathbb{R}}\mathbb{E}\left[\ \left|-X+\frac{t^2}{16\mu_0}\right|\mathbf{1}_{\{-X+\frac{t^2}{16\mu_0}<0\}}\right]\ e^{-\frac{t^2}{64\mu_0}}\,dt\\ &= \frac{\sqrt{\mu_0}}{2\sqrt{\pi}}\int_{\mathbb{R}}\left(2e^{-\frac{t^2}{32\mu_0}}\right)\ e^{-\frac{t^2}{64\mu_0}}\,dt\\ &= \frac{8\mu_0}{\sqrt{3}}.\end{aligned}\tag{25}$$

By combining Equations (18), (22), (23), (24) and (25), we obtain Formula (14).
Finally, we turn to the calculation of the expectation of the number of minima and maxima in B_ρ. We know that:

$$\mathcal{N}_\rho^e = \mathcal{N}_\rho^{min} + \mathcal{N}_\rho^{max}$$

so $\mathbb{E}[\mathcal{N}_\rho^e] = \mathbb{E}[\mathcal{N}_\rho^{min}] + \mathbb{E}[\mathcal{N}_\rho^{max}]$.

By symmetry of the Gaussian field ψ, we have the following equality: $\mathcal{N}_\rho^{max}(-\psi) \overset{\mathcal{L}}{=} \mathcal{N}_\rho^{max}(\psi) = \mathcal{N}_\rho^{min}(-\psi)$ for $-\psi \overset{\mathcal{L}}{=} \psi$, therefore $\mathbb{E}[\mathcal{N}_\rho^{min}] = \mathbb{E}[\mathcal{N}_\rho^{max}]$.
Finally, we obtain

$$\mathbb{E}[\mathcal{N}_\rho^{min}] = \mathbb{E}[\mathcal{N}_\rho^{max}] = \frac{1}{2}\mathbb{E}[\mathcal{N}_\rho^e].$$

5. Second order

In this section, we will study the asymptotic behaviour of the second factorial moment of $\mathcal{N}_\rho^c$ when ρ goes to zero. The following theorem is the main result of this paper. Given two quantitites α_ρ, β_ρ, write $\alpha_\rho \asymp \beta_\rho$ if for two constants $0 < c < c' < \infty$, we have $c\alpha_\rho \leqslant \beta_\rho \leqslant c'\alpha_\rho$ for ρ sufficiently small, and $\alpha_\rho \sim \beta_\rho$ if $\alpha_\rho/\beta_\rho \to 1$, with the convention $0/0 = 1$.

Theorem 5.1. *Let ψ be an isotropic Gaussian field $\mathbb{R}^2 \to \mathbb{R}$ that satisfies Assumption 2.1. The* repulsion factor $\mathsf{R}_c := \mathsf{R}_{\mathcal{X}_c}$ *is given by*

$$\mathsf{R}_c = \frac{\sqrt{3}}{8}\left(5\frac{\sigma'''(0)\sigma'(0)}{(\sigma''(0))^2}-3\right).$$

As $\rho \to 0$, we have the following asymptotic equivalent expression for the second factorial moment of the number of critical points

$$\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c-1)] \sim \mathsf{R}_c\lambda_c^2|B_\rho|^2 \asymp \rho^4.\tag{26}$$

Depending on the law of ψ, R_c can take any prescribed value in $[\frac{1}{8\sqrt{3}}, \infty)$, and $\frac{1}{8\sqrt{3}}$ is the minimal possible value, it is reached iff ψ is a shifted Gaussian random wave (Example 1).

For the numbers of extrema, saddles in a ball of radius ρ, we have as $\rho \to 0$

$$\mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] \asymp \rho^7, \tag{27}$$

$$\mathbb{E}[\mathcal{N}_\rho^s(\mathcal{N}_\rho^s - 1)] \asymp \rho^7 \ln(\rho), \tag{28}$$

$$\mathbb{E}[\mathcal{N}_\rho^e \mathcal{N}_\rho^s] \sim \mathsf{R}_c \lambda_c^2 |B_\rho|^2. \tag{29}$$

Remark 3. The repulsion factor terminology comes from $\lambda_c|B_\rho| \sim \mathbb{E}(\mathcal{N}_\rho)$ and by the heuristic explanation after (1).

Remark 4. By truncating the expansion of the type (38) at a lower order, one could prove that Expression (26) is valid under the weaker assumption that Γ is of class $\mathcal{C}^{6+\beta}$ (and ψ is of classe $\mathcal{C}^3$).

Example 2 (Bargmann Fock field)**.** Consider the Bargmann-Fock field with parameter k, which is the stationary isotropic Gaussian field with reduced covariance function

$$\sigma(r) = \exp(-kr), r \geqslant 0.$$

According to Proposition 3, we have for the first order

$$\begin{aligned}
\sigma'(0) &= -k \\
\sigma''(0) &= k^2 \\
\sigma'''(0) &= -k^3 \\
\mathbb{E}[\mathcal{N}_\rho^c] &= \frac{4}{\sqrt{3}} k\, \rho^2.
\end{aligned}$$

Hence the attraction factor is

$$\mathsf{R}_c = \frac{\sqrt{3}}{4} < 1,$$

which means that the process of critical points is locally weakly repulsive. It logically does not depend on the scaling factor k.

Example 3 (Gaussian random waves)**.** Consider the Gaussian random wave introduced at Example 1 by $\sigma(x) = J_0(k\sqrt{x}), x \geqslant 0$. We have :

$$\begin{aligned}
\sigma'(0) &= -\frac{k^2}{4} \\
\sigma''(0) &= \frac{k^4}{2^5} \\
\sigma''(0) &= -\frac{k^6}{3 \cdot 2^7} \\
\lambda_c &= \frac{k^2}{2\sqrt{3}\pi} \text{ i.e. } \mathbb{E}[\mathcal{N}_\rho^c] = \frac{k^2}{2\sqrt{3}}\, \rho^2,
\end{aligned}$$

Hence the attraction factor takes the smallest possible value

$$\mathsf{R}_c = \frac{1}{8\sqrt{3}} < 1,$$

which means that the process of critical points is locally weakly repulsive. We retrieve the second factorial moment of Beliaev, Cammarota and Wigman [BCW19]

$$\mathbb{E}(\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)) \sim \frac{k^4}{2^6 3\sqrt{3}}\rho^4.$$

Example 4. Consider the centered stationary Gaussian random field ψ with spectral measure

$$F(d\lambda) = |\lambda|^{-7}\mathbf{1}_{\{|\lambda|\geqslant 1\}}d\lambda.$$

One has by Proposition 2,$|\sigma'(0)| < \infty, \sigma''(0) < \infty, -\sigma'''(0) = \infty$, hence $\mathsf{R}_c = \infty$, but Theorem 5.1 does not apply precisely because F's higher moments are infinite, meaning that ψ is not of class $\mathcal{C}^3$. Hence we consider $F_t(d\lambda) = \mathbf{1}_{\{|\lambda|\leqslant t\}}F(d\lambda)(\int_0^t dF)^{-1}$ for $t > 1$. We have as $t \to \infty$

$$\frac{\eta_0\nu_0}{\mu_0^2} \sim c\int_1^t r^6 r^{-7} r dr \asymp t.$$

It implies that the repulsion factor of F_t can reach arbitrarily high values. In particular, this parametric model provides processes of critical points that are weakly locally attractive.

