

# Generalized affine signal analysis with time-delay threshold

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GENERALIZED AFFINE SIGNAL ANALYSIS  
WITH TIME-DELAY THRESHOLDS

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ABSTRACT

A class of functions recently introduced by Dash and Paul to describe signals containing components with various onset-arrival threshold positions in time is placed within the context of the affine group formalism of Aslaksen, Klauder, Grossmann, and Morlet generalized to include a time-translation degree of freedom. Each function or "wavelet" has a causal structure, being zero before its own threshold time position, and contains parameters defining a power-law turn-on, ringing, decay, and some possibility for phase modulation, along with an overall magnitude and phase. Signals are expanded in rapidly convergent series of these functions. Techniques for parameter estimation are discussed. Some possibilities for practical applications are mentioned.

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## I. INTRODUCTION

A promising development in signal processing and pattern recognition has been the realization that group-theoretic and Hilbert-space analytic tools combined with suitable sets of functions or "wavelets" can provide a useful expansion framework for a large class of signals. Historically, Prony analysis [1] was the first example of an signal expansion in terms of an overcomplete set of continuous functions along with a prescription for the expansion using a discrete number of constraints from a set of measurements in time. The idea provides a special case of Pade approximants [2], and independent later developments included applications in high energy physics as Regge pole theory [3] and in electromagnetic scattering theory as SEM theory [4]. The formal description in terms of the affine group and related coherent states was made by Aslaksen and Klauder [5] and independently by Grossmann and Morlet [6]. In physical terms, each wavelet is zero before a given fixed point in time (call it  $t_0 = 0$ ), and then oscillates with a fixed frequency and decays in time with an exponential envelope. The affine group parameters of translation and dilation are, in this case, associated with the frequency of oscillation and the exponential decay scale.

An extension of Prony analysis which allows an extra degree of freedom as a time translation was suggested by Dash [7] on the basis of work incorporating finite scales into Regge theory [8] and by Paul [9]. The class of functions used was the class of Prony functions discussed above, but translated by an arbitrary amount in time. The two translations of frequency and time are thus incorporated as in the usual coherent states related to the

Weyl-Heisenberg group [10] in addition to the dilations of affine coherent states. This generalization provides an economical description of data from some situations using only a few terms, but which require an infinite number of terms in a standard Prony expansion. Ref. [7] also discusses the physical origin of the time-delays of the wavelets due to causality when measurements are made at a given point in a spatially extended system which scatters an incident electromagnetic wave.

The general form for the wavelet of Ref. [7] is

$$f(Y) = \text{Re} \left\{ \frac{|A|e^{i\phi}}{\Gamma(z+1)} \Theta[Y]Y^z e^{\alpha Y} \right\}$$

Here  $Y = t - t_0$  is the translated time variable,  $\Theta[Y]$  is the Heaviside function defined as zero for  $Y < 0$ ,  $1/2$  at  $Y = 0$ , and  $1$  for  $Y > 0$ . Also,  $\alpha = \alpha_R + i\alpha_I$  defines the damping parameter  $\alpha_R < 0$  and the resonance-ringing frequency  $\alpha_I$ . The power take-off parameter  $z$  can be complex, but  $\text{Re}(z) \geq 0$  is required. Finally  $|A|$ ,  $e^{i\phi}$  are the overall magnitude and phase. The Fourier transform of  $f(Y)$  is

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} f(Y) dt \\ &= \frac{|A|}{2} e^{i\omega t_0} \left[ \frac{e^{i\phi}}{(-i\omega - \alpha)^{(z+1)}} + \frac{e^{-i\phi}}{(-i\omega - \alpha^*)^{(z^*+1)}} \right] \end{aligned}$$

The time delay parameter  $t_0$  manifests itself in the phase factor  $e^{i\omega t_0}$  which is not present in the Prony analysis, since  $t_0 = 0$  is assumed there. Because  $f(Y)$  is real, singularities at  $\omega = i\alpha$ ,  $i\alpha^*$  are present. If  $\omega$ ,  $\alpha_I > 0$  the dominant term is the second one at  $\omega = i\alpha^*$ . The width of the peak at  $\omega = \alpha_I$  for real  $\omega > 0$  is given by  $-2\alpha_R$ , while the falloff at the tail of the peak is given by  $z$ , as required by the usual Fourier transform arguments. With  $z = n$ , poles of order  $(n+1)$  are generated; if  $z$  is nonintegral the singularities are branch points.

