First order Mean Field Games on networks

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Abstract

We study deterministic mean field games in which the state space is a network. Each agent controls its velocity; in particular, when it occupies a vertex, it can enter in any edge incident to the vertex. The cost is continuous in each closed edge but not necessarily globally in the network. We shall follow the Lagrangian approach studying relaxed equilibria which describe the game in terms of a probability measure on admissible trajectories. The first main result of this paper establishes the existence of a relaxed equilibrium. The proof requires the existence of optimal trajectories and a closed graph property for the map which associates to each point of the network the set of optimal trajectories starting from that point.
Each relaxed equilibrium gives rise to a cost for the agents and consequently to a value function. The second main result of this paper is to prove that such a value function solves an Hamilton-Jacobi problem on the network.

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1 Introduction

The theory of mean field games (MFGs) has been investigated more and more since the pioneering works [19, 20, 21] of Lasry and Lions. Its purpose is to study the asymptotic behaviour of differential games, either deterministic or stochastic, as the number of players tends to infinity. It is well known that, when the dynamics are deterministic, the mathematical modeling of the MFG reveals a system of PDEs coupling a continuity equation for the density of the distribution of states (forward in time) and a Hamilton–Jacobi (HJ) equation for the optimal value of a representative agent (backward in time); see [14]. The major part of the literature on deterministic MFG addresses situations when the dynamics of each player is strongly controllable and the players are not constrained to remain in a given set. Nevertheless, under a strong controllability assumption, following ideas contained in [9, 10, 15], Cannarsa *et al.* in [11, 12, 13] study some models in which the

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agents are constrained to remain in a given compact set $\bar{\Omega}$, where Ω is a regular bounded open domain of the Euclidean space. In this setting, the distribution of states can become singular so they introduce a relaxed notion of equilibrium, which is defined in a Lagrangian setting rather than with PDEs: the evolution of the game is described in terms of probability measures defined on a set of admissible trajectories instead as a time-dependent measures on the set of states. In the same vein, Mazanti and Santambrogio [23] obtain the existence of MFG equilibria for minimal time MFGs; in their problem, each agent aims to exit a given closed subset of a general compact metric space in minimal time with a bounded speed. (For some generalizations in the Euclidean setting, see also [16]). Moreover, in [3], Achdou *et al.* prove the existence of a relaxed equilibrium in the case of deterministic MFG with control on the acceleration with state constraints; in this case, the strong controllability condition does not hold and the Hamiltonian fails to be convex or coercive.

The aim of the present paper is to prove similar results for MFGs where the state space is a network. A network is a rather irregular set made of edges and vertices; here we will focus on the model case of a junction where N half-lines are glued at a point, for instance the origin, but we remark that all the results hold also for a general network with a finite number of vertices and of edges. We assume that the agents control their velocity; in particular, when an agent is in the junction, it can choose either to enter in any edge or to remain in the junction. Moreover, the costs (running and terminal) can change from edge to edge and other costs can appear for the times when the trajectories stay in the origin. All the costs depend on the distribution of agents in a nonlocal, regularizing, manner. If the time evolution of the distribution of the players is known, then each agent has to solve an optimal problem with finite time horizon, a quadratic cost on the velocity and a bounded cost which is continuous in the state inside each edge but not in the vertices.
The study of deterministic control problem on junctions, networks or stratified sets is rather recent (see [2, 18, 17, 8, 22, 25]) and this topics still displays a lot of interesting open problems. Let us recall that the aforementioned paper [23] on minimal time MFGs also applies to networks. In the recent preprint [5] Gomes *et al.* study a class of stationary MFG on networks that can be reformulated in terms of Wardrop equilibria.
The results of the present paper are the first step of a more general research project on deterministic MFGs on networks which we intend to pursue. In the first part of this paper we study the well posedness of the optimal control problem on networks (without dependence on the distribution of states): we establish that, for each initial state, there exists an optimal trajectory and that the map which associates to each point the set of optimal trajectories starting from that point fulfills a closed graph property. We recall the definition of the associated HJ problem on the network and of generalized viscosity solution. We prove that the value function of our optimal control problem is a generalized viscosity solution. From the point of view of the optimal control, the HJ at the junction is motivated by two features of the problem. When it is in the junction, the agent may remain there or enter in some edge (in particular, it can move only in the directions pointing *inward* some edge and it cannot move in the opposite direction) and it has to consider also a cost that appears only at the junction.

The second part deals with equilibria for MFG constrained in a network. The main result of this paper is the existence of such a MFG equilibrium; the proof relies on the Kakutani's fixed point theorem and on a closed graph property. With such an equilibrium at hand, the costs for the agents are well defined so we can introduce the value function. Using the results obtained in the first part, we obtain that this value function is a generalized viscosity solution to the associated HJ problem.

Our work is a first attempt to prove the existence of a MFG equilibrium in the model case of a junction with costs depending separately on the control (velocity) and on the state, which may change from edge to edge and with another cost for the sole vertex.

We point out that in this article we have no formulation of the distribution of the states of the agents in terms of a differential equation. Let us just observe that, in our setting, singular measures (as Dirac measures) may appear; see Example 3.1 below.

In a forthcoming paper we will prove similar results for optimal control problems (and MFGs) where the agents are constrained to remain in a network and control only their acceleration, also in the case when the control set is bounded.

This paper is organized as follows. In the rest of this section we introduce the geometry of a junction and some notations. Section 2 is devoted to optimal control problem on the junction; in particular, we establish the existence of an optimal trajectory for every starting point, a closed graph property for the map that associates to each point the set of optimal trajectories and some properties of the value function (mainly, that it solves a HJ problem on the junction). Section 3 concerns deterministic MFGs on the junction; taking advantage of the results of previous section, we prove the existence of a MFG equilibrium and that the value function of the associated optimal control problem solves a HJ problem.

1.1 Notations

The geometry of a junction. We shall adopt the notations of the paper [4]. We focus on the model case of a junction in $\mathbb{R}^d$ with N ($N > 1$) semi-infinite straight edges, denoted by $(J_i)_{i=1,\dots,N}$. The edge J_i is the closed half-line $\mathbb{R}^+ e_i$. The vectors e_i are two by two distinct unit vectors in $\mathbb{R}^d$. The half-lines J_i are glued at the origin O to form the *junction* $\mathcal{G}$:

$$\mathcal{G} = \bigcup_{i=1}^{N} J_i.$$

The geodetic distance $d(x, y)$ between two points x, y of $\mathcal{G}$ is

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same edge } J_i \\ |x| + |y| & \text{if } x, y \text{ belong to different edges } J_i \text{ and } J_j. \end{cases}$$

If φ is a function on J_i, with a small abuse of notations, we still denote φ the function $\mathbb{R}^+ \ni \bar{x} \mapsto \varphi(\bar{x} e_i)$ defined on $\mathbb{R}^+$.

Gradient of a function. We denote $C^1(\mathcal{G})$ the set of continuous functions $\varphi \in C(\mathcal{G})$ such that, for every $i = 1, \dots, N$ their restriction to the edge J_i, $\varphi_{|J_i}$ belongs to $C^1(\mathbb{R}^+)$; moreover, for $\varphi \in C^1(\mathcal{G})$, we set

$$D\varphi(x) = \begin{cases} D\varphi_{|J_i} & \text{if } x \in J_i \setminus \{O\} \\ \left(D\varphi_{|J_1}, \dots, D\varphi_{|J_N}\right) & \text{if } x = O; \end{cases} \tag{1.1}$$

note that $D\varphi(x)$ is 1-dimensional when the point x is "inside" some edge while it is N-dimensional when the point x coincides with the vertex O. In a similar manner, we denote $C^1(\mathcal{G} \times [0, T])$ the set of continuous functions $\varphi \in C(\mathcal{G} \times [0, T])$ such that their restriction to each edge $\varphi_{|J_i \times [0,T]}$ belongs to $C^1(J_i \times [0, T])$.

2 The optimal control problem

For $T > 0$, we consider optimal control problems with horizon T which have different running costs in the edges and at the vertex. We are going to describe the assumptions on the dynamics and costs in each edge J_i. The sets of controls are denoted by A_i and the system is driven by a dynamics $\tilde{f}_i$ and the running cost is given by $\tilde{\ell}_i$. For the sake of simplicity, we shall focus our study on the case where the agent chooses directly its velocity and the cost depend separately on the control, namely: $\tilde{f}_i = \alpha$ and $\tilde{\ell}_i(x,t,\alpha) = \ell_i(x,t) + |\alpha|^2/2$. However, let us stress that all our calculations may be easily extended to more general settings such as: networks (instead of a simple junction), running costs which are strongly convex and coercive in the control and depend separately on the control and dynamics strongly controllable, at most linear with respect to the control and with unbounded velocity. Our main assumptions are as follows

[H0] For $i = 0, \ldots, N$, in order to avoid confusion among the control sets, we set $A_i = \{i\} \times \mathbb{R}$; hence, the sets A_i are disjoint. We set $A := \bigcup_{i=0}^N A_i$. For $a = (i, \bar{a}) \in A$, we denote $|a| = |\bar{a}|$. We identify a and $\bar{a}e_i$ so, with some abuse of notations, we shall write $\bar{a}e_i$ instead of $(i, \bar{a})$ and viceversa.
We set $f_i : J_i \times A_i \to \mathbb{R}$, with $f_i(x,a) = \bar{a}$ for $a = (i, \bar{a})$ and we will use the notation $F_i(x)$ for the set $\{f_i(x,a)e_i, a \in A_i\} = \mathbb{R}e_i$ for $x \in J_i$ $(i = 1, \ldots, N)$ and $F_0(O) = \{0\}$.

[H1] For $i = 1, \ldots, N$, the functions $\ell_i : J_i \times [0,T] \to \mathbb{R}$ are continuous and bounded functions. We introduce $\ell_* : [0,T] \to \mathbb{R}$ continuous and bounded.
For $i = 1, \ldots, N$, the functions $g_i : J_i \to \mathbb{R}$ are continuous and bounded functions. Let g_* be a fixed number.

We now provide a general version of Filippov implicit function lemma, which will be useful to prove Theorem 2.2 below. For the proof, we refer the reader to [24].

Theorem 2.1 *Let I be an interval of $\mathbb{R}$ and $\gamma : I \to \mathbb{R}^d \times \mathbb{R}^d$ be a measurable function. Let A be a metric space. Let K be a closed subset of $\mathbb{R}^d \times A$ and $\Psi : K \to \mathbb{R}^d \times \mathbb{R}^d$ be continuous. Assume that $\gamma(I) \subset \Psi(K)$, then there is a measurable function $\Phi : I \to K$ with*

$$\Psi \circ \Phi(t) = \gamma(t) \quad \text{for a.a. } t \in I.$$

Let us introduce the set

$$M = \{(x,a);\ x \in \mathcal{G}, \quad a \in A_i \text{ if } x \in J_i \backslash \{O\}, \text{ and } a \in A \text{ if } x = O\} \tag{2.1}$$

and note that it is closed; moreover, since the sets A_i are disjoint, for each $(x,a) \in M$ there exist a unique $i \in \{1, \ldots, N\}$ and a unique $\bar{a} \in \mathbb{R}$ such that $(x,a) = (x,(i,\bar{a}))$. We also define the function f on M by

$$\forall (x,a) \in M, \qquad f(x,a) = \begin{cases} f_i(x,a)e_i & \text{if } x \in J_i \backslash \{O\}, \\ f_i(O,a)e_i & \text{if } x = O \text{ and } a \in A_i,\ i \neq 0, \\ 0 & \text{if } x = O \text{ and } a \in A_0. \end{cases}$$

The function f is continuous on M because the sets A_i are disjoint. Let $\tilde{F}(x)$ be defined by

$$\tilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \in J_i \backslash \{O\} \\ \cup_{i=0}^N F_i(O) & \text{if } x = O. \end{cases}$$

For $x \in \mathcal{G}$, the set of admissible curves starting from x is

$$Y_{x,0} = \left\{ y_x \in W^{1,2}([0,T];\mathcal{G}) \; : \left| \begin{array}{ll} \dot{y}_x(t) \in \tilde{F}(y_x(t)), & \text{for a.e. } t > 0, \\ y_x(0) = x, & \end{array} \right. \right\}. \tag{2.2}$$

Theorem 2.2 *Assume [H0] and [H1]. Then*

1. *For any $x \in \mathcal{G}$, $Y_{x,0}$ is nonempty.*
2. *For any $x \in \mathcal{G}$, for any $y_x \in Y_{x,0}$, there exists a measurable function $\Phi : [0,T] \to M$, $\Phi = (\phi_1, \phi_2)$ such that*

$$\begin{array}{c} \phi_2 = (i, \bar{\phi}_2), \text{ with } \bar{\phi}_2 \in \mathbb{R}, \text{ when } \phi_1 \in J_i \setminus \{O\} \\ (y_x(s), \dot{y}_x(s)) = (\phi_1(s), f(\phi_1(s), \phi_2(s))), \quad \text{for a.e. } s, \end{array}$$

   which means in particular that y_x is a continuous representation of ϕ_1

3. *Almost everywhere in $[0,T]$,*

$$\dot{y}_x(s) = \sum_{i=1}^{N} 1_{\{y_x(s) \in J_i \setminus \{O\}\}} \bar{\phi}_2(s) e_i.$$

4. *Almost everywhere on $\{s : y_x(s) = O\}$, $f(O, \phi_2(s)) = 0$.*

Proof. The proof of point 1 is easy, because $0 \in \tilde{F}(x)$ for every $x \in \mathcal{G}$.
The proof of point 2 is a consequence of Theorem 2.1, with $K = M$, $I = [0,T]$, $\gamma(s) = (y_x(s), \dot{y}_x(s))$ and $\Psi(x,a) = (x, f(x,a))$.
From point 2, we deduce

$$\dot{y}_x(s) = \sum_{i=1}^{N} 1_{\{y_x(s) \in J_i \setminus \{O\}\}} \bar{\phi}_2(s) e_i + 1_{\{y_x(s) = O\}} f(O, \phi_2(s)),$$

and from Stampacchia's theorem, $f(O, \phi_2(s)) = 0$ almost everywhere in $\{s : y_x(s) = O\}$.
This yields points 3 and 4. $\square$

Remark 2.1 *It is worth noticing that in Theorem 2.2, a solution y_x can be associated with several control laws ϕ_2 which may be different even on sets with positive measure. Actually, for a.e. $s \in \{s \in [0,T] \mid y_x(s) \in J_i \setminus \{O\}\}$, the control $\phi_2(s)$ is uniquely defined as $\phi_2(s) = \dot{y}_x(s)$ and belongs to $\mathbb{R}e_i$ (for $i = 1, \ldots, N$). On the other hand, for a.e. $s \in \{s \in [0,T] \mid y_x(s) = O\}$, by Stampacchia theorem, the control $\phi_2(s)$ is null and it can be arbitrarily chosen in any A_i, for $i = 0, \ldots, N$.*

For any $x \in \mathcal{G}$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we introduce the set of admissible *trajectories* (namely, couples of controls and curves) on the interval $[t_1, t_2]$ which start from x at t_1:
(2.3)

$$\Gamma_{t_1,t_2}[x] = \left\{ \begin{array}{ll} (y_x, \alpha) \in L^2([t_1,t_2], M) : & y_x \in W^{1,2}([t_1,t_2];\mathcal{G}), \\ & y_x(s) = x + \displaystyle\int_{t_1}^{s} f(y_x(\tau), \alpha(\tau)) d\tau \quad \text{in } [t_1,t_2] \end{array} \right\};$$

for simplicity, when $t_2 = T$, we just write $\Gamma_{t_1}[x]$ instead of $\Gamma_{t_1,T}[x]$ and, when $t_2 = T$ and $t_1 = 0$, we drop the subscript: $\Gamma[x] = \Gamma_{0,T}[x]$. Moreover, we define the set of all admissible trajectories starting at time $t = 0$ as follows:

$$\Gamma := \bigcup_{x\in\mathcal{G}} \Gamma[x]. \tag{2.4}$$

Remark 2.2 *The concatenation of two admissible trajectories is still an admissible trajectory. In other words, for $0 \le t_1 \le t_2 \le t_3 \le T$ and $x \in \mathcal{G}$, if $(y_1, \alpha_1) \in \Gamma_{t_1,t_2}[x]$ and $(y_2, \alpha_2) \in \Gamma_{t_2,t_3}[y_1(t_2)]$, the trajectory $(\tilde{y}, \tilde{\alpha})$ defined by*

$$\tilde{y}(s) := \begin{cases} y_1(s) & \text{for } s \in [t_1, t_2] \\ y_2(s) & \text{for } s \in [t_2, t_3] \end{cases} \qquad \text{and} \qquad \tilde{\alpha}(s) := \begin{cases} \alpha_1(s) & \text{for } s \in [t_1, t_2] \\ \alpha_2(s) & \text{for } s \in [t_2, t_3] \end{cases}$$

belongs to $\Gamma_{t_1,t_3}[x]$.

