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Diagonally dominant matrices for cryptography

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Abstract. Diagonally dominant lattices have already been used in cryptography, notably in the GGH and DRS schemes. This paper further studies the possibility of using diagonally dominant matrices in the context of lattice-based cryptography. To this end we study geometrical and algorithmic properties of lattices generated by such matrices. We prove novel bounds for the first minimum and the covering radius with respect *to the max norm* and study the quality reached by a specific solver for the Approximate Closest Vector Problem. Using these new results, we propose an decryption failure free encryption scheme using diagonally dominant matrices. We then propose solutions to patch the DRS signature scheme, in particular using matrices with negative noise.

Keywords: Diagonally dominance · Euclidean lattices · Algorithmic · Statistical attacks.

1 Introduction

1.1 Context and motivation

Diagonally dominant matrices. Diagonally dominant matrices have been an interesting object of study for over a century, starting at least from the Lévy-Desplanques theorem (1881) [4], with several links to general matrix theory with research spanning up to today [31,12,56]. Numerous applications of diagonal dominance can be found in various fields such as numerical linear algebra [39], Markov chains, graphs Laplacians, perturbation theory[5]. On the other hand, lattices generated by diagonally dominant matrices fitting the Lévy-Desplanques theorem was not investigated. Such lattices seemed to have found some application in cryptography on few specific instances [53,59] where in both papers the focus was more in the matrix generation than a study of the resulting lattice. On the other hand, when strict dominance is not required (i.e not fitting the Lévy-Desplanques theorem), "large diagonals" saw some uses in cryptography [32,43,54] as well as in modular arithmetic [6].

[4] A history of this theorem through the ages can be seen in [61]
[5] [20] lists some applications.

Euclidean lattices. The study of computational problems on lattices in general is also an old and very studied topic [48,14,5]. Classical problems such as computing a shortest vecor – named the Shortest Vector Problem (SVP) – and computing the closest lattice vector from a target vector – the Closest Vector Problem (CVP) – can be proven to be NP-hard in the general case [1,42]. As a matter of fact, relaxed version of these problems stay hard. Notably, even if we authorise exponential preprocessing computations, the CVP is also NP-hard for small approximation factors [3]. The hardness of these problems over Euclidean lattices motivated cryptographers to consider them as building blocks for cryptographic schemes [33,55], which led to extensive study of Euclidean lattices in the past decades.

Lattice-based cryptography. The first example of schemes using Euclidean lattices were using generic lattices and use a trapdoor one-way function whose hardness to invert is based on the CVP. One can cite the Goldreich-Goldwasser-Halevi (GGH) scheme [33] or constructions using the plain Learning With Errors (LWE) problem such as FRODO [11]. Note that their security can also be linked to the hardness of the SVP. For efficiency reasons one tends to consider *algebraic lattices*, meaning lattices which can be described by means of polynomial rings. Some of the noticeable constructions are NTRU [35] or the schemes based on the Ring Learning With Errors (RING-LWE) or the Module Learning With Errors (MODULE-LWE) problems. Their security can be linked to the SVP on the restricted classes of *ideal lattices* – also called the Ideal Shortest Vector Problem (IDEAL-SVP) – or *module lattices* – also called the Module Shortest Vector Problem (MODULE-SVP). One may wonder whether the additional algebraic structure can be used to solve the SVP more efficiently. Thus, the study of the IDEAL-SVP has gathered sustained attention in the past few years. First it was shown that the intermediate problem of recovering short generators of principal ideals can be solved in quantum polynomial time over cyclotomic fields [15] and even classical polynomial time over multiquadratic [8] or multicubic fields [38]. Then Cramer, Ducas and Wesolowski extended the analysis of [15] to the IDEAL-SVP and showed that one could obtain a subexponential approximation factor in quantum polynomial time [16]. With a slightly different approach, this result can be generalized to all number fields provided an exponential pre-processing phase [52], which might be an artifact of the proof if we refer to experimental results obtained in [9,10]. Thus the IDEAL-SVP seems to be strictly weaker than the SVP. Even though the RING-LWE or MODULE-LWE problems are harder than the IDEAL-SVP, there is no guarantee that algebraic attacks mentioned previously cannot be used to tackle them.

Thus, studying other types of trapdoors or constructions is still an interesting and important research direction, recently explored in [28] or [23,25] for example.

Digital signatures with lattices. In order to build digital signatures schemes with lattices, one can follow the *hash-then-sign* paradigm. In this setting, the hash of the message $H(m)$ is a random vector of the space and a valid signature is then a lattice vector close to $H(m)$. The security of the scheme is guaranteed

as soon as solving the CVP is hard. The original GGH and NTRU signature schemes were originally following a naive version of this paradigm, using the so-called Babai round-off algorithm to produce the signature. However Nguyen and Regev successfully used the observation that the difference between the message and a valid signature lie within the fundamental parallelepiped of the secret basis to recover the latter [49]. Ducas and Nguyen showed that this statistical attack could be extended to more complex structures than bases which allowed them to break potential counter-measures in practice [22]. The same kind of attack [40] has recently been applied to break the PEREGRINE signature scheme [57].

In order to prevent the attack, Plantard, Win and Susilo [54] described how to produce a hash-then-sign scheme based on the max norm in the hope that the signatures lie in a space independent of the secret basis. Their work rely on matrices of the form $\mathbf{B} = \mathbf{D} + \mathbf{N}$ where $\mathbf{D}$ and $\mathbf{N}$ are such that the spectral radius $\rho(\mathbf{D}^{-1} \cdot \mathbf{N}) < 1$. Then this work has been adapted for DRS, a candidate of the first round of the NIST call for standardization [53], relying on the fact that the matrices used as lattice bases are diagonally dominant. This allows the γ-Guaranteed Distance Decoding (GDD_γ) to be solved with an algorithm adapted from [54]. This scheme has known a learning attack by Ducas and Yu [24]. One has to note that this attack differs from the previous ones and that it *does not break completely* the second version of the scheme [60]. However, it remains a serious attack with around 30 bits of security loss for the first set of parameters, using 2^{30} signatures only.

1.2 Our Contributions

This work is composed of three parts.

1. In Section 3 we improve our theoretical knowledge of diagonally dominant lattices by giving two new bounds on the key lattice invariants in the context of cryptography *for the max norm*, one for the covering radius and one for the first minimum.
   More precisely, we start by giving a lower bound on the size of the shortest vector in infinity norm. Guessing the size of the shortest vector or even an approximation is known to be NP-hard [19], thus we believe providing a tighter upper bound for any specific family of lattices is an interesting result in itself. Then we give an improved study of the reduction algorithm of [54] for diagonally dominant matrices and prove a stronger reduction capability than previously proven for such lattices [59]. We also prove that our aforementioned algorithms operate at most a polynomial (in the dimension and the size of its entries) amount of vector additions or multiplications by a scalar. Consequently, both results give novel upper and lower bounds on the size of the covering radius for such lattices
2. Secondly, using this new results, we are able to provide a decryption failure free cryptosystem relying on diagonally dominant matrices. It follows a framework close the GGH encryption schemes [33,7]. We discuss formal security and the steps to take towards IND-CCA security, using standard

techniques or transformations [30,18]. We also evaluate the practical security of the scheme using common cryptanalytic techniques to assess lattice-based constructions. We show that it is asymptotically secure.
3. Finally, building upon our results of Section 3 again, we explore solutions for patching the DRS signature scheme against Ducas and Yu's statistical attack [24]. In particular, our experiments tend to show that using secret keys with negative noise only mitigate the impact of the leak.

