Unified Generalized H2 Nonlinear Parameter Varying Observer: Application to Automotive Suspensions

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***Abstract*—This paper extends the unified observer design problem to the class of Nonlinear Parameter Varying (NLPV) systems with parameter dependence in both the dynamics and the control input matrices. First, parameterization of the observer matrices, herein generalized for the NLPV case, allows us to decouple the input disturbance from the estimation error. Then, the vanishing disturbance caused by the nonlinearity is bounded by the Lipschitz property and the effect of measurement noise on the error is minimized using the generalized $\mathcal{H}_2$ condition. Both objectives are combined into a single framework thanks to the $\mathcal{S}$-procedure. Furthermore, the asymptotic stability of the error is tackled using a parameter-dependent Lyapunov function, then a grid-based Linear Matrix Inequalities (LMIs) solution is provided, which reduces conservatism. The efficiency of this observer is illustrated and compared with an LPV observer through the damper force estimation problem, a crucial topic in semi-active suspensions.**

***Index Terms*—Nonlinear parameter varying systems, observers, suspension systems.**

I. INTRODUCTION

IT has been known from the literature that under particular conditions and assumptions, the Linear Parameter Varying (LPV) representation of a nonlinear system is interesting thanks to its linear-like stability analysis and controller/observer design, including as well the $\mathcal{H}_\infty$ and/or generalized $\mathcal{H}_2$ (g$\mathcal{H}_2$) performance criteria [1].

However, "linearizing" nonlinear systems using the LPV language, known as the quasi-LPV representation, comes with the cost of reducing the generality of the system representation. Recall that to be assigned as a scheduling parameter, a (nonlinear) function of the state x must be known or estimated and bounded at least in the region where x remains. This condition, even if it can be satisfied, would certainly increase conservatism. Often, by strategically choosing to maintain a certain level of nonlinearity in the system representation (rather than render it linear thanks to the LPV technique), we can benefit from interesting properties of nonlinear functions, e.g., Lipschitz conditions, which reduce conservatism and lead to more realistic results.

This idea leads to the so-called Nonlinear Parameter Varying (NLPV) class of systems that has emerged as a potential research trend in the LPV community [2]. In the literature, the observers of this new class take various structures, from the classical form [3], the descriptor form [4], [5] to the two-DOF [6] or the generalized one [7], [8]. The main objective of this work is to extend the unified formulation of unknown input observers [9], [10] for NLPV systems. Such unified observers are indeed interesting since they provide a general parameterization of a set of observers while ensuring the decoupling of the input disturbance or its minimization through the $\mathcal{H}_\infty$ condition. Here, the effect of the system's nonlinearity on the estimation error is still considered and bounded by the Lipschitz condition, thus decreasing conservatism compared to merely bounding it by a constant such as in [11]. This condition has also been well studied for nonlinear observer design such as [12], [13], [14], [15], or [16].

In this paper, we propose a unified g$\mathcal{H}_2$ observer to the state estimation problem for a class of NLPV systems including a Lipschitz nonlinearity, where the state and input matrices do depend on the parameter vector. The solution is obtained by solving a set of Linear Matrix Inequalities (LMIs). Our main contributions are as follows:

- Observer parameterization is generalized from the time-invariant case [10] into a parameter-dependent form (as in Section III), maximizing the adaptability of the scheduled observer with respect to the system's variation. This generalization also allows us to have parameter dependence even in the system's dynamics matrix (as in (1));
- Compared to [8] where the authors assumed the existence of *constant* arbitrary matrices satisfying all the rank conditions and where the problem was solved using a polytopic approach, which can be overly conservative, parameter-dependent arbitrary matrices and the grid-based methodology [17] are used instead throughout this work, for which these rank conditions are assumed almost everywhere but verified at the grid points only;
- The g$\mathcal{H}_2$ condition is used to minimize the effects of the measurement noise on the estimation error whereas it was the $\mathcal{H}_\infty$ condition in [10]. As sensor noise usually appears in the high-frequency domain, the g$\mathcal{H}_2$ norm is more effective to handle this issue;
- For illustration, the method is applied for damper force estimation in vehicle semi-active (SA) suspensions with the nonlinear model in [18] and compared with the LPV observer there, in both the frequency and time domains.

This paper is outlined as follows. In Section II, the observer design problem is formulated. Section III presents the observer parameterization. The design method is detailed in Section IV. The application is illustrated in Section V. Finally, the conclusion is drawn in Section VI.

Gia Quoc Bao Tran was with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, 38000 Grenoble, France. He is now with Centre Automatique et Systèmes (CAS), Mines Paris, Université PSL, 75006 Paris, France (e-mail: gia-quoc-bao.tran@minesparis.psl.eu).

Thanh-Phong Pham is with the University of Danang—University of Technology and Education, 550000 Danang, Vietnam (e-mail: pt-phong@ute.udn.vn).

Olivier Sename is with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, 38000 Grenoble, France (e-mail: olivier.sename@grenoble-inp.fr).

This research is funded by Funds for Science and Technology Development of the University of Danang under project number B2020-DN06-21.

Notations. Let $\underline{w}$ (resp. $\overline{w}$) denote the minimum (resp. maximum) value of the variable w. Let $W^\top$ be the transpose of the matrix W and W^+ be its Moore-Penrose inverse. Last, let W^{-1} denote the inverse of the invertible square matrix W.

II. Problem Formulation

Consider an NLPV system of the form

$$\begin{cases} \dot{x} = A(\rho)x + B(\rho)\Phi(x,u) + D_1\omega_r \\ y = Cx + D_2\omega_n, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state; $u \in \mathbb{R}^{n_u}$ is the control input; $\omega_r \in \mathbb{R}^{n_r}$ is the unknown input disturbance; $\omega_n \in \mathbb{R}^{n_n}$ is the measurement noise; $y \in \mathbb{R}^{n_y}$ is the measured output; $\rho \in \mathbb{R}^{n_\rho}$ is the vector of continuous-time varying parameters. As usually assumed for LPV systems, ρ is known and bounded, i.e., $\rho \in [\underline{\rho}, \overline{\rho}]$ element-wise; all matrices $A(\rho)$, $B(\rho)$, D_1, C, and D_2 are known and defined (continuous) for any value of $\rho \in [\underline{\rho}, \overline{\rho}]$. As input disturbance and measurement noise usually enter NLPV systems through constant matrices, without loss of generality, we consider D_1 and D_2 to be constant. The control input is always bounded, i.e., $u \in [\underline{u}, \overline{u}]$ element-wise.

