Symplectic structure for elastic and chiral conducting cosmic string models

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This article is based on the covariant canonical formalism and corresponding symplectic structure on phase space developed by Witten, Zuckerman, and others in the context of field theory. After recalling the basic principles of this procedure, we construct the conserved bilinear symplectic current for generic elastic string models. These models describe current carrying cosmic strings evolving in an arbitrary curved background spacetime. Particular attention is paid to the special case of the chiral string for which the worldsheet current is null. Different formulations of the chiral string action are discussed in detail, and as a result the integrability property of the chiral string is clarified.

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I. INTRODUCTION

Following a preliminary study [1] whose application was restricted to nonconducting Nambu-Goto strings, in this article we focus on conducting cosmic string models in arbitrary curved spacetimes, and in this context discuss the general principles of the covariant canonical variational analysis. As emphasised by Witten, Zuckerman, and others [2–9] in the context of relativistic field theories including gravity, the potential utility of this covariant analysis is as a starting point for covariant quantization.

The analysis culminates in the construction of a corresponding symplectic structure which is locally representable as a current. This symplectic current—which should not be confused with the worldsheet current of the conducting string—is defined as an antisymmetric bilinear functional of a pair of independent perturbations, and it is conserved whenever both perturbations are on shell (in the sense of satisfying the relevant dynamical field equations). The Poisson brackets on classical phase space (the space of solutions of the classical equations of motion) are then expressed in a manifestly covariant form in terms of the symplectic structure [3,5], and as usual one quantizes by replacing the Poisson brackets with commutators.

Despite this, we stress that our aim is not to quantize. Rather, we carry out the first and already rather complicated step in this direction by simply calculating the symplectic current for conducting cosmic string models. As such, this will not give us any further insights into the evolution of current carrying strings in curved spacetimes nor of their quantization. However, in the process we will be able to formulate the relevant string actions in a rather more convenient form which may aid further analysis.

The task of extending the covariant analysis from ordinary fields to branes—meaning systems with support confined to a lower dimensional world sheet—was recently taken up by Cartas-Fuentevilla [10,11]. (See also an earlier paper [12] in which the method was used to quantize Nambu strings in three-dimensional Minkowski space.) The necessary analysis is facilitated by the relatively new development [13–15] of suitably covariant methods of geometrical analysis, which have already been shown to be far more efficient than the more cumbersome (and error-prone) frame-dependent methods used in earlier work for treating other problems, such as the divergences arising from self-interaction [16–18].

The earlier article [1] demonstrates the agreement of the canonical approach, as developed by Cartas-Fuentevilla [10,11], with the result of an earlier and rather different approach [13] to the construction of the symplectic surface current. Both these analyses focused on strings with no internal physical structure on the worldsheet, that is, on actions of the simple Dirac-Nambu-Goto type which are proportional to the worldsheet surface measure.

The present work deals with the more general case of elastic string models of the kind [19] appropriate for macroscopic applications, such as the strings used for musical instruments since the time of Pythagoras, and in particular for the macroscopic description of the effect in cosmic strings of mechanisms of the various (fermionic and bosonic superconducting) kinds originally proposed by Witten [20]. The earliest work on the consequences of the Witten mechanism emphasized effects due to electromagnetic coupling, but it was recognized by Davis and Shellard [21] that the most important consequence would be the formation of vortons, meaning centrifugally supported equilibrium configurations of loops in which effects of electromagnetic coupling (if present at all) are relatively unimportant. While it is also possible to have noncircular vorton configurations [22], the simplest possibility is that of circular configurations. In this case (provided electromagnetic effects are absent or negligible) both the static equilibrium and full dynamical evolution are particularly amenable to an exact mathematical treatment: this was provided by a recent study [23] whose notation scheme will be followed here.

This paper is set up in the following way. In Sec. II we review the generic elastic string model and the limit case of the chiral model, and derive the relevant equations of motion. Different formulations of the chiral string action are

discussed in detail in Sec. II A, and as a result we are able to reinterpret more simply the integrability property of the chiral string. The purpose of Sec. III is to summarize the basic steps of the canonical analysis of Witten and others, and this is done in the context of a general worldsheet action density. In Sec. IV we discuss in which reference system the symplectic current will be evaluated; the result for the general elastic string is given in Sec. V and for the chiral string in Sec. VI. Finally, this paper contains an appendix in which we use Dirac's Hamiltonian method for constrained systems in order to write the chiral string action in a linearized Polyakov-like form.

II. ELASTIC AND CHIRAL STRING MODELS

The string models we consider are governed by an action integral of the form

$$\mathcal{I}=\int L\|\gamma\|^{1/2}d^2\sigma \tag{1}$$

over a supporting worldsheet with internal coordinates σ^i $(i=0,1)$ and induced metric $\gamma_{ij}=g_{\mu\nu}x^\mu_{,i}x^\nu_{,j}$, in a background with coordinates x^μ $(\mu=0,1,...,d)$ $(d\geqslant 2)$ and (flat or curved) spacetime metric $g_{\mu\nu}$. In the Nambu-Goto case, the scalar Lagrangian action density L is just a constant. For the more general category [23] of elastic string models considered here, L depends on the magnitude of the gradient of a freely variable phase field φ. More explicitly, the equation of state is $L\{w\}$ where

$$w=\psi^2\gamma^{ij}\varphi_{,i}\varphi_{,j} \tag{2}$$

with ψ a normalization parameter with fixed value given by

$$\psi^2=\kappa_0. \tag{3}$$

The constant κ_0 may be fixed according to convenience without loss of generality of the model: the standard convention [23], however, is to choose κ_0 such that the derived quantity $\mathcal{K}$ defined by

$$\mathcal{K}^{-1}=-2\frac{dL}{dw} \tag{4}$$

tends to unity in the null limit $w\to 0$ [see Eq. (17) and Sec. II A].