We deem that the asymptotical security of GGH-like schemed using diagonally dominant matrices can be achieved, however our work tends to show that the practical security at a level comparable to other schemes like SQUIRRELS or [25] is difficult to achieve for suitable dimensions. Thus, we deem that trying to achieve such a goal is still an interesting and challenging research direction. Another option that we plan to explore is to use the good decoding properties of such matrices in other framework such as the Lattice Isomorphism Problem [23].

2 Background

We assume the readers know what is the set of integers $\mathbb{Z}$, the set of integral matrices with n rows and m columns $M_{n,m}(\mathbb{Z})$, the determinant, norms and other basics of linear algebra. We refer readers to [45,46] for a more complete background of lattice theory.

Definition 1 (Lattice).
We define an integral lattice $\mathcal{L}$ as a subgroup of $\mathbb{Z}^n$. A basis $\mathbf{B}$ of an integral lattice $\mathcal{L}$ is a basis of $\mathcal{L}$ as a $\mathbb{Z}$-module, and denote by $\mathcal{L}(\mathbf{B})$ the lattice generated by the rows of a basis $\mathbf{B}$. We write the volume (or determinant) of the lattice and compute it as $\det(\mathcal{L}) = \sqrt{\det(\mathbf{B} \cdot \mathbf{B}^T)}$.

While an integral lattice can potentially have an infinity of basis, a lattice only admits an unique basis in Hermite Normal Form (HNF).

Definition 2 (HNF).
Let $\mathcal{L}$ be a full-rank integral lattice of dimension n and $H \in M_{n,n}(\mathbb{Z})$ a basis of $\mathcal{L}$. H is said to be in HNF *if, and only if,*

$$\forall 1 \leq i,j \leq d,\ H_{i,j} \begin{cases} = 0 & if\ i > j \\ \geq 0 & if\ i \leq j \\ < H_{j,j} & if\ i < j \end{cases}$$

In this paper we only consider full-rank integral lattices
Lattices have some important invariant with strong computational property.

Definition 3 (Minima of a lattice). *We denote by $\lambda_k^{(l)}(\mathcal{L})$ the smallest value r such that a ball centered in zero and of radius r in norm l contains k linearly independent vectors of $\mathcal{L}$.*

Definition 4 (Covering radius). *Given a lattice $\mathcal{L}$, we define its covering radius $\mu^{(l)}(\mathcal{L})$ as the smallest value such that for any $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{v} \in \mathcal{L}$ such that $\|\mathbf{x} - \mathbf{v}\|_l < \mu^{(l)}(\mathcal{L})$.*

There is some relation between all those invariants. For example, for any lattice $\frac{1}{2}\lambda_1^{(2)}(\mathcal{L}) \leq \mu^{(2)}(\mathcal{L}) \leq \frac{\sqrt{n}}{2}\lambda_n^{(2)}(\mathcal{L})$ (See [45]).

While many computational problems on lattices exist, we define only the lattice problems useful for the comprehension of the paper.

Definition 5 (Approximate Shortest Vector Problem ($\mathbf{SVP}_\gamma$)). *Given a basis of a lattice $\mathcal{L}$ of dimension n and an* approximation factor $\gamma \in \mathbb{R}_+$*, find $\mathbf{v} \in \mathcal{L} \setminus \{0\}$ such that $\|v\ Vert \leqslant \gamma \cdot \lambda_1(\mathcal{L})$.*

Definition 6 (Approximate Closest Vector Problem ($\mathbf{CVP}_\gamma$)). *Given a basis of a lattice $\mathcal{L}$ of dimension n, a target vector $\mathbf{t} \in \mathbb{R}^n$ and an approximation factor $\gamma \in \mathbb{R}_+$, find $\mathbf{v} \in \mathcal{L}$ such that $\forall \mathbf{w} \in \mathcal{L}, \|\mathbf{t} - \mathbf{v}\| \leq \gamma \cdot \|\mathbf{t} - \mathbf{w}\|$.*

The first minimum $\lambda_1^{(l)}(\mathcal{L})$ and the covering radius $\mu^{(l)}(\mathcal{L})$ offers some natural bounds which transform the generic problem CVP in some useful variant, especially for cryptographic applications.

Definition 7 ($\mathbf{GDD}_\gamma$). *Given a lattice $\mathcal{L}$, and a bound $\gamma \geq 1$, for any target $\mathbf{t} \in \mathbb{R}^n$ find a lattice vector $\mathbf{v} \in \mathcal{L}$ such that $\|\mathbf{t} - \mathbf{v}\| < \gamma \cdot \mu^{(l)}(\mathcal{L})$.*

There exists another variant of CVP; if the first variant, GDD_γ, is key for lattice based signature scheme, the second variant, Bounded Distance Decoding (BDD), us key for lattice based encryption scheme.

Definition 8 (BDD). *Given a lattice $\mathcal{L}$, and a bound $\alpha \leq 1$, for any target $\mathbf{t} \in \mathbb{R}^n$ such there exist a vector $\mathbf{v} \in \mathcal{L}$ with $\|\mathbf{t} - \mathbf{v}\| < \alpha \cdot \lambda_1^{(l)}(\mathcal{L})$, find $\mathbf{v}$.*

Those problems are usually tackled with the combination of a "good" basis, e.g. LLL-reduced [37] or BKZ-reduced [13], together with an appropriate algorithm such as Babai's round-off or nearest plane algorithms [5]. For example, that is the approach proposed by Klein [36] for solving BDD for some α.

Remark 1. Note that a CVP_γ algorithm can be used as a GDD_γ solver as long as the approximation factor γ ensures that any target has a solution. Remark also that solving the GDD_γ is equivalent to computing a *short* coset representative of $\mathbf{t}$ mod $\mathcal{L}$. We will often consider algorithms solving this "short coset representative" problem, that we will call *reduction algorithms* and write `Reduce` for a generic algorithm. In this context the approximation factor γ of Definition 7 will be called the *reduction radius*.

In this paper, we consider a specific family of "good" lattice bases, allowing us to tackle the above problems more easily. Thus, we can use them as secret trapdoors for cryptographic constructions.

Definition 9 (Diagonally Dominant Matrix).
Let a matrix $\mathbf{B} \in M_n(\mathbb{Z})$*, we write* $\delta_i(\mathbf{B})$*,*

$$\delta_i(\mathbf{B}) = \mathbf{B}_{i,i} - \sum_{\substack{j=1 \\ i \neq j}}^{n} |\mathbf{B}_{i,j}|$$

and we will call $\mathbf{B}$ Diagonally Dominant *if, and only if,*

$$\forall i \in [\![1, n]\!], \quad \delta_i(\mathbf{B}) > 0.$$

Furthermore, we will note the dominance level

$$\Delta(\mathbf{B}) = \min \delta_i(\mathbf{B}).$$

It follows from the Lévy-Desplanques theorem that a diagonally dominant matrix is always full-rank.

