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Ibrahim Soufi. On the existence of two solutions for a boundary value problem on the half-line. 2022. ⟨hal-03731350⟩

**HAL Id: hal-03731350**

**<https://hal.science/hal-03731350v1>**

Preprint submitted on 21 Jul 2022

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# On the existence of two solutions for a boundary value problem on the half-line

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## Abstract

In this note we consider Dirichlet boundary value problem on a half line. Using critical point theory and Mountain Pass Theorem, we prove the existence of two nontrivials solutions.

## Keywords

Dirichlet boundary value problem, half line, critical point, mountain pass.

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## I INTRODUCTION

In this work is concerned with the following boundary value problem (BVP for short) on the half-line

$$\begin{cases} -(p(t)u'(t))' = \lambda f(t, u(t)), & \text{a.e. } t > 0, \\ u(0) = u(+\infty) = 0, \end{cases} \quad (1)$$

where  $\lambda > 0$  be a numerical parameter and  $f : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function. The coefficient  $p : [0, +\infty) \rightarrow (0, +\infty)$  satisfies  $\frac{1}{p} \in L^1(0, +\infty)$ , and

$$\int_0^{+\infty} \left( \int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

This paper is devoted to the generalization of problem in [9], when we consider the problem in the half-line.

Define the space

$$H_{0,p}^1(0, +\infty) = \{u \in AC([0, +\infty), \mathbb{R}) \mid u(0) = u(+\infty) = 0, \sqrt{p}u' \in L^2(0, +\infty)\}.$$

Take  $v \in H_{0,p}^1(0, +\infty)$ , multiply the equation in problem (1) by  $v$ , and then integrate over  $(0, +\infty)$ , we get

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = \lambda \int_0^{+\infty} f(t, u(t))v(t)dt.$$

This leads to the natural concept of weak solution for problem (1).

**Definition 1:** We say that a function  $u \in H_{0,p}^1(0, +\infty)$  is a weak solution of problem (1) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt - \lambda \int_0^{+\infty} f(t, u(t))v(t)dt = 0,$$

for all  $v \in H_{0,p}^1(0, +\infty)$ .

In order to study problem (1), we consider the functional  $J : H_{0,p}^1(0, +\infty) \longrightarrow R$  defined by

$$J(u) = \frac{1}{2} \int_0^{+\infty} p(t)u'^2(t)dt - \lambda \int_0^{+\infty} F(t, u(t))dt, \quad (2)$$

where

$$F(t, u) = \int_0^u f(t, s)ds.$$

**Lemma 1.**  $H_{0,p}^1(0, +\infty)$  embeds in  $L^2(0, +\infty)$ .

**Proof.** For  $u \in H_{0,p}^1(0, +\infty)$ , we have

$$|u(t)| = \left| \int_t^{+\infty} u'(s)ds \right| = \left| \int_t^{+\infty} \sqrt{p(s)}u'(s) \frac{1}{\sqrt{p(s)}}ds \right|.$$

Then, by the Cauchy-Schwartz inequality

$$\begin{aligned} |u(t)|^2 &\leq \left( \int_t^{+\infty} p(s)u'^2(s)ds \right) \left( \int_t^{+\infty} \frac{1}{p(s)}ds \right) \\ &\leq \left( \int_0^{+\infty} p(s)u'^2(s)ds \right) \left( \int_t^{+\infty} \frac{1}{p(s)}ds \right). \end{aligned}$$

Hence

$$\int_0^{+\infty} |u(t)|^2 dt \leq \left( \int_0^{+\infty} \left( \int_t^{+\infty} \frac{1}{p(s)}ds \right) dt \right) \left( \int_0^{+\infty} p(s)|u'(s)|^2 ds \right),$$

that is

$$\|u\|_{L^2} \leq \sqrt{M} \|\sqrt{p}u'\|_{L^2}.$$

**Theorem 1.** (a) The operator

$$\begin{aligned} T : H_{0,p}^1(0, +\infty) &\longrightarrow T(H_{0,p}^1(0, +\infty)) \subset L^2(0, +\infty) \times L^2(0, +\infty) : = L_2^2(0, +\infty) \\ u &\longrightarrow T(u) = (u, \sqrt{p}u') \end{aligned}$$

is an isometric isomorphism.