For any such elastic string action there is a corresponding *chiral* string action which is obtained by relaxing the condition that ψ is fixed and letting it instead be a freely varying auxiliary field. On application of the variation principle, it follows that the on-shell configurations satisfy the same dynamical equations as in the corresponding elastic model, although there is now the further constraint that $w=0$. It can be verified that this null constraint is consistent with the dynamical equations: if $w=0$ at an initial time, it will automatically be preserved by the evolution of the system.

This chiral string is of interest [24] since in a number of different and cosmologically relevant cases [25,26] Witten's fermionic zero mode mechanism indeed gives rise to purely left- (or purely right-) moving modes and hence to a null current. Furthermore, it was recently recognized [27–29] that the chiral string equations of motion are exactly integrable in a flat background (see also further comments on this in Sec. II A). This has led to a number of analyses of the cosmological consequences of chiral cosmic strings [28,30]. The chiral string can also be thought of as a useful approximation for the treatment of nearby solutions of the less tractable generic elastic class.

As in the simple Nambu-Goto case $(\varphi=0)$, the full set of dynamical equations for the generic elastic case is given by the local surface energy-momentum conservation equation, which takes the standard form

$$\bar{\nabla}_\mu T^{\mu\nu}=0. \tag{5}$$

Here the surface covariant differentiation operator $\bar{\nabla}$ is defined in terms of the fundamental tensor $\eta^{\mu\nu}$ by

$$\bar{\nabla}_\mu=\eta^\nu_\mu\nabla_\nu, \quad \eta^{\mu\nu}=\gamma^{ij}x^\mu_{,i}x^\nu_{,j}, \tag{6}$$

and the surface stress-momentum-energy density is defined by

$$T^{\mu\nu}=2\|\gamma\|^{-1/2}\frac{\partial(L\|\gamma\|^{1/2})}{\partial g_{\mu\nu}}. \tag{7}$$

The motion of the worldsheet is governed by the orthogonally projected part of Eq. (5), which takes the form

$$T^{\mu\nu}K_{\mu\nu}{}^\rho=0, \tag{8}$$

where $K_{\mu\nu}{}^\rho$ is the second fundamental tensor as defined [19] by

$$K_{\mu\nu}{}^\rho=\eta^\sigma_\nu\bar{\nabla}_\mu\eta^\rho_\sigma. \tag{9}$$

For the Nambu-Goto string $T^{\mu\nu}$ is simply proportional to the first fundamental tensor $\eta^{\mu\nu}$, and the extrinsic evolution equation (9) by itself constitutes the complete set of dynamical equations. However, in the general case, the surface stress-energy tensor will have the slightly more complicated form

$$T^{\mu\nu}=L\,\eta^{\mu\nu}+\mathcal{K}c^\mu c^\nu, \tag{10}$$

involving a surface current c^μ given by

$$c^\mu=\frac{\psi}{\mathcal{K}}\bar{\nabla}^\mu\varphi=\frac{\psi}{\mathcal{K}}x^\mu_{,i}\gamma^{ij}\varphi_{,j}. \tag{11}$$

Now the full set of dynamical equations is given by the extrinsic evolution equation (8), together with the surface current conservation law

$$\bar{\nabla}_\mu c^\mu=0. \tag{12}$$

For a generic state of an elastic string model, the first fundamental tensor and the worldsheet stress energy tensor

will be given [19] in terms of a preferred orthonormal pair of, respectively, timelike and spacelike unit worldsheet tangent vectors u^μ and $\bar{u}^\mu$ by

$$\eta^{\mu\nu} = -u^\mu u^\nu + \tilde{u}^\mu \tilde{u}^\nu \tag{13}$$

and

$$T^{\mu\nu} = U u^\mu u^\nu - T \tilde{u}^\mu \tilde{u}^\nu, \tag{14}$$

in which U and T are, respectively, interpretable as the local rest frame energy density and the tension of the string. When w is negative, u^μ will be aligned with the current, which will be expressible in the form $c^\mu = \psi^{-1} \nu u^\mu$, where ν is a conserved number density, and we shall be able to make the identifications $T = -L$, $U = T - w\mathcal{K}^{-1}$ so that the effective mass (or chemical potential) defined by $\mu = dU/d\nu$ will be given directly by $\mu^2 = -\gamma^{ij}\varphi_{,i}\varphi_{,j}$. On the other hand, when w is positive, it is $\tilde{u}^\mu$ that will be aligned with the current, which will then be given by $c^\mu = \psi^{-1}\mu\tilde{u}^\mu$, and we shall be able to make the identification $U = -L$, $T = U - w\mathcal{K}^{-1}$, for a winding number density $\nu = -dT/d\mu$ given by $\nu^2 = \gamma^{ij}\varphi_{,i}\varphi_{,j}$.

In so far as its internal dynamics is concerned, such an elastic string is the one-dimensional analogue of an ordinary barotropic perfect fluid, whose mass density ρ is specified just as a function of a conserved number density ν or equivalently of the corresponding pressure $P = \mu\nu - \rho$, with $\mu = d\rho/d\nu$, by an ''equation of state'' function which provides the corresponding sound velocity by the well-known differential formula $c_L^2 = dP/d\rho$. In the elastic string case U plays the role of ρ, while the tension T plays the role of $-P$, so the speed c_L say (relative to the preferred rest frame) of longitudinal ''woggle'' (sound type) perturbations will be given [19,31] by

$$c_L^2 = -dT/dU, \tag{15}$$

whose right-hand side must be positive (to avoid local instability) and less than unity (to avoid causality violation) for any physically admissible ''equation of state.'' However, the kind of ''equation of state'' that is physically admissible for an elastic string is qualitatively different from that of an ordinary—positive pressure—perfect fluid, as exemplified by the polytropic case defined for a fixed index $\Gamma \geqslant 1$ by $P = \kappa\nu^\Gamma$ with a positive value of the constant coefficient κ. This gives $\rho = P/W + m\nu$, where the reduced polytropic index is given by $W = \Gamma - 1$, and the mass parameter m is a constant of integration (which may be neglected in the ultrarelativistic limit that is relevant in the usual cosmological applications). The reason why such a polytropic ansatz is inapplicable is that, in order to be admissible, a string ''equation of state'' must provide stability not just with respect to longitudinal perturbations but also with respect to transverse ''wiggle'' type perturbations, whose propagation velocity c_E is given [19,31] by

$$c_E{}^2 = T/U. \tag{16}$$

(For a Nambu-Goto string $c_E = 1$, while c_L has no analogue.) Subject to the usual presumption that the mass density can never be negative, it follows that in the case of a string (as opposed to a fluid) it is not the pressure but the tension T that has to be positive.