A remark about overcompleteness is in order. Any function in an overcomplete set can be expanded in terms of the others. Therefore, no expansion is unique. An overcomplete set can be reduced in size to a complete set by suitable discretization, cutting out the "extra" elements. However, we shall regard overcompleteness as a virtue, allowing reasonably accurate expansions with only a few terms due to the richness of the set of functions from which we can draw. While nonunique, such an expansion can often be distinguished from other expansions by the number of terms needed for a given accuracy, and we shall use this as one criterion for selecting out a preferred expansion.

A very interesting case of practical importance to which we think the formalism we present could be applicable is voiced speech, which consists of repeated quasi-periodic segments [11]. Applications of the affine formalism to seismic signal processing using different wavelets are described in Ref. [12].

The set-up of the rest of this paper is as follows. In Section II we briefly describe the formalism. In Section III we give some possible suggestions for initial parameter estimation. The appendix contains some technical details.

## II. FORMALISM

In this section we give a summary of the formalism contained in Refs. [xxx]. We denote by  $G_A$  the affine group of dilations  $\lambda = -\alpha_R$  and translations  $\alpha_I$ . We choose a "fiducial wavelet"  $f_0(t) = \Theta[t]t^z e^{-t}$  where now we require that  $\text{Re}(z) > 0$ . We will construct all other wavelets from  $f_0$  by the operations of translations and dilations. Define  $Q, P$  as the co-ordinate and momentum operators obeying  $[Q, P] = i$  with eigenvalues  $q$  and  $p$ . The dilation operator is  $B = (QP + PQ)/2$ . We take  $P$  as corresponding to multiplication by  $t$ , so  $Q = i\partial/\partial t$ , and we also take  $p = t_0$ ,  $q = \alpha_I$ . Unitary representations for the affine group  $G_A$  and the Weyl-Heisenberg group  $G_{WH}$  are then given by operations on  $f_0(t)$ , viz

$$f^{(g)A}(t) = c_A [U_A(g)f_0](t) = c_A e^{-iqP} e^{-i/n(\lambda)B} f_0(t) = c_A \lambda^{1/2} e^{-i\alpha_I t} f_0(\lambda t)$$

$$f^{(g)WH}(t) = c_{WH} [U_{WH}(g)f_0](t) = c_{WH} e^{i(-qP + pQ)} f_0(t) = e^{-i\alpha_I t} f_0(t - t_0)$$

Here we have included normalization constants  $c_A, c_{WH}$  for convenience where  $c_A^{-2} = 2\pi \|f_0\|^2$  with  $\|f_0\|^2 = \int_0^\infty (dx/x) f_0^2(x)$ , and  $c_{WH} = e^{-i\alpha_I t_0/2}$ .

Evidently the wavelets of Ref. [7] in eq. 1.1 are given by a combination of the affine and Weyl-Heisenberg operators acting on  $f_0$ . We will be expanding signals in a series of such functions. Our strategy for getting each wavelet will involve choosing an element of the Weyl-Heisenberg group, thus

specifying  $t_0$ , along with an element of the affine group specifying  $\alpha_I, \lambda = -\alpha_R$ . A value for  $z$  must also be fixed. The relative contribution to the expansion of each such term will be given by the formalism.

In general we can call  $f^{(g)}(t)$  a coherent state, and the set of all  $f^{(g)}(t)$  is an overcomplete set which is dense in the Hilbert space  $H$  of square-integrable functions ~~on the group~~.

The inner product on  $H$  is  $\int \bar{\psi}_1(t) \psi_2(t) dt$ . ~~The inner product on  $H$  is  $(f_1, f_2) = \int d\mu(g) f_1^*(g) f_2(g)$ . There is a resolution of the identity  $1 = \int d\mu(g) |f^{(g)}\rangle \langle f^{(g)}|$ . Specifically this takes the form~~

$$\int d\mu(g) f^{(g)}(t) [f^{(g)}]^*(t') = \int d\mu(g) [U(g)f_0](t) [U(g)f_0]^*(t') = \delta(t - t')$$

For  $G_A$ , the (left-invariant) measure is  $d\mu(g) = d\alpha_I d\lambda/\lambda^2$  with  $\alpha_I \in (-\infty, \infty), \lambda \in (0, \infty)$ . This equation is easily shown by integrating over  $\alpha_I$  and then scaling the  $\lambda$  integral by  $x = \lambda t$ .