(b)  $H_{0,p}^1(0, +\infty)$  is a reflexive space.

**Proof.** (a) It is clear that  $T$  is a linear operator and that  $T$  conserves norms, i.e.,

$$\forall u \in H_{0,p}^1(0, +\infty), \quad \|Tu\|_{L_2^2} = \|u\|_p.$$

Indeed

$$\begin{aligned} \|Tu\|_{L_2^2} &= \|(u, \sqrt{p}u')\|_{L_2^2} \\ &= \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2} \\ &= \|u\|_p. \end{aligned}$$

(b) Since  $L^2(0, +\infty)$  is a reflexive Banach space, the cartesian product  $L^2_2(0, +\infty)$  is also a reflexive Banach space with respect to the norm

$$\|u\|_{L^2_2} = \|u_1\|_{L^2} + \|u_2\|_{L^2}, \quad \text{where } u = (u_1, u_2) \in L^2_2(0, +\infty).$$

From part (a),  $T(H^1_{0,p}(0, +\infty))$  is a closed subspace of  $L^2_2(0, +\infty)$ , the space  $T(H^1_{0,p}(0, +\infty))$  is reflexive. Consequently  $H^1_{0,p}(0, +\infty)$  is also reflexive.

Notice that  $H^1_{0,p}(0, +\infty)$  is a Banach space equipped with the norm

$$\|u\|_{0,p} = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt},$$

or the equivalent norm

$$\|u\|_p = \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2}.$$

Moreover the space  $H^1_{0,p}(0, +\infty)$  is reflexive.

**Theorem 2.** [8] On  $H^1_{0,p}(0, +\infty)$ , the quantity  $\|u\| = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt}$  is a norm which is equivalent to the  $H^1_{0,p}(0, +\infty)$ -norm.

**Lemma 2.**  $(H^1_{0,p}(0, +\infty), \|\cdot\|)$  embeds in  $(C_0[0, +\infty), \|u\|_\infty)$ , where  $C_0[0, +\infty) = \{u \in C([0, +\infty), \mathbb{R}) \mid \lim_{t \rightarrow +\infty} u(t) = 0\}$  and  $\|u\|_\infty = \sup_{t \in [0, +\infty)} |u(t)|$ , and we have

$$\|u\|_\infty \leq M\|u\| \quad \text{where } M = \sqrt{\left\|\frac{1}{p}\right\|_{L^1}}. \quad (3)$$

**Theorem 3.** The embedding

$$H^1_{0,p}(0, +\infty) \hookrightarrow C_0[0, +\infty) \quad (4)$$

is compact.

Let  $E$  be a real reflexive Banach space, and write  $B_\rho$  for the closed ball centered at 0 with radius  $\rho$  in  $E$  and  $\partial B_\rho = \{u \in E \mid \|u\| = \rho\}$  for the boundary of  $B_\rho$ .

We will employ the following conditions.

(H<sub>1</sub>)  $\Phi : E \rightarrow \mathbb{R}$  is a convex, continuously Fréchet differentiable functional

with derivative  $\varphi : E \rightarrow E^*$ ;

(H<sub>2</sub>)  $H : E \rightarrow \mathbb{R}$  is a continuously Fréchet differentiable functional with

derivative  $h : E \rightarrow E^*$ ;

(H<sub>3</sub>) the operator  $h : E \rightarrow E^*$  is compact;

(H<sub>4</sub>) there exist constants  $\alpha, \alpha_1 > 1$ ,  $\gamma > 0$  such that

$$\gamma\|v\|_E^\alpha \leq \langle \varphi(v), v \rangle \quad \text{for all } v \in E$$

and

$$\lim_{\|u\|_E \rightarrow +\infty} \sup \frac{\Phi(u)}{\|u\|_E^{\alpha_1}} = +\infty;$$

(H<sub>5</sub>)  $H$  is a convex functional.