In the chiral case the form (14) will no longer be valid, but Eqs. (10) and (11) still apply, although they must be supplemented by the constraint $w = 0$, which means that the current c^μ is null. In this null current limit, the Lagrangian L tends to a fixed value specified by the relevant Kibble mass scale m, say, and for the standard choice of κ_0 in Eq. (3) the quantity $\mathcal{K}$ tends to unity according to the specifications

$$L\{0\} = -m^2, \quad \mathcal{K}\{0\} = 1 \tag{17}$$

(see Sec. II A). Thus in the chiral case the stress-energy tensor will take the form

$$T^{\mu\nu} = -m^2 \eta^{\mu\nu} + c^\mu c^\nu, \tag{18}$$

subject to the nullity condition

$$c^\mu c_\mu = 0, \tag{19}$$

where the current is given simply by

$$c^\mu = \psi \bar{\nabla}^\mu \varphi = \psi x^\mu_{,i} \gamma^{ij} \varphi_{,j}. \tag{20}$$

Recall that, whereas ψ is held constant in the analogous formula (11) for the elastic case, it may vary in Eq. (20) for the chiral case. However, the on-shell variations of ψ are severely restricted since on a two-dimensional worldsheet the nullity condition (19) (i.e., $w = 0$) automatically imposes that the phase field must satisfy the harmonicity condition

$$(\|\gamma\|^{1/2}\gamma^{ij}\varphi_{,i})_{,j} = 0. \tag{21}$$

It can thus be seen from the internal coordinate version of the current conservation law (12), namely,

$$(\|\gamma\|^{1/2}\gamma^{ij}\psi\varphi_{,i})_{,j} = 0, \tag{22}$$

that the auxiliary field ψ must satisfy the dynamical evolution condition

$$\gamma^{ij}\varphi_{,i}\psi_{,j} = 0, \tag{23}$$

which implies that ψ will be restricted on shell to be a function of just φ.

A. Different actions for the chiral string model

Whereas different equations of state for $L\{w\}$ give qualitatively different elastic string models, the chiral model on the other hand is *unique* (just as in the Nambu-Goto case), modulo the choice of the fixed mass scale m. Thus there is no loss of generality in taking the chiral string equation of state to have the simple linear form originally proposed by Witten [20], namely,

$$L = -m^2 - \frac{1}{2}w \Rightarrow \mathcal{K} = 1. \tag{24}$$

However, it is instructive to consider the many different allowed alternatives which may, for instance, make the integrability property of the chiral string mentioned above more transparent. Indeed, one can see that the equations of motion (19) and (22) as well as the stress-energy tensor (18) for the chiral string can be obtained from any action of the form

$$L=-m^2-\frac{1}{2}w+a_2w^2+a_3w^3+\cdots, \tag{25}$$

where the a_n $(n=2,3,...)$ are arbitrary numerical constants. These additional terms do not contribute on shell since in the equations of motion they lead to terms proportional to w^{n-1} which vanish in the chiral limit $w=0$. Off shell, however, they are important. As we now explain, a particularly useful version of this action for many reasons is the nonlinear form

$$L=-m^2(1+w/m^2)^{1/2}\Rightarrow\mathcal{K}=(1+w/m^2)^{1/2}, \tag{26}$$

which may be described as the ''square root'' action.

First consider the action (26) for arbitrary (generally nonzero) w. From the corresponding stress-energy tensor it follows that the energy density U and tension T satisfy

$$UT=\text{const.} \tag{27}$$

Hence the two sound speeds c_E and c_L defined for the action (26) by Eqs. (15) and (16) will be equal,

$$c_L=c_E. \tag{28}$$

This characterizes the transonic elastic string model that was already known to be integrable for any w [32]. Thus, turning now to the chiral case, once one has realized the equivalence of actions (24) and (26), it comes as no surprise that the chiral action is also integrable [as was first proved in Ref. [27] though on the basis of the action in the form (24)].

A second feature of the action corresponding to the Lagrangian given in Eq. (26) is that it can be viewed as a Kaluza-Klein projection of a Nambu-Goto action in a spacetime of one higher dimension [33,34]. Indeed,

$$\mathcal{I}=-m^2\int d^2\sigma\|\gamma\|^{1/2}(1+w/m^2)^{1/2}=-m^2\int d^2\sigma\|\Gamma\|^{1/2}, \tag{29}$$

where Γ_{ij} is now the projection of the $(d+2)$-dimensional metric

$$G_{\mathrm{AB}}=\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \psi^2/m^2 \end{pmatrix} \tag{30}$$

with the brane embedding given by

$$q^{\mathrm{A}}=\begin{pmatrix} x^\mu(\sigma^i) \\ \varphi(\sigma^i) \end{pmatrix}, \tag{31}$$

so that $\Gamma_{ij}=\partial_i q^{\mathrm{A}}\partial_j q^{\mathrm{B}}G_{\mathrm{AB}}$.

Finally, it is interesting to observe that the action coming from Eq. (26) can be put into a Polyakov-like form—that is, a form which depends linearly on γ_{ij}—but this is not possible for the action coming from Eq. (24): in the Appendix we demonstrate this using Dirac's Hamiltonian formalism for constrained systems [35,36] and also discuss the algebra of constraints. As expected, the result for the Polyakov action is

$$\mathcal{I}=-m^2\int d^2\sigma\|\gamma\|^{1/2}(1+w/m^2)^{1/2} \tag{32}$$

$$=-\frac{m^2}{2}\int d^2\sigma\|h\|^{1/2}h^{ij}\left(g_{\mu\nu}\partial_i x^\mu\partial_j x^\nu+\frac{\psi^2}{m^2}\partial_i\varphi\partial_j\varphi\right), \tag{33}$$

where h^{ij} is now an auxilliary metric field (again see the Appendix). Note that this Polyakov action is reparametrization and scale invariant so that coordinates can be chosen such that $h_{ij}=\eta_{ij}$ where η_{ij} is the Minkowski metric. As for the Nambu string [37], this could be a starting point to discuss a possible quantization of the chiral string [38].