5.1. **Discussion and related litterature.** The equivalence (26) generalises the results of [BCW19], and shows that locally, the random planar wave model yields the more repulsive critical points. We also show that for a general process ψ, the subprocesses formed by extrema and saddle points experience locally a strong repulsion with three more orders of magnitude for ρ. It confirms the idea that close to a large portion of saddle points, there is an extremal point nearby, and conversely, but that the closest point of the same type (extrema or saddle) is typically much further away.

A current novelty is also to derive the precise asymptotic repulsion for the extrema process and the saddle process. Hence we are able to state that the ratio between the internal repulsion forces among extremal points and among saddle points tends to infinity as the radius of the observation ball goes to 0.

Azais and Delmas [AD22] derived upper bounds about such quantities in any dimension. In particular, their results are consistant with ours in the critic-critic, extrema-extrema and extrema-saddle cases.

6. Proofs

6.1. **Conditioning.** The proofs of all formulas of Theorem 5.1 are based on the Kac-Rice formula in Theorem 3.1, for instance if $B = B' = \mathbb{R}^2$, we have the second factorial moment of the number of critical points

$$\mathbb{E}\left[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)\right] = \int\int_{B_\rho \times B_\rho} \mathrm{K}_2(z,w)\, dzdw, \tag{30}$$

where K_2 is the 2-point correlation function : :

$$\begin{aligned}\mathrm{K}_2(z,w) = {} & \phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0),(0,0)) \\ & \times \mathbb{E}\left[\, |\det H_\psi(z)|\, |\det H_\psi(w)| \,\middle|\, \nabla\psi(z) = \nabla\psi(w) = 0 \,\right].\end{aligned} \tag{31}$$

Let us briefly introduce where the difficulty comes from and why higher order differentiability is required. For z, w close from 0, if $\nabla\psi(z) = \nabla\psi(w) = 0$, then the second order derivatives are also small, and the determinant is dominated by third order differentials. When one imposes additional constraints on the determinant signs, it yields other cancellations within third order derivatives, requiring fourth order differentiability.

Thanks to the stationarity and isotropy of ψ, it suffices to compute $\mathrm{K}_2(z,w)$ for $z=(r,0)$ and $w=(-r,0)$ for all $r>0$. To evaluate $\mathbb{E}\left[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c-1)\right]$, the idea is to change the conditioning in $\mathrm{K}_2(z,w)$. To symmetrize the problem, we introduce some notations for r near 0, $r\neq 0$, exploiting Proposition 1 and Definition 1,

$$\begin{cases} a) & \Delta_i(r):=\frac{1}{2}\partial_i\psi(z)+\frac{1}{2}\partial_i\psi(w) \qquad \text{implies } \Delta_i(r)=\partial_i\psi(0)+\frac{r^2}{2}\partial_{i11}\psi(0)+\mathcal{O}_{\mathbb{P}}(r^4) \\ b) & \Delta_{i1}(r):=\frac{1}{2r}(\partial_i\psi(z)-\partial_i\psi(w)) \quad \text{implies } \Delta_{i1}(r)=\partial_{i1}\psi(0)+\frac{r^2}{6}\partial_{i111}\psi(0)+\mathcal{O}_{\mathbb{P}}(r^4). \end{cases}, \tag{32}$$

The crucial point is that $\nabla\psi(z)=\nabla\psi(w)=0$ is equivalent to $\Delta_i(r)=0,\Delta_{i1}(r)=0, i=1,2$. Let us introduce

$$Y_r=(\Delta_1(r),\Delta_2(r),\Delta_{11}(r),\Delta_{12}(r)), r>0$$

so that $Y_r=0$ is equivalent to $\nabla\psi(z)=\nabla\psi(w)=0$, and

$$Y_0:=(\partial_1\psi(0),\partial_2\psi(0),\partial_{11}\psi(0),\partial_{21}\psi(0)).$$

We will see later that Y_0 is non-degenerate, hence Y_r is also non-degenerate for r small enough, by continuity of the covariance matrix.

We denote the conditionnal probability and expectation with respect to $Y_r=0$ by

$$\mathbb{P}^{(r)}(\cdot)=\mathbb{P}(\cdot\mid Y_r=0), \qquad \mathbb{E}^{(r)}(\cdot)=\mathbb{E}(\cdot\mid Y_r=0), r\geqslant 0.$$

Remark 5. Let (X,Y) be a Gaussian vector with Y non-degenerate. If M is non-singular matrix and if φ is a measurable function with polynomial bounds, then

$$E(\varphi(X)|Y=0)=E(\varphi(X)|MY=0).$$

So, since obviously

$$\nabla\psi(z)=\nabla\psi(w)=0 \iff Y_r=0,$$

the 2-point correlation function $\mathrm{K}_2(z,w)$ becomes :

$$\mathrm{K}_2(z,w)=\phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0),(0,0))\times\mathbb{E}^{(r)}[\,|\det H_\psi(z)|\,|\det H_\psi(w)|]. \tag{33}$$

Using (12) we can evaluate the density in 0, and the previous expression becomes

$$K_2(z,w)=\frac{1}{(2\pi)^2 2^5\sqrt{3}(-\eta_0)r^2}\mathbb{E}^{(r)}[\,|\det H_\psi\psi(0)|\,|\det H_\psi(r)|](1+o(1)).$$

It remains to express the product of determinants under the conditioning in function of $\Delta_i(r)=\Delta_{i1}(r)=0$, this involves higher order derivatives (see(32)).

Lemma 1. Assume ψ satisfies Assumption 2.1 for some $\beta>0$ and let $0<\alpha<\beta/2$. For $z=(r,0)$ and $w=(-r,0)$, we have if $\Delta_i=\Delta_{1i}=0, i=1,2$,

$$\det H_\psi(z) = r(A_1+rB_0+\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})) \tag{34}$$

$$\det H_\psi(w) = r(-A_1+rB_0+\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})) \tag{35}$$

$$\det H_\psi(z)\det H_\psi(w) = r^2[-{A_1}^2+g(r)] \tag{36}$$

where

$$\begin{cases} A_1=\partial_{22}\psi(0)\partial_{111}\psi(0) \\ B_0=\partial_{221}\psi(0)\,\partial_{111}\psi(0)-\partial_{211}\psi(0)^2+\frac{1}{3}\partial_{22}\psi(0)\partial_{1111}\psi(0) \\ g(r)=r^2B_0^2+\mathcal{O}_{\mathbb{P}}(A_1r^{1+\alpha})+\mathcal{O}_{\mathbb{P}}(r^{2+\alpha}) \end{cases}. \tag{37}$$

Proof. Define

$$\begin{cases} a) & \frac{r^2}{3!}\Delta_{1111}(\pm r) = \partial_{11}\psi(\pm r,0) - \Delta_{11}(\pm r) - (\pm\, r\partial_{111}\psi(0)) & \text{implies } \Delta_{1111}(\pm r) = \partial_{1111}\psi(0) + \mathcal{O}_{\mathbb{P}}(r^\alpha) \\ b) & \frac{r^2}{3!}\Delta_{2111}(\pm r) = \partial_{21}\psi(\pm r,0) - \Delta_{21}(\pm r) - (\pm\, r\partial_{211}) & \text{implies } \Delta_{2111}(\pm r) = \partial_{2111}\psi(0) + \mathcal{O}_{\mathbb{P}}(r^\alpha) \\ c) & \frac{r^2}{2!}\Delta_{2211}(\pm r) = \partial_{22}\psi(\pm r,0) - \partial_{22}\psi(0) - (\pm\, r\partial_{221}) & \text{implies } \Delta_{2211}(\pm r) = \partial_{2211}\psi(0) + \mathcal{O}_{\mathbb{P}}(r^\alpha). \end{cases}$$

