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Abstract

In this note we consider Dirichlet boundary value problem on a half line. Using critical point theory and Mountain Pass Theorem, we prove the existence of two nontrivials solutions.

Keywords

Dirichlet boundary value problem, half line, critical point, mountain pass.

I INTRODUCTION

In this work is concerned with the following boundary value problem (BVP for short) on the half-line

$$\left\{\begin{array}{rcl} -\left(p(t)u'(t)\right)' & = & \lambda f(t,u(t)), \quad \text{a.e. } t>0, \\ u(0)=u(+\infty) & = & 0, \end{array}\right. \tag{1}$$

where $\lambda>0$ be a numerical parameter and $f:[0,+\infty)\times R\longrightarrow R$ is a Carathéodory function. The coefficient $p:[0,+\infty)\longrightarrow(0,+\infty)$ satisfies $\dfrac{1}{p}\in L^1(0,+\infty)$, and

$$\int_0^{+\infty}\left(\int_t^{+\infty}\frac{1}{p(s)}ds\right)dt<+\infty.$$

This paper is devted to the generalization of problem in [9], when we consider the problem in the half-line.
Define the space

$$H^1_{0,p}(0,+\infty)=\left\{u\in AC([0,+\infty),R)\mid u(0)=u(+\infty)=0,\ \sqrt{p}u'\in L^2(0,+\infty)\right\}.$$

Take $v\in H^1_{0,p}(0,+\infty)$, multiply the equation in problem (1) by v, and then integrate over $(0,+\infty)$, we get

$$\int_0^{+\infty}p(t)u'(t)v'(t)dt=\lambda\int_0^{+\infty}f(t,u(t))v(t)dt.$$

This leads to the natural concept of weak solution for problem (1).

Definition 1: We say that a function $u\in H^1_{0,p}(0,+\infty)$ is a weak solution of problem (1) if

$$\int_0^{+\infty}p(t)u'(t)v'(t)dt-\lambda\int_0^{+\infty}f(t,u(t))v(t)dt=0,$$

for all $v \in H^1_{0,p}(0,+\infty)$.

In order to study problem (1), we consider the functional $J : H^1_{0,p}(0,+\infty) \longrightarrow R$ defined by

$$J(u) = \frac{1}{2}\int_0^{+\infty} p(t)u'^2(t)dt - \lambda \int_0^{+\infty} F(t,u(t))dt, \tag{2}$$

where

$$F(t,u) = \int_0^u f(t,s)ds.$$

Lemma 1. *$H^1_{0,p}(0,+\infty)$ embeds in $L^2(0,+\infty)$.*

Proof. For $u \in H^1_{0,p}(0,+\infty)$, we have

$$|u(t)| = \left|\int_t^{+\infty} u'(s)ds\right| = \left|\int_t^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}ds\right|.$$

Then, by the Cauchy-Schwartz inequality

$$\begin{aligned} |u(t)|^2 &\leq \left(\int_t^{+\infty} p(s)u'^2(s)ds\right)\left(\int_t^{+\infty}\frac{1}{p(s)}ds\right) \\ &\leq \left(\int_0^{+\infty} p(s)u'^2(s)ds\right)\left(\int_t^{+\infty}\frac{1}{p(s)}ds\right). \end{aligned}$$

Hence

$$\int_0^{+\infty}|u(t)|^2dt \leq \left(\int_0^{+\infty}(\int_t^{+\infty}\frac{1}{p(s)}ds)dt\right)\left(\int_0^{+\infty}p(s)|u'(s)|^2ds\right),$$

that is

$$\|u\|_{L^2} \leq \sqrt{M}\|\sqrt{p}u'\|_{L^2}.$$

Theorem 1. *(a) The operator*

$$\begin{aligned} T : H^1_{0,p}(0,+\infty) &\longrightarrow T(H^1_{0,p}(0,+\infty)) \subset L^2(0,+\infty)\times L^2(0,+\infty) :\ = L^2_2(0,+\infty) \\ u &\longrightarrow T(u) = (u, \sqrt{p}u') \end{aligned}$$

is an isometric isomorphism.
(b) $H^1_{0,p}(0,+\infty)$ is a reflexive space.