For clarity reasons, we will mainly consider diagonally dominant matrices such that for any $i \in [\![1, n]\!]$, $\mathbf{B}_{i,i} = D$ for some fixed $D \in \mathbb{Z}$ and $\mathbf{N}_{i,i} = 0$. However, all results and their proofs can be modified easily to the case where $\mathbf{B} = \mathbf{D} + \mathbf{N}$ with $\mathbf{D}$ a general diagonal matrix and $\mathbf{N}$ with non-zero diagonal matrix.

3 Results on fundamental values for diagonally dominant lattices

In this section we analyze diagonally dominant lattices with respect to *the max norm*. We improve our knowledge on both the covering radius and the first minimum which are cryptographically relevant lattice invariants. We present those results in Theorem 1 and Theorem 2. We also lower the bound for the covering radius for matrices with negative noise $\mathbf{N}$, see Section 3.3.

3.1 Tighter bound on Diagonally Dominant Lattice Covering Radius

The results proven in this section will prove the following theorem.

Theorem 1. *Consider* $\mathbf{B} \in M_n(\mathbb{Z})$ *a matrix and* $\mathcal{L} = \mathcal{L}(\mathbf{B})$*. There is an algorithm* `PSW` *(Alg. 1) such that for any vector* $\mathbf{v} \in \mathbb{R}^n$*, it returns in polynomial time a vector* $\mathbf{w}$ *respecting*

$$\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}, \quad \|\mathbf{w}\|_\infty \leqslant D - \frac{\Delta(\mathbf{B})}{2}$$

i.e.

$$\mu^{(\infty)}(\mathcal{L}) \leqslant D - \frac{\Delta(\mathbf{B})}{2}.$$

The proof of this theorem is done by proving an upper bound on the convergence radius of a reduction algorithm which we will prove to terminate within a polynomial number of arithmetic operations.

The PSW reduction algorithm was first introduced in [54] and is a known approximation of Babaï's Round-off algorithm [5] in the case of matrices of the form $\mathbf{D}-\mathbf{N}$ where $\mathbf{N}\cdot\mathbf{D}^{-1}$ have a spectral radius lower than 1. It was then used a second time in cryptography [53] within the DRS scheme. The algorithm was proven to finish for with $\|\mathbf{w}\|_\infty < D$ in [53], but did not take into account the leeway $\Delta(\mathbf{B})$ A slight modification of the reduction proof given in [59] gives us a tighter bound by changing the loop condition in line 2 of the algorithm to a comparison with a value $R_i \geqslant D - \delta(\mathbf{B}, i)/2$ for every index i. This gives us the modified version, described in Algorithm 1.

Algorithm 1 PSW reduction

Require: $\mathbf{v} \in \mathbb{R}^n, \mathbf{B}$ a diagonally dominant matrix, a bound vector $R \in \mathbb{N}^n$.
Ensure: $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(B)$ and $\forall i \in [\![1, n]\!], \mathbf{w}_i < R_i$.

1: $\mathbf{w} \leftarrow \mathbf{v}$
2: **while** $\bigvee_{j=1}^{n}(|\mathbf{w}_j| > R_j)$ **do**
3: $\quad i \leftarrow$ any index such that $|\mathbf{w}_i| > R_i$
4: $\quad$ **if** $|\mathbf{w}_i| \geqslant D$ **then**
5: $\quad\quad q \leftarrow \mathrm{sign}(\mathbf{w}_i) \cdot \lfloor |\mathbf{w}_i| / D \rfloor$
6: $\quad$ **else**
7: $\quad\quad q \leftarrow \mathrm{sign}(\mathbf{w}_i)$
8: $\quad$ **end if**
9: $\quad \mathbf{w} \leftarrow \mathbf{w} - q \cdot \mathbf{B}_i$ {Reduce $|\mathbf{w}_i|$}
10: **end while**
11: **return** $\mathbf{w}$

Correctness. The following lemma states that for a given R, the algorithm terminates given that values R_i are above a certain bound which varies for each index.

Lemma 1 (Tighther bound in PSW-reduction algorithm). *For input $\mathbf{v} \in \mathbb{Z}^n$, a diagonally dominant matrix $\mathbf{B}$ and $R \in \mathbb{R}_+^n$ such that $\forall i \in [\![1, n]\!], R_i \geqslant D - \delta_i(\mathbf{B})/2$, the PSW reduction (alg. 1) terminates and outputs $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$ where $\forall i,\ |\mathbf{w}_i| \leq R_i$.*

Proof. Let $S(\mathbf{v}, R) \stackrel{\text{def}}{=} \{i \in [\![1, n]\!] \mid |\mathbf{v}_i| > R_i\}$ and f be the function defined on $\mathbb{Z}^n \times [\![1, n]\!]$ by $f : (\mathbf{w}, i) \mapsto \mathbf{w} - \mathrm{sign}(|\mathbf{w}_i|) \cdot \left\lfloor \frac{\mathbf{w}_i}{D} \right\rfloor \cdot \mathbf{B}_i$. In order to show that Algorithm 1 ends and outputs a correct vector, we will prove the following:

$$\bigvee_{j=1}^{n}(|\mathbf{w}_j| > R_j) \implies \forall i \in S(\mathbf{w}, R), \|f(\mathbf{w}, i)\|_1 < \|\mathbf{w}\|_1. \tag{1}$$

First remark that if the left side of (1) is verified, then f modifies $\mathbf{w}$. Now let us show that (1) is true. First assume that there exists $i \in S(\mathbf{w}, R)$ such that $|\mathbf{w}_i| > D$. Then $f(\mathbf{w}, i)_i$ has the same sign than $\mathbf{w}_i$, therefore $|f(\mathbf{w}, i)| = |\mathbf{w}_i| - \lfloor |\mathbf{w}_i| / D \rfloor \cdot D$. Moreover we have

$$\forall j \in [\![1, n]\!] \setminus \{i\}, |\mathbf{w}_j| \leqslant |\mathbf{w}_j| + \left\lfloor \frac{|\mathbf{w}_i|}{D} \right\rfloor \cdot |\mathbf{B}_{i,j}|,$$

which gives

$$\|f(\mathbf{w}, i)\|_1 \leqslant |f(\mathbf{w}, i)_i| + \sum_{\substack{j=1 \\ j \neq j}}^{n} |f(\mathbf{w}, i)_j| \leqslant |\mathbf{w}_i| - \left\lfloor \frac{|\mathbf{w}_i|}{D} \right\rfloor \cdot D + \sum_{\substack{j=1 \\ j \neq i}}^{n} |\mathbf{w}_j| + \left\lfloor \frac{|\mathbf{w}_i|}{D} \right\rfloor \cdot |\mathbf{B}_{i,j}|.$$

This leads to

$$\|f(\mathbf{w}, i)\|_1 \leqslant \|\mathbf{w}\|_1 + \left\lfloor \frac{|\mathbf{w}_i|}{D} \right\rfloor \cdot \delta_i(\mathbf{B}) \leqslant \|\mathbf{w}\|_1 - \left\lfloor \frac{|\mathbf{w}_i|}{D} \right\rfloor < \|\mathbf{w}\|_1.$$