$\langle \cdot, \cdot \rangle$  denotes the action of a derivative on a suitable element or else a duality pairing. We will determine such a value  $\lambda^* > 0$  that for each  $\lambda \in (0, \lambda^*]$  the corresponding Euler action functional  $J : E \longrightarrow \mathbb{R}$

$$J(u) = \Phi(u) - \lambda H(u),$$

has a critical point on  $B_\rho$ . This implies the solvability of

$$\varphi(u) = \lambda h(u), \quad u \in E. \quad (5)$$

**Theorem 4.** [9] Assume that  $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_5$  are satisfied. Fix some  $\lambda^* > 0$  and let  $u_0, v \in E$  be such that

$$J(u_0) \leq J(v) \quad \text{and} \quad \varphi(v) = \lambda^* h(u_0). \quad (6)$$

Then  $u_0$  is a critical point to  $J$ , and thus it solves (5).

**Theorem 5.** [9] Let  $E$  be an infinite dimensional reflexive Banach space and let  $B_\rho$  be fixed. Assume that  $\mathbf{H}_1 - \mathbf{H}_5$  are satisfied. Then there exists  $\lambda^* > 0$  such that for each  $\lambda \in (0, \lambda^*]$  there exists  $u_0 \in B_\rho$  with

$$J(u_0) = \inf_{u \in B_\rho} J(u) \quad (7)$$

and such that  $u_0$  is a critical point to  $J$ , and thus it solves (5).

**Theorem 6.** [9] Let  $E$  be an infinite dimensional reflexive Banach space. Assume that  $\mathbf{H}_1 - \mathbf{H}_5$  are satisfied. Take some  $\rho > 0$ . Then there exists  $\lambda^* > 0$  such that for each  $\lambda \in (0, \lambda^*]$  there exists  $u_0 \in B_\rho$  with

$$J(u_0) = \inf_{u \in B_\rho} J(u) \quad (8)$$

and such that  $u_0$  is a critical point of  $J$ , and thus it solves (5). If for some  $v \in B_\rho$  it holds that  $J(v) < 0$  and  $J(0) = 0$  or else  $h(0) \neq 0$  and  $J(0) = 0$ , then  $u_0$  is non-trivial.

Assume additionally that there exists  $\lambda_1^* \leq \lambda^*$  such that for all  $\lambda \in (0, \lambda_1^*)$

(a)  $J$  satisfies the PS-condition,

(b)  $J(0) < \inf_{u \in \partial B_{\rho_1}} J(u)$  for some  $\rho_1 > \|u_0\|_E$ ,

(c) there exists  $w \in E \setminus B_{\rho_1}$  with  $J(w) \leq 0$ .

Then for all  $\lambda \in (0, \lambda_1^*)$  the functional  $J$  has two nontrivial critical points, namely  $u_0$  and another non-zero critical point  $z_0$  different from  $u_0$ .

## II APPLICATIONS

We will employ the following hypotheses.

(A<sub>1</sub>) There exist positive functions  $a, b$  with  $a, b \in L^1(0, +\infty)$  and  $\sigma > 1$  such that

$$|f(t, u)| \leq a(t)|u|^\sigma + b(t), \quad \text{for a.e. } t \in [0, +\infty) \text{ and all } u \in \mathbb{R},$$

(A<sub>2</sub>) there exists functions  $c_1, c_2 : [0, +\infty) \longrightarrow (0, +\infty)$  with  $c_1, c_2 \in L^1(0, +\infty)$ , and  $\theta > 2$  such that

(a)  $\theta F(t, u) \leq u f(t, u)$ , for a.e.  $t \in [0, +\infty)$  and all  $u \in \mathbb{R} \setminus \{0\}$ ,

(b)  $F(t, u) \geq c_1(t)|u|^\theta - c_2(t)$ , for a.e.  $t \geq 0$  and all  $u \in \mathbb{R}$ ,

(A<sub>3</sub>) the function  $u \longmapsto F(t, u)$  is convex on  $\mathbb{R}$  for a.e.  $t \in [0, +\infty)$ .