Proof. (a) It is clear that T is a linear operator and that T conserves norms, i.e.,

$$\forall\, u \in H^1_{0,p}(0,+\infty), \quad \|Tu\|_{L^2_2} = \|u\|_p.$$

Indeed

$$\begin{aligned} \|Tu\|_{L^2_2} &= \|(u,\sqrt{p}u')\|_{L^2_2} \\ &= \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2} \\ &= \|u\|_p. \end{aligned}$$

(b) Since $L^2(0,+\infty))$ is a reflexive Banach space, the cartesian product $L^2_2(0,+\infty))$ is also a reflexive Banach space with respect to the norm

$$\|u\|_{L^2_2} = \|u_1\|_{L^2} + \|u_2\|_{L^2}, \quad \text{where } u = (u_1, u_2) \in L^2_2(0,+\infty).$$

From part (a), $T(H^1_{0,p}(0,+\infty))$ is a closed subspace of $L^2_2(0,+\infty)$, the space $T(H^1_{0,p}(0,+\infty))$ is reflexive. Consequently $H^1_{0,p}(0,+\infty)$ is also reflexive.

Notice that $H^1_{0,p}(0,+\infty)$ is a Banach space equipped with the norm

$$\|u\|_{0,p} = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt},$$

or the equivalent norm

$$\|u\|_p = \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2}.$$

Moreover the space $H^1_{0,p}(0,+\infty)$ is reflexive.

Theorem 2. *[8] On $H^1_{0,p}(0,+\infty)$, the quantity $\|u\| = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt}$ is a norm which is equivalent to the $H^1_{0,p}(0,+\infty)$-norm.*

Lemma 2. *$(H^1_{0,p}(0,+\infty), \|\cdot\|)$ embeds in $(C_0[0,+\infty), \|u\|_\infty)$, where $C_0[0,+\infty) = \{u \in C([0,+\infty), R) \mid \lim_{t\to+\infty} u(t) = 0\}$ and $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$, and we have*

$$\|u\|_\infty \le M\|u\| \quad \textit{where} \quad M = \sqrt{\left\|\frac{1}{p}\right\|_{L^1}}. \tag{3}$$

Theorem 3. *The embedding*

$$H^1_{0,p}(0,+\infty) \hookrightarrow C_0[0,+\infty) \tag{4}$$

is compact.

Let E be a real reflexive Banach space, and write B_ρ for the closed ball centered at 0 with radius ρ in E and $\partial B_\rho = \{u \in E | \ \|u\| = \rho\}$ for the boundary of B_ρ.
We will employ the following conditions.
$(\mathbf{H_1})$ $\Phi : E \longrightarrow R$ is a convex, continuously Fréchet differentiable functional

with derivative $\varphi : E \longrightarrow E^*$;
$(\mathbf{H_2})$ $H : E \longrightarrow R$ is a continuously Fréchet differentiable functional with

derivative $h : E \longrightarrow E^*$;
$(\mathbf{H_3})$ the operator $h : E \longrightarrow E^*$ is compact;
$(\mathbf{H_4})$ there exist constants $\alpha, \alpha_1 > 1$, $\gamma > 0$ such that

$$\gamma\|v\|^\alpha_E \le \langle \varphi(v), v\rangle \quad \text{for all} \quad v \in E$$

and

$$\lim_{\|u\|_E\to+\infty} \sup \frac{\Phi(u)}{\|u\|^{\alpha_1}_E} = +\infty;$$

$(\mathbf{H_5})$ H is a convex functional.

$\langle ., \ . \rangle$ denotes the action of a derivative on a suitable element or else a duality pairing.
We will determine such a value $\lambda^* > 0$ that for each $\lambda \in (0, \lambda^*]$ the corresponding Euler action functional $J : E \longrightarrow R$

$$J(u) = \Phi(u) - \lambda H(u),$$

has a critical point on B_ρ. This implies the solvability of

$$\varphi(u) = \lambda h(u), \quad u \in E. \tag{5}$$

Theorem 4. *[9] Assume that $\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_5}$ are satisfied. Fix some $\lambda^* > 0$ and let $u_0, v \in E$ be such that*

$$J(u_0) \leq J(v) \ \text{ and } \ \varphi(v) = \lambda^* h(u_0). \tag{6}$$

Then u_0 is a critical point to J, and thus it solves (5).

Theorem 5. *[9] Let E be an infinite dimensional reflexive Banach space and let B_ρ be fixed. Assume that $\mathbf{H_1} - \mathbf{H_5}$ are satisfied. Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*]$ there exists $u_0 \in B_\rho$ with*

$$J(u_0) = \inf_{u \in B_\rho} J(u) \tag{7}$$

and such that u_0 is a critical point to J, and thus it solves (5).