Assumption 1: The map $\Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_\Phi}$ (where n_Φ is the dimension of the codomain of Φ) is globally Lipschitz over the set of the state vector x [19], [20], i.e.,

$$\|\Phi(x,u) - \Phi(\hat{x},u)\| \leq \|\Gamma(x-\hat{x})\|, \forall (x,\hat{x}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}, \tag{2}$$

where Γ is the known Lipschitz constant matrix that is defined and computed assuming a bounded input $u \in [\underline{u}, \overline{u}]$. For the interested readers, the even more general case where Φ is only locally Lipschitz is analyzed later in Remark 4.

Consider a self-scheduled NLPV observer of the form

$$\begin{cases} \dot{z} = N(\rho)z + J(\rho)y + H(\rho)\Phi(\hat{x},u) + M(\rho)v \\ \dot{v} = P(\rho)z + Q(\rho)y + G(\rho)v \\ \hat{x} = R(\rho)z + S(\rho)y, \end{cases} \tag{3}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ is the estimated state and $z, v \in \mathbb{R}^{n_x}$ are the state and auxiliary vectors of the observer. By designing the observer, we determine all the parameter-dependent matrices $N(\rho)$, $J(\rho)$, $H(\rho)$, $M(\rho)$, $P(\rho)$, $Q(\rho)$, $G(\rho)$, $R(\rho)$, and $S(\rho)$ such that all the conditions related to decoupling the bounded unknown input disturbance and minimizing the effects of random measurement noise on the estimation error (under Assumption 1) detailed later are satisfied.

Define the dynamic error as $\epsilon = z - T(\rho)x \in \mathbb{R}^{n_x}$, where $T(\rho) \in \mathbb{R}^{n_x \times n_x}$ is a parameter-dependent arbitrary matrix. Differentiating ϵ with respect to time, using (1) and (3), and denoting $\zeta = (\epsilon \quad v)^\top \in \mathbb{R}^{2n_x}$, we obtain

$$\begin{cases} \dot{\zeta} = \begin{pmatrix} N(\rho) & M(\rho) \\ P(\rho) & G(\rho) \end{pmatrix} \zeta + \begin{pmatrix} H(\rho) - T(\rho)B(\rho) \\ 0 \end{pmatrix} \Phi(\hat{x},u) \\ \quad + \begin{pmatrix} T(\rho)D_1 \\ 0 \end{pmatrix} \omega_r + \begin{pmatrix} N(\rho)T(\rho) - T(\rho)A(\rho) + J(\rho)C \\ P(\rho)T(\rho) + Q(\rho)C \end{pmatrix} x \\ \quad + \begin{pmatrix} -T(\rho)B(\rho) \\ 0 \end{pmatrix} \Delta\Phi + \begin{pmatrix} J(\rho)D_2 \\ Q(\rho)D_2 \end{pmatrix} \omega_n \\ \hat{x} = \begin{pmatrix} R(\rho) & 0 \end{pmatrix} \zeta + (R(\rho)T(\rho) + S(\rho)C)\, x + S(\rho)D_2\omega_n, \end{cases} \tag{4}$$

where $\Delta\Phi = \Phi(x,u) - \Phi(\hat{x},u)$ satisfies (2).

If the algebraic conditions

$$N(\rho)T(\rho) - T(\rho)A(\rho) + J(\rho)C = 0, \tag{5}$$

$$T(\rho)D_1 = 0, \tag{6}$$

$$H(\rho) - T(\rho)B(\rho) = 0, \tag{7}$$

$$P(\rho)T(\rho) + Q(\rho)C = 0, \tag{8}$$

$$R(\rho)T(\rho) + S(\rho)C = I, \tag{9}$$

are satisfied for all $\rho \in [\underline{\rho}, \overline{\rho}]$, the system in (4) is reduced to

$$\begin{cases} \dot{\zeta} = \begin{pmatrix} N(\rho) & M(\rho) \\ P(\rho) & G(\rho) \end{pmatrix} \zeta + \begin{pmatrix} -T(\rho)B(\rho) \\ 0 \end{pmatrix} \Delta\Phi \\ \quad + \begin{pmatrix} J(\rho)D_2 \\ Q(\rho)D_2 \end{pmatrix} \omega_n \\ e = \begin{pmatrix} R(\rho) & 0 \end{pmatrix} \zeta + S(\rho)D_2\omega_n, \end{cases} \tag{10}$$

where $e = \hat{x} - x$ is the estimation error. It is worth noting from (10) that the disturbance ω_r has been decoupled from the error e.

The observer design problem is, therefore, under Assumption 1, to find the matrices $N(\rho)$, $J(\rho)$, $H(\rho)$, $M(\rho)$, $P(\rho)$, $Q(\rho)$, $G(\rho)$, $R(\rho)$, and $S(\rho)$ of the observer (3) such that

- The system (10) is asymptotically stable for $\omega_n(t) = 0$;
- $\|e(t)\|_\infty < \gamma\|\omega_n(t)\|_2$ for $\omega_n(t) \neq 0$; γ is minimized.

III. Observer Parameterization

It is worth noting that solving equations (5)-(9) with parameter variations is not an easy task. In this Section, the grid-based methodology is used for observer parameterization, which means the algebraic and rank conditions are assumed *almost* everywhere but checked only at the (high enough number) of frozen values (detailed in Remark 2). Therefore, ρ will be considered here only at these grid points. This means, when there is no confusion, ρ denotes its frozen values at N_g grid points in a finite set of values in $[\underline{\rho}, \overline{\rho}]$.

In order to determine the observer matrices $N(\rho)$, $J(\rho)$, $H(\rho)$, $M(\rho)$, $P(\rho)$, $Q(\rho)$, $G(\rho)$, $R(\rho)$, and $S(\rho)$ satisfying all the conditions (5)-(9), parameterization [9], [10] is performed using the general solution of (5)-(9).