Now consider $i \in S(\mathbf{w}, R)$ such that $|\mathbf{w}_i| < D$. Then the signs of $\mathbf{w}_i$ and $f(\mathbf{w}, i)_i$ are different. Moreover if we write $|\mathbf{w}_i| = R_i + t$ with $t \in [\![1, D - R_i]\!]$, we obtain $|f(\mathbf{w}, i)_i| = |R_i - D + t| = D - R_i - t$. Therefore we have

$$|f(\mathbf{w}, i)_i| = |\mathbf{w}_i| - 2(R_i + t) + D.$$

Following the same reasoning as before to bound $\|f(\mathbf{w}, i)\|_1$, we have

$$\|f(\mathbf{w}, i)\|_1 \leqslant \|\mathbf{w}\|_1 - 2(R_i + t) + D + D - \delta_i(\mathbf{B})$$

and noting that $R_i \geqslant D - \delta_i(\mathbf{B})/2$ we obtain

$$\|f(\mathbf{w}, i)\|_1 \leqslant \|\mathbf{w}\|_1 - 2(R_i + t) + 2R_i < \|\mathbf{w}\|_1.$$

□

Algorithm 1 uses a linear memory and does not need to store much more than the size of the target and the matrix. This is an advantage compared to Babai's nearest plane algorithm which needs the GSO or Babaï's rounding-off algorithm which requires a matrix inverse. [6]

Worst-case complexity. The average-case time-complexity of Algorithm 1 was briefly experimented in [54], however a proper worst-case analysis was not provided and does not seem to have been done in the literature.

Lemma 2. *Let $\mathbf{B} \in M_n(\mathbb{Z})$ be a diagonally dominant matrix and $\mathbf{v} \in \mathbb{Z}^n$, and denote by b the value $\frac{n(D+1)}{n(D+1)-\Delta(\mathbf{B}))}$. An upper bound on the complexity of vector operations done by Algorithm 1 is in*

$$O\left(\log_b\left(\frac{\|\mathbf{v}\|_1}{nD}\right) + \frac{nD}{2}\right).$$

[6] maybe add that one does not require to use floats, we can do everything with integral arithmetic.

Proof. Let us consider the reduction of $\|\mathbf{w}\|_1$ to count the number of reduction steps, using the results and the reasoning of Lemma 1.

First assume $\|\mathbf{w}\|_1 > nD$ which guarantees $\|\mathbf{w}\|_\infty > D$. Thus the coefficient q is greater 1. Denote by $\mathbf{w}'$ the value of the vector after the update in f Algorithm 1. Then $\|\mathbf{w}\|_1$ is updated as

$$\|\mathbf{w}'\|_1 \leqslant \|\mathbf{w}\|_1 - q \cdot \Delta(\mathbf{B}).$$

From $\|\mathbf{w}\|_\infty \leqslant \|\mathbf{w}\|_1 \leqslant n\|\mathbf{w}\|_\infty$ we obtain $q \geqslant \frac{\|\mathbf{w}\|_1}{n(D+1)}$. Thus we get

$$\|\mathbf{w}'\|_1 \leq \|\mathbf{w}\|_1 - \frac{\|\mathbf{w}\|_1}{n(D+1)} \cdot \Delta(\mathbf{B}) = \|\mathbf{w}\|_1 \cdot \left(\frac{n(D+1) - \Delta(\mathbf{B})}{n(D+1)}\right).$$

If we use this inequality and we write k the number of steps necessary to reach the condition $\|w\|_1 \leqslant nD$, i.e to reach the second case, using the worst assumptions we obtain:

$$\|\mathbf{w}\|_1 = \left(\frac{n(D+1) - \Delta(\mathbf{B})}{n(D+1)}\right)^k \cdot \|\mathbf{v}\|_1 \leqslant nD.$$

This gives a $O\left(\log_b\left(\frac{\|\mathbf{v}\|_1}{n(D+1)}\right)\right)$ number of vectors operations to reach $\|\mathbf{w}\|_1 \leqslant nD$.

We can now focus on the case $\|\mathbf{w}\|_1 \leqslant nD$. Note that $\|\mathbf{w}\|_1 \leqslant nD$ still do not give us much information about $\|\mathbf{w}\|_\infty$, so we continue our analysis using $\|\mathbf{w}\|_1$. We proceed by counting the least untactful possible reduction of $\|\mathbf{w}\|_1 \leqslant nD$ per step until $\|\mathbf{w}\|_1 = 0$: each step reduces $\|\mathbf{w}\|_1$ of at least $2t$ (2 with $t = 1$). Therefore, we upper-bound the amount of loop iterations left by $\frac{\|\mathbf{w}\|_1}{2} \leqslant \frac{nD}{2}$. □

By approximating $\log(b) = -\log(1 - \frac{\Delta}{nD}) \approx \frac{\Delta(\mathbf{B})}{nD}$ ($\frac{\Delta(\mathbf{B})}{nD}$ is close to 0 so the approximation holds) and setting $\|\mathbf{v}\|_1 = nD^n$ (i.e each coefficient to an approximate of the determinant), we can obtain the simpler formula ignoring constants:

$$O\left(n^2 D \frac{\log(D)}{\Delta(\mathbf{B})}\right)$$

In addition, if we set $D = n$ and $\Delta(\mathbf{B}) = 1$ as in the different versions of the DRS scheme [53,59,60], we obtain $O(n^3 \log n)$.

Remark 2. This complexity bound obtained in Lemma 2 is not tight and does not reflect at all the significantly faster experimental results reported in [54,59,53], which is understandable: the probability to trigger a *single* least-impactful iteration is $2^{-(n-1)}$, i.e as probable as solving a $\{0,1\}$-knapsack problem with $n-1$ entries randomly. However, our result still proves polynomial operation complexity and constant memory (besides input memory) as far as vector operations (i.e. fixed dimension) are concerned.

3.2 Result on Diagonally Dominant Lattice First Minimum

The importance of $\Delta(\mathbf{B})$ for the quality of the lattice have been exposed in the previous section. In this section, we present a second result linking once again $\Delta(\mathbf{B})$ with an invariant of the lattice. However, this time we are able to bound the first minima of the lattice. This is the first result in this direction which alleviate the complexity of using diagonally dominant matrix for encryption, especially if one wants to avoid any probability of decryption failure.

Theorem 2. *Let $\mathbf{B} \in M_n(\mathbb{Z})$ be a diagonally dominant matrix of diagonal D. Then $\lambda_1^{(\infty)}(\mathcal{L}(\mathbf{B})) \geqslant \Delta(\mathbf{B})$.*

Proof. Consider $l \in \mathbb{Z}^n$ and write $\mathbf{v} = l \cdot \mathbf{B}$. Then write $l' = (|l_i|)_{i \in [\![1,n]\!]}$. There exists $\mathbf{B}' \in M_n(\mathbb{Z})$ a matrix such that $|B'_{i,j}| = |B_{i,j}|$ for any pair $(i,j) \in [\![1,n]\!]^2$, and for all $i \in [\![1,n]\!], \mathbf{B}'_{i,i} = D$ and $\mathbf{v}_i = \pm(l' \cdot \mathbf{B}')_i$. Thus $\mathbf{B}'$ is a diagonally dominant matrix such that $\delta_i(\mathbf{B}') = \delta_i(\mathbf{B})$ for all $i \in [\![1,n]\!]$. Now let us show that $\|\mathbf{v}\|_\infty \geqslant \Delta(\mathbf{B})$. We will first bound the taxicab norm, and then use the classic norm inequality

$$\|\mathbf{v}\|_\infty \leqslant \|\mathbf{v}\|_1 \leqslant n\|\mathbf{v}\|_\infty. \tag{2}$$