III. GENERAL CANONICAL SYMPLECTIC STRUCTURE

Our aim is to subject the elastic and chiral string models to a canonical analysis of the kind described in the earlier article [1]. However, before doing so, we summarize in this section the main features of the canonical analysis. The following discussion is general and is not restricted to a particular string model.

Consider the general case of a worldsheet action density

$$\mathcal{L}=\|\eta\|^{1/2}L, \tag{34}$$

in which the Lagrangian density L depends on a set of field components q^{A} and on their surface derivatives $q^{\mathrm{A}}_{,i}=\partial_i q^{\mathrm{A}}=\partial q^{\mathrm{A}}/\partial\sigma^i$ where the σ^i are the worldsheet coordinates. In this discussion we do not restrict the worldsheet dimension to 2, so that the formalism presented here will be applicable not just to strings but also to higher dimensional branes. The field variables q^{A} can be of internal or external kind, the most obvious example of the latter kind being the background coordinates x^μ themselves. [In the case of the elastic string $q^{\mathrm{A}}=(x^\mu,\varphi)$.]

Subject to the understanding that the internal coordinates are held fixed,

$$\delta\sigma^i=0, \tag{35}$$

the generic action variation

$$\delta\mathcal{L}=\mathcal{L}_{\mathrm{A}}\delta q^{\mathrm{A}}+p^i_{\mathrm{A}}\delta q^{\mathrm{A}}_{,i} \tag{36}$$

specifies a set of partial derivative components $\mathcal{L}_{\mathrm{A}}$ and an associated set of generalized momentum components p^i_{A}, which can be used to construct a corresponding pseudo-Hamiltonian scalar density

$$\tilde{\mathcal{H}}=p^i_{\mathrm{A}}q^{\mathrm{A}}_{,i}-\mathcal{L}. \tag{37}$$

[The covariance of such a pseudo-Hamiltonian distinguishes it from the ordinary kind of Hamiltonian, which depends on

the introduction of some preferred time foliation, as, for instance, in the Appendix of this article, Eq. (A8).]

According to the variational principle, the dynamically admissible ''on-shell'' configurations are those characterized by the vanishing of the Eulerian derivative given by

$$\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}=\mathcal{L}_{A}-p^{i}_{\mathrm{A},i}\,. \tag{38}$$

For an on-shell configuration, i.e., when the dynamical equations

$$\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}=0 \tag{39}$$

are satisfied, the pseudo-Hamiltonian variation will take the form

$$\delta\tilde{\mathcal{H}}=q^{\mathrm{A}}_{,i}\delta p^{i}_{\mathrm{A}}-p^{i}_{\mathrm{A},i}\delta q^{\mathrm{A}}. \tag{40}$$

Thus the Lagrangian variation can be written as a pure surface divergence

$$\delta\mathcal{L}=\vartheta^{i}_{,i}\,, \tag{41}$$

where ϑ^{i} is the generalized Liouville one-form (on the configuration space cotangent bundle) defined by

$$\vartheta^{i}=p^{i}_{\mathrm{A}}\delta q^{\mathrm{A}}. \tag{42}$$

Equation (41) shows that the Liouville one-form is interpretable as a surface current that will be conserved (in the sense of having vanishing surface divergence) *provided* it is constructed from a perturbation that generates a local symmetry of the Lagrangian density, i.e., such that $\delta\mathcal{L}=0$. In the general case it is not conserved.

We can go on to construct a surface current that will always be conserved when the relevant dynamical equations are satisfied. This is done by taking the exterior differential of the Liouville form, i.e., by evaluating the commutator of a pair of successive independent variations, in the manner described in detail in the previous article [1]. This exterior variation procedure provides us with a closed (since manifestly exact) symplectic two-form expressible as

$$\varpi^{i}=\delta\wedge\vartheta^{i}=\delta p^{i}_{\mathrm{A}}\wedge\delta q^{\mathrm{A}}, \tag{43}$$

where we have used the wedge symbol $\wedge$ to indicate antisymmetrization with respect to the two independent variations involved. (Many authors prefer to use an extreme kind of abbreviation scheme in which the wedge symbol is omitted, but—as discussed in [1]—the use of such ultraconcise notation can lead to confusion in cases involving symmetrization as well as antisymmetrization.) It is easy to verify that, whenever both perturbations satisfy the relevant perturbed field equations

$$\delta\left(\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\right)=0, \tag{44}$$

the symplectic two-form will be interpretable as a conserved worldsheet current in the sense that it will satisfy

$$\varpi^{i}_{,i}=0. \tag{45}$$

We now follow the strategy of [13] and (as far as possible) work with quantities that are purely tensorial with respect to the background space. Hence we translate the surface current densities, whose components ϑ^{i} and ϖ^{i} depend on the choice of the internal coordinates σ^{i}, into corresponding quantities which have stricly vectorial background coordinate components. These are given by

$$\Theta^{\nu}=\|\gamma\|^{-1/2}x^{\nu}_{,i}\vartheta^{i},\quad \Omega^{\nu}=\|\gamma\|^{-1/2}x^{\nu}_{,i}\varpi^{i}. \tag{46}$$

The divergence law (41) is now rewritten in terms of the vectorial version of the Liouville form as

$$\overline{\nabla}_{\nu}\Theta^{\nu}=\|\eta\|^{-1/2}\delta(L\|\gamma\|^{1/2}). \tag{47}$$

Similarly the conservation law (44) simply becomes

$$\overline{\nabla}_{\nu}\Omega^{\nu}=0. \tag{48}$$

IV. COMOVING REFERENCE SYSTEM FOR THE SIMPLY ELASTIC CASE

We now apply the previous formalism to the elastic string for which $q^{\mathrm{A}}=(x^{\mu},\varphi)$ and $p^{\mathrm{A}}=(p^{\mu},\pi)$ where $p_{\mu}=\delta\mathcal{L}/\delta\dot{x}^{\mu}$, $\pi=\delta\mathcal{L}/\delta\dot{\varphi}$.