We can explicitly write the expression of $\det H_\psi(z) \det H_\psi(w)$:

$$\begin{aligned} \det H_\psi(z) &= \partial_{11}\psi(r,0)\partial_{22}\psi(r,0) - (\partial_{21}\psi(r,0))^2 \\ &= r(A_1 + rB_r + r^2C_r), \end{aligned} \tag{38}$$

with

$$\begin{aligned} B_r &= \partial_{111}\psi(0)\partial_{221}\psi(0) + \frac{1}{3}\Delta_{1111}(r)\ \partial_{22}\psi(0) - \partial_{211}\psi(0)^2 = B_0 + \mathcal{O}_{\mathbb{P}}(r^\alpha) \\ C_r &= \frac{\Delta_{1111}(r)}{3}\ \partial_{221}\psi(0) + \frac{\Delta_{2211}(r)}{2}\partial_{111}\psi(0) - \frac{2}{3}\Delta_{2111}(r)\partial_{211}\psi(0) = \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

and

$$\begin{aligned} \det H_\psi(w) &= \partial_{11}\psi(-r,0)\partial_{22}\psi(-r,0) - (\partial_{21}\psi(-r,0))^2 \\ &= r(-A_1 + rB'_r + r^2C'_r) \end{aligned} \tag{39}$$

with

$$\begin{aligned} B_{r'} =& \partial_{111}\psi(0)\partial_{221}\psi(0) + \frac{1}{3}\Delta_{1111}(-r)\ \partial_{22}\psi(0) - \partial_{211}\psi(0)^2 = B_0 + \mathcal{O}_{\mathbb{P}}(r^\alpha) \\ C'_r =& -\frac{\Delta_{1111}(-r)}{3}\ \partial_{221}\psi(0) - \frac{\Delta_{2211}(-r)}{2}\partial_{111}\psi(0) + \frac{2}{3}\Delta_{2111}(-r)\partial_{211}\psi(0) = \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

Combining Equations (38) and (39), an elementary calculus leads to

$$\det H_\psi(z) \det H_\psi(w) = r^2[-{A_1}^2 + g(r)]$$

where

$$\begin{aligned} g(r) =& rA_1(-B_r + B_{r'}) + r^2B_rB_{r'} - r^2A_1(C_r - C'_r) \\ =& rA_1(\mathcal{O}_{\mathbb{P}}(r^\alpha)) + r^2B_rB_{r'} - r^2A_1(C_r - C'_r) \\ =& \mathcal{O}_{\mathbb{P}}(A_1r^{1+\alpha}) + r^2B_0^2 + \mathcal{O}_{\mathbb{P}}(A_1r^{1+\alpha}) + \mathcal{O}_{\mathbb{P}}(r^{2+\alpha}) + \mathcal{O}_{\mathbb{P}}(r^{2+2\alpha}) \\ =& r^2B_0^2 + \mathcal{O}_{\mathbb{P}}(A_1r^{1+\alpha}) + \mathcal{O}_{\mathbb{P}}(r^{2+\alpha}). \end{aligned}$$

□

As a consequence from Lemma 1, the 2-point correlation function given by (33) becomes

$$\mathrm{K}_2(z,w) = \frac{1}{(2\pi)^2 2^5\sqrt{3}(-\eta_0)\mu_0}\mathbb{E}^{(r)}\left[|{A_1}^2 - g(r)|\right](1+o(1)). \tag{40}$$

6.2. **Dependency of derivatives.** In view of the previous result, we will have to estimate quantities related to the random vectors

$$X = (\partial_{22}\psi(0), \partial_{111}\psi(0), \partial_{122}\psi(0), \partial_{112}\psi(0), \partial_{1111}\psi(0))$$

and Y_r. We must consider the case where (X, Y_0) is degenerate. Examining Remark 1, we can split the variables involved in several groups that are mutually independent, there are for instance only two groups of size 3,

$$\{\partial_1\psi(0), \partial_{122}\psi(0), \partial_{111}\psi(0)\} \text{ and } \{\partial_{22}\psi(0), \partial_{1111}\psi(0), \partial_{11}\psi(0)\}.$$

Other groups, such as $\{\partial_{112}\psi(0), \partial_2\psi(0)\}$, have less members, and in the isotropic case they won't be in a linear relation because of (5):

$$\text{Cov }(\partial_{112}\psi(0), \partial_2\psi(0))^2 = m_{2,2}^2 < m_{2,0}m_{4,2} = \text{Var}(\partial_2)\text{Var}(\partial_{112}).$$

There is actually no other case to consider. Let us elucidate what can happen within the two bigger groups.

Proposition 4. Assume the spectral measure F is isotropic and not reduced to a Dirac mass in 0. There is $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that $\alpha\partial_1\psi(0) + \beta\partial_{111}\psi(0) + \gamma\partial_{122}\psi(0) = 0$ a.s. iff $\beta = \gamma$ and F is uniformly spread along a circle of radius $\sqrt{\alpha/\beta}$, i.e. if ψ is a SGRW with parameter $\sqrt{\alpha/\beta}$.

There is no $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that a.s. $\alpha\partial_{11}\psi(0) + \beta\partial_{22}\psi(0) + \gamma\partial_{1111}\psi(0) = 0$.

Proof. Using (3) and recalling the symmetry $m_{a,b} = m_{b,a}$

$$\begin{aligned}\text{Var}(\alpha\partial_1\psi(0) + \beta\partial_{111}\psi(0) + \gamma\partial_{122}\psi(0)) =& \alpha^2 m_{2,0} + \beta^2 m_{6,0} + \gamma^2 m_{2,4} + 2\alpha\beta m_{4,0} + 2\alpha\gamma m_{2,2} + 2\beta\gamma m_{4,2} \\ =& \int(-\alpha^2\lambda_1^2 - \beta^2\lambda_1^6 - \gamma^2\lambda_1^2\lambda_2^4 + 2\alpha\beta\lambda_1^4 + 2\alpha\gamma\lambda_1^2\lambda_2^2 - 2\beta\gamma\lambda_1^4\lambda_2^2)F(d\lambda) \\ =& -\int(-\alpha\lambda_1 + \beta\lambda_1^3 + \gamma\lambda_1\lambda_2^2)^2 F(d\lambda).\end{aligned}$$

Hence, dF-a.s., either $\lambda_1 = 0$ or $\gamma\lambda_1^2 + \beta\lambda_1^2 = \alpha$. By isotropy, it implies that $\gamma = \beta$ and that F's support is concentrated on zero and the circle with radius $\sqrt{\alpha/\beta}$. It corresponds to the GRW with radius $\sqrt{\alpha/\beta}$ plus an additional constant term.