Theorem 6. *[9] Let E be an infinite dimensional reflexive Banach space. Assume that $\mathbf{H_1} - \mathbf{H_5}$ are satisfied. Take some $\rho > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*]$ there exists $u_0 \in B_\rho$ with*

$$J(u_0) = \inf_{u \in B_\rho} J(u) \tag{8}$$

and such that u_0 is a critical point of J, and thus it solves (5). If for some $v \in B_\rho$ it holds that $J(v) < 0$ and $J(0) = 0$ or else $h(0) \neq 0$ and $J(0) = 0$, then u_0 is non-trivial.
Assume additionally that there exists $\lambda_1^* \leq \lambda^*$ such that for all $\lambda \in (0, \lambda_1^*)$

(a) J satisfies the PS-condition,

(b) $J(0) < \inf_{u \in \partial B_{\rho_1}} J(u)$ for some $\rho_1 > \|u_0\|_E$,

(c) there exists $w \in E \setminus B_{\rho_1}$ with $J(w) \leq 0$.
Then for all $\lambda \in (0, \lambda_1^*)$ the functional J has two nontrivial critical points, namely u_0 and another non-zero critical point z_0 different from u_0.

II APPLICATIONS

We will employ the following hypotheses.
$(\mathbf{A_1})$ There exist positive functions a, b with $a, \ b \in L^1(0, +\infty)$ and $\sigma > 1$ such that

$$|f(t,u)| \leq a(t)|u|^\sigma + b(t), \ \text{for a.e. } t \in [0, +\infty) \text{ and all } u \in R,$$

$(\mathbf{A_2})$ there exists functions $c_1, c_2 : [0, +\infty) \longrightarrow (0, +\infty)$ with $c_1, c_2 \in L^1(0, +\infty)$, and $\theta > 2$ such that

(a) $\theta F(t,u) \leq u f(t,u)$, for a.e. $t \in [0, +\infty)$ and all $u \in R \backslash \{0\}$,

(b) $F(t,u) \geq c_1(t)|u|^\theta - c_2(t)$, for a.e. $t \geq 0$ and all $u \in R$,

$(\mathbf{A_3})$ the function $u \longmapsto F(t,u)$ is convex on R for a.e. $t \in [0, \ +\infty)$.

Theorem 7. *Let $\lambda > 0$ be fixed. The functional J is well-defined and continuously differentiable on $H^1_{0,p}(0,+\infty)$. The derivative of J at any $u \in H^1_{0,p}(0,+\infty)$ has the following form*

$$\langle J'(u), v\rangle = \int_0^{+\infty} p(t)u'(t)v'(t)dt - \lambda \int_0^{+\infty} f(t,u(t))v(t)dt, \ \ \forall v \in H^1_{0,p}(0,+\infty).$$

In particular, $u \in H^1_{0,p}(0,+\infty)$ is a solution of problem (1) if and only if u is a critical point of J.

2.1 Existence of solution by the mountain pass lemma

Lemma 3. *Suppose that $(\mathbf{A_1}), (\mathbf{A_2(a)})$ hold. Then for any $\lambda > 0$, the functional J given by (2) satisfies the PS-condition.*

proof. Let us take a sequence $\{u_k\}_{k=1}^{\infty} \subset H^1_{0,p}(0,+\infty)$ such that$\{J(u_k)\}_{k=1}^{\infty}$ is bounded and $J'(u_k) \to 0$, as $k \to \infty$. We shall show that $\{u_k\}_{k=1}^{k=\infty}$ has a convergent subsequence.

Since $J'(u_k) \longrightarrow 0$, we see that for some $\epsilon > 0$, there exists k_0 with $\|J'(u_k)\| \leq \epsilon$ for $k \geq k_0$. Note that for $k \geq k_0$

$$|\langle J'(u_k),\ u_k\rangle| \leq \epsilon\|u_k\|$$

and because we have

$$\langle J'(u_k), u_k\rangle = \int_0^{+\infty} p(t)\left(u_k'(t)\right)^2 dt - \lambda \int_0^{+\infty} f(t,u_k(t))u_k(t)dt,$$

then

$$-\lambda \int_0^{+\infty} F(t,u_k(t))dt \ \ \geq \ \ \frac{-\epsilon}{\theta}\|u_k\| - \frac{1}{\theta}\|u_k\|^2.$$

Since $\{J(u_k)\}_{k=1}^{\infty}$ is bounded, there exists a constant C such that $|J(u_k)| \leq C,\ \forall k \in N$. Using the above estimates, we obtain

$$C \geq \left(\frac{\theta-2}{2\theta}\right)\|u_k\|^2 - \frac{\epsilon}{2}\|u_k\|.$$