**Theorem 7.** Let  $\lambda > 0$  be fixed. The functional  $J$  is well-defined and continuously differentiable on  $H_{0,p}^1(0, +\infty)$ . The derivative of  $J$  at any  $u \in H_{0,p}^1(0, +\infty)$  has the following form

$$\langle J'(u), v \rangle = \int_0^{+\infty} p(t) u'(t) v'(t) dt - \lambda \int_0^{+\infty} f(t, u(t)) v(t) dt, \quad \forall v \in H_{0,p}^1(0, +\infty).$$

In particular,  $u \in H_{0,p}^1(0, +\infty)$  is a solution of problem (1) if and only if  $u$  is a critical point of  $J$ .

## 2.1 Existence of solution by the mountain pass lemma

**Lemma 3.** Suppose that  $(A_1), (A_2(a))$  hold. Then for any  $\lambda > 0$ , the functional  $J$  given by (2) satisfies the PS-condition.

**proof.** Let us take a sequence  $\{u_k\}_{k=1}^\infty \subset H_{0,p}^1(0, +\infty)$  such that  $\{J(u_k)\}_{k=1}^\infty$  is bounded and  $J'(u_k) \rightarrow 0$ , as  $k \rightarrow \infty$ . We shall show that  $\{u_k\}_{k=1}^\infty$  has a convergent subsequence.

Since  $J'(u_k) \rightarrow 0$ , we see that for some  $\epsilon > 0$ , there exists  $k_0$  with  $\|J'(u_k)\| \leq \epsilon$  for  $k \geq k_0$ . Note that for  $k \geq k_0$

$$|\langle J'(u_k), u_k \rangle| \leq \epsilon \|u_k\|$$

and because we have

$$\langle J'(u_k), u_k \rangle = \int_0^{+\infty} p(t) (u_k'(t))^2 dt - \lambda \int_0^{+\infty} f(t, u_k(t)) u_k(t) dt,$$

then

$$-\lambda \int_0^{+\infty} f(t, u_k(t)) dt \geq \frac{-\epsilon}{\theta} \|u_k\| - \frac{1}{\theta} \|u_k\|^2.$$

Since  $\{J(u_k)\}_{k=1}^\infty$  is bounded, there exists a constant  $C$  such that  $|J(u_k)| \leq C$ ,  $\forall k \in N$ . Using the above estimates, we obtain

$$C \geq \left( \frac{\theta - 2}{2\theta} \right) \|u_k\|^2 - \frac{\epsilon}{2} \|u_k\|.$$

Since  $\theta > 2$ , then the sequence  $(u_k)$  is bounded in  $H_{0,p}^1(0, +\infty)$  i.e.,  $\exists M_2 > 0$  such that  $\|u_k\| \leq M_2$ ,  $\forall k \in N$ . Next, we prove that  $(u_k)$  converges strongly to some  $\bar{u}$  in  $H_{0,p}^1(0, +\infty)$ . Since  $(u_k)$  is bounded in the reflexive Banach space  $H_{0,p}^1(0, +\infty)$ , there exists a subsequence of  $(u_k)$  still denoted  $(u_k)$  such that  $(u_k)$  converges weakly to some  $\bar{u}$  in  $H_{0,p}^1(0, +\infty)$ , and  $\|\bar{u}\| \leq M_2$ . Then  $(u_k)$  converges to  $\bar{u}$  in  $C_0[0, +\infty)$  by Lemma 2.