The cost functional. The cost associated to the trajectory $(y_x, \alpha) \in \Gamma_t[x]$ is

$$\begin{aligned} J_t(x; (y_x, \alpha)) = \int_t^T \left[\sum_{i=1}^N \ell_i(y_x(\tau), \tau) \mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau) \mathbf{1}_{y_x(\tau)=O} \right] d\tau \\ + \int_t^T \frac{|\alpha(\tau)|^2}{2}\, d\tau + g(y_x(T)) \end{aligned} \tag{2.5}$$

where (recall that g_* and ℓ_* are introduced in assumption (H_1))

$$\begin{aligned} \ell_O(\tau) &= \min\{\ell_*(\tau), \min_{i=1,\dots,N} \ell_i(O, \tau)\} \\ g(y) &= \sum_{i=1}^N g_i(y) \mathbf{1}_{y\in J_i\setminus\{O\}} + \min\{g_*, \min_{i=1,\dots,N} g_i(O)\} \mathbf{1}_{y=O} \end{aligned} \tag{2.6}$$

For brevity, we introduce

$$L(x, t) := \sum_{i=1}^N \ell_i(x, t) \mathbf{1}_{x\in J_i\setminus\{O\}} + \ell_O(t) \mathbf{1}_{x=O} \qquad \forall (x, t) \in \mathcal{G} \times [0, T] \tag{2.7}$$

so to write

$$J_t(x; (y_x, \alpha)) = \int_t^T \left(L(y_x(\tau), \tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau + g(y_x(T)).$$

Remark 2.3 *Using the same arguments, we can also tackle the case of a cost of the form*

$$\begin{aligned} J_t(x; (y_x, \alpha)) = \int_t^T \left[\sum_{i=1}^N \ell_i(y_x(\tau), \tau) \mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} + \sum_{i=0}^N \ell_i(O, \tau) \mathbf{1}_{y_x(\tau)=O, \alpha(\tau)\in A_i} \right] d\tau \\ + \int_t^T \frac{|\alpha(\tau)|^2}{2}\, d\tau + g(y_x(T)) \end{aligned}$$

where $\ell_0(O, \tau) = \ell_*(\tau)$.

The value function. The value function of the optimal control problem is

$$u(x,t) = \inf_{(y,\alpha)\in\Gamma_t[x]} J_t(x;(y,\alpha)). \tag{2.8}$$

We denote

$$\Gamma_t^{\rm opt}[x] = \{(y,\alpha)\in\Gamma_t[x] \ :\ J_t(x;(y,\alpha)) = \min_{(\hat y,\hat\alpha)\in\Gamma_t[x]} J_t(x;(\hat y,\hat\alpha))\} \tag{2.9}$$

and, for simplicity, when $t=0$ we drop the subscript: $\Gamma^{\rm opt}[x]=\Gamma_t^{\rm opt}[x]$.

Remark 2.4 *The optimal controls are uniformly bounded in L^2 and the value function u is bounded. Indeed, by the boundedness of the costs ℓ_i and of g, testing the cost with the trajectory given by the control $\alpha\equiv 0$, we deduce that there exists a constant C such that, for every $(y,\alpha)\in\Gamma_t^{\rm opt}[x]$, there holds $\|\alpha\|_2\le C$. Again by the boundedness of the costs, we deduce that u is bounded.*

Remark 2.5 *The restriction of an optimal trajectory is still an optimal trajectory: for $(y,\alpha)\in\Gamma_t^{\rm opt}[x]$, $(y_{|[\bar t,T]},\alpha_{|[\bar t,T]})\in\Gamma_{\bar t}^{\rm opt}[y(\bar t)]$ for every $\bar t\in[t,T]$. Indeed, assume by contradiction that there exists a trajectory $(\bar y,\bar\alpha)\in\Gamma_{\bar t}^{\rm opt}[y(\bar t)]$ such that $J_{\bar t}(y(\bar t);(\bar y,\bar\alpha)) < J_{\bar t}(y(\bar t);(y_{|[\bar t,T]},\alpha_{|[\bar t,T]}))$. Then, by Remark 2.2, the concatenation $(\tilde y,\tilde\alpha)$ of (y,α) with $(\bar y,\bar\alpha)$, defined by*

$$\tilde y(s) = \begin{cases} y(s) & \text{for } s\in[t,\bar t] \\ \bar y(s) & \text{for } s\in[\bar t,T]\end{cases}, \qquad \tilde\alpha(s) = \begin{cases} \alpha(s) & \text{for } s\in[t,\bar t] \\ \bar\alpha(s) & \text{for } s\in[\bar t,T]\end{cases}$$

belongs to $\Gamma_t[x]$ and consequently there holds

$$\begin{aligned} u(x,t) &= J_t(x;(y,\alpha)) = \int_t^{\bar t}\left(L(y(\tau),\tau)+\frac{|\alpha(\tau)|^2}{2}\right)d\tau + J_{\bar t}(y(\bar t);(y_{|[\bar t,T]},\alpha_{|[\bar t,T]})) \\ &> \int_t^{\bar t}\left(L(y(\tau),\tau)+\frac{|\alpha(\tau)|^2}{2}\right)d\tau + J_{\bar t}(y(\bar t);(\bar y,\bar\alpha)) = J_t(x;(\tilde y,\tilde\alpha)) \end{aligned}$$

which contradicts the optimality of (y,α).

2.1 Existence of optimal trajectory

Let us now establish that for every starting point $(x,t)\in\mathcal G\times[0,T]$ there exists an optimal trajectory.

Proposition 2.1 *For each point $(x,t)\in\mathcal G\times[0,T]$, there exists an optimal trajectory, namely there exists $(y_x,\alpha)\in\Gamma_t[x]$ such that $u(x,t)=J_t(x;(y_x,\alpha))$. In particular, $\Gamma^{\rm opt}[x]\neq\emptyset$.*

Proof. Fix $(x,t)\in\mathcal G\times[0,T]$ and consider a minimizing sequence $(y^n,\alpha^n)\in\Gamma_t[x]$, namely such that $u(x,t)=\lim_{n\to\infty}J_t(x;(y^n,\alpha^n))$. By Remark 2.4, there exists a constant C, independent of n, such that

$$\int_t^T \frac{|\alpha^n(\tau)|^2}{2}\,d\tau \le C. \tag{2.10}$$

Moreover, from

$$y^n(s) = x + \int_t^s \sum_{i=1}^N 1_{\{y^n(\tau)\in J_i\setminus\{O\}\}}\bar{\alpha}^n(\tau)e_i\, d\tau, \tag{2.11}$$

we infer that the curves y^n are uniformly bounded and uniformly 1/2-Hölder continuous.

Possibly passing to subsequences (that we still denote by α^n and y^n), there exist a $\alpha \in L^2([t,T];\mathbb{R}^d)$ and a curve $y_x \in C^{1/2}([t,T];\mathbb{R}^d)$ such that: $\sum_{i=1}^N 1_{\{y^n(\cdot)\in J_i\setminus\{O\}\}}\bar{\alpha}^n(\cdot)e_i$ converge to α in the weak topology of $L^2([t,T];\mathbb{R}^d)$ and y_n uniformly converge to y_x. In particular, letting $n\to\infty$ in (2.11), we obtain

$$y_x \in W^{1,2}([t,T];\mathbb{R}^d) \quad \text{with } y_x(s) = x + \int_t^s \alpha(\tau)\, d\tau \in \mathcal{G}. \tag{2.12}$$

We now claim

$$(y_x, \alpha) \in \Gamma_t[x]. \tag{2.13}$$

In order to prove this property, it suffices to prove that α is an admissible control namely that $(y_x(s), \alpha(s))$ belongs to M for a.e. $s \in (t,T)$. To this end, we split our arguments according to the fact that $y_x(s)$ coincides or not with O.
Consider $s \in (t,T)$ such that $y_x(s) \in J_i \setminus \{O\}$ for some $i = 1,\dots,N$. Since the y^n are uniformly 1/2-Holder continuous and uniformly converge to y_x, we deduce that, for $\varepsilon > 0$ sufficiently small and for any n sufficiently large, there holds

$$y^n(\tau) \in J_i \setminus \{O\} \qquad \forall \tau \in (s-\varepsilon, s+\varepsilon).$$

In particular, for n sufficiently large, $\alpha^n(\tau) = \bar{\alpha}^n(\tau)e_i$ for $\tau \in (s-\varepsilon, s+\varepsilon)$. Letting $n\to\infty$, we conclude that $\alpha(\tau)$ is parallel to e_i for $\tau \in (s-\varepsilon, s+\varepsilon)$.
Consider $s \in E := \{s \in (t,T) : y_x(s) = O\}$. By (2.12), from Stampacchia's theorem, we get $\alpha = 0$ a.e. in $s \in E$. So, for instance, we can write $\alpha = 0e_1$ in E without modifying the curve y_x. Hence, our claim (2.13) is completely proved.

Let us now prove that (y_x, α) is an optimal trajectory namely that there holds

$$u(x,t) = J_t(x; (y_x, \alpha)). \tag{2.14}$$

Indeed, we have

$$u(x,t) \quad = \quad \lim_{n\to\infty}\left[\int_t^T \frac{|\alpha^n(\tau)|^2}{2} d\tau + \sum_{i=1}^5 I_i\right] \tag{2.15}$$

where

$$\begin{aligned}
I_1 &= \int_t^T \sum_{i=1}^N \ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} d\tau\\
I_2 &= \int_t^T \sum_{i=1}^N \ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y_x(\tau)\in \mathcal{G}\setminus J_i} d\tau\\
I_3 &= \int_t^T \sum_{i=1}^N \ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y_x(\tau)=O} d\tau\\
I_4 &= \int_t^T \ell_O(\tau)\mathbf{1}_{y^n(\tau)=O} d\tau\\
I_5 &= g(y^n(T)).
\end{aligned}$$

We shall study separately the contributions in the right hand side of (2.15). By standard theory, the convergence in the weak topology of $L^2([t,T];\mathbb{R}^d)$ entails

$$\int_t^T \frac{|\alpha(\tau)|^2}{2} d\tau \leq \liminf_{n\to\infty} \int_t^T \frac{|\alpha^n(\tau)|^2}{2} d\tau. \tag{2.16}$$

In I_1, the uniform convergence of y^n to y_x as $n \to \infty$ and the continuity of ℓ_i ensure that, for each i, there holds

$$\ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} \to \ell_i(y_x(\tau),\tau)\mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} \qquad \text{as } n\to\infty;$$

since the ℓ_i's are bounded, the dominated convergence theorem yields

$$I_1 \to \int_t^T \sum_{i=1}^N \ell_i(y_x(\tau),\tau)\mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} d\tau \qquad \text{as } n\to\infty. \tag{2.17}$$

In I_2, again the uniform convergence of y^n to y_x and the continuity of ℓ_i ensure that the integrand tends to zero as $n \to \infty$; again by the dominated convergence theorem, we infer:

$$I_2 \to 0 \qquad \text{as } n\to\infty. \tag{2.18}$$

Let us now consider the term I_5 and we split our arguments according to the fact that $y_x(T)$ coincides or not coincides with O. If $y_x(T) \in J_i \setminus \{O\}$ for some $i \in \{1,\dots,N\}$ then, the uniform convergence of y^n to y_x and the continuity of g_i entail $g(y^n(T)) = g_i(y^n(T)) \to g_i(y_x(T)) = g(y_x(T))$ as $n\to\infty$. If $y_x(T) = O$, again by the uniform convergence of y^n to y_x and by the definition of g in (2.6), for any $\varepsilon > 0$, we get $g(y^n(T)) \geq g(O) - \varepsilon = g(y_x(T)) - \varepsilon$ for n sufficiently large. In both cases, we obtain

$$\liminf_{n\to\infty} I_5 \geq g(y_x(T)). \tag{2.19}$$

On the other hand, we observe

$$\begin{aligned} I_3 + I_4 &= \int_t^T \left[\sum_{i=1}^N \ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y^n(\tau)=O}\right]\mathbf{1}_{y_x(\tau)=O} d\tau \\ &\quad + \int_t^T \ell_O(\tau)\mathbf{1}_{y^n(\tau)=O}\mathbf{1}_{y_x(\tau)\neq O} d\tau. \end{aligned}$$

We note $\mathbf{1}_{y^n(\cdot)=O}\mathbf{1}_{y_x(\cdot)\neq O} \to 0$ a.e. as $n\to\infty$; hence, by the dominated convergence theorem, we obtain

$$\int_t^T \ell_O(\tau)\mathbf{1}_{y^n(\tau)=O}\mathbf{1}_{y_x(\tau)\neq O} d\tau \to 0 \quad \text{as } n\to\infty.$$

Assume for the moment, that

$$\begin{aligned} \liminf_{n\to\infty} \int_t^T \left[\sum_{i=1}^N \ell_i(y^n(\tau),\tau)\mathbf{1}_{y^n(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y^n(\tau)=O}\right]\mathbf{1}_{y_x(\tau)=O} d\tau \\ \geq \int_t^T \ell_O(\tau)\mathbf{1}_{y_x(\tau)=O} d\tau. \end{aligned} \tag{2.20}$$

Replacing the relations (2.16)-(2.20) in (2.15), we obtain

$$u(x,t) \geq \int_t^T \left[\frac{|\alpha(\tau)|^2}{2} + \sum_{i=1}^N \ell_i(y_x(\tau),\tau)\mathbf{1}_{y_x(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y_x(\tau)=O} \right] d\tau + g(y_x(T))$$

which is equivalent to our statement (2.14).