First remark that we have the following:

$$\|\mathbf{v}\|_1 = \sum_{j=1}^{n} |(l' \cdot \mathbf{B}')_j| \geqslant \left| \sum_{j=1}^{n} \sum_{i=1}^{n} l_i \cdot \mathbf{B}'_{i,j} \right|.$$

Moreover for any $i \in [\![1,n]\!]$, $l'_i \geqslant 0$ and $\delta_i(B) > 0$, so we have

$$\left| \sum_{j=1}^{n} \sum_{i=1}^{n} l_i \cdot \mathbf{B}'_{i,j} \right| = \sum_{j=1}^{n} \sum_{i=1}^{n} l_i \cdot \mathbf{B}'_{i,j} \geqslant \sum_{i=1}^{n} l'_i \delta_i(\mathbf{B}).$$

Therefore, if $k = |\{i \in [\![1,n]\!] \mid l_i \neq 0\}|$ we obtain $\|\mathbf{v}\|_1 \geqslant k\Delta(\mathbf{B})$.
If $k = n$ then Equation (2) gives

$$\|\mathbf{v}\|_\infty \geqslant \Delta(\mathbf{B}).$$

Now consider the case with $k < n$. Without any loss of generality, assume $\forall i \in [\![1,k]\!], l_i \neq 0$. Denote by l'' the tuple $(l'_1, \ldots, l'_k)$ and $\mathbf{B}''$ the top left $k \times k$ submatrix of $\mathbf{B}'$. Then $\mathbf{B}''$ is diagonally dominant and $\forall i \in [\![1,k]\!], \delta_i(\mathbf{B}'') \geqslant \delta_i(\mathbf{B}') = \delta_i(\mathbf{B})$. We have

$$\forall i \in [\![1,k]\!], (l \cdot \mathbf{B})_i = (l' \cdot \mathbf{B}')_i = (l'' \cdot \mathbf{B}'')_i.$$

Then, since $|\{i \in [\![1,k]\!] \mid l''_i \neq 0\}| = k$, we can apply the previous result to l'' and $\mathbf{B}''$, therefore $\|l'' \cdot \mathbf{B}''\|_\infty \geqslant \Delta(\mathbf{B}'')$ and $\exists i_0 \in [\![1,k]\!], |(l'' \cdot \mathbf{B}'')_{i_0}| = \|l'' \cdot \mathbf{B}''\|_\infty$. Finally we get

$$|(l \cdot \mathbf{B})_{i_0}| = |(l' \cdot \mathbf{B}')_{i_0}| = |(l'' \cdot \mathbf{B}'')_{i_0}| \geqslant \Delta(\mathbf{B}'') \geqslant \Delta(\mathbf{B}') = \Delta(\mathbf{B}).$$

3.3 Diagonally Dominant with negative noise

One can obtain better results when considering more specific structures. In this section we consider diagonally dominant matrices $\mathbf{B} = \mathbf{D} + \mathbf{N}$ where the noise matrix $\mathbf{N}$ is such that $\forall (i,j) \in [\![1,n]\!], \mathbf{N}_{i,j} \leqslant 0$.

Lemma 3. *The bound on $\lambda_1^\infty(\mathcal{L})$ is tight, i.e. there is $\mathbf{B}$ such that $\lambda_1^\infty(\mathcal{L}(\mathbf{B})) = \Delta(\mathbf{B})$.*

Proof. Consider $\mathbf{B} = D \cdot \mathrm{Id}_n + \mathbf{N}$ such that $\mathbf{N}_{i,i+1} = 1 - D$ and $\mathbf{N}_{i,j} = 0$ whenever $j \neq i+1$. Then the vector $v \stackrel{\text{def}}{=} [1, \cdots, 1] \cdot \mathbf{B}$ satisfies the desired equality. □

Lemma 4. *Consider $\mathbf{B}$ a diagonally dominant matrix with negative noise. Then there is an algorithm – that we will denote by* `neg-PSW` *– that reduces any vector $\mathbf{v} \in \mathbb{R}_+^n$ to an equivalent vector $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(B)$ such that $\mathbf{w} \in [0, D[^n$.*

Proof. Let $\mathbf{v}$ be a vector and $\mathbf{w} \stackrel{\text{def}}{=} v - q \cdot \mathbf{B}_i$ for some $i \in [\![1,n]\!]$. Then remark that if $\mathbf{v}_i \geqslant qD$, we have $0 \leqslant \mathbf{w}_i < D$ and $\mathbf{w}_j \geqslant \mathbf{v}_j$ for all $j \neq i$. Moreover it is clear that $\|\mathbf{w}\|_1 = \|\mathbf{v}\|_1 - q\Delta(\mathbf{B})$. Thus it is clear that the algorithm will stop and that the outputted vector will lie in the claimed space. □

Remark 3. Note that one can easily shift the result to the centered hypercube $[\![-D/2, D/2[\![^n$ so that for any $\mathbf{v} \in \mathbb{N}^n$ there is $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$ with $\mathbf{w} \in [\![-D/2, D/2[\![^n$.

One can note that the reduction radius is smaller (by a factor up to 2) that for generic diagonally dominant matrices. Moreover, the covering radius does no depends anymore of $\Delta(\mathbf{B})$. An advantage which can be passed when diagonally dominant matrices are used for cryptography.

4 Diagonally Dominant Matrix Encryption

In this section we will describe an encryption scheme using diagonally dominant matrices, the we call DRE as a callback to DRS. First we describe in Section 4.1 the general framework of our construction based on a GDD_γ solver. We provide conditions on the matrices used as private keys to ensure the correctness of the scheme within this framework in Section 4.2. To this end we use the results on $\lambda_1^{(\infty)}$ and $\mu^{(\infty)}$ proven in Section 3 and summed-up in Theorem 1. Then we give an instantiation of this general framework in Section 4.3 and discuss security in Section 4.4.

4.1 General framework

Let us now describe the framework for the encryption scheme we are considering. As mentioned previously, it is based on the max norm l_∞. We fix as parameters $(D, n, M) \in \mathbb{N}^2$. Let us denote $\mathcal{L}$ the lattice generated by a diagonally dominant matrix $\mathbf{B} = D \cdot \mathrm{Id}_n + \mathbf{N}$. Let R be the radius in which we can find for any

$\mathbf{c} \in \mathbb{Z}^n$ a vector $\mathbf{m} \equiv \mathbf{c} \in \mathcal{L}$ s.t. $\|\mathbf{m}\|_\infty < R$. Algorithms 1 and 6 offers us parametrisable radii R directly from a parametrisable $\mathbf{B}$. Evidently, $\mathbf{B}$ is kept as a secret trapdoor as it allows for decryption. Let M be the upper bound of the max norm of the vector messages we wish to recover, such that if the vectors associated to the valid messages belong to a set $\mathcal{M}$, then $\mathcal{M} \subseteq [-M, M]^n$. Here, we consider that each message is associated to a vector $\mathbf{m} \in \mathbb{Z}^n$ we wish to recover, and that the encryption of $\mathbf{m}$ is associated to a ciphertext vector $\mathbf{c} = \mathbf{m} + \mathbf{v}$ where $\mathbf{v} \in \mathcal{L}(\mathbf{B})$. In summary we consider the following framework:

- The secret key $\mathbf{S}_K = \mathbf{B} \in M_n(\mathbb{Z})$ is a diagonally dominant matrix with diagonal coefficient D, and the public key $\mathbf{P}_K$ is $\mathbf{H} = \mathrm{HNF}(\mathbf{B})$.
- The message space is $\mathcal{M} \subseteq [\![-M, M]\!]^n$.
- The encryption function will be $\texttt{Encrypt}(\mathbf{m}, \mathbf{P}_K) = s \cdot \mathbf{H} + \mathbf{m}$, for some $s \in \mathbb{Z}^n$.
- The decryption function will be $\texttt{Decrypt}(\mathbf{c}, \mathbf{S}_K) = \texttt{Reduce}(\mathbf{c}, \mathbf{B})$, where $\texttt{Reduce}$ is a GDD_γ solver. Its convergence radius will be denoted by R.

With a similar approach to [32], we first show how one can use our results to guarantee correctness of decryption. Second, we discuss potential security concerns.

4.2 Guaranteeing decryption of valid messages (i.e. correctness)

In order to obtain a *correct* scheme we need to determine parameters ensuring the correctness of the decryption. The first condition that they need to satisfy is $M \leqslant R$ so that $\texttt{Reduce}(\mathbf{c}, \mathbf{B})$ is indeed a valid message. Then one needs to ensure unicity, meaning $\texttt{Reduce}(\texttt{Encrypt}(\mathbf{m}, \mathbf{P}_K)) = \mathbf{m}$. This is satisfied as soon as

$$R + M \leqslant \lambda_1^{(\infty)}(\mathcal{L}). \tag{3}$$

In particular for diagonally dominant matrices, we can use Algorithm 1 for $\texttt{Reduce}$ and Theorem 1 ensures that Equation (3) can be simply satisfied for $\mathbf{B}$ such that

$$\Delta(\mathbf{B}) > \frac{2}{3}(D + M), \tag{4}$$

which are straightforward to construct.

If we focus on matrices with negative noise only, then we can obtain larger bounds. Indeed, in this case $R = D/2$ so (3) becomes $\lambda_1^{\infty}(\mathcal{L}) \geqslant D/2 + M$ which gives $\Delta(\mathbf{B}) > D/2 + M$.

Thus, we could use smaller dominance levels for a fixed M or larger message spaces for the same value $\Delta(\mathbf{B})$.

4.3 Instantiation of the encryption scheme

To instantiate our encryption scheme, we first need to fix some public parameters as the diagonal coefficient D and the dimension n. We assume the message space

is composed of vectors over $\mathcal{M} = \{-1, 0, 1\}^n$, but we showed earlier that could also be subject to change.

From an external point of view, our scheme is is close to knapsack problem, such as the first proposition of Merkle-Hellman [41]. The major difference is within the setup and the decryption, which are details that are hidden to the messages senders.

Setup The setup is composed of two steps. For the secret key, we generate a diagonally dominant matrix with our chosen parameters (D, n). Since the message space is $[\![-1, 1]\!]^n$, following Equation (4), we will fix $\Delta(\mathbf{B}) = \frac{2}{3}(D+1)$.

For the public key, we compute the HNF of the secret key, assuming it has perfect form, i.e. $\mathcal{L}(\mathbf{B})$ is a co-cyclic lattice. If the HNF does not hold a perfect form, we can choose to discard the key or use a permutation to attempt obtaining a perfect HNF as reported in [58].

The public key is then the resulting HNF, with a small advantage: since the HNF holds a perfect form, only the last column vector needs to be sent.

Algorithm 2 `DRE-Setup`

Require: $(D, n) \in \mathbb{N}^2$.
Ensure: (P_K, S_K) the public and secret keys
1: $\Delta(\mathbf{B}) \leftarrow \frac{2}{3}(D+1)$
2: $\mathbf{B} \leftarrow \texttt{RDDgen}(D, n, \Delta(\mathbf{B}))$
3: $\mathbf{H} \leftarrow \text{HNF}(\mathbf{B})$
4: **while** $\texttt{IsPerfect}(\mathbf{H}) = \text{false}$ **do**
5: $\quad \mathbf{B} \leftarrow \texttt{RDDgen}(D, n, \Delta(\mathbf{B}))$
6: $\quad \mathbf{H} \leftarrow \text{HNF}(\mathbf{B})$
7: **end while**
8: $\mathbf{h} \leftarrow \mathbf{H}[1..n, n]$
9: **return** $(\mathbf{B}, \mathbf{h})$

Encryption For the encryption, we just sum or substract the corresponding values of the public key $\mathbf{P}_K$ according to our message $\mathbf{m}$. The resulting integer is our ciphertext c.

Because the keys $(\mathbf{B}, \mathbf{H})$ are chosen such that $\mathbf{H} = \text{HNF}(\mathbf{B})$ is perfect and $\mathbf{h}$ is the last column of $\mathbf{H}$, the output of `DRE-Encrypt` as described in Algorithm 3 is the last coefficient of a vector of the form $[0, \dots, 0, c] = \mathbf{m} + \mathbf{v}$ with $\mathbf{v} \in \mathcal{L}(B)$.

Algorithm 3 `DRE-Encrypt`

Require: A plaintext $\mathbf{m} \in [\![-1, 1]\!]^n$ and the public key $\mathbf{P}_K = \mathbf{h} \in \mathbb{Z}^{n-1}$.
Ensure: A ciphertext $\mathbf{c}$
1: $c \leftarrow 0$
2: **for** $i = 1$ to $n - 1$ **do**
3: $\quad c \leftarrow c - m_i \cdot h_i$
4: **end for**
5: $c \leftarrow c + m_n$
6: **return** c

Indeed, if one reduces the vector $\mathbf{m}$ with $\mathbf{H}$, as follows

$$\left[\begin{array}{cccc|c} m_1 & \dots & \dots & m_{n-1} & m_n \\ \hline 1 & 0 & \dots & 0 & h_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & h_{n-1} \\ 0 & \dots & \dots & 0 & \det(\mathbf{B}) \end{array}\right],$$

using the first $n - 1$ rows of $\mathbf{H}$ we can remark that the first vector will be transformed into

$$[0, \dots, 0, m_n - \sum_{i=1}^{n-1} m_i h_i] = \mathbf{m} - m \cdot \mathbf{H} + m_n \cdot [0, \dots, 0, \det(\mathbf{B})].$$

Note that this approach is very similar to the one chose in the SQUIRRELS scheme [28] recently submitted to the NIST call for proposals for quantum-resistant digital signature algorithms [51].

Decryption We can use the reduction algorithms studied earlier to recover $\mathbf{m}$ from c. From our study, Algorithm 4 will output the correct plaintext $\mathbf{m}$.