In the same way,

$$0 = \text{Var}(\alpha\partial_{22}\psi(0) + \beta\partial_{11}\psi(0) + \gamma\partial_{1111}\psi(0)) = \int(\alpha\lambda_2^2 + \beta\lambda_1^2 + \gamma\lambda_1^4)^2 F(d\lambda)$$

implies that F is trivial if F is isotropic. □

In conclusion, the only non-trivial linear relations possibly satisfied by the derivatives involved in (X, Y_0) is $\partial_{111}\psi(0) + \partial_{122}\psi(0) = \alpha\partial_1\psi(0), \alpha > 0$ and can only be satisfied by a SGRW. In the light of these results, functionals of interest only depend on the law of the vector X' under the conditioning $Y_0 = 0$, where

$$X' = \begin{cases}(\partial_{22}\psi(0), \partial_{111}\psi(0), \partial_{112}\psi(0), \partial_{1111}\psi(0)) \text{ if } \psi \text{ is a shifted GRW} \\ X \text{ otherwise}\end{cases}$$

because if ψ is a shifted GRW and $Y_0 = 0$, $\partial_{111}\psi(0) + \partial_{122}\psi(0) = -\alpha\partial_1\psi(0) = 0$ a.s. hence $\partial_{122}\psi(0)$ is directly expressible in function of $\partial_{111}\psi(0)$.

Lemma 2. The conditional density f_r of X' knowing Y_r converges pointwise to the density f_0 of X' knowing Y_0. There is furthermore $\sigma_1, \sigma_2, c_1, c_2 > 0$ such that for r sufficiently small,

$$c_1 g_{\sigma_1} \leqslant f_r \leqslant c_2 g_{\sigma_2}$$

where g_σ is the density of iid Gaussian variables $Z^\sigma = (Z_i^\sigma)_i$ with common variance σ^2. Hence for any non-negative functional φ_r

$$c_1\mathbb{E}(\varphi_r(\sigma_1 Z^1)) \leqslant \mathbb{E}^{(r)}(\varphi_r(X')) \leqslant c_2\mathbb{E}(\varphi_r(\sigma_2 Z^1)).$$

Proof. Since the vector (X', Y_0) is non-degenerate, by continuity of the covariance matrix, the vector (X', Y_r) is non-degenerate either for r sufficiently small, and the density of the former converges pointwise to the density of the latter. Hence the conditionnal density f_r of $(X' \mid Y_r)$ converges to f_0 the non-degenerate multivariate conditional Gaussian density of $(X \mid Y_0)$.
Let Γ_r be the covariance matrix of the conditional vector $(X' \mid Y_r)$. For $1 \leqslant i \leqslant d$, we denote by $\lambda_{i,}(r)$ the i-th eigenvalue of the matrix Γ_r. Since $\lambda_i(r) \to \lambda_i(0) > 0$, there exists constant $\sigma_1 \geqslant \sigma_2 > 0$ such that for r sufficiently small

$$\frac{1}{2\pi\sigma_2^2} \geqslant \lambda_i(r) \geqslant \frac{1}{2\pi\sigma_1^2}. \tag{41}$$

Hence $f_r(x)$ is bounded between $c\exp(-\sum_i x_i^2/(2\pi\sigma_1^2))$ and $c'\exp(-\sum_i x_i^2/(2\pi\sigma_2^2))$ for some $c, c' > 0$, which gives the desired claims. □

6.3. **Proof of (26) in Theorem 5.1.** From (40), we have

$$\mathrm{K}_2(z,w) = \frac{r^2}{(2\pi)^2 2^5\sqrt{3}(-\eta_0)\mu_0(1+o(1))}\mathbb{E}^{(r)}\left[|{A_1}^2 - g(r)|\right].$$

According to Lemma 2, the conditional density f_r of X' knowing Y_r converges pointwise to the non-degenerate density f_0 of X' knowing Y_0, and $\varphi_r(X') := A_1^2 - g(r)$ is uniformly bounded by a polynomial $P(X')$, Lebesgue's Theorem then yields

$$\lim_{r\to 0}\mathbb{E}^{(r)}\left[\mid A_1^2 - g(r) \mid\right] = \int\varphi_r(x)f_r(x)dx \to \int\lim_r \varphi_r(x)f_0(x)dx = \mathbb{E}^{(0)}\left[A_1^2\right]. \tag{42}$$

To compute the conditionnal law of $Z := (\partial_{22}\psi(0), \partial_{111}\psi(0))$, recall that in virtue of (3) and Remark 1 $\partial_{22}\psi(0)$ and $\partial_{111}\psi(0)$ are independent, and the covariance matrix of Y_0 is

$$\Gamma(Y_0) = \begin{pmatrix} m_{2,0} & 0 & 0 & 0 \\ 0 & m_{2,0} & 0 & 0 \\ 0 & 0 & m_{4,0} & 0 \\ 0 & 0 & 0 & m_{2,2} \end{pmatrix}.$$

The covariance matrix of Z and Y_0 is

$$\Gamma(Z, Y_0) = \begin{pmatrix} m_{2,1} & m_{0,3} & m_{2,2} & m_{1,3} \\ m_{4,0} & m_{3,1} & m_{5,0} & m_{4,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_{2,2} & 0 \\ m_{4,0} & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the conditional covariance of Z knowing Y_0 is

$$\begin{aligned}\Gamma(Z \mid Y_0) = \Gamma(Z) - \Gamma(Z, Y_0)\Gamma(Y_0)^{-1}\Gamma(Z, Y_0)^t &= \begin{pmatrix} m_{0,4} - m_{4,0}^{-1}m_{22}^2 & 0 \\ 0 & m_{6,0} - m_{2,0}^{-1}m_{4,0}^2 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Var}(\partial_{22}\psi(0)|\partial_{111}\psi(0)) & 0 \\ 0 & \mathrm{Var}(\partial_{111}\psi(0)|\partial_1\psi(0)) \end{pmatrix}\end{aligned}$$

the diagonal terms are positive in virtue of (5). By conditionnal independence of $\partial_{22}\psi(0)$ and $\partial_{111}\psi(0)$, and using Proposition 2

$$\begin{aligned}\mathbb{E}^{(0)}(A_1^2) = \mathrm{Var}(\partial_{22}\psi(0)|\partial_{11,1}\psi(0))\mathrm{Var}(\partial_{11,1}\psi(0)|\partial_1\psi(0)) &= (m_{0,4} - m_{4,0}^{-1}m_{22}^2)(m_{6,0} - m_{2,0}^{-1}m_{4,0}^2) > 0\\ &= \frac{2^5}{3}\mu_0(\frac{3\cdot 2^3}{\eta_0}(-5\nu_0\eta_0 + 3\mu_0^2))\\ &= 2^8\mu_0\frac{(-5\nu_0\eta_0 + 3\mu_0^2)}{\eta_0}.\end{aligned}$$

Combining Equation (42) , (40) we obtain

$$\lim_{r\to 0}\mathrm{K}_2(z,w) = \frac{1}{(2\pi)^2\ 2^5\sqrt{3}(-\eta_0)\mu_0(1+o(1))}\ 2^8\mu_0\frac{(-5\nu_0\eta_0 + 3\mu_0^2)}{\eta_0} = \frac{10\ \nu_0\eta_0 - 6\mu_0^2}{\pi^2\ \sqrt{3}\eta_0^2(1+o(1))} =: a(1+o(1)). \tag{43}$$

Finally, the second factorial moment of $\mathcal{N}_\rho^c$ when $\rho \to 0$, is given by

$$\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] = \int\int_{B_\rho\times B_\rho}\mathrm{K}_2(z,w)\,\mathrm{d}z\ \mathrm{d}w = a|B_\rho|^2(1+o(1)).$$

Recalling that

$$\lambda_c = \frac{4}{\sqrt{3}}\frac{\mu_0}{-\eta_0\pi}$$

yields indeed $a = \lambda_c^2\mathsf{R}_c$.