We now define overlap integrals for a given function  $\psi(t)$ . These give a "continuous representation" of  $H$ , a sort of a generalized Fourier transform,

$$\hat{\psi}(g) = \int [f^{(g)}]^*(t) \psi(t) dt$$

which is a sort of generalized Fourier transform. The inverse is

$$\psi(t) = \int d\mu(g) f^{(g)}(t) \hat{\psi}(g)$$

The set of overlap integrals forms another Hilbert space  $\mathcal{H}$ , this time of functions  $\hat{F}$  on the group  $G$ , with scalar product  $(\hat{f}_1, \hat{f}_2) = \int d\mu(g) \hat{f}_1^*(g) \hat{f}_2(g)$



Suppose now we let  $\psi(t) = f^{(g')}(t)$ . This defines the function  $K(g, g')$ ,

$$K(g, g') = \int [f^{(g)}]^*(t) f^{(g')}(t) dt$$

We also define functions  $e_g$  on the group by

$$e_{g'}(g) = K(g, g') = e_g^*(g')$$

These functions enjoy the following properties,

$$\int d\mu(g) K(g, g') K(g', g'') = K(g, g'')$$

$$(e_g, \hat{h}) = \hat{h}(g)$$

for any function  $\hat{h}(g)$  of the form of an overlap integral in eq xx. Thus,  $K(g, g')$  is a reproducing kernel, and  $e_g$  a reproducing vector on the group. Note that while  $e_g$  is not a delta function, it essentially acts like one. The dilations prevent the delta function from emerging, providing instead the scaled integral  $\|f_0\|^2$ . The above formulae are easily proved using the resolution of the identity eq. xx.

The specific form of  $K(g, g')$  is, fixing  $t_0$ ,

$$\begin{aligned}
 K(g, g') &= c_g^2 (\lambda \lambda')^{1/2} \int_{t_0}^{\infty} dt f_0[\lambda(t - t_0)] f_0[\lambda'(t - t_0)] e^{i(t - t_0)(\alpha_I - \alpha_I')} \\
 &= c_g^2 \Gamma(2z + 1) \left[ \frac{\lambda \lambda'}{A^2} \right]^{z + 1/2}
 \end{aligned}$$

where  $A = \lambda + \lambda' + i(\alpha_I' - \alpha_I)$ . If two different values for  $t_0$  are chosen for the functions in the integrand,  $K(g, g')$  is expressible in terms of confluent hypergeometric functions.

Now the reproducing kernels are used in the following interpolation function  $F_{INT}^{(N)}$  defined on the group. Suppose we have  $N$  constraints of the form

$$F_{INT}^{(N)}(g_j) = \zeta_j \equiv \hat{\psi}(g_j)$$

where  $g_j$  are discrete points on the group manifold,  $j = 1, \dots, N$ . Then expanding  $F_{INT}^{(N)}$  in terms of the reproducing vectors

$$F_{INT}^{(N)}(g) = \sum_{i=1}^N b_i e_{g_i}(g)$$

and imposing the constraints, we find that  $F_{INT}^{(N)}(g) = \sum_i K(g, g_i) (K^{-1} \zeta)_i$ , or

$$F_{INT}^{(N)}(g) = - \frac{1}{\det(K_{ij})} \begin{vmatrix} 0 & K(g, g_1) & K(g, g_2) & \dots & K(g, g_N) \\ \mathcal{J}_1 & & & & \\ \mathcal{J}_2 & & & & \\ \vdots & & & & \\ \mathcal{J}_N & & & & \end{vmatrix} \quad (K_{ij})$$

where  $K_{ij} = K(g_i, g_j)$ . Because of the reproducing kernel property, it is easy to show that  $\|F_{INT}^{(N)}\|$  is minimal. That is,  $F_{INT}^{(N)}$  is the function on the group of smallest norm that agrees with the  $N$  constraints. For, expanding  $F_{INT}^{(N)}(g)$  as above and differentiating the quantity  $S$  where

$$S \equiv (F_{INT}^{(N)}, F_{INT}^{(N)}) - 2 \operatorname{Re} \left\{ \sum_{i=1}^N \Lambda_i [F_{INT}^{(N)}(g_i) - \zeta_i] \right\} \quad \text{where } \Lambda_i \text{ are}$$

Lagrange multipliers produces the solution  $\Lambda_i = b_i^*$  while the  $b_i = (K^{-1}\zeta)_i$  are determined from the  $N$  constraint equations as above. This also means that if  $F_{EXACT}$  is the function of  $g$  we want to describe with  $F_{EXACT}(g_i) = \zeta_i$  then  $E(g) = F_{EXACT} - F_{INT}^{(N)}$  has minimal norm on  $G_A$ , so in a sense  $E(g)$  is orthogonal to the subspace of functions spanned by the  $N$  reproducing vectors  $e_{g_i}$  in the expansion of  $F_{INT}^{(N)}$ .