Algorithm 4 `DRE-Decrypt`

Require: A ciphertext $c = \texttt{DRE-Encrypt}(\mathbf{m}, \mathbf{h}) \in \mathbb{Z}$ and the secret key $S_K = \mathbf{B}$.
Ensure: The plaintext $\mathbf{m}$
1: $\Delta(\mathbf{B}) \leftarrow \frac{2}{3}(D + 1)$
2: $R \leftarrow D - \Delta(\mathbf{B})/2 \cdot [1, \dots, 1]$
3: $m \leftarrow c \bmod \det(\mathbf{B})$
4: $\mathbf{m} \leftarrow [0, \dots, 0, m]$
5: $\mathbf{m} \leftarrow \texttt{RSR}(\mathbf{m}, \mathbf{B}, R)$
6: **return** $\mathbf{m}$

4.4 Security concerns

Formal security. The scheme defined by the algorithms presented in Algorithm 2 is guaranteed to be correct but is not secure. Since it is deterministic, it is not even IND-CPA. In the following, we discuss the necessary milestones to reach in the path towards IND-CCA security. Note that, for example, the key encapsulation mechanism BAT [29] follows essentially the same steps.

One-wayness. The first level of security to achieve is one-wayness, i.e. that one cannot recover a message $\mathbf{m}$ given only the public key $\mathbf{P}_K$ and a random ciphertext $c = \texttt{DRE\_Encrypt}(\mathbf{P}_K, \mathbf{m})$. Obviously, an adversary is allowed to produce as many pairs of plaintext-ciphertext as he wants.

IND-CPA from one-wayness. Assume that the scheme achieve OW-CPA security. Then, following [26,29] one can be made IND-CPA security in the Random Oracle Model (ROM) with the following transformations of the encryption and decryption functions :

$$(\mathbf{m}, \mathbf{s}, \mathbf{P}_K, H) \mapsto [\mathbf{m} \oplus H(\mathbf{s}) \| \texttt{DRE-Encrypt}(\mathbf{P}_K, \mathbf{s})]$$

and

$$(\mathbf{c}_1, \mathbf{c}_2, \mathbf{S}_K) \mapsto H(\texttt{DRE-Decrypt}(\mathbf{S}_K, \mathbf{c}_2)) \oplus \mathbf{c}_1,$$

where $\mathbf{s}$ is a random vector and H a hash function modelised as a random oracle.

IND-CPA to IND-CCA Finally, famous transformations permit to reach IND-CCA security such as the Fujisaki-Okamoto (F.-O.) transform [30,18].

In the end, we see that the main difficulty is to obtain a OW-CPA version of DRE. One option could be to adapt the proof from [29, Theorem 2] by considering a class of random co-cyclic lattices whose HNF are hard to distinguish from the ones of diagonally dominant matrices.

We believe that this could be achieved through more extensive study of the determinant, which can be an easy sorting criterion.

We could also choose to hide the determinant, i.e. remove the last coeffeficient from the sent vector $\mathbf{h}$ before sharing it. In order to construct a setting where the public key is indistinguishable from random co-cyclic lattices, one could also randomize the public key by adding multiples of $\det(\mathbf{B})$ to its entries, so that it is close to uniform in a certain range $[\![2^{l-1}, 2^l]\!]$.

Note that it has been over 20 years that a similar structure, the GGH encryption of Micciancio [43] remains unbroken. We conjecture that the problem of distinguishing co-cyclic lattices with diagonally dominant bases (with similar parameters otherwise) from generic co-cyclic lattices is hard.

Concrete security. There are several security concerns that one needs to address if planning to build a cryptosystem. One of them is to ensure that deciphering $\mathbf{c}$ into $\mathbf{m}$ is not trivial without the secret key. Heuristically, if $\mathbf{c}$ is large

enough, the problem of recovering $\mathbf{m}$ from $\mathbf{c}$ can be seen as a specific instance of the CVP, which is known to be hard. With that in mind, what is left is the security of the public key. Since [43], it makes sense to provide a basis of $\mathcal{L}(\mathbf{B})$ as a Hermite Normal Form for the public key, however other choices might be possible. It might not even be necessary to provide a basis of $\mathcal{L}(\mathbf{B})$ in the first place. Let us assume the public key is chosen as another basis of the same lattice: in the last decades, it seemed that pure key recovery attacks on diagonally dominant matrices [53,58] or close structures [32,47] are rather unsuccessful. The weaknesses were mostly on signature scheme instances [49,22,24] which do not concern this section. Note that [49] also consider that the *encryption* approach of [32] is still secure, and to the extent of our knowledge this claim has not been challenged yet.

Key recovery

Naive attack. The most naive attack is to reduce the public key in order to recover the secret key or a basis with an equivalent quality. As a matter of fact, we will consider only the complexity of computing *one* short vector. In the case of DRS, it amounts to solve the SVP_γ for a small constant approximation factor. Note also that diagonally dominant lattices have unusually short vectors. Indeed, the secret key $\mathbf{B}$ is composed of vectors such that $D \leqslant \|\mathbf{B}_i\|_2 \leqslant \sqrt{2}D$ which is smaller than what is predicted by the Gaussian heuristic by a factor in $O(\sqrt{n})$. Thus, the situation is similar to what happens for the HAWK cryptosystem based on the $\mathbb{Z}^n$-LIP. Following the analysis done in [25], the required blocksize to recover a secret vector should satisfy $\sqrt{\beta/n} \approx \delta_\beta^{2\beta-n-1}$ with $\delta_\beta \approx (\beta/(2\pi e))^{1/2(\beta-1)}$ which gives $\beta \in O(n/2) + o(n)$.

Attack by BDD-uSVP. Apart from reducing the public key, one can use the fact that $\mathbf{B}$ is diagonally dominant. Indeed, each vector of the secret basis is then of the form $D \cdot \mathbf{e}_i + \mathbf{n}_i$ with $\|\mathbf{n}_i\|_1 < D$. Then solving a BDD instance with respect to $\mathcal{L}(\mathbf{B})$ and the target vector $D \cdot \mathbf{e}_i$ would yield the secret vector $\mathbf{B}_i$. The cost of such an attack – without any additional knowledge – can be estimated following [4,2]. It is mentioned in [24] that recovering $\mathbf{B}_i$ can be done with BKZ-β when

$$\sqrt{\beta/(n+1)} \cdot \|\mathbf{B}_i - D \cdot \mathbf{e}_i\| \approx \delta_\beta^{2\beta-n-1} \cdot D^{n/(n+1)}. \tag{5}$$

If the dominance level is $\alpha \cdot D$ then $\|\mathbf{B}_i - D \cdot \mathbf{e}_i\| \approx \sqrt{(1-\alpha) \cdot D}$, which gives the broad condition $\sqrt{(1-\alpha) \cdot \beta} \approx \delta_\beta^{2\beta-n-1} \cdot n^{n/(n+1)}$, considering that $D = n$. This ensures asymptotical security.

4.5 Message recovery

For message recovery, one needs to compute $\mathbf{m}$ from c, where c corresponds to a vector $\mathbf{c} \equiv \mathbf{m} \bmod \mathcal{L}(\mathbf{B})$. Thus $\mathbf{v} = \mathbf{c} - \mathbf{m}$ is a lattice vector such that

$d(\mathbf{c}, \mathbf{v}) = \|\mathbf{m}\|$. Since $\mathbf{m}$ is particularly short, this amounts to solving a BDD instance with $\|\mathbf{m}\| \approx \sqrt{n/2}$.