It remains to prove inequality (2.20). To this end, we note that the set E is compact. Consider any $\varepsilon > 0$. Since y^n uniformly converge to y_x, there holds $\|y^n - y_x\|_{L^\infty(E)} \leq \varepsilon$ for n sufficiently large. Hence, by the continuity of ℓ_i, we have that, for n sufficiently large, there holds

$$\ell_i(y^n(\tau),\tau) > \ell_O(\tau) - \varepsilon \quad \forall \tau \in E$$

which easily implies inequality (2.20). $\square$

2.2 Closed graph property

In this subsection, we establish that $\Gamma^{\text{opt}}[x]$, defined in (2.9), verifies the closed graph property.

Proposition 2.2 *Fix $x \in \mathcal{G}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \mathcal{G}$ and $x_n \to x$ as $n \to \infty$. Consider $(y_n,\alpha_n) \in \Gamma^{\text{opt}}[x_n]$ for any $n \in \mathbb{N}$. Assume that, as $n \to \infty$, y_n uniformly converge to a curve y. Then, the curve y belongs to $Y_{x,0}$ (defined in* (2.2)*). Moreover, there exists a measurable function α such that (y,α) belongs to $\Gamma^{\text{opt}}[x]$, defined in* (2.9).

To this end, we first establish a result on the approximation of admissible trajectories whose proof is postponed at the end of this subsection.

Lemma 2.1 *Fix $x \in \mathcal{G}$ and $(y,\alpha) \in \Gamma[x]$; consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in \mathcal{G}$ such that $\delta_n := d(x_n,x) \to 0$ as $n \to \infty$. Then, there exists a sequence $\{(y_n,\alpha_n)\}_{n\in\mathbb{N}}$ such that, for any $n \in \mathbb{N}$, $(y_n,\alpha_n) \in \Gamma[x_n]$,*

$$\begin{array}{llll} & (i) & \sup_{[0,T]} d(y_n(\cdot),y(\cdot)) \leq \delta_n + \|\alpha\|_2\sqrt{\delta_n} \qquad \textit{with} \qquad y_n(T) = y(T) \\ (2.21) & (ii) & \|\alpha_n\|_2^2 \leq \|\alpha\|_2^2 + \delta_n\left(1 + \frac{\|\alpha\|_2^2}{T-\delta_n}\right) \\ & (iii) & \lim\limits_{n\to\infty} J_0(x_n;(y_n,\alpha_n)) = J_0(x;(y,\alpha)). \end{array}$$

Proof of Proposition 2.2. Consider x, x_n, (y,α) and y as in the statement. We have to prove that there exists a control α such that

i) (y,α) belongs to $\Gamma[x]$, namely it is admissible for x,

ii) (y,α) is optimal for J_0, namely: $J_0(x,(y,\alpha)) \leq J_0(x,(\hat{y},\hat{\alpha}))$ for every $(\hat{y},\hat{\alpha}) \in \Gamma[x]$.

Fix any $(\hat{y},\hat{\alpha}) \in \Gamma[x]$. Lemma 2.1 ensures that there exists a sequence $\{(\hat{y}_n,\hat{\alpha}_n)\}_{n\in\mathbb{N}}$ such that $(\hat{y}_n,\hat{\alpha}_n) \in \Gamma[x_n]$ and

$$(2.22) \qquad \begin{array}{c} \hat{y}_n \to \hat{y} \text{ uniformly in } [0,T] \text{ as } n \to \infty, \qquad \|\hat{\alpha}_n\|_2 \leq \|\hat{\alpha}\|_2 + o_n(1), \\ \limsup\limits_{n\to\infty} J_0(x_n;(\hat{y}_n,\hat{\alpha}_n)) \leq J_0(x;(\hat{y},\hat{\alpha})) \end{array}$$

where $o_n(1)$ is a sequence such that $\lim_n o_n(1) = 0$. On the other hand, the optimality of (y_n,α_n) yields

$$(2.23) \qquad J_0(x_n;(y_n,\alpha_n)) \leq J_0(x_n;(\hat{y}_n,\hat{\alpha}_n)).$$

From the last relations we deduce that $J_0(x_n;(y_n,\alpha_n))$ are uniformly bounded and, in particular that there exists a constant C, independent of n, such that $\int_t^T |\alpha_n(\tau)|^2\,d\tau \le C$. Hence, repeating the same arguments as those in the proof of Proposition 2.1 (in particular, those of (2.13)), we deduce that $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to some control α in the weak topology of $L^2([0,T],\mathbb{R}^d)$ and $(y,\alpha)\in\Gamma[x]$. Hence, the proof of point (i) is achieved. Performing the $\liminf_n$ in (2.23) and using (2.22), we also deduce $J_0(x,(y,\alpha)) \le J_0(x,(\hat{y},\hat{\alpha}))$. By the arbitrariness of $(\hat{y},\hat{\alpha})\in\Gamma[x]$, the proof of point (ii) is achieved. □

Proof of Lemma 2.1. Without any loss of generality (possibly passing to a subsequence that we still denote $\{x_n\}$), we assume that, for n sufficiently large, all the points x and x_n belong to the same edge (for simplicity, say the first one J_1); indeed, for $x = O$, we can argue edge by edge using the fact that the edges incident in O are finite. We write: $x = \bar{x}e_1$, $x_n = \bar{x}_n e_1$ for $\bar{x},\bar{x}_n \in \mathbb{R}^+$. We note $\delta_n = d(x,x_n) = |\bar{x}-\bar{x}_n|$. We now introduce a control α_n such that the corresponding curve y_n is admissible (namely, it remains in the network). To this end, we introduce the control

$$\alpha_n(s) = \begin{cases} \left\{\begin{array}{ll} e_1 & \text{if } \bar{x}_n \le \bar{x} \\ -e_1 & \text{if } \bar{x}_n > \bar{x} \end{array}\right\} & \text{for } s\in[0,\delta_n] \\ \frac{T}{T-\delta_n}\alpha\left((s-\delta_n)\frac{T}{T-\delta_n}\right) & \text{for } s\in(\delta_n,T] \end{cases}$$

(note that here the structure $A_i = \{i\}\times\mathbb{R}$ plays a crucial role) and we denote y_n the curve starting from the point x_n and obeying to the control α_n, namely

$$y_n(s) = x_n + \int_t^s \alpha_n(\tau)\,d\tau.$$

We observe that, for $s\in[0,\delta_n]$ we have

$$y_n(s) = x_n + (x-x_n)d(x,x_n)^{-1}s;$$

in particular, $y_n(\delta_n) = x$. For $s\in[\delta_n,T]$, by definition of α_n and a change of variable, we get

$$\begin{aligned} y_n(s) &= x + \int_{\delta_n}^s \frac{T}{T-\delta_n}\alpha\left((\tau-\delta_n)\frac{T}{T-\delta_n}\right)d\tau = x + \int_0^{(s-\delta_n)\frac{T}{T-\delta_n}} \alpha(\tau)\,d\tau \\ &= y\left((s-\delta_n)\frac{T}{T-\delta_n}\right); \end{aligned} \tag{2.24}$$

in particular, (y_n,α_n) is an admissible trajectory with

$$y_n(T) = y(T). \tag{2.25}$$

Note that y_n is the trajectory that starts in x_n, moves with speed 1 towards x, reaches the point x at time δ_n (clearly, in this time interval it always remains in the edge e_1) and, from time δ_n follows the trajectory y with a time rescaling such that $y_n(T) = y(T)$.

Let us now estimate $d(y(s),y_n(s))$. For $s\in[0,\delta_n]$, we have

$$\begin{aligned} d(y(s),y_n(s)) &\le d(y_n(s),x) + d(y(s),x) \le (\delta_n - s) + \int_0^s |\bar{\alpha}(\tau)|\,d\tau \\ &\le (\delta_n - s) + \|\alpha\|_2\sqrt{s} \end{aligned}$$

where the last inequality is due to Hölder inequality.
For $s \in [\delta_n, T]$, by (2.24), we have

$$\begin{aligned} d(y(s), y_n(s)) &= d\left(y(s), y\left((s-\delta_n)\frac{T}{T-\delta_n}\right)\right) \leq \int_{(s-\delta_n)\frac{T}{T-\delta_n}}^{s} |\bar{\alpha}(\tau)|\, d\tau \\ &\leq \|\alpha\|_2 \sqrt{\delta_n}\sqrt{\frac{T-s}{T-\delta_n}}. \end{aligned}$$

The last two inequalities easily imply the bound in (2.21)-(i).

Let us now prove the bound in (2.21)-(ii). By definition of α_n, we have

$$\begin{aligned} \|\alpha_n\|_2^2 &= \delta_n + \int_{\delta_n}^{s} \left(\frac{T}{T-\delta_n}\right)^2 \alpha\left((\tau-\delta_n)\frac{T}{T-\delta_n}\right)^2 d\tau = \delta_n + \int_0^T \frac{T}{T-\delta_n}\alpha(\tau)^2\, d\tau \\ &= \delta_n + \|\alpha\|_2^2 + \frac{\delta_n}{T-\delta_n}\|\alpha\|_2^2 \end{aligned}$$

which easily implies the bound of (2.21)-(ii).

We now prove (2.21)-(iii). Relation (2.25) entails: $g(y_n(T)) = g(y(T))$; hence, there holds

$$J_0(x_n; (y_n, \alpha_n)) - J_0(x; (y, \alpha)) = \sum_{i=1}^{4} I_i \tag{2.26}$$

where

$$\begin{aligned} I_1 &= \int_0^{\delta_n} \left[\sum_{i=1}^{N} \ell_i(y_n(\tau), \tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y_n(\tau)=O}\right] d\tau \\ I_2 &= \frac{\|\alpha_n\|_2^2 - \|\alpha\|_2^2}{2} \\ I_3 &= \int_{\delta_n}^{T} \left[\sum_{i=1}^{N} \ell_i(y_n(\tau), \tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y_n(\tau)=O}\right] d\tau \\ I_4 &= -\int_0^{T} \left[\sum_{i=1}^{N} \ell_i(y(\tau), \tau)\mathbf{1}_{y(\tau)\in J_i\setminus\{O\}} + \ell_O(\tau)\mathbf{1}_{y(\tau)=O}\right] d\tau. \end{aligned}$$

By the boundedness of ℓ_i, for $K = \sum_1^N \|\ell_i\|_\infty + \|\ell_*\|_\infty$, we have

$$|I_1| \leq K\delta_n.$$

Relation (2.21)-(ii) entails

$$|I_2| \leq \frac{\delta_n}{2}\left(1 + \frac{\|\alpha\|_2^2}{T-\delta_n}\right).$$

Moreover, taking advantage of relation (2.24) and our choice of α_n, we have

$$\begin{aligned} I_3 = \int_{\delta_n}^{T} \Bigg[\sum_{i=1}^{N} \ell_i\left(y\left((\tau-\delta_n)\frac{T}{T-\delta_n}\right), \tau\right) \mathbf{1}_{y\left((\tau-\delta_n)\frac{T}{T-\delta_n}\right)\in J_i\setminus\{O\}} \\ + \ell_O(\tau)\mathbf{1}_{y\left((\tau-\delta_n)\frac{T}{T-\delta_n}\right)=O}\Bigg] d\tau. \end{aligned}$$

Moreover, by a change of variable in the last integral, we obtain

$$I_3 = \int_0^T \left[\sum_{i=1}^N \ell_i \left(y(\xi), \frac{T-\delta_n}{T}\xi + \delta_n \right) \mathbf{1}_{y(\xi)\in J_i\setminus\{O\}} \right.$$
$$\left. + \ell_O \left(\frac{T-\delta_n}{T}\xi + \delta_n \right) \mathbf{1}_{y(\xi)=O} \right] \left(1 - \frac{\delta_n}{T} \right) d\xi.$$

Therefore, let $\mathcal{G}_1$ a bounded subset of $\mathcal{G}$ such that $y(s)$ belongs to $\mathcal{G}_1$ for all $s \in [0,T]$; let ω be a common modulus of continuity of the ℓ_i in $\mathcal{G}_1$. Last relation and definition of I_4 yield

$$|I_3 + I_4| \le \int_0^T (N+1)\omega \left(\delta_n \frac{T-\xi}{T} \right) d\xi + \delta_n (N+1) K \le (N+1) T \omega(\delta_n).$$

Replacing all these estimates in (2.26) and passing to the lim sup, we accomplish the proof of (iii). $\square$

2.3 Properties of the value function

We can now establish some useful properties of the value function: the dynamic programming principle and the continuity of u. Let us underline that the structure of our control set (in particular, that $\alpha = 0$ belongs to every A_i) plays a crucial role.

Proposition 2.3 (Dynamic programming principle) *Assume [H0] and [H1]. For any $(x,t) \in \mathcal{G} \times [0,T]$ and $\bar{t} \in [t,T]$, there holds*

$$u(x,t) = \inf_{(y,\alpha)\in\Gamma_{t,\bar{t}}[x]} \left\{ u(y(\bar{t}),\bar{t}) + \int_t^{\bar{t}} \left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau \right\}. \tag{2.27}$$

Proof. (i). For any $(y,\alpha) \in \Gamma_t[x]$, there holds

$$\begin{aligned} J_t(x,(y,\alpha)) &= \int_t^{\bar{t}} \left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau + J_{\bar{t}}(y(\bar{t}), (y_{|[\bar{t},T]}, \alpha_{|[\bar{t},T]})) \\ &\ge \int_t^{\bar{t}} \left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau + u(y(\bar{t}),\bar{t}) \end{aligned}$$

where $(y_{|[\bar{t},T]}, \alpha_{|[\bar{t},T]})$ is the restriction of the trajectory (y,α) in the interval $[\bar{t},T]$. Passing to the infumum over $(y,\alpha) \in \Gamma_t[x]$, we achieve relation (2.27) with the "$=$" replaced with "$\ge$".