Since  $\lim_{k \rightarrow +\infty} J'(u_k) = 0$  and  $(u_k)$  converges weakly to some  $\bar{u}$ , we get

$$\lim_{k \rightarrow +\infty} \langle J'(u_k) - J'(\bar{u}), u_k - \bar{u} \rangle \rightarrow 0 \quad (9)$$

and

$$\langle J'(u_k) - J'(\bar{u}), u_k - \bar{u} \rangle = \|u_k - \bar{u}\|^2 - \lambda \int_0^{+\infty} (f(t, u_k(t)) - f(t, \bar{u}(t))) (u_k(t) - \bar{u}(t)) dt.$$

Since  $u_k \rightarrow \bar{u}$  in  $C_0[0, +\infty)$ ,  $\forall t \in [0, +\infty)$ , then  $u_k(t) \rightarrow \bar{u}(t)$  for  $t \in [0, +\infty)$ , and since  $f$  is Carathéodory, we have  $f(t, u_k(t)) \rightarrow f(t, \bar{u}(t))$  as  $k \rightarrow +\infty$  p.p.  $t \in [0, +\infty)$ , and using  $(\mathbf{A}_1)$  we have

$$|f(t, u_k(t))| \leq M_2^\sigma M^\sigma a(t) + b(t). \quad (10)$$

Since:  $a \in L^1(0, +\infty)$ ,  $b \in L^1(0, +\infty)$ , then

$$M_2^\sigma M^\sigma a + b \in L^1(0, +\infty). \quad (11)$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k \rightarrow +\infty} \int_0^{+\infty} f(t, u_k(t)) dt = \int_0^{+\infty} f(t, \bar{u}(t)) dt. \quad (12)$$

Then (9) and (12) imply that  $\{u_k\}_{n=1}^\infty$  is strongly convergent.

**Lemma 4.** Suppose that  $(\mathbf{A}_1), (\mathbf{A}_2(\mathbf{b}))$  hold. There exists  $\lambda_1^* > 0$  such that for any  $\lambda \leq \lambda_1^*$  there exist numbers  $\rho_1, \alpha > 0$  such that

- (1) there are positive constants  $\rho_1, \alpha$  such that  $J(u) \geq \alpha$  for all  $u \in H_{0,p}^1(0, +\infty)$  with  $\|u\| = \rho_1$ .
- (2) There exists an element  $w \in H_{0,p}^1(0, +\infty)$  with  $\|w\| > \rho_1$

and such that  $J(w) < 0$ .

**proof.** Thus in (2)

$$J(u) = \frac{1}{2} \|u\|^2 - \lambda \int_0^{+\infty} F(t, u(t)) dt.$$

We first verify that  $J$  satisfies assumption (1) in Lemma 4.

Define  $\beta_1 := \frac{1}{\sigma + 1} M^\sigma \rho_1^\sigma \|a\|_{L^1} + \|b\|_{L^1}$ . Then we see that for any  $u \in \partial B(0, \rho_1)$  with  $\|u\| = \rho_1 = 2M$ , (2) and  $(\mathbf{A}_1)$  give

$$\left| \int_0^{+\infty} F(t, u(t)) dt \right| \leq M \rho_1 \beta_1.$$

Put  $\lambda_1^* = \frac{1}{\beta_1}$  and fix  $\lambda \in (0, \lambda_1^*)$ .

Moreover,

$$J(u) \geq 2M^2(1 - \lambda\beta_1),$$

this implies that assumption (1) holds if we take  $\rho_1 = 2M$  and  $\alpha = 2M^2(1 - \lambda\beta_1)$ .

Now (A2) guarantees that for some  $w_0 \in H_{0,p}^1(0, +\infty)$  with  $w_0 \neq 0$  and  $s \in R_+$ , we have the following estimation

$$J(sw_0) \leq \frac{1}{2}s^2\|w_0\|^2 - \lambda s^\theta \int_0^{+\infty} c_1(t)|w_0(t)|^\theta dt + \lambda \int_0^{+\infty} c_2(t)dt.$$

Since  $\theta > 2$  we see that  $J(sw_0) \rightarrow -\infty$  as  $s \rightarrow +\infty$ . Thus there is some  $s_0$  such that  $w = s_0 w_0$  we have  $J(w) < 0$ .

Therefore assumption (2) in Lemma 4 is also satisfied.

Now the mountain pass lemma allows us to formulate the following existence result.