First, (8) and (9) are rewritten as

$$\begin{pmatrix} P(\rho) & Q(\rho) \\ R(\rho) & S(\rho) \end{pmatrix} \begin{pmatrix} T(\rho) \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{11}$$

Note that (11) is solvable if and only if

$$\operatorname{rank}\begin{pmatrix} T(\rho) & C & 0 & I \end{pmatrix}^\top = \operatorname{rank}\begin{pmatrix} T(\rho) & C \end{pmatrix}^\top = n_x, \forall \rho \in [\underline{\rho}, \overline{\rho}]. \tag{12}$$

Let $E(\rho) \in \mathbb{R}^{n_x \times n_x}$ be a full row rank parameter-dependent arbitrary matrix such that

$$\operatorname{rank}\begin{pmatrix} E(\rho) & C \end{pmatrix}^\top = \operatorname{rank}\begin{pmatrix} T(\rho) & C \end{pmatrix}^\top = n_x, \forall \rho \in [\underline{\rho}, \overline{\rho}]. \tag{13}$$

Then, there always exists a parameter matrix $K(\rho)$ such that

$$\begin{pmatrix} T(\rho) \\ C \end{pmatrix} = \begin{pmatrix} I & -K(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} E(\rho) \\ C \end{pmatrix} \iff T(\rho) = E(\rho) - K(\rho)C. \tag{14}$$

As a result, (11) becomes

$$\begin{pmatrix} P(\rho) & Q(\rho) \\ R(\rho) & S(\rho) \end{pmatrix} \begin{pmatrix} I & -K(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} E(\rho) \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

whose exact solution is

$$\begin{aligned} \begin{pmatrix} P(\rho) & Q(\rho) \\ R(\rho) & S(\rho) \end{pmatrix} &= (\begin{pmatrix} 0 \\ I \end{pmatrix} \Sigma^+(\rho) \\ &\quad - Y_m(\rho)(I - \Sigma(\rho)\Sigma^+(\rho))) \begin{pmatrix} I & K(\rho) \\ 0 & I \end{pmatrix}, \end{aligned}$$

where $\Sigma(\rho) = \begin{pmatrix} E(\rho) \\ C \end{pmatrix}$ and $Y_m(\rho)$ is a free matrix of appropriate dimension.

This is equivalent to

$$\begin{aligned} P(\rho) &= -Y_{m1}(\rho)\beta_1(\rho), \\ Q(\rho) &= -Y_{m1}(\rho)\beta_2(\rho), \\ R(\rho) &= \alpha_1(\rho) - Y_{m2}(\rho)\beta_1(\rho), \\ S(\rho) &= \alpha_2(\rho) - Y_{m2}(\rho)\beta_2(\rho), \end{aligned} \tag{15}$$

where $Y_{m1}(\rho) = \begin{pmatrix} I & 0 \end{pmatrix} Y_m(\rho)$, $Y_{m2}(\rho) = \begin{pmatrix} 0 & I \end{pmatrix} Y_m(\rho)$, $\alpha_1(\rho) = \Sigma^+(\rho) \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\alpha_2(\rho) = \Sigma^+(\rho) \begin{pmatrix} K(\rho) \\ I \end{pmatrix}$, $\beta_1(\rho) = (I - \Sigma(\rho)\Sigma^+(\rho)) \begin{pmatrix} I \\ 0 \end{pmatrix}$, and $\beta_2(\rho) = (I - \Sigma(\rho)\Sigma^+(\rho)) \begin{pmatrix} K(\rho) \\ I \end{pmatrix}$.

Next, from (6) and (14), we obtain

$$K(\rho)CD_1 = E(\rho)D_1, \tag{16}$$

which is solvable if and only if

$$\begin{aligned} \operatorname{rank} \begin{pmatrix} E(\rho)D_1 \\ CD_1 \end{pmatrix} &= \operatorname{rank}(\begin{pmatrix} E(\rho) \\ C \end{pmatrix} D_1) \\ &= \operatorname{rank} D_1 = \operatorname{rank}(CD_1), \forall \rho \in [\underline{\rho}, \overline{\rho}], \end{aligned} \tag{17}$$

and then the exact solution of (16) is

$$K(\rho) = E(\rho)D_1(CD_1)^+.$$

From (7), one obtains

$$\begin{aligned} H(\rho) = T(\rho)B(\rho) &= (E(\rho) - K(\rho)C)B(\rho) \\ &= (E(\rho) - E(\rho)D_1(CD_1)^+C)B(\rho). \end{aligned} \tag{18}$$

Then, substituting (14) into (5), we obtain

$$\begin{aligned} N(\rho)(E(\rho) - K(\rho)C) - (E(\rho) - K(\rho)C)A(\rho) + J(\rho)C = 0 \\ \iff \begin{pmatrix} N(\rho) & J(\rho) - N(\rho)K(\rho) \end{pmatrix} \Sigma(\rho) \\ = (E(\rho) - E(\rho)D_1(CD_1)^+C)A(\rho), \end{aligned}$$

which can be parameterized as

$$\begin{pmatrix} N(\rho) & K_1(\rho) \end{pmatrix} \Sigma(\rho) = \Theta(\rho), \tag{19}$$

where

$$\begin{aligned} K_1(\rho) &= J(\rho) - N(\rho)K(\rho), \\ \Theta(\rho) &= (E(\rho) - E(\rho)D_1(CD_1)^+C)A(\rho), \end{aligned} \tag{20}$$

and the solution of (19) is given by

$$\begin{pmatrix} N(\rho) & K_1(\rho) \end{pmatrix} = \Theta(\rho)\Sigma^+(\rho) - Y_{m3}(\rho)(I - \Sigma(\rho)\Sigma^+(\rho)),$$

which is equivalent to

$$\begin{aligned} N(\rho) &= \alpha_3(\rho) - Y_{m3}(\rho)\beta_1, \\ K_1(\rho) &= \alpha_4(\rho) - Y_{m3}(\rho)\beta_3(\rho), \end{aligned} \tag{21}$$

where $Y_{m3}(\rho)$ is a free matrix of appropriate dimension, $\alpha_3(\rho) = \Theta(\rho)\Sigma^+(\rho) \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\alpha_4(\rho) = \Theta(\rho)\Sigma^+(\rho) \begin{pmatrix} 0 \\ I \end{pmatrix}$, and $\beta_3(\rho) = (I - \Sigma(\rho)\Sigma^+(\rho)) \begin{pmatrix} 0 \\ I \end{pmatrix}$.