The meaning of the convention (35) that the local variation δ should be evaluated at a fixed value of the internal coordinates σ^{i} depends on how these coordinates are chosen. For explicit solutions of the field equations it may be most convenient to choose coordinates that are constant along characteristics—as discussed in Sec. II A, this is particularly true in the transonic case for which the extrinsic and intrinsic characteristics coincide. However, in the generic elastic case, the most convenient option is to take the internal coordinates to be comoving with respect to the intrinsic material structure. For a generic p-brane this is specified in terms of a set of p independent scalar fields; for an ordinary perfect fluid or more general elastic solid in four-dimensional spacetime one has $p=3$; while in the elastic string case we are concerned with here one simply has $p=1$ with the field in question being the phase scalar φ. Assuming the absence of singularities where the phase gradient is not just null but actually vanishes, there will be no loss of generality in postulating that the internal coordinates are comoving with respect to φ. In other words, we postulate that there is a vanishing variation, $\delta\varphi=0$, and hence also vanishing gradient variation, $\delta\varphi_{,i}=0$, with respect to these coordinates.

Subject to the choice of such a comoving internal reference system, the only remaining independent field variables in the elastic string model are the background coordinates x^{μ}, so the generic variation (36) will take the specific form

$$\delta\mathcal{L}=\mathcal{L}_{\mu}\xi^{\mu}+p_{\mu}{}^{i}\xi^{\mu}_{,i}\,, \tag{49}$$

using the notation

$$\xi^{\mu}=\delta x^{\mu} \tag{50}$$

for the relevant displacement vector. The corresponding expression for the Liouville form (42) is given by

$$\vartheta^{i}=p_{\mu}{}^{i}\xi^{\mu}, \tag{51}$$

while the symplectic two-form (43) is given by the expression

$$\varpi^{i}=\delta_{\Gamma}p_{\mu}{}^{i}\wedge\xi^{\mu}. \tag{52}$$

Here the parallel variation $\delta_{\Gamma}p_{\mu}{}^{i}$ is given in terms of the simple momentum variation $\delta p_{\mu}{}^{i}$ by

$$\delta_{\Gamma}p_{\mu}{}^{i}=\delta p_{\mu}{}^{i}-\Gamma_{\mu}{}^{\nu}{}_{\rho}p_{\nu}{}^{i}\xi^{\rho}, \tag{53}$$

where $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ are the Riemannian connection components. In view of the symmetry of the latter it actually makes no difference whether $\delta_{\Gamma}p_{\mu}{}^{i}$ or $\delta p_{\mu}{}^{i}$ is used in Eq. (53), but use of the parallel variation is more convenient for our next step, which is the evaluation of the corresponding background tensorial formulas.

Since the background coordinate displacement will affect the Lagrangian only via the change of the induced metric, the resulting variation will be given simply by

$$\delta\mathcal{L}=\frac{1}{2}\|\eta\|^{1/2}T^{\mu\nu}\delta_{\mathrm{L}}g_{\mu\nu}, \tag{54}$$

where $T^{\mu\nu}$ is the surface stress-energy tensor defined in Eq. (7), and $\delta_{\mathrm{L}}g_{\mu\nu}$ is the Lagrangian variation of the metric, meaning the change with respect to a coordinate system that is comoving with respect to the displacement. In the absence of any Eulerian variation (meaning that the spacetime background is held fixed) the Lagrangian variation is just given by the corresponding Lie derivative:

$$\delta_{\mathrm{L}}g_{\mu\nu}=\vec{\xi}\pounds g_{\mu\nu}=2\nabla_{(\mu}\xi_{\nu)}. \tag{55}$$

Comparing Eq. (54) with the canonical variation formula (49), the partial derivatives involved can be read out as

$$\mathcal{L}_{\mu}=\|\gamma\|^{1/2}\Gamma_{\mu}{}^{\nu}{}_{\rho}T_{\nu}{}^{\rho} \tag{56}$$

and

$$p_{\mu}^{i}=\|\gamma\|^{1/2}T_{\mu\nu}\eta^{ij}x_{,j}^{\nu}. \tag{57}$$

It is thus immediately apparent that the pseudo-Hamiltonian density (37) will be given by

$$\|\gamma\|^{-1/2}\tilde{\mathcal{H}}=T_{\nu}{}^{\nu}-L, \tag{58}$$

and that the vectorial version (46) of the Liouville current will be given simply by

$$\Theta^{\nu}=T_{\mu}{}^{\nu}\xi^{\mu}. \tag{59}$$

V. EVALUATION OF THE SYMPLECTIC CURRENT

In order to evaluate the symplectic current a little more work is required. To start with (as in the earlier work [1]), it is convenient to go over from parallel variations to the corresponding Lagrangian variations using the relations

$$\delta_{\Gamma}x_{,i}^{\mu}=x_{,i}^{\nu}\nabla_{\nu}\xi^{\mu} \tag{60}$$

and

$$\delta_{\Gamma}T_{\mu}{}^{\nu}=\delta_{\mathrm{L}}T_{\mu}{}^{\nu}+T_{\rho}{}^{\nu}\nabla_{\mu}\xi^{\rho}-T_{\mu}{}^{\rho}\nabla_{\rho}\xi^{\nu}. \tag{61}$$