We now prove the other inequality. Consider $(y,\alpha) \in \Gamma_{t,\bar{t}}[x]$. For $(\bar{y},\bar{\alpha}) \in \Gamma_{\bar{t}}^{\text{opt}}[y(\bar{t})]$ (whose existence is ensured by Proposition 2.1), we have

$$u(y(\bar{t}),\bar{t}) + \int_t^{\bar{t}} \left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau = J_t(x,(\tilde{y},\tilde{\alpha})) \ge u(x,t)$$

where $(\tilde{y},\tilde{\alpha})$ is the concatenation of the trajectory (y,α) on $[t,\bar{t}]$ and of the trajectory $(\bar{y},\bar{\alpha})$ on $[\bar{t},T]$; we recall from Remark 2.2 that $(\tilde{y},\tilde{\alpha}) \in \Gamma_t[x]$. Passing to the infimum over $(\bar{y},\bar{\alpha}) \in \Gamma_{\bar{t}}[y(\bar{t})]$, we accomplish the proof. $\square$

We can now establish the continuity of u.

Proposition 2.4 (Continuity of the value function) *Assume [H0] and [H1]. Then, the function u is continuous in $\mathcal{G}\times[0,T)$.*

Proof. Consider $(x_1,t_1),(x_2,t_2)\in\mathcal{G}\times[0,T_1)$, with $T_1<T$ and $\delta:=d(x_1,x_2)<(T-T_1)/2$. Without any loss of generality, we can assume that both x_1 and x_2 belong to the same edge, say e_1. Consider $(y_2,\alpha_2)\in\Gamma^{\mathrm{opt}}_{t_2}[x_2]$. Consider the trajectory that starts in x_1 at time t_1 and obeying to the control

$$\alpha_1(s)=\begin{cases}\left\{\begin{array}{ll} e_1 & \text{if } x_1\le x_2\\ -e_1 & \text{if } x_1>x_2\end{array}\right\} & \text{for } s\in[t_1,t_1+\delta]\\ \frac{T-t_2}{T-t_1-\delta}\alpha_2\left(\frac{T-t_2}{T-t_1-\delta}s-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}\right) & \text{for } s\in(t_1+\delta,T];\end{cases}$$

hence, the corresponding curve is

$$y_1(s)=\begin{cases} x_1+\frac{x_2-x_1}{\delta}(s-t_1) & \text{for } s\in[t_1,t_1+\delta)\\ x_2 & \text{for } s=t_1+\delta\\ x_2+\int_{t_1+\delta}^s \frac{T-t_2}{T-t_1-\delta}\alpha_2\left(\frac{T-t_2}{T-t_1-\delta}\tau-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}\right)d\tau & \text{for } s\in[t_1+\delta,T].\end{cases}$$

We observe that, for $s\in[t_1+\delta,T]$, there holds

$$y_1(s)=x_2+\int_{t_2}^{\frac{T-t_2}{T-t_1-\delta}s-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}}\alpha_2(\xi))\,d\xi=y_2\left(\frac{T-t_2}{T-t_1-\delta}s-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}\right)$$

and we deduce that $(y_1,\alpha_1)\in\Gamma_{t_1}(x_1)$ with $y_1(T)=y_2(T)$. We also estimate

$$\begin{aligned}\|\alpha_1\|^2_{L^2(t_1,T)} &= \delta+\int_{t_1}^T\left(\frac{T-t_2}{T-t_1-\delta}\right)^2\left|\alpha_2\left(\frac{T-t_2}{T-t_1-\delta}s-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}\right)\right|^2d\tau\\ &= \delta+\frac{T-t_2}{T-t_1-\delta}\|\alpha_2\|^2_{L^2(t_2,T)}\\ &= \delta+\|\alpha_2\|^2_{L^2(t_2,T)}+\frac{t_1-t_2+\delta}{T-t_1-\delta}\|\alpha_2\|^2_{L^2(t_2,T)}.\end{aligned}\tag{2.28}$$

We now estimate (recall $y_1(T)=y_2(T)$)

$$J_{t_1}(x_1;(y_1,\alpha_1))-J_{t_2}(x_2;(y_2,\alpha_2))=\sum_{i=1}^4 I_i$$

where

$$\begin{array}{ll} I_1=\int_{t_1}^{t_1+\delta}L(y_1(\tau),\tau)\,d\tau & I_2=\frac{\|\alpha_1\|^2_{L^2(t_1,T)}-\|\alpha_2\|^2_{L^2(t_2,T)}}{2}\\ I_3=\int_{t_1+\delta}^T L(y_1(\tau),\tau)d\tau & I_4=-\int_{t_2}^T L(y_2(\tau),\tau)\,d\tau.\end{array}$$

By the boundedness of the running cost and (2.28), for some constant K, there hold: $|I_1|\le K\delta$ and

$$|I_2|\le\frac{\delta}{2}+\frac{\|\alpha_2\|^2_{L^2(t_2,T)}}{2}\frac{|t_1-t_2|+\delta}{T-T_1}.$$

Moreover, we have

$$\begin{aligned} I_3 &= \int_{t_1+\delta}^T L\left(y_2\left(\frac{T-t_2}{T-t_1-\delta}\tau-\frac{T(\delta+t_1-t_2)}{T-t_1-\delta}\right),\tau\right)d\tau\\ &= \frac{T-t_1-\delta}{T-t_2}\int_{t_2}^T L\left(y_2(\xi),\frac{T-t_1-\delta}{T-t_2}\xi+\frac{T(\delta+t_1-t_2)}{T-t_2}\right)d\xi\\ &= \frac{T-t_1-\delta}{T-t_2}\int_{t_2}^T L\left(y_2(\xi),\frac{T-t_1-\delta}{T-t_2}\xi+\frac{T(\delta+t_1-t_2)}{T-t_2}\right)d\xi\end{aligned}$$

where the last equality is due to the same arguments used in the proof of Lemma 2.1. We deduce

$$
\begin{aligned}
|I_3+I_4| &= \Big| \tfrac{t_2-t_1-\delta}{T-t_2} \int_{t_2}^{T} L\left(y_2(\xi), \tfrac{T-t_1-\delta}{T-t_2}\xi + \tfrac{T(\delta+t_1-t_2)}{T-t_2}\right) d\xi \\
&\quad + \int_{t_2}^{T} \left[L\left(y_2(\xi), \tfrac{T-t_1-\delta}{T-t_2}\xi + \tfrac{T(\delta+t_1-t_2)}{T-t_2}\right) - L(y_2(\xi),\xi)\right] d\xi \Big| \\
&\le \tfrac{|t_2-t_1|+\delta}{T-T_1} K + (N+1)\omega\left(\tfrac{|t_2-t_1|+\delta}{T-T_1} 2T\right)
\end{aligned}
$$

where ω is a modulus of continuity as the one in proof of Lemma 2.1. In conclusion, we have

$$
|J_{t_1}(x_1;(y_1,\alpha_1)) - J_{t_2}(x_2;(y_2,\alpha_2))| \le \tilde{K}(|t_2-t_1|+\delta) + \tilde{\omega}(|t_2-t_1|+\delta)
$$

for suitable constant $\tilde{K}$ (depending only on T_1) and a suitable modulus of continuity $\tilde{\omega}$ (depending on T_1, $|x_1|$ and $|x_2|$). By the optimality of (y_2,α_2), we deduce

$$
\begin{aligned}
u(x_1,t_1) &\le J_{t_1}(x_1;(y_1,\alpha_1)) \le J_{t_2}(x_2;(y_2,\alpha_2)) + \tilde{K}(|t_2-t_1|+\delta) + \tilde{\omega}(|t_2-t_1|+\delta) \\
&\le u(x_2,t_2) + \tilde{K}(|t_2-t_1|+\delta) + \tilde{\omega}(|t_2-t_1|+\delta).
\end{aligned}
$$

Reversing the role of (x_1,t_1) and (x_2,t_2), we get

$$
|u(x_1,t_1) - u(x_2,t_2)| \le \tilde{K}(|t_2-t_1|+\delta) + \tilde{\omega}(|t_2-t_1|+\delta)
$$

which is equivalent to the statement. □

Remark 2.6 *Observe that, when the running costs ℓ_i are θ-Hölder continuous with respect to time, the same arguments of the previous proof establish that the value function is θ-Hölder continuous.*

Let us also establish the next property for future reference which is not used in the rest of this paper.

Lemma 2.2 *Fix $(x,t) \in \mathcal{G}\times[0,T)$ and $(y,\alpha)\in\Gamma_t[x]$; consider a sequence $\{(x_n,t_n)\}_{n\in\mathbb{N}}$, such that $(x_n,t_n)\in\mathcal{G}\times[0,T)$, $\delta'_n := d(x_n,x)+|t_n-t| \to 0$ as $n\to\infty$. Then, there exists a sequence $\{(y_n,\alpha_n)\}_{n\in\mathbb{N}}$, such that $(y_n,\alpha_n)\in\Gamma_{t_n}[x_n]$*

$$
(2.29) \qquad
\begin{array}{ll}
(i) & \sup\limits_{[t_n\vee t, T]} d(y_n(\cdot),y(\cdot)) \le \delta_n + |t_n-t| + \|\alpha\|_2\sqrt{\delta'_n}, \qquad y_n(T)=y(T) \\
(ii) & \|\alpha_n\|_2^2 \le \|\alpha\|_2^2 + \delta'_n\left(1+\frac{\|\alpha\|_2^2}{T-\delta'_n}\right) \\
(iii) & \lim\limits_{n\to\infty} J_{t_n}(x_n;(y_n,\alpha_n)) = J_t(x;(y,\alpha)).
\end{array}
$$

Proof of Lemma 2.2. We adapt the arguments of the proof of Lemma 2.1; it suffices to consider that all the points x_n and x belong to the edge e_1 and the two cases where all the t_n's are smaller o greater than t. Set $\delta_n = d(x_n,x)$.

Case: $t_n\le t$, $\forall n\in\mathbb{N}$. We introduce the control

$$
\alpha_n(s) = \begin{cases}
0 & \text{for } s\in[t_n,t] \\
\begin{Bmatrix} e_1 & \text{if } \bar{x}_n \le \bar{x} \\ -e_1 & \text{if } \bar{x}_n > \bar{x} \end{Bmatrix} & \text{for } s\in[t,t+\delta_n] \\
\frac{T-t}{T-t-\delta_n}\alpha\left(s\frac{T-t}{T-t-\delta_n} - \delta_n\frac{T}{T-t-\delta_n}\right) & \text{for } s\in(t+\delta_n,T]
\end{cases}
$$

and we denote y_n the corresponding curve starting from x_n at time t_n; clearly, $(y_n, \alpha_n) \in \Gamma_{t_n}[x_n]$. We have

$$y_n(s) = \begin{cases} x_n & \text{for } s \in [t_n, t] \\ x_n + (x - x_n)\delta_n^{-1}(s - t) & \text{for } s \in [t, t + \delta_n] \\ y\left(s\frac{T-t}{T-t-\delta_n} - \delta_n \frac{T}{T-t-\delta_n}\right) & \text{for } s \in [t + \delta_n, T]. \end{cases}$$

By the same arguments as before, we obtain the bounds in (2.29)-(i) and (ii). Moreover, we have

$$J_{t_n}(x_n; (y_n, \alpha_n)) - J_t(x; (y, \alpha)) = \sum_{i=1}^{5} I_i$$

where, for $i = 1, \ldots, 4$, the I_i's are analogous to the corresponding ones in (2.26) while $I_5 = \int_{t_n}^{t} L(x_n, 0, \tau)\, d\tau$. By the boundedness of the costs ℓ_i, for a suitable constant K, we have $|I_5| \leq K|t - t_n|$. For the same calculations as in the proof of Lemma 2.1, we obtain the statement.

Case: $t_n \geq t$, $\forall n \in \mathbb{N}$. Note: $t + \delta_n' = t_n + \delta_n$. We introduce the control

$$\alpha_n(s) = \begin{cases} \left\{ \begin{array}{ll} e_1 & \text{if } \bar{x}_n \leq \bar{x} \\ -e_1 & \text{if } \bar{x}_n > \bar{x} \end{array} \right\} & \text{for } s \in [t_n, t + \delta_n'] \\ \frac{T-t}{T-t-\delta_n'}\alpha\left(s\frac{T-t}{T-t-\delta_n'} - \delta_n' \frac{T}{T-t-\delta_n'}\right) & \text{for } s \in (t + \delta_n', T] \end{cases}$$

and we denote y_n the corresponding curve starting from x_n at time t_n; clearly, $(y_n, \alpha_n) \in \Gamma_{t_n}[x_n]$. We have

$$y_n(s) = \begin{cases} x_n + (x - x_n)\delta_n'^{-1}(s - t) & \text{for } s \in [t, t + \delta_n'] \\ y\left(s\frac{T-t}{T-t-\delta_n} - \delta_n \frac{T}{T-t-\delta_n}\right) & \text{for } s \in [t + \delta_n', T]. \end{cases}$$

For the same calculations as in the proof of Lemma 2.1, we obtain the statement. □

2.4 Hamilton-Jacobi problem

We now introduce Hamilton-Jacobi operators on the network and the corresponding definition of viscosity solution and we prove that the value function solves the HJ problem. We shall following the approach by Imbert and Monneau [17].

HJ operators and HJ problem on the junction. We introduce the Hamilton-Jacobi operator associated to the optimal control problem: the Hamiltonian has a different formulation according to the fact that the point coincides or not coincides with the junction O. We also need to introduce the relaxed controls as in [2]. We set: for $x \in J_i$ with $i = 1, \ldots, N$,

$$FL_i(x, t) := \overline{\text{co}}\{(a, a^2/2) \ : \ a \in \mathbb{R}\}, \quad FL_i^{\searrow}(x, t) := FL_i(x, t) \cap \{(\zeta, \xi) \in \mathbb{R}^2 \ : \ \zeta \geq 0\}$$

where "co" stands for the convex hull, while, for $x = O$ we also introduce

$$FL(O, t) := \bigcup_{i=1}^{N} FL_i^{\searrow}(O, t).$$

We easily get

$$FL_i(x,t) = \{(\zeta,\xi) \in \mathbb{R}^2 \;:\; \xi \geq \zeta^2/2\}, \quad FL_i^{\searrow}(x,t) := \{(\zeta,\xi) \in \mathbb{R}^2 \;:\; \zeta \geq 0,\, \xi \geq \zeta^2/2\}.$$

We can now introduce our Hamilton-Jacobi operators as: for any $x \in J_i$ ($i = 1, \dots, N$), $t \in [0,T]$, $\bar{p} \in \mathbb{R}$, $p = (p_1, \dots, p_N) \in \mathbb{R}^N$

$$\begin{aligned}
H_i(x,t,\bar{p}) &:= \sup_{(\zeta,\xi)\in FL_i(x,t)} \{-\bar{p}\zeta - \xi - \ell_i(x,t)\} \\
H_i^{\searrow}(x,t,\bar{p}) &:= \sup_{(\zeta,\xi)\in FL_i^{\searrow}(x,t)} \{-\bar{p}\zeta - \xi - \ell_i(x,t)\} \\
H_O(t,p) &:= \max\left\{-\ell_*(t), \max_{i=1,\dots,N} \{-\ell_i(O,t)\}, \max_{i=1,\dots,N} \left\{H_i^{\searrow}(O,t,p_i)\right\}\right\} \\
&= \max\left\{-\ell_O(t), \max_{i=1,\dots,N} \left\{H_i^{\searrow}(O,t,p_i)\right\}\right\}.
\end{aligned}$$

By elementary calculus (see also [2, Remark 4.1] for a similar argument), we have

$$\begin{aligned}
H_i(x,t,\bar{p}) &= \sup_{a\in\mathbb{R}} \left\{-\bar{p}a - \frac{|a|^2}{2} - \ell_i(x,t)\right\} = \frac{|p|^2}{2} - \ell_i(x,t) \quad \forall x \in J_i \\
H_i^{\searrow}(O,t,\bar{p}) &= \max_{\bar{\alpha}\geq 0} \{-\bar{\alpha}\bar{p} - \ell_i(O,t) - |\bar{\alpha}|^2/2\} = \begin{cases} \frac{|\bar{p}|^2}{2} - \ell_i(O,t) & \text{if } \bar{p} \leq 0 \\ -\ell_i(O,t) & \text{if } \bar{p} > 0. \end{cases}
\end{aligned} \tag{2.30}$$

We introduce our Hamilton-Jacobi problem

$$\begin{cases} -\partial_t u + H_i(x,t,Du) = 0 & \text{if } x \in J_i \setminus \{O\} \\ -\partial_t u + H_O(t,Du) = 0 & \text{if } x = O \\ u(T,x) = g(x) & \text{on } \mathcal{G} \end{cases} \tag{2.31}$$

(recall that Du is introduced in (1.1) and it is 1-dimensional if $x \in J_i \setminus \{O\}$ while it is N-dimensional when $x = O$).