Since $\theta > 2$, then the sequence (u_k) is bounded in $H^1_{0,p}(0,+\infty)$ i.e., $\exists M_2 > 0$ such that $\|u_k\| \leq M_2,\ \forall k \in N$. Next, we prove that (u_k) converges strongly to some $\overline{u}$ in $H^1_{0,p}(0,+\infty)$. Since (u_k) is bounded in the reflexive Banach space $H^1_{0,p}(0,+\infty)$, there exists a subsequence of (u_k) still denoted (u_k) such that (u_k) converges weakly to some $\overline{u}$ in $H^1_{0,p}(0,+\infty)$, and $\|\overline{u}\| \leq M_2$. Then (u_k) converges to $\overline{u}$ in $C_0[0,+\infty)$ by Lemma 2.
Since $\lim\limits_{k\longrightarrow+\infty} J'(u_k) = 0$ and (u_k) converges weakly to some $\overline{u}$, we get

$$\lim_{k\longrightarrow+\infty} \langle J'(u_k) - J'(\overline{u}),\ u_k - \overline{u}\rangle \longrightarrow\ 0 \tag{9}$$

and

$$\langle J'(u_k) - J'(\overline{u}), u_k - \overline{u}\rangle = \|u_k - \overline{u}\|^2 - \lambda \int_0^{+\infty} \left(f(t,u_k(t)) - f(t,\overline{u}(t))\right)\left(u_k(t) - \overline{u}(t)\right)dt.$$

Since $u_k \to \overline{u}$ in $C_0[0,+\infty)$, $\forall t \in [0,+\infty)$, then $u_k(t) \to \overline{u}(t)$ for $t \in [0,+\infty)$, and since f is Carathéodory, we have
$f(t,u_k(t)) \to f(t,\overline{u}(t))\, as\ k \to +\infty$ p.p. $t \in [0,+\infty)$,
and using $(\mathbf{A_1})$ we have

$$|f(t,u_k(t))| \leq M_2^{\sigma} M^{\sigma} a(t) + b(t). \tag{10}$$

Since: $a \in L^1(0,+\infty)$, $b \in L^1(0,+\infty)$, then

$$M_2^{\sigma} M^{\sigma} a + b \in L^1(0,+\infty). \tag{11}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k \longrightarrow +\infty} \int_0^{+\infty} f(t,u_k(t))dt = \int_0^{+\infty} f(t,\overline{u}(t))dt. \tag{12}$$

Then (9) and (12) imply that $\{u_k\}_{n=1}^{\infty}$ is strongly convergent.

Lemma 4. *Suppose that* $(\mathbf{A_1}), (\mathbf{A_2(b)})$ *hold. There exists* $\lambda_1^* > 0$ *such that for any* $\lambda \leq \lambda_1^*$ *there exist numbers* $\rho_1, \alpha > 0$ *such that*
(1) *there are positive constants* ρ_1, α *such that* $J(u) \geq \alpha$ *for all* $u \in H^1_{0,p}(0,+\infty)$ *with* $\|u\| = \rho_1$.
(2) *There exists an element* $w \in H^1_{0,p}(0,+\infty)$ *with* $\|w\| > \rho_1$

and such that $J(w) < 0$.

proof. Thus in (2)

$$J(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^{+\infty} F(t,u(t))dt.$$

We first verify that J satisfies assumption (1) in Lemma 4.
Define $\beta_1 := \dfrac{1}{\sigma+1} M^{\sigma} \rho_1^{\sigma} \|a\|_{L^1} + \|b\|_{L^1}$. Then we see that for any $u \in \partial B(0,\rho_1)$ with $\|u\| = \rho_1 = 2M$, (2) and $(\mathbf{A_1})$ give

$$\left| \int_0^{+\infty} F(t,u(t))dt \right| \leq M\rho_1\beta_1.$$

Put $\lambda_1^* = \dfrac{1}{\beta_1}$ and fix $\lambda \in (0,\ \lambda_1^*)$.
Moreover,

$$J(u) \geq 2M^2(1-\lambda\beta_1),$$

this implies that assumption (1) holds if we take $\rho_1 = 2M$ and $\alpha = 2M^2(1-\lambda\beta_1)$.