Remark 1: If $P(\rho)$, $Q(\rho)$, $R(\rho)$, $S(\rho)$, $H(\rho)$, $N(\rho)$, and $J(\rho)$ can be chosen according to (15), (18), (21), and (20), respectively, then all the decoupling conditions (5)-(9) are satisfied. As a result, the system (4) is rewritten as (10) and the disturbance ω_r is effectively decoupled from the error e.

Remark 2: All the rank conditions in this Section, namely (12), (13), and (17), are assumed for *almost* all parameter values. The set of values of ρ that violate these, if any, must be strictly finite and isolated. As a result, these conditions are satisfied for all the grid points used for observer design.

After parameterization, the error system (10) is rewritten as

$$\begin{cases} \dot{\zeta} = \mathbb{A}(\rho)\zeta + \mathbb{W}(\rho)\Delta\Phi + \mathbb{B}(\rho)\omega_n \\ e = \mathbb{C}(\rho)\zeta + \mathbb{D}(\rho)\omega_n, \end{cases} \tag{22}$$

where, from the results of the parameterization above:

$$\begin{aligned} \mathbb{A}(\rho) &= \begin{pmatrix} N(\rho) & M(\rho) \\ P(\rho) & G(\rho) \end{pmatrix} = \mathbb{A}_1(\rho) - Z(\rho)\mathbb{A}_2(\rho), \\ \mathbb{W}(\rho) &= \begin{pmatrix} -T(\rho)B(\rho) \\ 0 \end{pmatrix}, \\ \mathbb{B}(\rho) &= \begin{pmatrix} J(\rho)D_2 \\ Q(\rho)D_2 \end{pmatrix} = \mathbb{B}_1(\rho) - Z(\rho)\mathbb{B}_2(\rho), \\ \mathbb{C}(\rho) &= \begin{pmatrix} R(\rho) & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1(\rho) - Y_{m2}(\rho)\beta_1(\rho) & 0 \end{pmatrix}, \\ \mathbb{D}(\rho) &= S(\rho)D_2 = (\alpha_2(\rho) - Y_{m2}(\rho)\beta_2(\rho))D_2, \end{aligned}$$

where $\mathbb{A}_1(\rho) = \begin{pmatrix} \alpha_3(\rho) & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{A}_2(\rho) = \begin{pmatrix} \beta_1(\rho) & 0 \\ 0 & -I \end{pmatrix}$, $\mathbb{B}_1(\rho) = \begin{pmatrix} \Theta(\rho)\Sigma^+(\rho) \begin{pmatrix} K(\rho) \\ I_{n_y} \end{pmatrix} D_2 \\ 0 \end{pmatrix}$, $\mathbb{B}_2(\rho) = \begin{pmatrix} \beta_2(\rho)D_2 \\ 0 \end{pmatrix}$, and $Z(\rho) = \begin{pmatrix} Y_{m3}(\rho) & M(\rho) \\ Y_{m1}(\rho) & G(\rho) \end{pmatrix}$.

All the matrices $\mathbb{A}_1(\rho)$, $\mathbb{A}_2(\rho)$, $\mathbb{W}(\rho)$, $\mathbb{B}_1(\rho)$, $\mathbb{B}_2(\rho)$, $\mathbb{C}(\rho)$, and $\mathbb{D}(\rho)$ are known and computing all the observer matrices reduces to finding $Z(\rho)$, which is discussed in Section IV.

IV. Observer Design

Following the problem formulation in Section II, with $\Delta\Phi$ bounded by Assumption 1, the observer design problem is now to find $Z(\rho)$ such that

- The system (22) is asymptotically stable for $\omega_n(t) = 0$;
- $\|e(t)\|_\infty < \gamma\|\omega_n(t)\|_2$ for $\omega_n(t) \neq 0$; γ is minimized.

First, to apply the g$\mathcal{H}_2$ condition, the parameterized error system has to be strictly proper, i.e., $\mathbb{D}(\rho) = 0, \forall \rho \in [\underline{\rho}, \overline{\rho}]$. We can then choose $Y_{m2}(\rho)$ for example as

$$Y_{m2}(\rho) = (\alpha_2(\rho)D_2)(\beta_2(\rho)D_2)^+. \tag{23}$$

Consequently, we have $\mathbb{C}(\rho) = \begin{pmatrix} \alpha_1(\rho) - (\alpha_2(\rho)D_2)(\beta_2(\rho)D_2)^+\beta_1(\rho) & 0 \end{pmatrix}$.

Now, since a parameter-dependent Lyapunov function is used later in this Section for the stability proof, Assumption 2 is added for the scheduling parameter derivative.

Assumption 2: The scheduling parameter's time derivative is bounded, i.e., there are constants ν_i such that $|\dot{\rho}_i|\leq \nu_i, i = 1, 2, \dots, n_\rho$.

Based on Theorem 1, the observer design problem is then solved using LMIs.

Theorem 1: Under Assumptions 1 and 2, the observer design problem is solved if there exist matrices $X(\rho) = X^\top(\rho) > 0$, $Y(\rho)$, and a scalar $\varepsilon_l > 0$ minimizing γ and satisfying for all $\rho \in [\underline{\rho}, \overline{\rho}]$, the set of LMIs

$$\begin{pmatrix} \Omega_{11a}(\rho) + \varepsilon_l \mathbb{C}^\top(\rho)\Gamma^\top\Gamma\mathbb{C}(\rho) & X(\rho)\mathbb{W}(\rho) & \Omega_{13}(\rho) \\ \mathbb{W}^\top(\rho)X(\rho) & -\varepsilon_l I & 0 \\ \Omega_{13}^\top(\rho) & 0 & -I \end{pmatrix} < 0,$$

$$\begin{pmatrix} X(\rho) & I \\ I & \gamma^2 I \end{pmatrix} > 0,$$

where $\Omega_{11a}(\rho) = \mathbb{A}_1^\top(\rho)X(\rho) + X(\rho)\mathbb{A}_1(\rho) - \mathbb{A}_2^\top(\rho)Y^\top(\rho) - Y(\rho)\mathbb{A}_2(\rho) \pm \sum_{i=1}^{n_\rho} \nu_i \frac{\partial X(\rho)}{\partial \rho_i}$ and $\Omega_{13}(\rho) = X(\rho)\mathbb{B}_1(\rho) - Y(\rho)\mathbb{B}_2(\rho)$. Then, $Z(\rho) = X^{-1}(\rho)Y(\rho)$.