Definition of viscosity solution. We introduce the definition of viscosity solution for the HJ problem (2.31) as follows.

Definition 2.1 *We say that a function $u \in C(\mathcal{G} \times [0,T])$ is a subsolution (resp., supersolution) of problem* (2.31) *if it fulfills: $u(T,\cdot) \leq g(\cdot)$ (resp., $u(T,\cdot) \geq g(\cdot)$) and, for every function $\varphi \in C^1(\mathcal{G} \times [0,T])$ touching u from above (resp., below) at $(x,t) \in \mathcal{G} \times [0,T]$, we have*

$$\begin{array}{ll} -\partial_t\varphi(x,t) + H_i(x,t,D\varphi) \leq 0 \quad (\textit{resp., } \geq 0) & \textit{if } x \in J_i \setminus \{O\} \\ -\partial_t\varphi + H_O(t,D\varphi) \leq 0 \quad (\textit{resp., } \geq 0) & \textit{if } x = O. \end{array}$$

We say that a function $u \in C(\mathcal{G} \times [0,T])$ is a solution of problem (2.31) *if it is both a subsolution and a supersolution of the problem.*

Main result. We can now establish the main result of this section, namely that the value function is a viscosity solution to the HJ problem.

Theorem 2.3 *Assume [H0] and [H1]. Then, the value function u defined in* (2.8) *is a solution to problem* (2.31).

Proof. We shall borrow some arguments of the proof of [17, Theorem 6.4]. Clearly, by standard theory on viscosity solutions, it suffices to prove that u is a supersolution and a subsolution on the junction O.

`Step 1:` u `is a supersolution at` O. Let $\varphi \in C^1(\mathcal{G}\times[0,T])$ be a function touching u from below at $(O,\bar{t})$, for some $\bar{t}\in(0,T)$; wlog, since u is bounded, we can assume that $u-\varphi$ attains a global minimum at $(O,\bar{t})$, namely, $\varphi(x,t)\le u(x,t)\ \forall(x,t)\in\mathcal{G}\times[0,T]$ and $\varphi(O,\bar{t}) = u(O,\bar{t})$. Let $(y,\alpha)\in\Gamma_{\bar{t}}[O]$ be an optimal trajectory for $u(O,\bar{t})$. The Dynamic Programming Principle in Proposition 3.2-(i) and Remark 2.5 ensure

$$u(O,\bar{t}) = u(y(s),s) + \int_{\bar{t}}^{s}\left[L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right]d\tau \qquad \forall s\in[\bar{t},T].$$

which entails

$$\varphi(y(s),s) - \varphi(O,\bar{t}) + \int_{\bar{t}}^{s}\left[L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right]d\tau \le 0 \qquad \forall s\in[\bar{t},T].$$

By the same arguments of [17, Theorem 6.4 (proof)], we deduce

$$\int_{\bar{t}}^{s}\left[\partial_t\varphi(y(\tau),\tau) + D\varphi(y(\tau),\tau)\cdot\alpha(\tau) + L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right]d\tau \le 0 \quad \forall s\in[\bar{t},T] \tag{2.32}$$

with the convention $D\varphi(\tau,y(\tau))\cdot\alpha(\tau)=0$ for a.e. $\tau\in\{\tau\in[\bar{t},T]\ :\ y(\tau)=O\} =: \mathcal{T}_0$ which makes sense because the Stampacchia theorem yields $\alpha(\tau)=0$ for a.e. $\tau\in\mathcal{T}_0$. Recall from Remark 2.4 the uniform bound of the optimal control in L^2 so there holds

$$d(y(\tau),O) \le \int_{\bar{t}}^{\tau}|\alpha(s)|\,ds \le C(\tau-\bar{t})^{1/2} \qquad \forall\tau\in[\bar{t},T];$$

hence, by the regularity of φ, there exists a constant K such that

$$|\psi(y(\tau),\tau) - \psi(O,\bar{t})| \le K(\tau-\bar{t})^{1/2} \qquad \forall\tau\in[\bar{t},T] \qquad \text{for } \psi=\varphi,\partial_t\varphi, D\varphi. \tag{2.33}$$

It is expedient to introduce the sets of times

$$\mathcal{T}_0^s := \{\tau\in(\bar{t},s)\ :\ y(\tau)=O\}, \qquad \mathcal{T}_i^s := \{\tau\in(\bar{t},s)\ :\ y(\tau)\in J_i\backslash\{O\}\} \quad \text{for } i=1,\dots,N.$$

Note that $\mathcal{T}_O^s$ is a closed set while the $\mathcal{T}_i^s$'s are open sets with $(\bar{t},s)=\bigcup_{i=0}^N\mathcal{T}_i^s$ so relation (2.32) can be written as

$$\sum_{i=0}^{N}\int_{\mathcal{T}_i^s}\xi(\tau)\,d\tau \le 0 \qquad \forall s\in[\bar{t},T] \tag{2.34}$$

where

$$\xi(\tau) := \partial_t\varphi(y(\tau),\tau) + D\varphi(y(\tau),\tau)\cdot\alpha(\tau) + L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}.$$

We shall deal with the contributions of $i=0$ and separately with the ones of $i=1,\dots,N$ in (2.34).

Consider $i=0$. For a.e. $\tau\in\mathcal{T}_0^s$ there hold: $\alpha=0$ (again by Stampacchia theorem) and $L(y(\tau),\tau)=\ell_O(\tau)$. Hence, we have

$$\int_{\mathcal{T}_0^s}\xi(\tau)\,d\tau = \int_{\mathcal{T}_0^s}\left[\partial_t\varphi(O,\tau)+\ell_O(\tau)\right]d\tau \ge \int_{\mathcal{T}_0^s}\left[\partial_t\varphi(O,\bar{t})+\ell_O(\bar{t})\right]d\tau - (s-\bar{t})\omega(s-\bar{t})$$

where the inequality is due to (2.33) and to the continuity of ℓ_O and ω is a modulus of continuity which depends on the constant K of (2.33) and on the modulus of continuity of ℓ_O. Moreover, we observe that the definition of H_O guarantees

$$\ell_O(\bar t) \geq -H_O(\bar t, p) \qquad \forall p \in \mathbb{R}^N.$$

By the last two relations, we deduce

$$\int_{\mathcal{T}_0^s} \xi(\tau)\, d\tau \geq |\mathcal{T}_0^s| \left(\partial_t \varphi(O,\bar t) - H_O(\bar t, D\varphi(\bar t, O))\right) - (s-\bar t)\omega(s-\bar t). \tag{2.35}$$

Consider now $i \in \{1,\dots,N\}$; recall that, for a.e. $\tau \in \mathcal{T}_i^s$, the control α has the form $\alpha(\tau) = \bar\alpha(\tau) e_i$ with $\bar\alpha(\tau) \in \mathbb{R}$. By relation (2.33), Remark 2.4 and the continuity of ℓ_i, there exists a modulus of continuity ω such that

$$\begin{aligned}
\int_{\mathcal{T}_i^s} \xi(\tau)\, d\tau &= \int_{\mathcal{T}_i^s} \left[\partial_t \varphi(y(\tau),\tau) + D\varphi_{|J_i}(y(\tau),\tau)\bar\alpha(\tau) + \ell_i(y(\tau),\tau) + \frac{|\bar\alpha(\tau)|^2}{2}\right] d\tau \\
&\geq \int_{\mathcal{T}_i^s} \left[\partial_t \varphi(O,\bar t) + D\varphi_{|J_i}(O,\bar t)\bar\alpha(\tau) + \ell_i(O,\bar t) + \frac{|\bar\alpha(\tau)|^2}{2}\right] d\tau - (s-\bar t)\omega(s-\bar t). \qquad (2.36)
\end{aligned}$$

Exploiting the convexity of the set FL_i, by the same arguments as those in [17, eq.(6.22)] (as a matter of facts, in this case it is enough to use Jensen inequality), we infer that there exists $(\zeta_i, \xi_i) \in FL_i(O,\bar t)$ such that

$$\begin{array}{l}
\int_{\mathcal{T}_i^s} D\varphi_{|J_i}(O,\bar t)\bar\alpha(\tau)\, d\tau = D\varphi_{|J_i}(O,\bar t) \int_{\mathcal{T}_i^s} \bar\alpha(\tau)\, d\tau = |\mathcal{T}_i^s| D\varphi_{|J_i}(O,\bar t)\zeta_i \\
\int_{\mathcal{T}_i^s} |\bar\alpha(\tau)|^2/2\, d\tau = |\mathcal{T}_i^s| \xi_i.
\end{array}$$

Moreover, note that the curve $y(\cdot)$ occupies the vertex O at time $\bar t$ and during the interval $(\bar t, s)$ may enter, and exit, many edges. However, if $y(s) \in J_i \setminus \{O\}$ for $s \in (t_1,t_2)$ and $y(t_1) = y(t_2) = O$ (namely, it occupies the vertex at times t_1 and t_2 and is inside edge J_i in (t_1,t_2)), then, there holds

$$\int_{t_1}^{t_2} \bar\alpha(\tau)\, d\tau = 0$$

and consequently

$$\int_{\mathcal{T}_i^s} \bar\alpha(\tau)\, d\tau = \begin{cases} y(s) & \text{if } y(s) \in J_i \\ 0 & \text{otherwise} \end{cases}$$

which implies: $\zeta_i \geq 0$. We deduce

$$\begin{aligned}
\int_{\mathcal{T}_i^s} \left[D\varphi_{|J_i}(O,\bar t)\bar\alpha(\tau) + \ell_i(O,\bar t) + \frac{|\bar\alpha(\tau)|^2}{2}\right] d\tau &= |\mathcal{T}_i^s| \left[D\varphi_{|J_i}(O,\bar t)\zeta_i + \ell_i(O,\bar t) + \xi_i\right] \\
&\geq -|\mathcal{T}_i^s| H_i^{\searrow}(O,\bar t, D\varphi_{|J_i}(O,\bar t)).
\end{aligned}$$

Replacing last inequality in (2.36), we obtain

$$\begin{aligned}
\int_{\mathcal{T}_i^s} \xi(\tau)\, d\tau &\geq |\mathcal{T}_i^s| \left[\partial_t \varphi(O,\bar t) - H_i^{\searrow}(O,\bar t, D\varphi_{|J_i}(O,\bar t))\right] - (s-\bar t)\omega(s-\bar t) \\
&\geq |\mathcal{T}_i^s| \left[\partial_t \varphi(O,\bar t) - H_O(\bar t, D\varphi(O,\bar t))\right] - (s-\bar t)\omega(s-\bar t).
\end{aligned}$$

Replacing last inequality and relation (2.35) in (2.34), we achieve

$$\begin{aligned}(N+1)(s-\bar{t})\omega(s-\bar{t}) &\geq \left(\sum_{i=0}^{N}|\mathcal{T}_i^s|\right)|\left[\partial_t\varphi(O,\bar{t}) - H_O(\bar{t}, D\varphi(O,\bar{t}))\right] \\ &\geq (s-\bar{t})\left[\partial_t\varphi(O,\bar{t}) - H_O(\bar{t}, D\varphi(O,\bar{t}))\right]\end{aligned}$$

where the last relation is due to $(\bar{t}, s) = \cup_{i=0}^{N}\mathcal{T}_i^s$ with $\mathcal{T}_i^s \cap \mathcal{T}_j^s = \emptyset$ for $i \neq j$. Dividing last relation by $(s-\bar{t})$ and letting $s \to \bar{t}^+$, we conclude

$$-\partial_t\varphi(O,\bar{t}) + H_O(\bar{t}, D\varphi(O,\bar{t})) \geq 0$$

which is equivalent to our statement.

`Step 2:` u `is a subsolution at` O. Let $\varphi \in C^1(\mathcal{G} \times [0,T])$ be a function touching u from above at $(O,\bar{t})$, for some $\bar{t} \in (0,T)$, namely, $\varphi(x,t) \geq u(x,t)\ \forall (x,t) \in \mathcal{G} \times [0,T]$ and $\varphi(O,\bar{t}) = u(O,\bar{t})$. The Dynamic Programming Principle in Proposition 2.3 ensures, for any $s \in (\bar{t},T)$ and any $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$:

$$u(O,\bar{t}) \leq u(y(s),s) + \int_{\bar{t}}^{s}\left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right)d\tau;$$

we deduce that, for any $s \in (\bar{t},T)$ and any $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$, there holds

$$\varphi(y(s),s) - \varphi(O,\bar{t}) + \int_{\bar{t}}^{s}\left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right)d\tau \geq 0. \tag{2.37}$$

Note that relation (2.37) can be written as: for any $s \in (\bar{t},T)$ and any $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$, there holds

$$\varphi(y(s),s) - \varphi(O,\bar{t}) + \sum_{i=0}^{N}\int_{\mathcal{T}_i^s}\left(L(y(\tau),\tau) + \frac{|\alpha(\tau)|^2}{2}\right)d\tau \geq 0 \tag{2.38}$$

where, for $i \in \{0,\dots,N\}$, the sets $\mathcal{T}_i^s$ as defined as in the previous step (and depend on the trajectory (y,α)). We now split our arguments choosing arbitrarily some trajectories $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$ whose curve $y(\cdot)$ remains in O or enters in some edge J_i.