The interpolation function is then inserted into the inversion formula to obtain the approximate form  $\psi_{APP}^{(N)}(t)$  of the original signal  $\psi(t)$ ,

$$\begin{aligned} \psi_{APP}^{(N)}(t) &= \int d\mu(g) [U(g)f_0](t) F_{INT}^{(N)}(g) \\ &= \sum_{i,j=1}^N f^{(g_i)}(t) (K^{-1})_{ij} \int [f^{(g_j)}]^*(t') \psi(t') dt' \end{aligned}$$

Note as  $N \rightarrow \infty$ ,  $\psi_{APP}^{(N)}(t) \rightarrow \psi(t)$  through the formula

$$\sum_{i,j=1}^{\infty} f^{(g_i)}(t) (K^{-1})_{ij} [f^{(g_j)}]^*(t') = \delta(t - t')$$

which can formally be verified by multiplying both sides by  $[f^{(g_1)}]^*(t) f^{(g_m)}(t')$  and integrating in  $t, t'$  over all values to get  $K_{lm}$ . Detailed conditions on the nature of the lattice in the group parameter space necessary to produce this result are discussed in the Appendix.

Calling  $K_{ij} = \langle ij \rangle$  and  $\zeta_j = \langle j\psi \rangle$  where the  $\langle \rangle$  brackets indicate the integrals over  $t$ , and setting  $N = 2$ , we obtain the suggestive form

$$\psi_{APP}^{(N)}(t) = \frac{f^{(g_1)}[\langle 22 \rangle \langle 1\psi \rangle - \langle 12 \rangle \langle 2\psi \rangle] + f^{(g_2)}[\langle 11 \rangle \langle 2\psi \rangle - \langle 21 \rangle \langle 1\psi \rangle]}{\langle 11 \rangle \langle 22 \rangle - \langle 12 \rangle \langle 21 \rangle}$$

This shows clearly that if in fact  $\psi(t) = f^{(g_1)}(t)$ ,  $f^{(g_2)}(t)$ , or a linear combination of the two, then the approximation is exact,  $\psi_{APP}^{(N)}(t) = \psi(t)$ .

This statement generalizes to  $N > 2$ .

### III. PARAMETER ESTIMATION

In this section we give some consideration to the parameter estimation mentioned above. Questions of numerical accuracy, stability, and speed will not be addressed here. The essential idea is that we shall choose  $t_0$  according to some procedure along with an estimate of  $\alpha_I, \lambda$ , the affine group parameters, for each wavelet. Then, points in the affine-group parameter space around each  $\alpha_I, \lambda$  will be taken and used in the interpolation-approximation formula. Perhaps only a few such points will be needed. These points will define a little cluster of wavelets around each  $\alpha_I, \lambda$ .

To some extent the procedure for choosing the  $t_0$  depends on what we are after. A fast, even though physically unmotivated, choice would be simply to take  $t_0^{(n)} = t_0^{(0)} + n\delta t$  at equally spaced intervals of width  $\delta t$ . A choice of the various wavelets would be defined in successive time intervals, subtracting out the contributions from the wavelets in previous intervals. A physically motivated procedure would be to calculate the partial norms or differential

"energies"  $E_{12} = \int_{t_1}^{t_2} dt |\psi(t)|^2$  and compare  $E_{12}$  for various values of  $t_2$  for fixed  $t_1$  to see if any appreciable new effect enters.

A somewhat more sophisticated idea involves prior wavelet parameter estimation. Consider the Fourier transform of a wavelet  $\bar{f}(\omega)$  in eq. xx written in the form  $\bar{f}(\omega) = e^{i\omega t_0} \bar{b}(\omega)$ , where  $\bar{b}(\omega)$  is the rest of the expression with a bump at  $\omega = \alpha_I$  with width  $-2\alpha_R$  and falloff given by the parameter  $z$ . Rewriting,

$$e^{i\omega t_0} = \frac{\tilde{f}(\omega)}{\tilde{b}(\omega)}$$

If we could estimate  $\tilde{b}(\omega)$  approximately, we could test for  $e^{i\omega t_0}$  counting the oscillations in  $\omega$  in the real or imaginary parts of  $\tilde{f}(\omega)/\tilde{b}(\omega)$  around  $\omega = \omega_0 = \alpha_I$ . Alternatively writing  $\tilde{f}(\omega) = |\tilde{f}(\omega)|\exp[i\Phi_s(\omega)]$  we have near  $\omega \approx \omega_0$ ,