Proof: Considering the parameter-dependent Lyapunov function candidate $V(e, \rho) = e^\top X(\rho)e$ where $X(\rho) = X^\top(\rho) > 0$, we denote $\eta = \begin{pmatrix} e & \Delta\Phi & \omega_n \end{pmatrix}^\top$ and derive

$$\begin{aligned} \dot{V}(e, \rho) &= \dot{e}^\top X(\rho)e + e^\top X(\rho)\dot{e} + e^\top \left(\sum_{i=1}^{n_\rho} \dot{\rho}_i \frac{\partial X(\rho)}{\partial \rho_i} \right) e \\ &= e^\top \Big[\mathbb{A}^\top(\rho)X(\rho) + X(\rho)\mathbb{A}(\rho) + \sum_{i=1}^{n_\rho} \dot{\rho}_i \frac{\partial X(\rho)}{\partial \rho_i} \Big] e \\ &\quad + e^\top X(\rho)\mathbb{W}(\rho)\Delta\Phi + \Delta\Phi^\top \mathbb{W}^\top(\rho)X(\rho)e \\ &\quad + e^\top X(\rho)\mathbb{B}(\rho)\omega_n + \omega_n^\top \mathbb{B}^\top(\rho)X(\rho)e \\ &= \eta^\top \begin{pmatrix} \Omega_{11b}(\rho) & X(\rho)\mathbb{W}(\rho) & \Omega_{13}(\rho) \\ \mathbb{W}^\top(\rho)X(\rho) & 0 & 0 \\ \Omega_{13}^\top(\rho) & 0 & 0 \end{pmatrix} \eta \\ &:= \eta^\top \mathcal{Q}_1(\rho)\eta, \end{aligned}$$

where, by using $\mathbb{A}(\rho) = \mathbb{A}_1(\rho) - Z(\rho)\mathbb{A}_2(\rho)$ and $\mathbb{B}(\rho) = \mathbb{B}_1(\rho) - Z(\rho)\mathbb{B}_2(\rho)$ then introducing the intermediate variable $Y(\rho) = X(\rho)Z(\rho)$, we obtain $\Omega_{11b}(\rho) = \mathbb{A}_1^\top(\rho)X(\rho) + X(\rho)\mathbb{A}_1(\rho) - \mathbb{A}_2^\top(\rho)Y^\top(\rho) - Y(\rho)\mathbb{A}_2(\rho) + \sum_{i=1}^{n_\rho} \dot{\rho}_i \frac{\partial X(\rho)}{\partial \rho_i}$ and $\Omega_{13}(\rho) = X(\rho)\mathbb{B}_1(\rho) - Y(\rho)\mathbb{B}_2(\rho)$.

From Assumption 1, the Lipschitz condition (2) gives

$$\Delta\Phi^\top \Delta\Phi \leq e^\top \mathbb{C}^\top(\rho)\Gamma^\top\Gamma\mathbb{C}(\rho)e. \tag{24}$$

This condition (independent of ω_n) is then rewritten as

$$\eta^\top \mathcal{Q}_2(\rho)\eta \leq 0, \tag{25}$$

where $\mathcal{Q}_2(\rho) = \begin{pmatrix} -\mathbb{C}^\top(\rho)\Gamma^\top\Gamma\mathbb{C}(\rho) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Now, define the following terms

$$\begin{aligned} J_1 &= -\omega_n^\top \omega_n = \eta^\top \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} \eta := \eta^\top \mathcal{Q}_3 \eta, \\ J_2(\rho) &= \gamma_2^2 e^\top X(\rho)e - e^\top e = e^\top \underbrace{(\gamma^2 X(\rho) - I)}_{\mathcal{Q}_4(\rho)} e. \end{aligned}$$

The g$\mathcal{H}_2$ condition then gives (note that the system (22) is now strictly proper)

$$\begin{cases} \dot{V}(e, \rho) + J_1 < 0 \\ J_2(\rho) > 0. \end{cases} \tag{26}$$

As we apply the $\mathcal{S}$-procedure [21] to the Lipschitz constraint (25) and the conditions (26), $\dot{V}(e, \rho) < 0$ if there exists a scalar $\varepsilon_l > 0$ such that

$$\begin{aligned} &\begin{cases} \dot{V}(e, \rho) - \varepsilon_l \eta^\top \mathcal{Q}_2(\rho)\eta + J_1 < 0 \\ J_2(\rho) > 0 \end{cases} \\ \iff &\begin{cases} \eta^\top(\mathcal{Q}_1(\rho) - \varepsilon_l \mathcal{Q}_2(\rho) + \mathcal{Q}_3)\eta < 0 \\ e^\top \mathcal{Q}_4(\rho)e > 0 \end{cases} \\ \iff &\begin{cases} \mathcal{Q}_1(\rho) - \varepsilon_l \mathcal{Q}_2(\rho) + \mathcal{Q}_3 < 0 \\ \mathcal{Q}_4(\rho) > 0, \end{cases} \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} \Omega_{11b}(\rho) + \varepsilon_l \mathbb{C}^\top(\rho)\Gamma^\top\Gamma\mathbb{C}(\rho) & X(\rho)\mathbb{W}(\rho) & \Omega_{13}(\rho) \\ \mathbb{W}^\top(\rho)X(\rho) & -\varepsilon_l I & 0 \\ \Omega_{13}^\top(\rho) & 0 & -I \end{pmatrix} < 0,$$

$$\gamma^2 X(\rho) - I > 0.$$

Schur's lemma is then applied to the second condition, making it an LMI. Under Assumption 2, these resulting LMIs are satisfied if and only if the ones in Theorem 1 are satisfied [17]. Finally, (26) implies the g$\mathcal{H}_2$ performance, which is $\|e(t)\|_\infty^2 < \gamma^2 \|\omega_n(t)\|_2^2$. The proof is completed. □

Remark 3: When the error system (22) has no external inputs, we get $\dot{V}(e, \rho) < 0$, which guarantees the asymptotic stability of (22). Then, with $X(\rho)$ admitting positive bounds on the compact set $[\underline{\rho}, \overline{\rho}]$, the error e asymptotically converges to 0 since $\Delta\Phi$ vanishes as $\hat{x}$ approaches x.