Case (a): the curve y remains in O. For any $s \in (\bar{t},T]$, consider the trajectory $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$ with $\alpha(\cdot) = 0$; clearly, $y(\cdot) = O$ and $(\bar{t},s) = \mathcal{T}_0^s$. Relation (2.38) becomes

$$\varphi(O,s) - \varphi(O,\bar{t}) + \int_{\bar{t}}^{s}\ell_O(\tau)d\tau \geq 0.$$

By the continuity of ℓ_O with respect to t, for some modulus of continuity ω, we deduce

$$\varphi(O,s) - \varphi(O,\bar{t}) + (s-\bar{t})\ell_O(\bar{t}) \geq -(s-\bar{t})\omega(s-\bar{t}).$$

Dividing the previous relation by $(s-\bar{t})$, letting $s \to \bar{t}^+$ and taking into account the regularity of φ, we infer

$$\partial_t\varphi(O,\bar{t}) + \ell_O(\bar{t}) \geq 0. \tag{2.39}$$

Case (b): the curve y enters in an edge J_i. Fix $i \in \{1,\dots,N\}$. For any $m \in \mathbb{N}\setminus\{0\}$, fix $\bar{a} \in (0,m)$. For any $s \in (\bar{t},T]$, consider the trajectory $(y,\alpha) \in \Gamma_{\bar{t},s}[O]$ with $\alpha(\tau) = \bar{a}e_i$ for

$\tau \in (\bar{t}, s)$; clearly, $\alpha(\tau) \in A_i$ and $y(\tau) \in J_i \setminus \{O\}$ for $\tau \in (\bar{t}, s)$ and consequently $(\bar{t}, s) = \mathcal{T}_i^s$. (Note that here the unboudedness of J_i is not essential; indeed, if J_i has length l_i, then it is enough to choose $s \le \bar{t} + l_i/\bar{a}$). By the same arguments of Step 1 (see equation (2.32)), inequality (2.38) can be written as

$$\int_{\bar{t}}^{s} \left[\partial_t \varphi(y(\tau), \tau) + D\varphi_{|J_i}(y(\tau), \tau)\bar{a} + \ell_i(y(\tau), \tau) + \frac{\bar{a}^2}{2} \right] d\tau \ge 0 \quad \forall s \in [\bar{t}, T].$$

As in Step 1, taking into account Remark 2.4, estimate (2.33) and the uniform continuity of ℓ_i in any neighbourhood of O, we get

$$\int_{\bar{t}}^{s} \left[\partial_t \varphi(O, \bar{t}) + D\varphi_{|J_i}(O, \bar{t})\bar{a} + \ell_i(O, \bar{t}) + \frac{\bar{a}^2}{2} \right] d\tau \ge -(s - \bar{t})\omega(s - \bar{t}) \quad \forall s \in [\bar{t}, T]$$

for some suitable modulus of continuity ω. Dividing the previous inequality by $(s - \bar{t})$ and letting $s \to \bar{t}^+$, we obtain

$$\partial_t \varphi(O, \bar{t}) + D\varphi_{|J_i}(O, \bar{t})\bar{a} + \ell_i(O, \bar{t}) + \frac{\bar{a}^2}{2} \ge 0.$$

By the arbitrariness of $\bar{a} \in (0, m)$, we deduce

$$\partial_t \varphi(O, \bar{t}) - \sup_{\bar{a} \in (0,m)} \left\{ -D\varphi_{|J_i}(O, \bar{t})\bar{a} - \ell_i(O, \bar{t}) - \frac{\bar{a}^2}{2} \right\} \ge 0$$

and, by arbitrariness of m, also

$$\partial_t \varphi(O, \bar{t}) - \sup_{\bar{a} \ge 0} \left\{ -D\varphi_{|J_i}(O, \bar{t})\bar{a} - \ell_i(O, \bar{t}) - \frac{\bar{a}^2}{2} \right\} \ge 0.$$

By equation (2.30), we get

$$\partial_t \varphi(O, \bar{t}) - H_i^{\searrow}(O, \bar{t}, D\varphi_{|J_i}(O, \bar{t})) \ge 0.$$

By arbitrariness of i in the last relation and by inequality (2.39), we conclude

$$\partial_t \varphi(O, \bar{t}) - \max \left\{ \max \left\{ -\ell_*(\bar{t}), \max_{i=1,\dots,N} \{-\ell_i(O, \bar{t})\} \right\}, \max_{i=1,\dots,N} \left\{ H_i^{\searrow}(x, t, D\varphi_{|J_i}(O, \bar{t})) \right\} \right\} \ge 0$$

which is equivalent to our statement. □

3 Mean Field Games equilibrium

This section concerns the MFG problem. Taking advantage of the results established in Section 2, we prove that there exists a MFG equilibrium and that the value function of the associated optimal control problem is a viscosity solution to a HJ problem.

3.1 Setting and notations

Probability sets and evaluation map. Let $\mathcal{P}(\mathcal{G})$ denote the set of probability measures on $\mathcal{G}$ endowed with the narrow topology. For $t \in [0, T]$, the evaluation map $e_t : \Gamma \to \mathcal{G}$ is defined by $e_t(y_x, \alpha) = y_x(t)$. For any $\mu \in \mathcal{P}(\Gamma)$ and $t \in [0, T]$, define the Borel probability measure $m^\mu(t)$ on $\mathcal{G}$ by $m^\mu(t) = e_t \sharp \mu$.

Costs. We introduce the running cost and the terminal cost dependent on the distribution of the population. We consider costs $L_i \in C(\mathcal{P}(\mathcal{G}); C(\mathcal{G} \times [0,T]))$, for $i = 1, \ldots, N$, and $L_* \in C(\mathcal{P}(\mathcal{G}); C([0,T])$. Similarly, let $G_i : \mathcal{P}(\mathcal{G}) \to C(\mathcal{G})$, $i = 1, \ldots, N$, and $G_* : \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ be continuous functions. $L_i[m](\cdot, \cdot)$ and $G_i[m](\cdot)$ denote the images of $m \in \mathcal{P}(\mathcal{G})$ by L_i and respectively by G_i and analogously for L_* and G_*.

We assume

$$(H_1^{\mathrm{MFG}}) \quad K = \max\Bigg(\sup_{m\in\mathcal{P}(\mathcal{G})} \|L_*[m]\|_{L^\infty}, \max_{i=1,\ldots,N} \sup_{m\in\mathcal{P}(\mathcal{G})} \|L_i[m]\|_{L^\infty}, \sup_{m\in\mathcal{P}(\mathcal{G})} \|G_*[m]\|_{L^\infty}, \max_{i=1,\ldots,N} \sup_{m\in\mathcal{P}(\mathcal{G})} \|G_i[m]\|_{L^\infty} \Bigg) \in \mathbb{R}^+.$$

For simplicity of notations, for $x \in \mathcal{G}$ and $t \in [0,T]$, we shall write

$$\begin{aligned} L[m](x,t) &= \sum_{i=1}^N L_i[m](x,t)\mathbf{1}_{x\in J_i\setminus\{O\}} + L_O[m](t)\mathbf{1}_{x=O} \\ G[m](x) &= \sum_{i=1}^N G_i[m](x)\mathbf{1}_{x\in J_i\setminus\{O\}} + \min\{G_*[m], \min_{i=1,\ldots,N} G_i[m](O)\}\mathbf{1}_{x=O} \end{aligned} \tag{3.1}$$

where

$$L_O[m](\tau) = \min\{L_*[m](\tau), \min_{i=1,\ldots,N} L_i[m](O,\tau)\}.$$

Admissible curves. We introduce the sets of admissible *curves*

$$\tilde{\Gamma}_C[x] := \{y \in Y_{x,0} \ : \ d(y(s), O) \le C, \ \forall s \in [0,T], \ \|\dot{y}\|_2 \le C\}, \quad \tilde{\Gamma}_C := \bigcup_{x\in\mathcal{G}} \tilde{\Gamma}_C[x] \tag{3.2}$$

and we endow $\tilde{\Gamma}_C$ with the topology of uniform convergence. Note that a *curve* is the sole $y \in Y_{y(0),0}$ while a *trajectory* is formed by the couple $(y, \alpha) \in \Gamma$.

Lemma 3.1 *For every positive constant C, the set $\tilde{\Gamma}_C$ is compact.*

Proof. Fix $C > 0$ and consider a sequence $\{y_n\}_{n\in\mathbb{N}}$, with $y_n \in \tilde{\Gamma}_C$. Possibly passing to a subsequence (that we still denote y_n), the sequence $\{\dot{y}_n\}_n$ converges in the weak topology of $L^2([0,T], \mathbb{R}^d)$ to some $\alpha \in L^2([0,T], \mathbb{R}^d)$, with $\|\alpha\|_2 \le C$. Then, $\{y_n\}_n$ is uniformly convergent to some curve $y \in C([0,T], \mathcal{G})$. Clearly, $\alpha = \dot{y}$. Moreover, arguing as in the proof of Proposition 2.1 we obtain that the curve y is admissible, namely $y \in Y_{y(0),0}$, and consequently that y belongs to $\tilde{\Gamma}_C$. □

The set $\mathcal{P}(\tilde{\Gamma}_C)$ and the associated costs. Let $\mathcal{P}(\tilde{\Gamma}_C)$ denote the set of probability measures on $\tilde{\Gamma}_C$ endowed with the narrow topology. For $t \in [0,T]$, the evaluation map $e_t : \tilde{\Gamma}_C \to \mathcal{G}$ is defined by $e_t(y) = y(t)$. For any $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$ and $t \in [0,T]$, define the Borel probability measure $m^\mu(t)$ on $\mathcal{G}$ by $m^\mu(t) = e_t\sharp\mu$. Clearly, $\mathrm{supp}(m^\mu(t)) \subset \{x \in \mathcal{G} : d(x,O) \le C\}$. It is possible to prove that, if $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$, then the map $t \mapsto m^\mu(t)$ belongs to $C^{1/2}([0,T], \mathcal{P}(\mathcal{G}))$; see Lemma 3.4 below. Hence, for all $(y, \alpha) \in \Gamma$, the functions $t \mapsto F_i[m^\mu(t)](y(t))$ are continuous and bounded by the constant K introduced in (H_1^{MFG}). With $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$ and $(y, \alpha) \in \Gamma[x]$ we associate the cost

$$J^\mu(x; (y, \alpha)) = \int_0^T \left(L[m^\mu(\tau)](y(\tau), \tau) + \frac{|\alpha(\tau)|^2}{2} \right) d\tau + G[m^\mu(T)](y(T)). \tag{3.3}$$

Remark 3.1 *We recall from Theorem 2.2 and Remark 2.1 that for each $y \in \tilde{\Gamma}_C[x]$ there exists $\alpha \in L^2([0,T], \mathbb{R}^d)$ such that $(y, \alpha) \in \Gamma[x]$. Such a control α is unique for a.e. $t \in \{t \in [0,T] : y(t) \neq O\}$ and it is not unique in $\{t \in [0,T] : y(t) = O\}$. However, the associated cost is independent of the choice of this control, namely: for any $y \in \tilde{\Gamma}_C[x]$, there holds*

$$J^{\mu}(x; (y, \alpha_1)) = J^{\mu}(x; (y, \alpha_2)) \qquad \forall (y, \alpha_1), (y, \alpha_2) \in \Gamma[x].$$

For every $y \in \tilde{\Gamma}_C[x]$, we define α_y the control such that $(y, \alpha_y) \in \Gamma[x]$ and $\alpha_y(t) = 0 \in A_0$ for a.e. $t \in \{t \in [0,T] : y(t) = O\}$. Note that this control is uniquely defined up to a set of null measure.

Optimal trajectories. Fix $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$; for any $x \in \mathcal{G}$, let us set

$$\Gamma^{\mu,\mathrm{opt}}[x] = \left\{ (y, \alpha) \in \Gamma[x] \ : \ J^{\mu}(x; (y, \alpha)) = \min_{(\widetilde{y}, \widetilde{\alpha}) \in \Gamma[x]} J^{\mu}(x; (\widetilde{y}, \widetilde{\alpha})) \right\} \tag{3.4}$$

where J^{μ} is defined in (3.3). Proposition 2.1 entails that for each $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$ and $x \in \mathcal{G}$, the set $\Gamma^{\mu,\mathrm{opt}}[x]$ of optimal trajectories starting from x is not empty.

Remark 3.2 *By our assumption (H_1^{MFG}), there exists a positive constant $\tilde{C}$ such that: for every $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$, $x \in \mathcal{G}$ and $(y, \alpha) \in \Gamma^{\mu,\mathrm{opt}}[x]$, there holds $\|\alpha\|_2 \leq \tilde{C}$. In particular, if $m_0 \in \mathcal{P}(\mathcal{G})$ has compact support, (eventually increasing the constant $\tilde{C}$), for every $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$, $x \in supp(m_0)$ and $(y, \alpha) \in \Gamma^{\mu,\mathrm{opt}}[x]$, there holds $y \in \tilde{\Gamma}_{\tilde{C}}[x]$.*

The set $\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$. We assume

$$(H_2^{\mathrm{MFG}}) \qquad\qquad m_0 \in \mathcal{P}(\mathcal{G}) \qquad \text{has compact support.}$$

Let $\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$ denote the set of measures $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$ such that $e_0 \sharp \mu = m_0$. Clearly, $\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$ may be empty; however, we have the following result

Lemma 3.2 *Under assumptions (H_1^{MFG}) and (H_2^{MFG}), for C sufficiently large, $\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$ is not empty.*

Proof. We argue adapting the arguments of [11, Remark 3.2]. For $C \geq \tilde{C}$ (where $\tilde{C}$ is the constant introduced in Remark 3.2), we consider the map: $j : \mathrm{supp}(m_0) \to \tilde{\Gamma}_C$, $j(x)(t) = x$ for any $t \in [0,T]$. We denote $\tilde{m}_0 = m_{0|\mathrm{supp}(m_0)}$ the restriction of m_0 on its support. We observe $e_0 \# (j \# \tilde{m}_0) = m_0$ so $(j \# \tilde{m}_0) \in \mathcal{P}_{m_0}(\tilde{\Gamma}_C)$. □

We show an example where the distribution of agents may develop a singularity.