Let us show that $\mathsf{R}_c \geqslant \frac{1}{8\sqrt{3}}$. Given a measure μ on $\mathbb{R}_+$, denote by

$$m_k(\mu) = \int t^k\mu(dt).$$

Since F is isotropic, define μ as the radial part of F, yielding with a polar change of coordinates

$$\int_{\mathbb{R}^2}\lambda_1^a\lambda_2^b F(d\lambda) = \int_0^{2\pi}\cos(\theta)^a\sin(\theta)^b d\theta m_{a+b+1}(\mu).$$

Introduce the probability measure, for $A\subset\mathbb{R}_+$,

$$\widetilde{\mu}(A) = \frac{\int_A t\mu(dt)}{m_1(\mu)}.$$

Using the spectral representation in Proposition 2 yields for some $c>0$

$$\frac{\nu_0\eta_0}{\mu_0^2} = c\frac{m_3(\mu)m_1(\mu)}{m^2(\mu)^2} = c\frac{m_1(\mu)m_2(\widetilde{\mu})m_1(\mu)}{(m_1(\mu)m_1(\widetilde{\mu}))^2} = c\frac{m_2(\widetilde{\mu})}{m_1(\widetilde{\mu})^2} \geqslant c$$

by the Cauchy-Schwarz inequality. The ratio is minimal if the equality is obtained in the Cauchy-Schwarz inequality, i.e. when t^2 is proportionnal to t $\widetilde{\mu}$-a.e.. This is the case only if $F(d\lambda)$ is uniformly spread on a circle of $\mathbb{R}^2$, with perhaps also an additional atom in 0. This corresponds exactly to the class of fields derived in Example 1, which are the SGRW. For the precise computation of the constant $\frac{1}{8\sqrt{3}}$, see Example 3.

In example 4, we derive spectral measures $F_t, t>1$ which achieve repulsion factors R_c in an interval of the form (α_0,∞) for some $\alpha_0\in\mathbb{R}$. Therefore it remains to show that all values between $\frac{1}{8\sqrt{3}}$ and α_0 can be achieved. For that we use an interpolation

$$G_s := sF_{RW} + (1-s)F_2, s\in[0,1]$$

where F_{RW} is the spectral measure of a GRW and F_2 belongs to the parametric family $F_t, t \geqslant 1$. The ratio of moments

$$s \mapsto \frac{m_3(G_s)m_1(G_s)}{m^2(G_s)^2}$$

evolves continuously with s because all the members of the numerator and denominator do, hence the repulsion factor evolves continuously between $\frac{1}{8\sqrt{3}}$ and α_0 and achieves all intermediary values.

6.4. **Proof of (27) in Theorem 5.1.** To compute the second factorial moment of $\mathcal{N}_\rho^e = \mathcal{N}_\rho^{(0,\infty)}$ when $\rho \to 0$, we apply the Kac-rice formula of Theorem 3.1 in the case $B_1 = B_2 = (0,\infty)$

$$\mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] = \int\int_{B_\rho \times B_\rho} \mathrm{K}_2^{e,e}(z,w)\,\mathrm{d}z\,\mathrm{d}w, \tag{44}$$

where

$$\begin{aligned}\mathrm{K}_2^{e,e}(z,w) = &\,\phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0)),(0,0))\\ &\times \mathbb{E}^{(r)}\Big[\,|\det H_\psi(z)|\,|\det H_\psi(w)|\,\mathbf{1}_{\{\det H_\psi(z)>0\}}\;\mathbf{1}_{\{\det H_\psi(w)>0\}}\Big].\end{aligned}$$

It becomes in virtue of (34)

$$\mathrm{K}_2^{e,e}(z,w) = r^2\,\phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0),(0,0))\;a_r \tag{45}$$

where

$$\begin{aligned}a_r :=&\mathbb{E}^{(r)}\left[\,\left|{A_1}^2 - g(r)\right|\,I_r\right]\\ I_r :=&\mathbf{1}_{\{\det H_\psi(z)>0\}}\mathbf{1}_{\{\det H_\psi(w)>0\}}.\end{aligned}$$

To be able to prove (27), we need to establish an upper bound and a lower bound of a_r So the proof is separated into two parts. We first give in Lemma 3 an asymptotic expression of a_r to get rid of superfluous variables.

Lemma 3. Let $J_r := \mathbf{1}_{\{|A_1|<rB_0\}}$,

$$|a_r - \mathbb{E}^{(r)}(|A_1^2 - r^2B_0^2|J_r)| = \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'})$$

for $0 < \alpha' < \alpha$, and as $r \to 0$

$$a_r \asymp \mathbb{E}(|A_1^2 - r^2B_0^2|J_r). \tag{46}$$

Proof. From (38)-(39) in the proof of Lemma 1,

$$I_r \leqslant \mathbf{1}_{\{|A_1|<rD_r\}}$$

where

$$D_r :=|B_0| + |B_r| + |B_r'| + r(|C_r| + |C_r'|),$$

is a variable with Gaussian tail. Recall that $g(r) = r^2B_0^2 + \mathcal{O}_{\mathbb{P}}(A_r r^{1+\alpha} + r^{2+\alpha})$, hence using (37),

$$\begin{aligned}\mathbb{E}^{(r)}\left[(|A_1^2 - g(r)| - |A_1^2 - r^2B_0^2|)I_r)\right] \leqslant&\mathbb{E}^{(r)}(\mathcal{O}_{\mathbb{P}}(r^{1+\alpha}A_1 + r^{2+\alpha}))|\mathbf{1}_{\{|A_1|<rD_r\}})\\ \leqslant&r^{2+\alpha}\mathbb{E}^{(r)}(\mathcal{O}_{\mathbb{P}}(D_r+1)\mathbf{1}_{\{|A_1|<rD_r\}}).\end{aligned} \tag{47}$$

Let $p, q > 1, \eta > 0$ such that $p^{-1} + q^{-1} = 1$ and $\alpha + \frac{1-\eta}{q} > \alpha' + 1$, then Holder's inequality yields

$$\mathbb{E}^{(r)}(\mathcal{O}_{\mathbb{P}}(D_r+1)\mathbf{1}_{\{|A_1|<rD_r\}}) \leqslant \mathbb{E}^{(r)}(\mathcal{O}_{\mathbb{P}}(D_r+1)^p)^{\frac{1}{p}}\mathbb{P}^{(r)}(|A_1| < rD_r)^{\frac{1}{q}}.$$

The probability on the right hand member can be bounded by

$$\mathbb{P}^{(r)}(|A_1| < rD_r) \leqslant \mathbb{P}^{(r)}(|D_r| > r^{-\eta}) + \mathbb{P}^{(r)}(|A_1| < r^{1-\eta}).$$