The vectorial version (47) of the symplectic current is thereby obtained in the form

$$\Omega^{\nu}=\|\gamma\|^{-1/2}x_{,i}^{\nu}\delta_{\Gamma}p_{\mu}{}^{i}\wedge\xi^{\mu}, \tag{62}$$

with

$$\|\gamma\|^{-1/2}x_{,i}^{\nu}\delta_{\Gamma}p_{\mu}{}^{i}=\delta_{\mathrm{L}}T_{\mu}{}^{\nu}-T_{\rho}{}^{\nu}\nabla_{\mu}\xi^{\rho}+T_{\mu}{}^{\nu}\overline{\nabla}_{\rho}\xi^{\rho}. \tag{63}$$

The advantage of Lagrangian variations is their convenience in relating the relevant intrinsic physical quantities via the appropriate equations of state. Following the example of Friedman and Schutz [39] in the context of ordinary relativistic fluid dynamics, we use the second order derivative of the action with respect to the background metric to obtain the hyper Cauchy tensor (generalized elasticity tensor) according to the prescription

$$C^{\mu\nu\rho\sigma}=\|\gamma\|^{-1/2}\frac{\partial(T^{\mu\nu}\|\gamma\|^{1/2})}{\partial g_{\rho\sigma}}=C^{\rho\sigma\mu\nu}. \tag{64}$$

The Lagrangian variation of the surface stress-energy tensor will thus be obtained in the form

$$\delta_{\mathrm{L}}T^{\mu\nu}=\left(C^{\mu\nu\rho\sigma}-\frac{1}{2}T^{\mu\nu}\eta^{\rho\sigma}\right)\delta_{\mathrm{L}}g_{\rho\sigma}. \tag{65}$$

The symplectic current can thereby be expressed in purely tensorial form as

$$\Omega^{\nu}=(2C_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}\overline{\nabla}_{\sigma}\xi^{\rho}+T^{\nu\rho}\overline{\nabla}_{\rho}\xi_{\mu})\wedge\xi^{\mu}. \tag{66}$$

Using the explicit expression (10) for the surface stress-energy tensor in a generic elastic string model, the corresponding expression for the required hyper Cauchy tensor is obtainable from the definition (64) in the form

$$C^{\mu\nu\rho\sigma}=L\left(\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu(\rho}\eta^{\sigma)\nu}\right)+\frac{\mathcal{K}}{2}(\eta^{\mu\nu}c^{\rho}c^{\sigma}+\eta^{\rho\sigma}c^{\mu}c^{\nu}-4c^{(\mu}\eta^{\nu)(\rho}c^{\sigma)})+\mathcal{K}^{2}\frac{d\mathcal{K}}{dw}c^{\mu}c^{\nu}c^{\rho}c^{\sigma}. \tag{67}$$

Recall that the explicit meaning of the concise wedge product notation used in Eq. (66) is that for a pair of independent

(e.g., virtual and real) displacement fields ζ^μ and ξ^μ say, the corresponding symplectic current will be given by

$$\Omega\{\zeta,\xi\}^\nu = \xi^\mu(2C_\mu{}^\nu{}_\rho{}^\sigma\bar{\nabla}_\sigma\zeta^\rho + T^{\nu\rho}\bar{\nabla}_\rho\zeta_\mu) - \zeta^\mu(2C_\mu{}^\nu{}_\rho{}^\sigma\bar{\nabla}_\sigma\xi^\rho + T^{\nu\rho}\bar{\nabla}_\rho\xi_\mu). \quad (68)$$

VI. THE CHIRAL CASE

As remarked above, the only way in which the off-shell action for the chiral model differs from that of the generic elastic case in version (46) of the Liouville current is that instead of being held constant the auxiliary field ψ is treated as a free variable. However, as its gradient is not involved in the action, this extra variable will not give rise to any corresponding momentum contribution, so the formulas of the two preceding sections will remain valid for the chiral model as characterized on shell by the current nullity condition (19) and the corresponding restrictions (17).

It was remarked that, for a given value of the overall normalization as fixed by the mass scale m, the same unique chiral model with the same on-shell stress-energy tensor (18) is obtained independently of the equation of state. Notice, however, that due to the presence of the final term proportional to $d\mathcal{K}/dw$, the formula (67) for the hyper Cauchy tensor gives a result that *does* depend on the choice of equation of state even in the chiral limit for which $w=0$ and $\mathcal{K}=1$. The simplest possibility is provided by the choice [27] of Witten's simple linear equation of state (24), which gives $d\mathcal{K}/dw=0$, so that the final term in Eq. (67) will drop out altogether. However, for other choices such as the more useful one (26) discussed in Sec. II, there will be an extra term proportional to $c^\mu c^\nu c^\rho c^\sigma$ with an arbitrary proportionality constant. Although it is uniquely defined on shell, the reason the chiral model does not provide a unique specification of the on-shell hyper Cauchy tensor is that the range of variation in the relevant partial derivative formula (65) is restricted by the requirement that, in order to preserve the nullity condition (19), the allowable displacements must be such as to ensure that the Lagrangian metric variation satisfies the on-shell chiral variation condition

$$c^\rho c^\sigma \delta_{\mathrm{L}} g_{\rho\sigma} = 0. \quad (69)$$

However, for this same reason, the final term in Eq. (67) will provide no contribution to the on-shell value of the corresponding symplectic tensor, whose value for the chiral model will thus be given unambiguously by substitution in Eq. (66) of the expression obtained from the linear action (24), namely,

$$C^{\mu\nu\rho\sigma} = m^2\left(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\right) + \frac{1}{2}(\eta^{\mu\nu}c^\rho c^\sigma + \eta^{\rho\sigma}c^\mu c^\nu - 4c^{(\mu}\eta^{\nu)(\rho}c^{\sigma)}). \quad (70)$$

In terms of the orthogonal projector $\perp_\nu^\mu = g_\nu^\mu - \eta_\nu^\mu$, it can be seen that this leads to an expression giving the conserved symplectic current for the chiral string model in the form

$$\Omega^\nu = (c_\mu c_\rho \eta^{\nu\sigma} + 2T_\mu^{[\sigma}\eta_\rho^{\nu]} - T^{\nu\sigma}\perp_{\mu\rho})\xi^\mu \wedge \bar{\nabla}_\sigma\xi^\rho, \quad (71)$$

in which the chiral stress energy tensor is given by Eq. (18).