$$\frac{d}{d\omega} \Phi_s(\omega) \approx t_0 + \frac{z + 1}{-\alpha_R}$$

"Strong" estimates like these are in practice limited by resolution and are difficult to carry out with precision. We therefore propose some "weak" tests with probes designed as projections. We shall describe two ideas, one involving using the prolate spherical functions as a probe to test for the presence of a time delay exponential  $e^{i\omega t_0}$  over a bandlimited region in frequency around the peak corresponding to a wavelet, and the other a modified sort of Fourier transform with an exponential probe.

We recall that the sinc function defined by the integral over a bandlimited region in frequency

$$\int_{\omega_0 - \Omega}^{\omega_0 + \Omega} \frac{d\omega}{2\pi} e^{-i\omega \hat{t}} e^{i\omega t_0} = e^{i\omega_0(t_0 - \hat{t})} \text{sinc}[\Omega(t_0 - \hat{t})]$$

serves as the kernel for the spherical prolate functions  $S_n(t)$  over the time band  $(-T, T)$  with  $T$  an arbitrary but fixed parameter,

$$\int_{-T}^T dt' \operatorname{sinc}[\Omega(t_0 - t')] S_n(t') = \lambda_n S_n(t_0)$$

where  $\lambda_n$  is the corresponding eigenvalue. The spherical prolate functions are the eigenfunctions of an operator given by successive time and frequency bandlimiting projections. Reviews can be found in Ref. 13.

Consider now the following procedure. Suppose that we have a method for approximate calculation of the  $\bar{b}(\omega)$  part of a wavelet component of the signal through appropriate bandlimiting around the bump in the power spectrum of the signal  $|\tilde{\psi}(\omega)|^2$ . Then substituting  $\tilde{\psi}(\omega)/\bar{b}(\omega)$  for  $e^{i\omega t_0}$  as in eq. xx, we write for some choice of  $\hat{t}_0$  which will be our guess at  $t_0$  and various values of  $t'$

$$\int_{\omega_0 - \Omega}^{\omega_0 + \Omega} \frac{d\omega}{2\pi} \left[ e^{-i\omega \hat{t}_0} \frac{\tilde{\psi}(\omega)}{\bar{b}(\omega)} \right] e^{i(\omega - \omega_0)(t' - \hat{t}_0)} \approx e^{i\omega_0(t_0 - \hat{t}_0)} \operatorname{sinc}[\Omega(t' + t_0 - 2\hat{t}_0)]$$

Now also,

$$\int_{-T}^T dt' \operatorname{sinc}[\Omega(t' + t_0 - 2\hat{t}_0)] S_n(t') = \lambda_n S_n(2\hat{t}_0 - t_0)$$

The idea is then to vary  $\hat{t}_0$ , numerically matching up the two sides of the integral equation for  $S_n$  but with the integral in eq. xx inserted for the sinc kernel. We then use the resulting  $\hat{t}_0$  as our approximation to  $t_0$ .

A slightly simpler method may be to use the formula

$$e^{i\omega_0(t_0 - \hat{t}_0)} S_n(t_0) = e^{-i\omega_0 \hat{t}_0} \int_{\omega_0 - \Omega}^{\omega_0 + \Omega} \frac{d\omega}{2\pi} e^{i\omega t_0} (-i)^n \left( \frac{2\pi T}{\Omega \lambda_n} \right)^{1/2} S_n[(\omega - \omega_0)T/\Omega]$$

again with  $e^{i\omega t_0}$  replaced by  $\tilde{\psi}(\omega) / \bar{b}(\omega)$ , and varying  $\hat{t}_0$ .