**Theorem 8.** *Suppose also that (A<sub>1</sub>), (A<sub>2</sub>) hold. Then there exists  $\lambda_1^* > 0$  such that for all  $\lambda \in (0, \lambda_1^*)$ , problem (1) has at least one nontrivial solution.*

### Example 1

Consider the boundary value problem

$$\begin{cases} -(e^t u'(t))' &= \lambda \left( \frac{3u^3 + 2u}{(20 + t^2)\sqrt{u^2 + 1}} \right), \\ u(0) = u(+\infty) &= 0. \end{cases} \quad (13)$$

It can be easily checked that all conditions of Theorem 8 are satisfied with

$$f(t, u) = \frac{3u^3 + 2u}{(20 + t^2)\sqrt{u^2 + 1}}, \quad \sigma = 3, \quad \theta = 2.1,$$

$$a(t) = \frac{1}{1 + t^2}, \quad b(t) = \frac{5}{t^2 + 1}, \quad p(t) = e^t,$$

$$c_1(t) = \frac{1}{20 + t^2}, \quad c_2(t) = \frac{4}{10}e^{-t},$$

and

$$F(t, u) = \frac{u^2\sqrt{u^2 + 1}}{20 + t^2} - \frac{1}{10}e^{-t}$$

Then there is some  $\lambda_1^* > 0$  such that for all  $\lambda \in (0, \lambda_1^*)$ , problem (13) has at least one nontrivial solutions.

## 2.2 Existence of solution by the critical point theorem on a closed ball

**Theorem 9.** *Let  $B(0, \rho)$ , fixed. Assume that A<sub>1</sub>, A<sub>3</sub> are satisfied and that  $f(t, 0) \neq 0$ . Then there exist  $\lambda^* > 0$  such that for all  $\lambda \in (0, \lambda^*)$ , there exists  $u_0 \in B(0, \rho)$  with*

$$J(u_0) = \inf_{u \in B_\rho} J(u) \quad (14)$$

and such that  $u_0$  is a critical point, and thus it solves (1).



### 2.3 Existence of two solutions

Concerning the multiple solutions, we have the main result of this section where we need only that  $F$  is convex in addition to assumptions leading to a mountain pass solution.

**Theorem 10.** *Assume that  $A_1 - A_3$  are satisfied and that  $f(t, 0) \neq 0$ . Then there exist  $0 < \lambda_1^* \leq \lambda^*$  such that for all  $0 < \lambda \leq \lambda_1^*$ , problem (1) has at least two nontrivial solutions.*

### REFERENCES

- [1] A.M. Arthurs. *Complementary Variational Principles*. Clarendon Press, Oxford (1980).
- [2] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners*. Springer-Verlag, London, 2011
- [3] G. Bonanno, D. O'Regan, *A boundary value problem on the half-line via critical point methods*. Dynam. Systems Appl. 15 (2006), no. 3-4, 395–407.
- [4] B. E. Breckner, D. Repovas and C. Varga, *On the existence of three solutions for the Dirichlet problem on the Sierpinski gasket*, Nonlinear Anal. 73 (2010), no. 9, 2980–2990
- [5] H. BRÉZIS, *Analyse Fonctionnelle, Théorie et Applications*. Masson, Paris, 1983.
- [6] H. BRÉZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.
- [7] C. Corduneanu. *Integral Equations and Stability of Feedback Systems*. Cambridge University Press, Cambridge (1973). Iranian Math. Society, 43(1) 2017, 223-237.
- [8] M. Galewski., T. Moussaoui., I. Soufi, *On the existence of solutions for a boundary value problem on the half-line*, ejtde 2018, No. 12, 1 – 12.
- [9] M. Galewski, *Multiple Solutions to a Dirichlet Problem on the Sierpinski Gasket*. Taiwanese journal of mathematics 20(5) October 2016.
- [10] M. Galewski, *On a new multiple critical point theorem and some applications to anisotropic problems*. Taiwanese journal of mathematics 19(5) 2015. .
- [11] J. Mawhin, *Problèmes de Dirichlet variationnels non linéaires*, Les Presses de l'Université de Montréal, 1987.