VII. CONCLUSION

Our aim in this paper was first and foremost to calculate the symplectic current central to the covariant canonical analysis of Witten and others [2–9] for generic elastic and chiral string models. The purpose of this analysis was as a precursor to a covariant quantization. However, the symplectic current also provides potentially useful conservation laws for perturbations in a purely classical context, the most obvious example being provided when the background spacetime has a continuous symmetry generated by a solution k^μ, say, of the Killing equation $\nabla_{(\mu}k_{\nu)}=0$. Since such a Killing vector obviously displaces solutions into solutions, and thus trivially satisfies the relevant perturbed field equations (44), it can be substituted in place of ζ^μ in Eq. (68), so that for any other (nontrivial) solution ξ^μ of the perturbed field equations it can be seen that we shall obtain a corresponding conserved current given by

$$\Omega\{k,\xi\}^\nu = \xi^\mu T^{\nu\rho}\bar{\nabla}_\rho k_\mu - k^\mu(2C_\mu{}^\nu{}_\rho{}^\sigma\bar{\nabla}_\sigma\xi^\rho + T^{\nu\rho}\bar{\nabla}_\rho\xi_\mu). \quad (72)$$

Our work is based on the concise and efficient covariant analysis developed in [13–15], rather than the more cumbersome frame-dependent methods which have been used by others in the case of the simpler Nambu-Goto string. Our results are presented in Secs. V and VI for elastic and chiral strings, respectively. In the process, in Sec. II A and the Appendix, we also studied with care the different formulations of the chiral string model. We showed the equivalence of the Witten action (used in most studies to date), with the square root action (26) and in this way we were able to understand in a different way the integrability properties of the chiral string.

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APPENDIX

In this appendix we demonstrate how the Polyakov action (33) can be obtained from the original square root action (26) using Dirac's Hamiltonian formalism for constrained systems [35,36]. The bulk of this appendix is valid for arbitrary w and hence for general elastic string models: the special chiral limit $w=0$ will be discussed after Eq. (88) where the Poisson algebra of constraints—including the chiral one—is studied. (For a different approach to constrained superconducting membranes, see [40].)

Our starting point is the action (26)

$$\mathcal{I}=\int d^2\sigma\,\mathcal{L}=-m^2\int d^2\sigma\|\gamma\|^{1/2}(1+w/m^2)^{1/2} \quad \text{(A1)}$$

in an arbitrary background metric $g_{\mu\nu}$ and brane embedding x^μ. Let the two worldsheet coordinates be denoted by $\sigma^{(0,1)}\equiv(\tau,\sigma)$ with τ timelike, and let a prime indicate ∂_1 and an overdot ∂_0. The canonical momentum densities $p_\mu=\delta\mathcal{L}/\delta\dot{x}^\mu$, $\pi=\delta\mathcal{L}/\delta\dot{\varphi}$, and $p_\psi=\delta\mathcal{L}/\delta\dot{\psi}=0$ are given by

$$p_\mu=\frac{m^4}{\mathcal{L}}\left[(\dot{x}\cdot x')x'_\mu-x'^2\dot{x}_\mu+\frac{\psi^2}{m^2}(x'_\mu\dot{\varphi}\varphi'-\dot{x}_\mu\varphi'^2)\right], \quad \text{(A2)}$$

$$\pi=-\psi^2\frac{m^2}{\mathcal{L}}[x'^2\dot{\varphi}-(\dot{x}\cdot x')\varphi'], \quad \text{(A3)}$$

and the phase space of the system is $\{x^\mu,\varphi,p_\mu,\pi\}$. It follows directly from these definitions of p_μ and π that there are two primary constraints independent of $\dot{x}$ and $\dot{\varphi}$. One [see Eq. (A7) below] is straightforward to deduce. In order to find the other, it is easiest to express both p^2 and π^2 in terms of $w=\psi^2\gamma^{ij}\varphi_{,i}\varphi_{,j}$ defined in Eq. (2), leading to

$$-\frac{w}{m^6}[p^2+\psi^2\varphi'^2m^2]=x'^2+\frac{1}{m^4}[p^2+2\psi^2\varphi'^2m^2], \quad \text{(A4)}$$

$$w\left[x'^2\psi^2+\frac{\pi^2}{m^2}\right]=\psi^4\varphi'^2-\pi^2. \quad \text{(A5)}$$

Eliminating w between these equations gives the other primary constraint, so that the two constraints are

$$\mathcal{H}_1\equiv p^2+\frac{m^2\pi^2}{\psi^2}+m^4x'^2+m^2\psi^2\varphi'^2=0, \quad \text{(A6)}$$

$$\mathcal{H}_2\equiv p_\mu x'^\mu+\pi\varphi'=0. \quad \text{(A7)}$$

The Nambu-Goto limit is obtained when $\psi\to0$ (note that $\pi^2\sim\psi^4$) in which case these expressions reduce to the standard constraints [36] for the bosonic string.

Had we started instead from the linear Witten action (24) and calculated the new momenta, it would have followed that the second constraint (A7) still holds. On the other hand, a constraint of the first form linking p^2 and π^2 can no longer be obtained: although one can still express p^2 as a function of w in a manner analogous to Eq. (A4), there is no analogue of Eq. (A5) for the linear Witten action. Indeed, π^2 now depends on $\dot{\varphi}$ and $\dot{x}^\mu$ in a combination that can no longer be expressed solely in terms of w.