The LMIs in Theorem 1 must be satisfied for an infinite number of constraints over the trajectories of ρ. To relax this, the grid-based methodology [17] is considered here where these inequalities are solved for a set of a high enough number N_g of frozen values of ρ assumed belonging to a gridded domain of the varying parameter under Assumption 2, followed by post-analysis of the asymptotic stability of the solution using a much denser grid. The observer is designed by applying Theorem 1 to get $Z(\rho)$ then obtaining the frozen observers at the grid points. In its implementation, grid-based gain scheduling is performed using interpolation methods.

Remark 4: When Φ is only locally Lipschitz, we assume that the system solution x to be estimated remains in a compact set $\mathcal{X} \subset \mathbb{R}^{n_x}$ (remember that the control input u is always bounded in $[\underline{u}, \overline{u}] \subset \mathbb{R}^{n_u}$) and replace the Φ in the observer with $\tilde{\Phi} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_\Phi}$ such that $\tilde{\Phi} = \Phi$ on $\tilde{\mathcal{X}} \times [\underline{u}, \overline{u}]$ where $\tilde{\mathcal{X}} = \mathcal{X} + c$ for some $c > 0$ (the set of points lying within a distance c from all the points in $\mathcal{X}$, which is also compact) and $\tilde{\Phi}$ is bounded outside. This way, there exists $\Gamma > 0$ such that $\|\Phi(x, u) - \tilde{\Phi}(\hat{x}, u)\| \leq \|\Gamma(x - \hat{x})\|, \forall(x, \hat{x}) \in \mathcal{X} \times \mathbb{R}^{n_x}, \forall u \in [\underline{u}, \overline{u}]$. Therefore, by repeating all the steps until the system (4) and letting $\Delta\Phi = \Phi(x, u) - \tilde{\Phi}(\hat{x}, u)$, we can bound this nonlinearity exactly as above. The assumption on x is feasible as, in practice, the state is always limited by operating conditions, e.g., system dimensions and constraints.

Note that using Theorem 1 implies imposing parameterized forms of $X(\rho)$ and $Y(\rho)$ in terms of ρ, e.g., the polynomial form as seen later in Section V. Algorithm 1 summarizes the proposed observer design method.

Algorithm 1 Observer design

Input: The matrices $A(\rho)$, $B(\rho)$, C, D_1, and D_2, $\Phi(x, u)$, and the N_g gridded values of $\rho \in [\underline{\rho}, \overline{\rho}]$
Output: The observer matrices $N(\rho)$, $J(\rho)$, $H(\rho)$, $M(\rho)$, $P(\rho)$, $Q(\rho)$, $G(\rho)$, $R(\rho)$, and $S(\rho)$ at the grid points
Assumptions: The Lipschitz matrix Γ and the bound ν on $|\dot{\rho}|$

Step 1: Choose a structure for $E(\rho)$, then grid it and check (13) at the grid points
Step 2: Find $K(\rho)$, $T(\rho)$, and $\Theta(\rho)$ at the grid points, according to [9], (14), and (20)
Step 3: Find $H(\rho)$ at the grid points using (18)
Step 4: Deduce $\alpha_1(\rho)$, $\alpha_2(\rho)$, $\alpha_3(\rho)$, $\alpha_4(\rho)$, $\beta_1(\rho)$, $\beta_2(\rho)$, $\beta_3(\rho)$, (15), and (21)
Step 5: Deduce $Y_{m2}(\rho)$ according to (23), getting $\mathbb{C}(\rho)$, $\mathbb{D}(\rho)$, $R(\rho)$, and $S(\rho)$
Step 6: Find $\mathbb{A}_1(\rho)$, $\mathbb{A}_2(\rho)$, $\mathbb{W}(\rho)$, $\mathbb{B}_1(\rho)$, and $\mathbb{B}_2(\rho)$
Step 7: Fix the form of $X(\rho)$ and $Y(\rho)$ and solve for $Z(\rho)$ using Theorem 1, getting $M(\rho)$, $G(\rho)$, $Y_{m1}(\rho)$, and $Y_{m3}(\rho)$
Step 8: From $Y_{m1}(\rho)$, get $P(\rho)$ and $Q(\rho)$ using (15); from $Y_{m3}(\rho)$, get $N(\rho)$ and $J(\rho)$ using (21) and (20)

V. Application to Automotive SA Suspensions

To illustrate the proposed observer design method, we consider the damper force estimation problem in the SA suspension. Consider the quarter-car model made of the sprung and unsprung masses m_s and m_{us} [22]. From Newton's law of motion, the system dynamics around its equilibrium are

$$\begin{cases} m_s \ddot{z}_s = -F_s - F_d \\ m_{us} \ddot{z}_{us} = F_s + F_d - F_t, \end{cases} \tag{27}$$

where the spring force $F_s = k_s z_{def}$ with $z_{def} = z_s - z_{us}$ called the suspension deflection and $\dot{z}_{def}$ its velocity; the tire force $F_t = k_t(z_{us} - z_r)$. The damper force F_d consists of a passive part $F_{passive}$ and a controlled part F_{er} [10] as

$$\begin{cases} F_d = \underbrace{k_0 z_{def} + c_0 \dot{z}_{def}}_{F_{passive}} + F_{er} \\ \dot{F}_{er} = \frac{-1}{\tau(u)} F_{er} + \frac{f_c}{\tau(u)} \cdot u \cdot \tanh\left(k_1 z_{def} + c_1 \dot{z}_{def}\right), \end{cases} \tag{28}$$

where c_0, c_1, k_0, k_1, and f_c are constant parameters; $u \in [\underline{u}, \overline{u}]$ is the PWM control input signal. In the nonlinear damper model [18], $\tau(u)$ depends on the input u as