All variables involved in $\mathcal{O}_{\mathbb{P}}(D_r)$ have a Gaussian tail, hence

$$\mathbb{P}^{(r)}(\mathcal{O}_{\mathbb{P}}(|D_r|) > r^{-\eta}) = o(r^2).$$

By Lemma 2 with $\varphi_r(x) = \mathbf{1}_{\{|x_1x_2|<r^{1-\eta}\}}$, and Lemma 5-(i) (with $s = 0$),

$$\mathbb{P}^{(r)}(|A_1| < r^{1-\eta}) = \mathbb{E}^{(r)}(\varphi_r(X')) \leqslant c_2\mathbb{E}(\varphi_r(\sigma_2 Z)) < c' r^{1-\eta}\ln(r) \tag{48}$$

hence finally

$$r^{2+\alpha}\mathbb{E}^{(r)}(\mathcal{O}_{\mathbb{P}}(D_r+1)\mathbf{1}_{\{|A_1|<rD_r\}}) < c' r^{2+\alpha+\frac{1-\eta}{q}}\ln(r)^{1/q} = O(r^{3+\alpha'}). \tag{49}$$

To simplify indicators, remark that in virtue of (38),(39),

$$\begin{aligned} I_r &= \mathbf{1}_{\{A_1+rB_r+r^2C_r>0,-A_1+rB'_r+r^2C''_r>0\}} \\ J_r &= \mathbf{1}_{\{A_1+rB_0>0,-A_1+rB_0>0\}}. \end{aligned}$$

Both the events $\{I_r = 1\}, \{J_r = 1\}$ imply $|A_1| < rD_r$. If $I_r \neq J_r$, $A_1 + rB_r + r^2C_r$ has a sign different from $A_1 + rB_0$, or $-A_1 + rB'_r + r^2C'_r$ has a sign different from $-A_1 + rB_0$. In both cases it implies another event of magnitude $\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})$ because $B_r, B'_r = B_0 + \mathcal{O}_{\mathbb{P}}(r^\alpha), C_r, C'_r = \mathcal{O}_{\mathbb{P}}(1)$:

$$\begin{aligned} |I_r - J_r| &\leqslant 2\left(\mathbf{1}_{\{(A_1+rB_r+r^2C_r)\cdot(A_1+rB_0)<0\}} + \mathbf{1}_{\{(-A_1+rB'_r+r^2C'_r)\cdot(-A_1+rB_0)<0\}}\right) \\ &\leqslant 2\left(\mathbf{1}_{\{|A_1+rB_0|<\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\}} + \mathbf{1}_{\{|-A_1+rB_0|<\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\}}\right). \end{aligned}$$

Let now $p, q > 1, \eta > 0$ such that $(1 + \alpha - \eta)/q > 1 + \alpha'$. Since also $I_r - J_r \neq 0$ implies that either $I_r = 1$ or $J_r = 1$ and so $|A_1| < rD_r$, collecting (47),(49),

$$\begin{aligned} |a_r - &\mathbb{E}^{(r)}\left[|A_1^2 - r^2B_0^2|J_r\right]| \leqslant |\mathbb{E}^{(r)}(|A_1^2 - r^2B_0^2||I_r - J_r|)| + \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}) \\ &\leqslant |\mathbb{E}^{(r)}(|A_1^2 - r^2B_0^2|\mathbf{1}_{|A_1|<rD_r}|I_r - J_r|)| + \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}) \\ &\leqslant \mathbb{E}^{(r)}\left[(r^2D_r^2 + r^2B_0^2)\left(\mathbf{1}_{\{|A_1+rB_0|<\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\}} + \mathbf{1}_{\{|-A_1+rB_0|<\mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\}}\right)\right] + \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}) \\ &\leqslant \mathbb{E}^{(r)}[(r^2D_r^2 + r^2B_0^2)^p]^{\frac{1}{p}}\left[\mathbb{P}^{(r)}\left(|A_1 + rB_0| < \mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\right)^{\frac{1}{q}} + \mathbb{P}^{(r)}\left(|-A_1 + rB_0| < \mathcal{O}_{\mathbb{P}}(r^{1+\alpha})\right)^{\frac{1}{q}}\right] + \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}). \end{aligned}$$

We have

$$\mathbb{P}^{(r)}(|A_1 + rB_0| < \mathcal{O}_{\mathbb{P}}(r^{1+\alpha})) \leqslant \mathbb{P}^{(r)}(|A_1 + rB_0| < r^{1+\alpha-\eta}) + \mathbb{P}^{(r)}(\mathcal{O}_{\mathbb{P}}(1) > r^{-\eta}).$$

By an application of Lemma 2 similar to (48) with $\varphi_r(x)$ of the form $\mathbf{1}_{\{|x_1x_1+r\sum a_{i,j}x_ix_j|<r^{1+\alpha-\eta}\}}$ and Lemma 5-(i); the first member is in $r^{1+\alpha-\eta}\ln(r)$, hence finally for some $c < \infty$

$$|a_r - \mathbb{E}^{(r)}\left[|A_1^2 - r^2B_0^2|J_r\right]| \leqslant cr^2r^{(1+\alpha-\eta)/q}\ln(r)^{\frac{1}{q}} + \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}) = \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'}).$$

Finally, (46) follows from Lemma 2 with $\varphi_r(X') = A_1^2 - r^2B_0^2$. □

6.4.1. *Upper bound in* (27). According to the previous lemma it suffices to give an upper bound of $\mathbb{E}\left[\mid A_1^2 - r^2 B_0^2 | J_r\right]$. We stress that the crucial point that justifies the absence of a log term in the final result (compared to (28)) is the following inequality

$$J_r = \mathbf{1}_{\{|A_1|<rB_0\}} \leqslant \mathbf{1}_{\{|\partial_{22}\psi(0)|<2r\partial_{221}\}} + \mathbf{1}_{\{|\partial_{111}\psi(0)|<\frac{2}{3}r\partial_{1111}\psi(0)\}},$$

hence since B_0^2 is a polynomial in X' we can use Lemma 5-(iii) several times and get for some $c < \infty$

$$\mathbb{E}(|A_1^2 - r^2 B_0^2 | J_r) \leqslant 2\mathbb{E}(|r^2 B_0^2 | J_r) \leqslant cr^3. \tag{50}$$

Then, from (45),(50) and (40), we deduce that for some $c' < \infty$

$$\mathrm{K}_2^{e,e}(z,w) \leqslant c'\, r^3. \tag{51}$$

Finally, from (44) and (51), we deduce for some $c'' < \infty$

$$\mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] \leqslant c''\, \rho^7.$$

6.4.2. *Lower bound in* (27). Thanks to Lemma 3, it is sufficient to give a lower bound of $\mathbb{E}(|A_1^2 - r^2 B_0^2 | \mathbf{1}_{\{|A_1|\leqslant rB_0\}}|)$. Let us first assume that the Gaussian field ψ is not a SGRW (Example 1), hence the derivatives involved in X and Y_0 are not linearly linked. Define the event

$$\Omega = \{|\partial_{22}\psi(0)| < r, \frac{1}{2} < \partial_{111}\psi(0) < 1, |\partial_{211}\psi(0)| < 1, 8 < \partial_{122}, |\partial_{1111}\psi(0)| < 1\}.$$