Going back to the square root action, it follows from the above expressions that the canonical Hamiltonian density vanishes:

$$\mathcal{H}_c\equiv\pi\dot{\varphi}+\dot{x}_\mu p^\mu-\mathcal{L}=0. \quad \text{(A8)}$$

As explained by Dirac [35,36], this Hamiltonian is ambiguous since one is free to add arbitrary multiples of the vanishing primary and secondary constraints. These secondary constraints arise from imposing that the primary constraints should be preserved under time evolution [35]. Here, given that $\mathcal{H}_c=0$ and that the Poisson brackets of the $\mathcal{H}_{1,2}$ are closed [see Eqs. (A13)–(A15) below], it follows that there are no secondary constraints. Hence the ''total'' Hamiltonian density [35,36] which determines the dynamics of the system is simply a linear combination of the primary constraints,

$$\mathcal{H}\equiv\mathcal{H}_c+\frac{\lambda}{2m^2}\mathcal{H}_1+\mu\mathcal{H}_2=\frac{\lambda}{2m^2}\mathcal{H}_1+\mu\mathcal{H}_2, \quad \text{(A9)}$$

where μ and λ are dimensionless Lagrange multipliers. We now briefly turn away from the phase space approach to calculate the (Polyakov) Lagrangian

$$\mathcal{L}_P\equiv\dot{x}_\mu p^\mu+\dot{\varphi}\pi-\mathcal{H} \quad \text{(A10)}$$

corresponding to Eq. (A9). From the equation of motion $\dot{x}_\mu(\tau,\sigma')=\int d\sigma\{x_\mu(\tau,\sigma'),\mathcal{H}(\tau,\sigma)\}$ (with a similar one for φ) it follows that $p_\mu=(m^2/\lambda)(\dot{x}_\mu-\mu x'_\mu)$, and similarly $\pi=-(\psi^2/\lambda)(\mu\varphi'-\dot{\varphi})$. These enable all the momenta to be eliminated from $\mathcal{L}_P$ of Eq. (A10), and one finds in the particular case of the square root action

$$\mathcal{I}=\int d^2\sigma\,\mathcal{L}_P=-\frac{m^2}{2}\int d^2\sigma\|h\|^{1/2}h^{ij}\left(g_{\mu\nu}\partial_i x^\mu\partial_j x^\nu+\frac{\psi^2}{m^2}\partial_i\varphi\partial_j\varphi\right), \quad \text{(A11)}$$

where the components of h_{ij} are simply expressible in terms of λ and μ:

$$\sqrt{-h}h^{00}=-\frac{1}{\lambda},\quad \sqrt{-h}h^{01}=\frac{\mu}{\lambda},\quad \sqrt{-h}h^{11}=\lambda-\frac{\mu^2}{\lambda}. \quad \text{(A12)}$$

This is the result given in Eq. (33). Note that the action (A11) has reparametrization and Weyl scale invariance generated by the two primary constraints $\mathcal{H}_{1,2}$, and reflected in the two free parameters λ and μ. For instance, an appropriate choice of coordinates can set $h_{ij}=\eta_{ij}$ where η_{ij} is the Minkowski metric [or more simply one can choose $\lambda=1$ and $\mu=0$ in Eq. (A12)]. As for the Nambu string [37], this is an appropriate starting point with which to quantize the chiral string [38].

As noted above, this result is a special property of the square root action. For example, in the case of the linear (Witten) action, Eq. (24), only the primary constraint $\mathcal{H}_2$ exists so that the ''full'' Hamiltonian would now contain a single term $\mathcal{H}=\mu\mathcal{H}_2$. Following a similar procedure to that above shows that one would no longer be able to eliminate the momenta in order to calculate the corresponding Lagrangian. (The equations of motion do not give equations for p_μ and π but rather the equalities $\dot{x}_\nu=\mu x'_\nu$ and $\dot{\varphi}=\mu\varphi'$.)

We now make a comment about the algebra of constraints for the square root action. As for the Nambu string [36] (and also as previously noted), it follows from the equations of motion that generators $\mathcal{H}_1$ and $\mathcal{H}_2$ form a closed algebra (i.e., they are first class constraints):

$$\{\mathcal{H}_1(\sigma),\mathcal{H}_1(\sigma')\}\propto\mathcal{H}_2\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'), \tag{A13}$$

$$\{\mathcal{H}_1(\sigma),\mathcal{H}_2(\sigma')\}\propto\mathcal{H}_1\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'), \tag{A14}$$

$$\{\mathcal{H}_2(\sigma),\mathcal{H}_2(\sigma')\}\propto\mathcal{H}_2\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'). \tag{A15}$$

Is it also consistent to impose the chiral constraint at the algebraic level? As noted in Sec. II, the chiral constraint $\partial\mathcal{L}/\partial\psi=0$ imposes the nullity condition

$$w=0, \tag{A16}$$

which we would like to express on the same footing as the primary constraints (A6) and (A7)—that is, in terms of momenta only. Let this constraint be denoted by $\mathcal{H}_3$. From Eqs. (A4) and (A5), if therefore follows that

$$\mathcal{H}_3\equiv\alpha[m^4x'^2+p^2+2\psi^2\varphi'^2m^2]+\psi^2m^2\beta\left[\varphi'^2-\frac{\pi^2}{\psi^4}\right]=0 \tag{A17}$$

for initially arbitrary α and β. Recall, however (Sec. II), that the null constraint $w=0$ is consistent with the dynamical equations, being conserved under time evolution. At the level of constraints, we demand that the chiral constraint $\mathcal{H}_3$ should also be conserved under time evolution—in other words, that it does not lead to a secondary constraint. Equivalently, this imposes that the Poisson brackets of $\mathcal{H}_3$ with the other two constraints $\mathcal{H}_1$, $\mathcal{H}_2$ should be closed. In turn one can show that this is true only for $\alpha=1/2$ and $\beta=-\alpha(1\pm\sqrt{2})$, in which case

$$\{\mathcal{H}_3(\sigma),\mathcal{H}_1(\sigma')\}\propto\mathcal{H}_2\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'), \tag{A18}$$

$$\{\mathcal{H}_2(\sigma),\mathcal{H}_3(\sigma')\}\propto\mathcal{H}_3\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'), \tag{A19}$$

$$\{\mathcal{H}_3(\sigma),\mathcal{H}_3(\sigma')\}\propto\mathcal{H}_2\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'). \tag{A20}$$

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