$$\tau(u) = 0.3643u^2 + 0.1124u. \tag{29}$$

Choosing the state $x = (z_s - z_{us}, \dot{z}_s, z_{us} - z_r, \dot{z}_{us}, F_{er})^\top$, the measured output y as $(\ddot{z}_s, \ddot{z}_{us})^\top$ (note that measuring accelerations has cost and mounting advantages over measuring deflections [3]) with measurement noise ω_n, the input disturbance $\omega_r = \dot{z}_r$, and $\rho = \frac{1}{\tau(u)} \in [\frac{1}{\tau(\overline{u})}, \frac{1}{\tau(\underline{u})}]$ s^{-1} where $0 < \underline{u} < \overline{u}$, we get $\Phi(x, u) = u \cdot \tanh\left(k_1 z_{def} + c_1 \dot{z}_{def}\right) = u \cdot \tanh(\Lambda x)$ where $\Lambda = \begin{pmatrix} k_1 & c_1 & 0 & -c_1 & 0 \end{pmatrix}$. Therefore, the system can be written in the form (1), Φ being Lipschitz. In this work, the parameters for observer design and simulation correspond to a $1/5$-scale vehicle [10]. Also, for operation safety, the control bounds are taken as $\underline{u} = 0.1$ and $\overline{u} = 0.5$.

A. Application of Algorithm 1

We choose here $E(\rho) = E_0 + \rho E_1$, whereas more complex forms can be used in other applications. For LMI solving, we choose the first-order forms $X(\rho) = X_0 + \rho X_1 > 0$ and $Y(\rho) = Y_0 + \rho Y_1$ for the variables, and then solve the LMIs for the coefficients X_0, X_1, Y_0, and Y_1. LMI solution with 14 evenly spaced grid points and parameter variation bound $\nu = 2600$ s^{-2} using the LMI Lab toolbox has given $\gamma = 0.4230$. Post-analysis using a grid consisting of 140 evenly spaced points in the same range has shown that the maximum real part of the poles of the error system varies between $-2.7408 \cdot 10^{-6}$ at $\underline{\rho}$ and $-1.1049 \cdot 10^{-6}$ at $\overline{\rho}$. Therefore, we conclude that the obtained solution guarantees the error's asymptotic stability.

B. Observer Synthesis Results

Let us compare our observer with the polytopic LPV one designed in [18]. Figure 1 shows the upper bound $\overline{\sigma}(e/\omega_n)$ on the singular values of the transfer functions of the error systems frozen at the grid points, which decreases rapidly in the region of high-frequency noise, emphasizing the effectiveness of our observer in terms of noise attenuation.

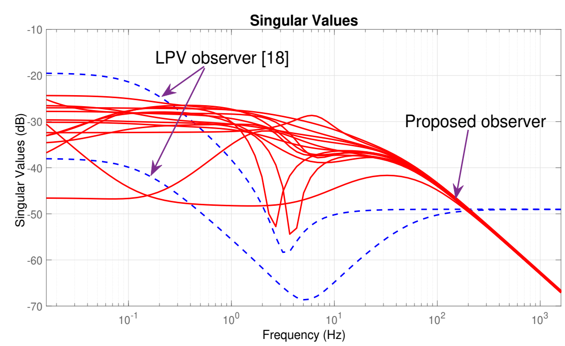

Fig. 1. Upper bound $\overline{\sigma}(e/\omega_n)$ on the singular values (blue—LPV observer at the vertices, red—proposed observer frozen at the grid points).

From Figure 2, it is evident that for each state component, the proposed observer gives a significantly better maximal attenuation level of sensor noise than the LPV one.

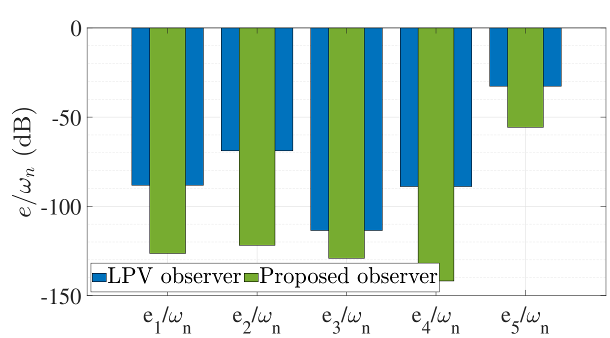

Fig. 2. Maximal attenuation level of sensor noise.

C. Damper Force Estimation Results

Both observers are then simulated with the nonlinear damper model above. We use two simulation scenarios with:

- An ISO road profile of type C (see [3] for a figure) and then a bump road profile of 2 cm (occurring at 0.5 s);
- A skyhook controller that gives the PWM signal;
- White noise only in the second scenario.

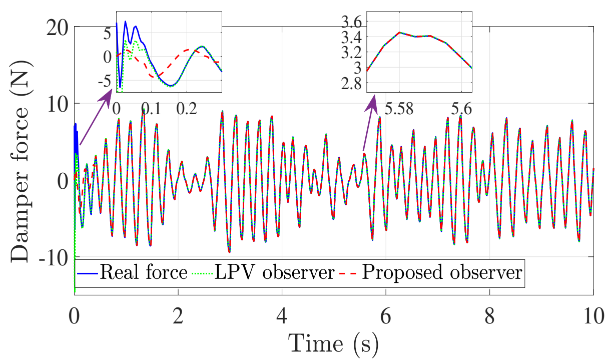

Fig. 3. Damper force estimation results (ISO road profile).

Damper force estimation results are shown in Figures 3 and 4. In the first scenario, the proposed observer converges slightly slower than the LPV one. But, with a bump road profile and realistic measurement noise, the proposed one gives an estimation error that is smaller and not noisy. Indeed, the proposed observer can minimize the effect of the measurement noise, while that of the road input is decoupled from the error.