We recall

$$\begin{aligned} A_1 &= \partial_{22}\psi(0)\partial_{111}\psi(0), \\ B_0 &= \partial_{221}\,\partial_{111}\psi(0) - \partial_{211}\psi(0)^2 + \frac{1}{3}\partial_{22}\psi(0)\,\partial_{1111}\psi(0) \\ Y_0 &= (\partial_1\psi(0), \partial_2\psi(0), \partial_{11}\psi(0), \partial_{12}\psi(0)). \end{aligned}$$

Hence under Ω

$$\begin{aligned} |A_1| &< r \\ B_0 &> 4 - 1 - \frac{r}{3} \end{aligned}$$

Hence for r sufficiently small, $B_0 > 2$, in particular $|A_1| \leqslant r|B_0|/2$ and we obtain

$$\begin{aligned} \mathbb{E}(|A_1^2 - r^2 B_0^2 | \mathbf{1}_{\{|A_1|\leqslant rB_0\}}) &\geqslant \mathbb{E}\left[\mathbf{1}_\Omega \mid r^2 B_0^2/4 \mid \mathbf{1}_{\{B_0 \geqslant 2A_1/r\}}\right] \\ &\geqslant r^2 \mathbb{P}(\Omega). \end{aligned}$$

Then since X is non-degenerate, its density is uniformly bounded and the proof is concluded with

$$\mathbb{P}(\Omega) \geqslant cr > 0$$

for some $c > 0$. In the degenerate case of the SGRW, $\partial_{122}\psi(0) = -\partial_{111}\psi(0)$ if $Y_0 = 0$ and we put instead

$$\Omega = \{|\partial_{111}\psi(0)| < r, \frac{1}{2} < \partial_{22}\psi(0) < 1, \partial_{1111}\psi(0) > 19, |\partial_{211}\psi(0)| < 1\}$$

If $Y_0 = 0$ and Ω is realised,

$$|A_1| < r$$
$$B_0 = -\partial_{111}\psi(0)^2 - \partial_{211}\psi(0)^2 + \frac{1}{3}\partial_{22}\partial_{1111}\psi(0) > -r^2 - 1 + \frac{19}{6}$$

hence $|A_1| < r < B_0 r/2$ for r small enough and the same method can be applied because X' has a bounded density. Therefore, it holds for some $c' > 0$

$$\mathbb{E}^{(0)}(|A_1^2 - r^2 B_0^2 | \mathbf{1}_{\{|A_1| \leqslant r B_0\}}|) \geqslant c'\, r^3. \tag{52}$$

From (45), (40)and (52), we get for some $c'' > 0$

$$\mathrm{K}_2^{e,e}(z,w) \geqslant\ c'' r^3. \tag{53}$$

Finally, from (44) and (53), we deduce that for some $c''' > 0$

$$\mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] \geqslant c''' \rho^7.$$

6.5. **Proof of (28) in Theorem 5.1.** Using Theorem 3.1 with $B_1 = B_2 = (-\infty, 0)$, the second factorial moment of $\mathcal{N}_\rho^s = N_\rho^{(-\infty,0)}$ is given by

$$\mathbb{E}[\mathcal{N}_\rho^s(\mathcal{N}_\rho^s - 1)] = \int\int_{B_\rho \times B_\rho} \mathrm{K}_2^{s,s}(z,w)\, \mathrm{d}z\ \mathrm{d}w,$$

where

$$\mathrm{K}_2^{s,s}(z,w) = r^2\ \phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0)),(0,0))\ \ \mathbb{E}^{(0)}\Big[\ |\det H_\psi(z)|\ |\det H_\psi(w)|\ \mathbf{1}_{\{\det H_\psi(z)<0\}}\ \ \mathbf{1}_{\{\det H_\psi(w)<0\}}\Big].$$

The difference is hence on the sign of the determinants, $\mathrm{K}_2^{s,s}(z,w)$ becomes

$$\mathrm{K}_2^{s,s}(z,w) = r^2\ \phi_{(\nabla\psi(z),\nabla\psi(w))}((0,0),(0,0))\ a'_r \tag{54}$$

where (see (39))

$$a'_r := \mathbb{E}^{(r)}\left[\ \left|{A_1}^2 - g(r)\right|\ I'_r\right]$$
$$I'_r := \mathbf{1}_{\{A_1 + rB_r + r^2 C_r < 0\}} \mathbf{1}_{\{-A_1 + rB'_r + r^2 C'_r < 0\}}.$$

The asymetry of the expression of the determinant yields a different estimate than in the previous case. To be able to prove (27), we need to establish an upper bound and a lower bound of a'_r as in the previous section (Lemma 3). We give in Lemma 4 an asymptotic expression of a'_r.

The proof is similar but there are also significant differences. The difference with respect to before is that the two signs of the determinants are negative, hence we replace J_r by

$$J'_r = \mathbf{1}_{\{A_1 + rB_0 < 0, -A_1 + rB_0 < 0\}} = \mathbf{1}_{\{|A_1| \leqslant -rB_0\}}$$

and emphasize that B_0 does not have the same law as $-B_0$.

Lemma 4. We have for $0 < \alpha' < \alpha$,

$$|a'_r - \mathbb{E}^{(r)}(|A_1^2 - r^2 B_0^2 | J'_r)| = \mathcal{O}_{\mathbb{P}}(r^{3+\alpha'})$$
$$a'_r \asymp \mathbb{E}(|A_1^2 - r^2 B_0^2 | J'_r)$$

The proof is omitted as the proof of Lemma 3 can be repdroduced verbatim, with resp. J'_r, I'_r, a'_r in place of resp. J_r, I_r, a_r.

6.5.1. *Upper bound.* The upper bound on J'_r is of different nature than that on J_r, in particular the third term

$$J'_r = \mathbf{1}_{\{|A_1|<-rB_0\}} \leqslant \mathbf{1}_{\{|\partial_{22}\psi(0)|<-6r\partial_{221}\psi(0)\}} + \mathbf{1}_{\{|\partial_{111}\psi(0)|<-2r\partial_{1111}\psi(0)\}} + \mathbf{1}_{\{|\partial_{22}\psi(0)\partial_{111}|<3r\partial_{211}\psi(0)^2\}}.$$

Then,

$$\mathbb{E}(|A_1 - r^2B_0^2|J'_r) \leqslant \mathbb{E}(2r^2B_0^2J_{r'})$$

hence we must use this time Lemma 5-(ii) for the last term of J'_r's bound,

$$\begin{aligned}\mathbb{E}(B_0^2\mathbf{1}_{|\partial_{22}\psi(0)\partial_{111}\psi(0)|<3r\partial_{211}\psi(0)^2}) \leqslant &\mathbb{E}(|\partial_{22}\psi(0)\partial_{111}\psi(0)|\mathbf{1}_{\{|\partial_{22}\psi(0)\partial_{111}\psi(0)|<3r\partial_{211}\psi(0)^2\}})\\ &+ \mathbb{E}(|\partial_{211}\psi(0)|^2\mathbf{1}_{\{|\partial_{22}\psi(0)\partial_{111}\psi(0)|<3r\partial_{211}\psi(0)^2\}}))\\ &+ \mathbb{E}(|\partial_{22}\psi(0)\partial_{1111}\psi(0)\mathbf{1}_{\{|\partial_{22}\psi(0)\partial_{111}\psi(0)|<3r\partial_{211}\psi(0)^2\}}))\\ \leqslant &cr\ln(r)\end{aligned}$$

for some $c < \infty$. The other terms are dealt with by Lemma 5-(iii) as in (50), hence the upper bound is in

$$\mathbb{E}(|A_1 - r^2B_0^2|J'_r) \leqslant c'r^3\ln(r)$$

for some $c' < \infty$, which yields (28) by (54) and Lemma 4.