Now $(\mathbf{A2})$ guarantees that for some $w_0 \in H^1_{0,p}(0,+\infty)$ with $w_0 \neq 0$ and $s \in R_+$, we have the following estimation

$$J(sw_0) \leq \frac{1}{2}s^2\|w_0\|^2 - \lambda s^\theta \int_0^{+\infty} c_1(t)|w_0(t)|^\theta dt + \lambda \int_0^{+\infty} c_2(t)dt.$$

Since $\theta > 2$ we see that $J(sw_0) \to -\infty$ as $s \longrightarrow +\infty$. Thus there is some s_0 such that $w = s_0 w_0$ we have $J(w) < 0$.
Therefore assumption (2) in Lemma 4 is also satisfied.

Now the mountain pass lemma allows us to formulate the following existence result.

Theorem 8. *Suppose also that* $(\mathbf{A_1}), (\mathbf{A_2})$ *hold. Then there existe* $\lambda_1^* > 0$ *such that for all* $\lambda \in (0, \lambda_1^*)$, *problem (1) has at least one nontrivial solution.*

Example 1

Consider the boundary value problem

$$\begin{cases} -\left(e^t u'(t)\right)' = \lambda\left(\dfrac{3u^3+2u}{(20+t^2)\sqrt{u^2+1}}\right), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{13}$$

It can be easily checked that all conditions of Theorem 8 are satisfied with

$$f(t,u) = \frac{3u^3+2u}{(20+t^2)\sqrt{u^2+1}}, \quad \sigma = 3, \quad \theta = 2.1,$$
$$a(t) = \frac{1}{1+t^2}, \quad b(t) = \frac{5}{t^2+1}, \quad p(t) = e^t,$$
$$c_1(t) = \frac{1}{20+t^2}, \quad c_2(t) = \frac{4}{10}e^{-t},$$

and

$$F(t,u) = \frac{u^2\sqrt{u^2+1}}{20+t^2} - \frac{1}{10}e^{-t}$$

Then there is some $\lambda_1^* > 0$ such that for all $\lambda \in (0, \lambda_1^*)$, problem (13) has at least one nontrivial solutions.

2.2 Existence of solution by the critical point theorem on a closed ball

Theorem 9. *Let* $B(0,\rho)$, *fixed. Assume that* $\mathbf{A_1}, \mathbf{A_3}$ *are satisfied and that* $f(t,0) \neq 0$, *Then there exist* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, *there existe* $u_0 \in B(0,\rho)$ *with*

$$J(u_0) = \inf_{u \in B_\rho} J(u) \tag{14}$$

and such that u_0 *is a critical point, and thus it solves (1)* .

2.3 Existence of two solutions

Concerning the multiple solutions, we have the main result of this section where we need only that F is convex in addition to assumptions leading to a mountain pass solution.

Theorem 10. *Assume that* $\mathbf{A_1} - \mathbf{A_3}$ *are satisfied and that* $f(t,0) \neq 0$. *Then there exist* $0 < \lambda_1^* \leq \lambda^*$ *such that for all* $0 < \lambda \leq \lambda_1^*$, *problem (1) has at least two nontrivial solutions.*

REFERENCES

[1] A.M. Arthurs. *Complementary Variational Principles.* Clarendon Press, Oxford (1980).

[2] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners.* Springer-Verlag, London, 2011

[3] G. Bonanno, D. O'Regan, *A boundary value problem on the half-line via critical point methods*. Dynam. Systems Appl. 15 (2006), no. 3-4, 395–407.

[4] B. E. Breckner, D. Repovas and C. Varga, *On the existence of three solutions for the Dirichlet problem on the Sierpinski gasket,* Nonlinear Anal. $73\ (2010),$ no. $9, 2980-2990$

[5] H. BRÉZIS, *Analyse Fonctionnelle, Théorie et Applications* . Masson, Paris, 1983.

[6] H. BRÉZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

[7] C. Corduneanu. *Integral Equations and Stability of Feedback Systems.* Cambridge University Press, Cambridge (1973). Iranian Math. Society, $43(1)$ $2017,$ 223-237.

[8] M. Galewski., T. Moussaoui., I. Soufi, *On the existence of solutions for a boundary value problem on the half-line,* ejtde $2018,$ No. $12,$ $1-12.$

[9] M. Galewski, *Multiple Solutions to a Dirichlet Problem on the Sierpinski Gasket. Taiwanese journal of mathematics* $20(5)$ *October* $2016.$

[10] M. Galewski, *On a new multiple critical point theorem and some applications to anisotropic problems. Taiwanese journal of mathematics* $19(5)$ $2015.$.

[11] J. Mawhin, *Problèmes de Dirichlet variationnels non linéaires,* Les Presses de l'Université de Montréal, 1987.