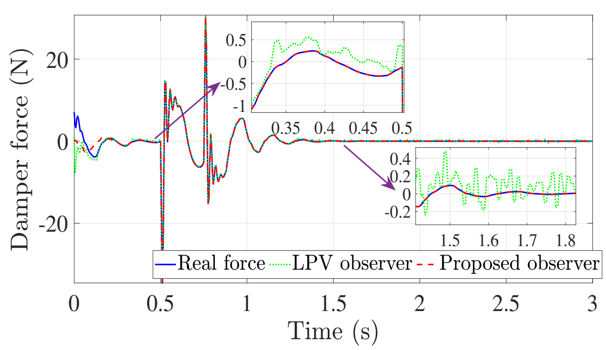

Fig. 4. Damper force estimation results (bump road profile).

VI. Conclusion

This paper designs a unified observer specifically for a class of NLPV systems with parameter dependence in both the dynamics and the input matrices and with a Lipschitz nonlinearity in the input. Following the grid-based methodology, parameterization is extended for the case of parameter dependence in the system's dynamics matrix, decoupling the effects of the input disturbance from the estimation error. The objectives of bounding the nonlinear disturbance term using the Lipschitz condition and bounding the measurement noise using the g$\mathcal{H}_2$ condition are combined into a single LMI framework thanks to the $\mathcal{S}$-procedure. Application to the SA suspension has illustrated the method in both the frequency and time domain and compared it with an LPV observer.

References

[1] J. Mohammadpour and C. W. Scherer, *Control of Linear Parameter Varying Systems with Applications*. Springer Science & Business Media, 2012.

[2] O. Sename and D. Rotondo, "Emerging Approaches for Nonlinear Parameter Varying Systems," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 17, pp. 8121–8123, 2021.

[3] T.-P. Pham, O. Sename, and L. Dugard, "A Nonlinear Parameter Varying Observer for Real-time Damper Force Estimation of an Automotive Electro-rheological Suspension System," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 17, pp. 8183–8205, 06 2021.

[4] G. O. Gordillo, M. Darouach, C. M. Astorga-Zaragoza, and L. Boutat-Baddas, "Generalized Dynamic Observer Design for Lipschitz Nonlinear Descriptor Systems," *IET Control Theory and Applications*, vol. 13, no. 14, pp. 2270–2280, Sept. 2019.

[5] M.-H. Do, D. Koenig, and D. Theilliol, "$\mathcal{H}_\infty$ Observer Design for Singular Nonlinear Parameter-varying System," in *2020 59th IEEE Conference on Decision and Control (CDC)*, 2020, pp. 3927–3932.

[6] Y. Wang, R. Rajamani, and D. M. Bevly, "Observer Design for Parameter Varying Differentiable Nonlinear Systems, With Application to Slip Angle Estimation," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1940–1945, 2017.

[7] A. Pérez-Estrada, G.-L. Osorio-Gordillo, M. Alma, M. Darouach, and V. Olivares Peregrino, "$\mathcal{H}_\infty$ Generalized Dynamic Unknown Inputs Observer Design for Discrete LPV Systems. Application to Wind Turbine," *European Journal of Control*, 09 2018.

[8] A. J. Perez-Estrada, G. L. Osorio Gordillo, M. Alma, M. Darouach, and V. H. Olivares-Peregrino, "Generalized Dynamic Observers for quasi-LPV Systems with Unmeasurable Scheduling Functions," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 17, pp. 5262–5278, Nov. 2018.

[9] N. Gao, M. Darouach, H. Voos, and M. Alma, "New Unified $\mathcal{H}_\infty$ Dynamic Observer Design for Linear Systems with Unknown Inputs," *Automatica*, vol. 65, pp. 43–52, 2016.

[10] T.-P. Pham, O. Sename, and L. Dugard, "Unified $\mathcal{H}_\infty$ Observer for a Class of Nonlinear Lipschitz Systems: Application to a Real ER Automotive Suspension," *IEEE Control Systems Letters*, vol. 3, no. 4, pp. 817–822, 2019.

[11] G. Q. B. Tran, T.-P. Pham, and O. Sename, "Multi-objective Unified qLPV Observer: Application to a Semi-active Suspension System," *IFAC-PapersOnLine*, vol. 54, no. 8, pp. 136–141, 2021, 4th IFAC Workshop on Linear Parameter Varying Systems.

[12] R. Rajamani, W. Jeon, H. Movahedi, and A. Zemouche, "On the Need for Switched-gain Observers for Non-monotonic Nonlinear Systems," *Automatica*, vol. 114, p. 108814, 2020.

[13] A. Zemouche and M. Boutayeb, "On LMI Conditions to Design Observers for Lipschitz Nonlinear Systems," *Automatica*, vol. 49, no. 2, pp. 585–591, 2013.

[14] D. Koenig, "Observers Design for Unknown Input Nonlinear Descriptor Systems via Convex Optimization," *IEEE Transactions on Automatic Control*, vol. AC-51, no. 06, pp. 1047–1052, June 2006.

[15] M. Darouach, L. Boutat-Baddas, and M. Zerrougui, "$\mathcal{H}_\infty$ Observers Design for a Class of Nonlinear Singular Systems," *Automatica*, vol. 47, pp. 2517–2525, 2011.

[16] F. Nemati, S. M. S. Hamami, and A. Zemouche, "A Nonlinear Observer-based Approach to Fault Detection, Isolation and Estimation for Satellite Formation Flight Application," *Automatica*, vol. 107, pp. 474–482, 2019.

[17] F. Wu, "Control of Linear Parameter Varying Systems," Ph.D. dissertation, University of California, Berkeley, 1995.

[18] T.-P. Pham, O. Sename, L. Dugard, and V. T. Vu, "LPV Force Observer Design and Experimental Validation from a Dynamical Semi-active ER Damper Model," *IFAC-PapersOnLine*, vol. 52, no. 17, pp. 60–65, 2019.

[19] G. Phanomchoeng and R. Rajamani, "Observer Design for Lipschitz Nonlinear Systems using Riccati Equations," in *Proceedings of the 2010 American Control Conference*, 2010, pp. 6060–6065.

[20] A. Zemouche and M. Boutayeb, "On LMI Conditions to Design Observers for Lipschitz Nonlinear Systems," *Automatica*, vol. 49, p. 585–591, 02 2013.

[21] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[22] S. Savaresi, C. Poussot-Vassal, C. Spelta, O. Sename, and L. Dugard, *Semi-Active Suspension Control for Vehicles*. Elsevier - Butterworth Heinemann, 2010.