Example 3.1 *In a junction with two edges, consider the costs: $L_1[m] \equiv -1$, $L_2[m] \equiv 1$, $L_*[m] = -1$ and $G_i[m] \equiv 0$ ($i = *, 1, 2$) for every $m \in \mathcal{P}(\mathcal{G})$. Assume that the initial distribution of agents is uniform on $[0, 1/2]e_1 \cup [0, 1/2]e_2$. Fix $\bar{x} \in (0, 1/2]$; let (y, α) be an optimal trajectory starting at $\bar{x}e_2$ at time $t = 0$. If we prove that (y, α) must reach O before time $t_{\bar{x}} = 5\bar{x}/4$ and stops there, then we have that the distribution of agents develops a singularity in the vertex O immediately after time $t = 0$.*

First of all, if the optimal trajectory (y,α) arrives in O it cannot leave O, otherwise taking $t_0 := \inf\{t\in[0,T]; y(t)=O\}\wedge T$, the trajectory $(\tilde{y},\tilde{\alpha})$, with

$$\tilde{\alpha}(t) := \begin{cases} \alpha(t) & \text{if } t\in[0,t_0] \\ 0 & \text{if } t\in(t_0,T] \end{cases}$$

does better than (y,α).
Moreover the optimal trajectory must reach O before time $t_{\bar{x}}$, if $\bar{x}\le 4T/5$. Indeed, for t_0 defined as before, there holds

$$J_0(\bar{x}e_2;(y,\alpha)) \ge -T+2t_0.$$

On the other hand, for the trajectory $(y_*(\cdot),\alpha_*(\cdot))$ with $\alpha_*(s) := \begin{cases} -e_2 & \text{if } s\in[0,\bar{x}] \\ 0 & \text{if } s\in(\bar{x},T] \end{cases}$, the cost is

$$J_0(\bar{x}e_2;(y_*,\alpha_*)) = -T+\frac{5\bar{x}}{2}.$$

So for $t_0 > 5\bar{x}/4$ we get a contradiction.
We deduce that, for any $t>0$, the distribution of players displays a singularity $c(t)\delta_O$ (δ_O is the Dirac delta in O) with $c(t)\ge(4t/5)\wedge(1/2)$. In particular, for $t>5/8$, we have $c\ge 1/2$.
Analogously, for $L_1=L_2=1$ and $L_*=-1$, a Dirac delta immediately appears in O and after the time $5/8$, the whole population is concentrated in O.

Example 3.2 *For another example of development of singular measures, see [1] which studies a problem arising in macroeconomics. In that case the singularity appears on the boundary of the domain.*

3.2 MFG equilibrium

Fix $\mu\in\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$; for any $x\in\mathcal{G}$, let us set

$$\tilde{\Gamma}_C^{\mu,\text{opt}}[x] = \left\{ y\in\tilde{\Gamma}_C[x] \;:\; J^\mu(x;(y,\alpha_y)) = \min_{(\widetilde{y},\widetilde{\alpha})\in\Gamma[x]} J^\mu(x;(\widetilde{y},\widetilde{\alpha}))\right\} \tag{3.5}$$

where J^μ is defined in (3.3) and α_y is a control such that $(y,\alpha_y)\in\Gamma[x]$ (see Remark 3.1).

Definition 3.1 *The probability measure $\mu\in\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$ is a constrained mean field game equilibrium associated with the initial distribution m_0 if*

$$\operatorname{supp}(\mu) \subset \bigcup_{x\in\operatorname{supp}(m_0)} \tilde{\Gamma}_C^{\mu,\text{opt}}[x]. \tag{3.6}$$

Theorem 3.1 *Assume (H_0), (H_1^{MFG}) and in (H_2^{MFG}); consider $C\ge\tilde{C}$ (where $\tilde{C}$ is the constant introduced in Remark 3.2). Then, there exists a constrained mean field game equilibrium $\mu\in\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$.*

The proof of the previous theorem is postponed to subsection 3.5.

3.3 Preliminary results

Lemma 3.3 *Let a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$, $\mu_n \in \mathcal{P}(\tilde{\Gamma}_C)$, be narrowly convergent to $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$ as $n \to \infty$. For all $t \in [0,T]$, the sequence $\{m^{\mu_n}(t)\}_{n\in\mathbb{N}}$ is narrowly convergent to $m^\mu(t)$.*

Proof. Adapting the arguments of [3, Lemma 3.1], we have that, for all $f \in C_b^0(\mathcal{G};\mathbb{R})$, there holds

$$\int_{\mathcal{G}} f(x)dm^{\mu_n}(t)(x) = \int_{\tilde{\Gamma}_C} f(y(t))d\mu_n(y) \to \int_{\tilde{\Gamma}_C} f(y(t))d\mu(y) = \int_{\mathcal{G}} f(x)dm^\mu(t)(x).$$

□

Lemma 3.4 *There holds*

$$\sup_{\mu\in\mathcal{P}(\tilde{\Gamma}_C)} Wass_1(m^\mu(t), m^\mu(s)) \le C|t-s|^{\frac{1}{2}} \qquad \forall t,s \in [0,T].$$

Proof. Consider any $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$. For any $t, s \in [0,T]$, there holds

$$\begin{aligned}\sup_\phi \int_{\mathcal{G}} \phi(x)[dm^\mu(t) - dm^\mu(s)](x) &= \sup_\phi \int_{\tilde{\Gamma}_C} [\phi(y(t)) - \phi(y(s))]\, d\mu(y) \\ &\le \int_{\tilde{\Gamma}_C} |y(t) - y(s)| d\mu(y) \le |t-s|^{\frac{1}{2}} \|\alpha\|_2\end{aligned}$$

where the supremum is performed over all the continuous 1-Lipschitz function. Owing to the definition of $\tilde{\Gamma}_C$ in (3.2) and to the arbitrariness of $\mu \in \mathcal{P}(\tilde{\Gamma}_C)$, last relation entails the statement. □

It is useful to recall the disintegration theorem:

Theorem 3.2 *Let X and Y be Radon metric spaces, $\pi : X \to Y$ be a Borel map, μ be a probability measure on X. Set $\nu = \pi\sharp\mu$. There exists a ν-almost everywhere uniquely defined Borel measurable family of probability measures $(\mu_y)_{y\in Y}$ on X such that*

$$\mu_y(X \setminus \pi^{-1}(y)) = 0, \quad \textit{for } \nu\textit{-almost all } y \in Y,$$

and for every Borel function $f : X \to [0,+\infty]$,

$$\int_X f(x)d\mu(x) = \int_Y \left(\int_X f(x)d\mu_y(x)\right) d\nu(y) = \int_Y \left(\int_{\pi^{-1}(y)} f(x)d\mu_y(x)\right) d\nu(y).$$

Recall that $(\mu_y)_{y\in Y}$ is a Borel family of probability measures if for any Borel subset B of X, $Y \ni y \mapsto \mu_y(B)$ is a Borel function from Y to $[0,1]$.

3.4 A closed graph property

Throughout this subsection, we assume $C \ge \tilde{C}$, where $\tilde{C}$ is the constant introduced in Remark 3.2. We first establish a closed graph property for the map $\tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$.

Proposition 3.1 *Consider $\mu \in \mathcal{P}_{m_0}(\tilde{\Gamma}_C)$ and $x \in supp(m_0)$. Consider also a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$, with $\mu_n \in \mathcal{P}_{m_0}(\tilde{\Gamma}_C)$, narrowly convergent to μ as $n \to \infty$ and a sequence of points $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in \mathcal{G}$ and $x_n \to x$ as $n \to \infty$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of curves such that $y_n \in \tilde{\Gamma}_C^{\mu_n,\mathrm{opt}}[x_n]$ and y_n uniformly converge to some curve y as $n \to \infty$. Then, y belongs to $\tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$, namely any trajectory (y, α_y) is an optimal trajectory for J^μ. In other words, the multivalued map $(x, \mu) \rightrightarrows \tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$ fulfills the closed graph property.*

Proof of Proposition 3.1. We borrow some arguments from the proof of Proposition 2.2 so we shall only detail the main novelties. We have to prove that

$$(i) \quad y \in \tilde{\Gamma}_C[x], \qquad (ii) \quad (y, \alpha_y) \text{ is optimal for } J^\mu.$$

By the definition of $\tilde{\Gamma}_C$, the controls α_{y_n} are uniformly bounded in L^2. Arguing as in the proof of Proposition 2.2, we deduce that, possibly passing to a subsequence (that we still denote α_{y_n}), the sequence $\{\alpha_{y_n}\}_n$ converges in the weak topology of $L^2((0,T), \mathbb{R}^d)$ to some control α_y, with $\|\alpha_y\|_2 \le C$, that $(y, \alpha_y) \in \Gamma[x]$ and $y \in \tilde{\Gamma}_C[x]$. The proof of point (i) is accomplished.
In order to prove (ii), it suffices to prove

$$J^\mu(x; (y, \alpha_y)) \le J^\mu(x; (\hat{y}, \hat{\alpha})) \qquad \forall (\hat{y}, \hat{\alpha}) \in \Gamma[x].$$

Fix any $(\hat{y}, \hat{\alpha}) \in \Gamma[x]$. Lemma 2.1 ensures that there exists a sequence $\{\hat{y}_n, \hat{\alpha}_n)\}_{n\in\mathbb{N}}$ such that $(\hat{y}_n, \hat{\alpha}_n) \in \Gamma[x_n]$, $\hat{y}_n(T) = \hat{y}(T)$ and

$$\hat{y}_n \to \hat{y} \text{ uniformly in } [0,T] \text{ as } n \to \infty, \qquad \|\hat{\alpha}_n\|_2 \le \|\hat{\alpha}\|_2 + o_n(1),$$

where $o_n(1)$ is a sequence such that $\lim_n o_n(1) = 0$. Since $y_n \in \tilde{\Gamma}_C^{\mu_n,\mathrm{opt}}[x_n]$, we have

$$J^{\mu_n}(x_n; (y_n, \alpha_{y_n})) \le J^{\mu_n}(x_n; (\hat{y}_n, \hat{\alpha}_n)). \tag{3.7}$$

We now study separately the two sides of previous inequality. For the right hand side of (3.7), the construction and the properties of $(\hat{y}_n, \hat{\alpha}_n)$ entail

$$J^{\mu_n}(x_n; (\hat{y}_n, \hat{\alpha}_n)) \quad \le \quad J^\mu(x; (\hat{y}, \hat{\alpha})) + \sum_{i=1}^{4} \bar{I}_i$$

where, for $\delta_n = d(x, x_n)$,

$$\begin{array}{ll} \bar{I}_1 = \int_0^{\delta_n} L[m^{\mu_n}(\tau)](\hat{y}_n(\tau), \tau) d\tau & \bar{I}_2 = \frac{\|\alpha_n\|_2^2 - \|\alpha\|_2^2}{2} \le o_n(1) \\ \bar{I}_3 = \int_{\delta_n}^{T} L[m^{\mu_n}(\tau)](\hat{y}_n(\tau), \tau) d\tau & \bar{I}_4 = -\int_0^T L[m^\mu(\tau)](\hat{y}(\tau), \tau) d\tau. \end{array}$$

By the boundedness of L, we have $\lim_{n\to\infty} \bar{I}_1 = 0$. Moreover, arguing as in the proof of Lemma 2.1, we have

$$\begin{aligned} \bar{I}_3 \quad = \quad & \int_0^T \left[\sum_{i=1}^{N} L_i \left[m^{\mu_n} \left(\frac{T - \delta_n}{T} \xi + \delta_n \right) \right] \left(\hat{y}(\xi), \frac{T - \delta_n}{T} \xi + \delta_n \right) \mathbf{1}_{\hat{y}(\xi) \in J_i \setminus \{O\}} \right. \\ & \left. + L_O \left[m^{\mu_n} \left(\frac{T - \delta_n}{T} \xi + \delta_n \right) \right] \left(\frac{T - \delta_n}{T} \xi + \delta_n \right) \mathbf{1}_{\hat{y}(\xi) = O} \right] \left(1 - \frac{\delta_n}{T} \right) d\xi \\ = \quad & \int_0^T L \left[m^{\mu_n} \left(\frac{T - \delta_n}{T} \xi + \delta_n \right) \right] \left(\hat{y}(\xi), \frac{T - \delta_n}{T} \xi + \delta_n \right) \left(1 - \frac{\delta_n}{T} \right) d\xi \end{aligned}$$

and consequently,

$$\bar{I}_3 + \bar{I}_4 = \bar{I}_5 + \bar{I}_6 + \bar{I}_7 + \bar{I}_8$$

where

$$\begin{array}{rcl}
\bar{I}_5 &=& -\frac{\delta_n}{T}\int_0^T L\left[m^{\mu_n}\left(\frac{T-\delta_n}{T}\xi+\delta_n\right)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right)\,d\xi \\
\bar{I}_6 &=& \int_0^T \Big(L\left[m^{\mu_n}\left(\frac{T-\delta_n}{T}\xi+\delta_n\right)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right) \\
&& \qquad -L\left[m^{\mu}\left(\frac{T-\delta_n}{T}\xi+\delta_n\right)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right)\Big)\,d\xi \\
\bar{I}_7 &=& \int_0^T \Big(L\left[m^{\mu}\left(\frac{T-\delta_n}{T}\xi+\delta_n\right)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right) - L\left[m^{\mu}(\xi)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right)\Big)\,d\xi \\
\bar{I}_8 &=& \int_0^T \Big(L\left[m^{\mu}(\xi)\right]\left(\hat{y}(\xi), \frac{T-\delta_n}{T}\xi+\delta_n\right) - L\left[m^{\mu}(\xi)\right]\left(\hat{y}(\xi),\hat{\alpha}(\xi),\xi\right)\Big)\,d\xi.
\end{array}$$

The boundedness of L entails: $|\bar{I}_5| = o_n(1)$. Owing to Lemma 3.3 and to our assumptions on L_i's, we have that $L_i[m^{\mu_n}(s)]$ uniformly converges to $L_i[m^{\mu}(s)]$ as $n\to\infty$, for every $i\in\{0,\dots,N\}$ and $s\in[0,T]$. Hence, by the dominated convergence theorem, we deduce: $|\bar{I}_6| = o_n(1)$. Moreover, by Lemma 3.4, again our assumptions on L_i's and dominated convergence theorem, we infer: $|\bar{I}_7| = o_n(1)$. Finally, since $\hat{y}$ is a bounded curve and $L_i[m]$ are continuous, we also get: $|\bar{I}_8| = o_n(1)$.