This last formula would be useful in the difficult (but sometimes encountered) case that several different parameters  $t_{0l}$   $l = 1, 2, \dots$  correspond to the same bump around  $\omega_0 = \alpha_1$ . Physically this can occur in electromagnetic scattering due to reflections of a wave from a metallic surface. The procedure follows from using the above formula with  $t_0$  replaced by the sum  $\sum_l \gamma_l e^{i\omega t_{0l}}$ , which in practice we again replace by  $\tilde{\psi}(\omega) / \bar{b}(\omega)$  as

before. Now however we consider different values of  $n$  and we form the sum

$$\sum_{n=0}^{\infty} \frac{S_n(t_{0l}) S_n(\hat{t}_0)}{\lambda_n} = \delta(t_{0l} - \hat{t}_0) \quad \text{for } 0 < t_{0l}, \hat{t}_0 \leq T$$

With some cutoff at  $N_{\max}$  past the "time-frequency bandwidth"  $\Omega T$  we get a numerical approximation to the delta function. This is necessary since



$\lambda_n \approx 1$  up to around  $n \approx \Omega T$  and decreases to 0 for  $n > \Omega T$ . If the factor  $1/\lambda_n$  is not included in the above equation, the right hand side is the sinc kernel. The idea is again to vary  $\hat{t}_0$  to isolate the different  $t_{0l}$  parameters.

Finally, a different idea for locating the  $t_0$  parameters would be to form the quantity

$$f^{(\epsilon)}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ e^{-i\omega \hat{t}_0} \tilde{f}(\omega) \right]^{i\epsilon \text{sgn}(\omega)}$$

where  $\text{sgn}(\omega) = \pm 1$  is the sign of  $\omega$  and  $\epsilon > 0$ . A simple calculation reveals that

$$f^{(\epsilon)}(t) = \int_0^{\infty} \frac{d\omega}{2\pi} e^{-\epsilon\omega(t_0 - \hat{t}_0)} \left\{ e^{-i\omega t} [\tilde{b}(\omega)]^{i\epsilon} + \text{c.c.} \right\}$$

where c.c. means complex conjugate. The dominant  $\epsilon$  dependence will be in the exponential in front of the brackets. The idea is then to fix  $\hat{t}_0$  and vary  $\epsilon$ . If the integral is cut off at some  $\omega_{\text{max}}$ ,  $\epsilon$  can be varied from large positive to large negative values. A good approximation  $\hat{t}_0 \approx t_0$  will result in the integral being roughly independent of  $\epsilon$ . This can be generalized to several  $t_{0l}$ . The idea is a little like employing the exponential time decay of a Wick-rotated 2-point Green function in quantum field theory or statistical mechanics to obtain the energy of a state [Ref. 14].

We now briefly discuss some ideas for estimation of the other parameters. Each prominent bump in the modulus squared (power spectrum)  $|\tilde{\psi}(\omega)|^2$  can be used to estimate an  $\alpha_I$ . The width of the bump around  $\omega = \omega_0 = \alpha_I$  approximately gives  $-2\alpha_R$ . The shape of the tail of the bump past the width at  $\omega = \pm x\alpha_R$  with  $x > 2$ , but still away from other bumps, gives  $z$ . Specifically, a given wavelet satisfies the following two equations which could be used to estimate  $\alpha_R$ ,  $z$ , with  $\tilde{\psi}(\omega) \approx \tilde{f}(\omega)$  near  $\omega = \omega_0$ ,

$$|\tilde{f}(\omega_0 \pm \alpha_R)|^2 = \frac{1}{2^{(z+1)}} |\tilde{f}(\omega_0)|^2$$

$$z + 1 = \frac{\ln \left[ |\tilde{f}(\omega_0)|^2 / |\tilde{f}(\omega_0 \pm x\alpha_R)|^2 \right]}{\ln \left[ 1 + x^2 \right]}$$

Finally it may be profitable to estimate the magnitude of the wavelet, subtract that out, and then perform the formal analysis on the difference. An estimate of the magnitude  $|A|$  can be given by the maximum value of  $\tilde{f}(\omega)$  at  $\omega_0$  or alternatively the integrated energy in the bump around  $\omega_0$ ,

$$|A| = 2[-\alpha_R]^{(z+1)} |\tilde{f}(\omega_0)|$$

$$|A|^2 \approx \frac{4[-\alpha_R]^{(2z+1)} \Gamma(z+1)}{\sqrt{\pi} \Gamma(z+1/2)} \int_{\text{bump}} |\tilde{f}(\omega)|^2 d\omega$$

The overall phase  $\phi$  estimate can be obtained from  $\tilde{g}(\omega) = ie^{-i\omega t_0} \tilde{f}(\omega)$

as

$$\tan(\phi) = \frac{\text{Re}[\tilde{g}(\omega_0)]}{\text{Im}[\tilde{g}(\omega_0)]}$$

Finally, partially overlapping wavelets can partially be accounted for by making a first estimate of parameters for all wavelets, and then redetermining parameters for a given peak  $\omega = \omega_{0k}$  by first subtracting off a background defined by evaluating the other wavelets at  $\omega = \omega_{0k}$ .

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