6.5.2. *Lower bound.* We recall the expression of A_1 and $-B_0$: $A_1 = \partial_{22}\psi(0)\partial_{111}\psi(0)$, $-B_0 = -\partial_{221}\psi(0)\ \partial_{111}\psi(0) + \partial_{211}\psi(0)^2 - \frac{1}{3}\partial_{22}\psi(0)\ \partial_{1111}\psi(0)$. The strategy is the same than at Section 6.4.2.

If ψ is a SGRW (Example 1), $\partial_{111}\psi(0) = -\partial_{122}\psi(0)$ if $Y_0 = 0$, let

$$\Omega = \{\partial_{211}\psi(0) > 2, |\partial_{22}\psi(0)\partial_{111}\psi(0)| < r, |\partial_{111}\psi(0)| < 1, |\partial_{22}\psi(0)| < 1, |\partial_{1111}\psi(0)| < 1\}.$$

Hence if $Y_0 = 0$ and Ω is realised

$$\begin{aligned}&A_1 < r,\\ -B_0 = \partial_{111}\psi(0)^2 + \partial_{211}\psi(0)^2 - \frac{1}{3}\partial_{22}\psi(0)\partial_{1111}\psi(0) &> 0 + 4 - \frac{1}{3} > 2A_1/r.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}(|A_1^2 - r^2B_0^2|\mathbf{1}_{\{|A_1|\leqslant -rB_0\}}) &\geqslant \mathbb{E}\left[\mathbf{1}_\Omega \mid r^2B_0^2/4 \mid \mathbf{1}_{\{B_0\geqslant 2A_1/r\}}\right]\\ &\geqslant\ r^2\mathbb{P}(\Omega).\end{aligned}$$

We must prove a converse to Lemma 5-(i) with $s = 0$. Since the density of X' is uniformly bounded from below on $[-3,3]^4$, for some $c > 0$,

$$\mathbb{P}(\Omega) \geqslant c\int_{[-1,1]^2}\mathbf{1}_{\{|x_1x_2|<r\}}dx_1dx_2 \asymp r\ln(r).$$

6.6. **Proof of (29) in Theorem 5.1.** We recall that $\mathcal{N}_\rho^c = \mathcal{N}_\rho^s + \mathcal{N}_\rho^e$ hence $\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1) = \mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1) + \mathcal{N}_\rho^s(\mathcal{N}_\rho^s - 1) + 2\mathcal{N}_\rho^e\mathcal{N}_\rho^s$.
So, we have:

$$\mathbb{E}[\mathcal{N}_\rho^e\mathcal{N}_\rho^s] = \frac{1}{2}\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] - \mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] - \mathbb{E}[\mathcal{N}_\rho^s(\mathcal{N}_\rho^s - 1)].$$

Combining this formula with previous estimates (26), (27) and (28), we obtain

$$\mathbb{E}[\mathcal{N}_\rho^e\mathcal{N}_\rho^s] = \frac{1}{2}\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] + o(1)$$

ending the proof of (29).

Lemma 5. Let $(Z_1, \dots, Z_k)$ be a non-degenerate Gaussian vector and $a_{i,j}$ real fixed coefficients. Then,

(i)

$$\mathbb{P}(|Z_1 Z_2 + s\sum_{i,j} a_{i,j} Z_i Z_j| < r) \leqslant Cr\ln(r) \tag{i}$$

for C depending on the law of the Z_i (and not on s or the $a_{i,j}$),

(ii) for $\alpha_i \geqslant 0$

$$\mathbb{E}(|Z_1^{\alpha_1} \dots Z_k^{\alpha_k}| \mathbf{1}_{\{|Z_1 Z_2| < r Z_3^2\}}) \leqslant C' \begin{cases} r\ln(r) \text{ if } \alpha_1 = \alpha_2 = 0 \\ r \text{ otherwise} \end{cases} \tag{ii}$$

for some $C' < \infty$.

(iii) Let some coefficients $\alpha_i \in \mathbb{N}$, $(Z_1, \dots, Z_q)$ be a Gaussian vector. Then, for some $C'' < \infty$,

$$\mathbb{E}(|Z_1^{\alpha_1} \dots Z_q^{\alpha_q}| \mathbf{1}_{\{|Z_1| \leqslant r Z_2\}}) \leqslant C'' r. \tag{iii}$$

Proof. (i) Assume first that the Z_i are iid Gaussian. Let us study for $a, b \in \mathbb{R}$, $Y_1 := Z_1 - as, Y_2 := Z_2 - bs$. Since Y_1, Y_2 have a density bounded by $\kappa < \infty$ (universal), we have for $c \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}(|Y_1 Y_2 - c| \leqslant r) \leqslant & \mathbb{P}(|Y_2| \leqslant r) + \mathbb{P}(|Y_1 - c/Y_2| < r/Y_2, |Y_2| > 1) + \mathbb{P}(|Y_1 - c/Y_2| < r/Y_2, |Y_2| \in [r, 1]) \\
\leqslant & \kappa r + \mathbb{P}(|Y_1 - c/Y_2| < r) + \mathbb{E}\left[\mathbb{P}(Y_1 \in [c/Y_2 \pm r/Y_2] \mid Y_2) \mathbf{1}_{\{r < |Y_2| < 1\}}\right] \\
\leqslant & \kappa r + \kappa r + \mathbb{E}(\kappa r / Y_2 \mathbf{1}_{\{r < |Y_2| < 1\}}) \\
\leqslant & 2\kappa r + \kappa r \int_r^1 \frac{1}{y_2} 2\kappa dy_2 \\
\leqslant & 2\kappa r + 2\kappa^2 r \ln(r),
\end{aligned}$$

uniformly on a, b, c, s. Then it remains to notice that

$$Z_1 Z_2 + s\sum_{i,j} a_{i,j} Z_i Z_j = (Z_1 - As)(Z_2 - Bs) - C_s$$

where A, B, C_s are independent of Z_1, Z_2. Then

$$\mathbb{P}(|Z_1 Z_2 + s\sum_{i,j} a_{i,j} Z_i Z_j| < r) = \mathbb{E}(\mathbb{P}(|(Z_1 - As)(Z_2 - Bs) - C_s| \mid A, B, C_s)) \leqslant Cr\ln r.$$

In the non-independent Gaussian case, the joint density $f(x_1, \dots, x_k)$ of $(Z_1, \dots, Z_k)$ is bounded by $\kappa \exp(-c\sum_i x_i^2)$ for some $c, \kappa > 0$ (c would be the smallest eigenvalue of the covariance matrix). From there on the conclusion is easy:

$$\int \mathbf{1}_{\{|x_1 x_2 + s\sum_{i,j} a_{i,j} x_i x_j| < r\}} f(x_1, \dots, x_k) dx_1 \dots dx_k \leqslant \kappa \int \mathbf{1}_{\{\dots\}} \exp(-c\sum_i x_i^2) dx_1 \dots dx_k$$

and the right hand member corresponds to the independent case, already treated.