In summary, there holds

$$\limsup_n J^{\mu_n}(x_n;(\hat{y}_n,\hat{\alpha}_n)) \le J^{\mu}(x;(\hat{y},\hat{\alpha})). \tag{3.8}$$

For the left hand side of (3.7), we borrow some arguments of the proof of Proposition 2.1. By definition of cost (3.3), there holds

$$J^{\mu_n}(x_n;(y_n,\alpha_{y_n})) \;=\; \int_0^T \frac{|\alpha_{y_n}(\tau)|^2}{2}d\tau + \sum_{i=1}^{5}\hat{I}_i \tag{3.9}$$

where

$$\begin{array}{rcl}
\hat{I}_1 &=& \displaystyle\int_0^T \sum_{i=1}^N L_i[m^{\mu_n}(\tau)](y_n(\tau),\tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y(\tau)\in J_i\setminus\{O\}}d\tau \\
\hat{I}_2 &=& \displaystyle\int_0^T \sum_{i=1}^N L_i[m^{\mu_n}(\tau)](y_n(\tau),\tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y(\tau)\in \mathcal{G}\setminus J_i}d\tau \\
\hat{I}_3 &=& \displaystyle\int_0^T \sum_{i=1}^N L_i[m^{\mu_n}(\tau)](y_n(\tau),\tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}}\mathbf{1}_{y(\tau)=O}d\tau \\
\hat{I}_4 &=& \displaystyle\int_0^T L_O[m^{\mu_n}(\tau)](\tau)\mathbf{1}_{y_n(\tau)=O}d\tau \\
\hat{I}_5 &=& G[m^{\mu_n}(\tau)](y_n(T)).
\end{array}$$

By standard theory, the convergence in the weak topology of $L^2([t,T];\mathbb{R}^d)$ entails

$$\int_0^T \frac{|\alpha(\tau)|^2}{2}d\tau \le \liminf_{n\to\infty}\int_0^T \frac{|\alpha_n(\tau)|^2}{2}d\tau.$$

Let us recall from Lemma 3.3 that, for each $t\in[0,T]$, the map $\mathcal{P}(\tilde{\Gamma}_C)\ni\mu\mapsto m^{\mu}(t)\in\mathcal{P}(\mathcal{G})$ is continuous; hence, by our assumption, for every $i\in\{1,\dots,N\}$, $L_i[m^{\mu_n}(t)](\cdot,\cdot)$ and

$G_i[m^{\mu_n}(T)](\cdot)$ converge uniformly respectively to $L_i[m^\mu(t)](\cdot,\cdot)$ and to $G_i[m^\mu(T)](\cdot)$ as $n\to\infty$. Therefore, owing to dominated convergence theorem, we deduce

$$\hat{I}_1 \to \int_0^T \sum_{i=1}^N L_i[m^\mu(\tau)](y(\tau),\tau)\mathbf{1}_{y(\tau)\in J_i\setminus\{O\}}d\tau \qquad \text{and} \qquad \hat{I}_2\to 0, \quad \text{as } n\to\infty.$$

By the same arguments of the proof of Proposition 2.1 and the definition of $G[m]$ in (3.1), we obtain

$$\liminf_{n\to\infty} \hat{I}_5 \geq G[m^\mu(T)](y(T)).$$

Furthermore, we have

$$\begin{aligned}\hat{I}_3+\hat{I}_4 \quad &= \quad \int_0^T \left[\sum_{i=1}^N L_i[m^{\mu_n}(t)](y_n(\tau),\tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}} + L_O[m^{\mu_n}(t)](\tau)\mathbf{1}_{y_n(\tau)=O}\right]\mathbf{1}_{y(\tau)=O}d\tau \\ &\quad + \int_0^T L_O[m^{\mu_n}(t)](\tau)\mathbf{1}_{y_n(\tau)=O}\mathbf{1}_{y(\tau)\neq O}d\tau.\end{aligned}$$

Again the dominated convergence theorem ensures

$$\int_0^T L_O[m^{\mu_n}(t)](\tau)\mathbf{1}_{y_n(\tau)=O}\mathbf{1}_{y(\tau)\neq O}d\tau \to 0 \qquad \text{as } n\to\infty.$$

Moreover, Fatou's Lemma and the boundedness of L_i yield

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T &\left[\sum_{i=1}^N L_i[m^{\mu_n}(\tau)](y_n(\tau),\tau)\mathbf{1}_{y_n(\tau)\in J_i\setminus\{O\}}\right.\\ &\left.+L_O[m^{\mu_n}(\tau)](\tau)\mathbf{1}_{y_n(\tau)=O}\right]\mathbf{1}_{y(\tau)=O}d\tau \geq \int_0^T L_O[m^\mu(\tau)](\tau)\mathbf{1}_{y(\tau)=O}d\tau.\end{aligned}$$

Replacing all these relations in (3.9), we conclude

$$\begin{aligned}\liminf_{n\to\infty} J^{\mu_n}(x_n;(y_n,\alpha_{y_n})) \quad &\geq \quad \int_0^T \left[\frac{|\alpha(\tau)|^2}{2}+\sum_{i=1}^N L_i[m^\mu(\tau)](y(\tau),\tau)\mathbf{1}_{y(\tau)\in J_i\setminus\{O\}}\right.\\ &\qquad \left.+L_O[m^\mu(\tau)](\tau)\mathbf{1}_{y(\tau)=O}\right]d\tau + G[m^\mu(T)](y(T)) \\ &= \quad J^\mu(x;(y,\alpha_y)).\end{aligned} \tag{3.10}$$

In conclusion, relations (3.7), (3.8) and (3.10) entail

$$J^\mu(x;(y,\alpha_y)) \leq J^\mu(x;(\hat{y},\hat{\alpha}));$$

by the arbitrariness of $(\hat{y},\hat{\alpha})\in\Gamma[x]$, we get $J^\mu(x;(y,\alpha_y)) = \min_{(\hat{y},\hat{\alpha})\in\Gamma[x]} J^\mu(x;(\hat{y},\hat{\alpha}))$ which is equivalent to (ii). □

3.5 Proof of Theorem 3.1

This subsection is entirely devoted to the proof of Theorem 3.1. To this end, we follow the arguments of [11].

Let us first recall some notations. For every $\mu\in\mathcal{P}_{m_0}(\tilde{\Gamma}_C)$, let J^μ be the associated cost as in (3.3); for any $x\in\mathcal{G}$, let $\tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$ be the set of optimal curves starting from x for the cost J^μ as in (3.4). Proposition 2.1 ensures: $\tilde\Gamma_C^{\mu,\mathrm{opt}}[x] \neq \emptyset$ for every $x \in \mathcal{G}$. It is worth to recall from Lemma 3.1 that the set $\tilde\Gamma_C$ is compact; by Prokhorov theorem [7, Theorem 5.1.3], also $\mathcal{P}(\tilde\Gamma_C)$ is compact.

We introduce the multivalued map $E : \mathcal{P}_{m_0}(\tilde\Gamma_C) \rightrightarrows \mathcal{P}_{m_0}(\tilde\Gamma_C)$ as

$$E(\mu) = \{\hat\mu \in \mathcal{P}_{m_0}(\tilde\Gamma_C) \ : \ \operatorname{supp}\hat\mu_x \subset \tilde\Gamma_C^{\mu,\mathrm{opt}}[x] \quad m_0-\text{a.e. } x \in \mathcal{G}\} \tag{3.11}$$

where $\{\hat\mu_x\}_{x\in\mathcal{G}}$ is the family of Borel probability measures on $\mathcal{P}_{m_0}(\tilde\Gamma_C)$ obtained applying the disintegration Theorem 3.2 with μ, X, Y and π replaced respectively by $\hat\mu$, $\mathcal{P}_{m_0}(\tilde\Gamma_C)$, $\mathcal{G}$ and e_0 (so, clearly, ν coincides with m_0). In order to achieve the statement, it suffices to prove that the map E admits a fixed point. Let us assume for the moment the following properties

(i) for every $\mu \in \mathcal{P}_{m_0}(\tilde\Gamma_C)$, the set $E(\mu)$ is not empty and convex

(ii) the map E fulfills the closed graph property.

Then, Kakutani fixed point theorem ensures that the map E admits a fixed point. It remains to prove the above properties.
(i). From Proposition 2.1 and Proposition 2.2, we recall that $\tilde\Gamma_C^{\mu,\mathrm{opt}}[x] \neq \emptyset$ for every $x \in \mathcal{G}$ and that the map $\tilde\Gamma_C^{\mu,\mathrm{opt}}[\cdot]$ fulfills the closed graph property. Therefore, the result [6, Theorem 8.1.4] guarantees that the map $\tilde\Gamma_C^{\mu,\mathrm{opt}}[\cdot]$ has a Borel measurable selection that we denote: $x \mapsto y_x^\mu$ for every $x \in \mathcal{G}$. We introduce a measure $\hat\mu$ on $\tilde\Gamma_C$ as follows:

$$\hat\mu(B) = \int_{\mathcal{G}} \delta_{y_x^\mu}(B)\, m_0(dx) \qquad \forall \text{ Borel } B \subset \tilde\Gamma_C,$$

where $\delta_{y_x^\mu}(\cdot)$ is the Dirac delta-function centered in y_x^μ. Note that $\hat\mu_x = \delta_{y_x^\mu}$ for m_0-a.e. $x \in \mathcal{G}$. Hence, $\hat\mu$ belongs to $E(\mu)$.
Let us now prove that $E(\mu)$ is convex. Fix $\mu^1, \mu^2 \in E(\mu)$ and $\lambda \in [0,1]$. By easy calculation, one obtains $\lambda\mu^1 + (1-\lambda)\mu^2 \in \mathcal{P}_{m_0}(\tilde\Gamma_C)$. On the other hand, for $i = 1, 2$, since $\mu^i \in E(\mu)$, by the disintegration theorem 3.2, there exist a Borel measurable family $\{\mu_x^i\}_{x\in\mathcal{G}}$ of probability measures (which is m_0-a.e. uniquely defined and "disintegrate" μ^i with respect to m_0) and a set $\mathcal{A}_i \subset \mathcal{G}$ such that $m_0(\mathcal{A}_i) = 0$ and $\operatorname{supp}\mu_x^i \subset \tilde\Gamma_C^{\mu,\mathrm{opt}}[x]$ for every $x \in \mathcal{G} \setminus \mathcal{A}_i$. Therefore, the measure $\lambda\mu^1 + (1-\lambda)\mu^2$ can be disintegrated as follows: for each Borel function f on $\tilde\Gamma_C$, we have

$$\int_{\tilde\Gamma_C} f(\gamma)(\lambda\mu^1 + (1-\lambda)\mu^2)\,(d\gamma) = \int_{\mathcal{G}} \left(\int_{\tilde\Gamma_C} f(\gamma)(\lambda\mu_x^1 + (1-\lambda)\mu_x^2)\,(d\gamma) \right) m_0(dx)$$

with $m_0(\mathcal{A}_1 \cup \mathcal{A}_2) = 0$ and

$$\operatorname{supp}(\lambda\mu_x^1 + (1-\lambda)\mu_x^2) \subset \tilde\Gamma_C^{\mu,\mathrm{opt}}[x] \qquad \forall x \in \mathcal{G} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2).$$

Hence, $\lambda\mu^1 + (1-\lambda)\mu^2$ belongs to $E(\mu)$, namely $E(\mu)$ is convex.
(ii). Consider a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures $\mu_n \in \mathcal{P}_{m_0}(\tilde\Gamma_C)$ which narrowly converges to some $\mu \in \mathcal{P}_{m_0}(\tilde\Gamma_C)$ as $n \to \infty$. Consider also a sequence $\{\hat\mu_n\}_{n\in\mathbb{N}}$, with $\hat\mu_n \in E(\mu_n)$ for any $n \in \mathbb{N}$, which narrowly converges to some $\hat\mu \in \mathcal{P}_{m_0}(\tilde\Gamma_C)$ as $n \to \infty$. Our aim is to prove that $\hat\mu$ belongs to $E(\mu)$.
By the disintegration theorem, there exists a m_0-a.e. uniquely defined Borel measurable

family of measures $\{\hat{\mu}_x\}_{x\in\mathcal{G}}$ on $\tilde{\Gamma}_C$ and $\mathcal{A}\subset\mathcal{G}$ such that: $m_0(\mathcal{A})=0$, $\hat{\mu}_x(\tilde{\Gamma}_C\setminus e_0^{-1}(\{x\}))=0$ for every $x\in\mathcal{G}\setminus\mathcal{A}$ and

$$\int_{\tilde{\Gamma}_C} f(y)\hat{\mu}(dy)=\int_{\mathcal{G}}\left(\int_{\tilde{\Gamma}_C[x]} f(y)\hat{\mu}_x(dy)\right)m_0(dx).$$

Consider $x\in\mathcal{G}\setminus\mathcal{A}$ and $\hat{y}\in\operatorname{supp}\hat{\mu}_x$. The Kuratowski theorem ([7, Proposition 5.1.8]) ensures that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$, with $y_n\in\operatorname{supp}\hat{\mu}_n$, which converges to $\hat{y}$ in the topology of $\tilde{\Gamma}_C$. Let $x_n=e_0(y_n)$. Since $\hat{\mu}_n\in E(\mu_n)$, there holds: $y_n\in\tilde{\Gamma}_C^{\mu_n,\mathrm{opt}}[x_n]$. By Proposition 3.1, we infer $\hat{y}\in\tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$. By the arbitrariness of $\hat{y}\in\operatorname{supp}\hat{\mu}_x$, we obtain $\operatorname{supp}\hat{\mu}_x\subset\tilde{\Gamma}_C^{\mu,\mathrm{opt}}[x]$ and consequently, by the arbitrariness of $x\in\mathcal{G}\setminus\mathcal{A}$, that $\hat{\mu}$ belongs to $E(\mu)$. □

3.6 Mild solutions

Theorem 3.1 ensures the existence of a MFG equilibrium $\mu\in\mathcal{P}_{m_0}(\Gamma)$. For simplicity of notations, we write

$$\ell_i(x,t)=L_i[m^\mu(t)](x,t)\qquad\text{and}\qquad g_i(x)=G_i[m^\mu(T)](x)\qquad\forall(x,t)\in\mathcal{G}\times[0,T]\tag{3.12}$$

and we shall use the abridged notation of L as in (2.7). By Lemma 3.4, these functions ℓ_i fulfills assumption (H_1). Note also that these costs ℓ_i are the ones payed by the agents in our MFG. We can consider the value function naturally associated to this equilibrium μ:

$$u(x,t)=\inf_{(y_x,\alpha)\in\Gamma_t[x]} J_t^\mu(x;(y_x,\alpha))\tag{3.13}$$

where J_t^μ is the cost defined in (3.3). The purpose of this section is to establish several properties of the value function using the results of Section 2.

As preliminary step, invoking Proposition 2.3, Proposition 2.4, Remark 2.6 and Lemma 3.4, we have that the value function verifies the following properties.

Proposition 3.2 *The value function u defined in* (3.13) *fulfills the following properties*

(i) (Dynamic programming principle)

$$u(x,t)=\inf_{(y_x,\alpha)\in\Gamma_{t,\bar{t}}[x]}\left\{u(y_x(\bar{t}),\bar{t})+\int_t^{\bar{t}}\left(L(y_x(\tau),\tau)+\frac{|\alpha(\tau)|^2}{2}\right)d\tau\right\}$$

where

$$\Gamma_{t,\bar{t}}[x]=\left\{\begin{array}{ll}(y_x,\alpha)\in L^2([t,\bar{t}],M): & y_x\in W^{1,2}([t,\bar{t}];\mathcal{G}),\\ & y_x(s)=x+\displaystyle\int_t^s\alpha(\tau)d\tau\quad \textit{in } [t,\bar{t}]\end{array}\right\};$$

(ii) the value function is continuous in $\mathcal{G}\times[0,T)$.

Applying Theorem 2.3, we can now prove that u solves the HJ problem associated with the costs ℓ_i.

Theorem 3.3 *The value function u defined in* (3.13) *is a solution to problem* (2.31) *with the costs ℓ_i defined in* (3.12) *and $g=G[m^\mu(T)]$.*

Remark 3.3 *Under classical monotonicity assumptions for L and G, see e.g. [11, Theorem 4.1 and Remark 4.1], the mild solution is unique.*

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