As for the key recovery, we can use estimation on BKZ to evaluate the cost of such type of attack,

$$\sqrt{\beta/(n+1)} \cdot \|\mathbf{m}\| \approx \delta_\beta^{2\beta-n-1} \cdot D^{n/(n+1)}. \tag{6}$$

Therefore, we can extract a block size for BKZ to recover the message, $\sqrt{\frac{\beta}{2}} \approx \delta_\beta^{2\beta-n-1} \cdot n^{n/(n+1)}$.

5 Heuristic patch of the DRS scheme

In this section we will study possible patches for the DRS signature scheme [53,59] against Ducas and Yu statistical attacks [24]. Exploiting the new reduction radii that we obtained in Sections 3.1 and 3.3, we first explore in Section 5.2 how changing this radius in the DRS scheme impacts the *practical* efficiency of the attack. Then in Section 3.3 we study a new version of the DRS scheme based on diagonally dominant matrices with *negative noise* that we call NEGATIVE-DRS. We analyse its security under statistical attacks. Finally, as an extra layer of security, we look into the possibility of adding a mask to the signature, see Section 5.4.

5.1 Quick recap of the DRS scheme and attacks

The signature scheme called DRS was a submission to the first round of the NIST standardization process for signature scheme [50] using diagonally dominant lattices. The main idea of DRS is to follow a framework close the one of GGH [33] but using the diagonal dominance property to sign within an hypercube independent of the secret key, hoping to prevent leaking the secret key as in [49] for example. This was first presented by Plantard et al. in [54]. However the original DRS scheme has been subject to a learning attack from Ducas and Yu [63], which was then extended to the second version of the scheme (the so-called DRSv2 [60]) [24].

The main idea behind this learning attack is that a signature $\mathbf{s}$ obtained from the signature algorithm is of the form $\mathbf{s} = \mathbf{s}' \pm \mathbf{B}_i$, where $\mathbf{B}$ is the secret diagonally dominant matrix and $\mathbf{s}'$ is the vector we have just before the algorithm stops. This relation introduces a correlation between the coefficients of the row B_i and the ones of $\mathbf{s}$. Then by collecting lots of signatures and using learning techniques, one can make an educated guess on a key. Typically, for the ith basis vector, one guess $\mathbf{B}'_i$ which is *close* to the secret $\mathbf{B}_i$.

One direction to counter this learning attack would be to remove the link between the signatures and the secret key; typically one can think that the signatures are not good enough, i.e., the signature $\mathbf{s}$ is too large. Indeed, a perfect signature would be $\mathbf{s} \equiv \mathbf{m} \bmod \mathcal{L}$ such that $\mathbf{s} - \mathbf{m}$ is a lattice vector closest to $\mathbf{m}$, or equivalently such that $\mathbf{s}$ has minimal norm.

Remark that one could obtain a signature scheme with a security proof within the GPV framework by restraining to co-cyclic lattices, similarly to what is done in the SQUIRRELS scheme recently submitted to the additional call for standardization for signature schemes [28]. It is unclear however whether the structure of diagonally dominant matrices would allow for improvements compared to existing schemes.

5.2 Changing the reduction radius

From Section 3, we know that the reduction radius of D as taken in the DRS schemes [53,60] can be lowered to $D - \Delta(\mathbf{B})/2$. Remark that this modification alone would not change much. Indeed, for DRS the noise is such that $\Delta(\mathbf{B}) = 1$, which gives a reduction radius of $D-1$ instead of D. Thus, one needs to increase $\Delta(\mathbf{B})$ as well to have a potential impact.

Impact of the dominance level We study how the dominance level $\Delta(\mathbf{B})$ associated with a reduction radius $D-\Delta(\mathbf{B})/2$ impact the efficiency of the attack described in [24]. One can find in Figure 1 the data recovered in our experiments.

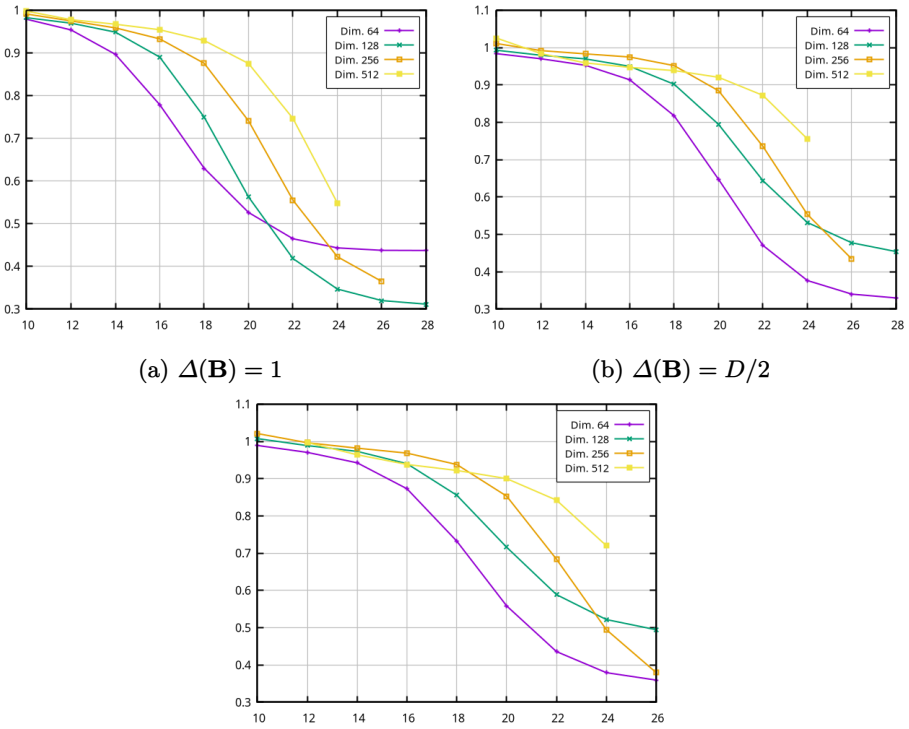
Figure 1a corresponds to the setting of the original DRSv2 scheme and the attack reported by Ducas and Yu [24]. Remark that we obtain smaller factors $r(n, N)$ that they did, which is certainly due to the fact that we slightly modified their code, notably the key generation algorithm (for simplicity reasons). However, since we are interested in worsening the best attack possible on the scheme, we can safely take our results as reference. Our goal is then to obtain larger factors than the ones plotted in Figure 1a.

Note that decreasing the $\Delta(\mathbf{B})$ tends to slightly flatten the curves, especially for higher dimensions. Consequently the factors tend to converge to higher values. However this modification is clearly not sufficient to obtain a factor $r(n, N)$ sufficiently large for the attack to be deemed patched.

The special case of negative noise We mentioned previously that if the noise matrix $\mathbf{N}$ contains only negative values then one can go beyond the convergence radius $D - \Delta(\mathbf{B})/2$. We proved in Section 3.3 that it is possible to reduce any vector $\mathbf{v}$ to a coset representative $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$ such that $\mathbf{w} \in [\![0, D[\![^n$. Experimentally, one can obtain signed coefficients with a reduction in the hypercube $[\![-D/2, D/2[\![^n$ which we consider a better choice since it fits the setting of the original attack. Let us look into the impact that negative noises have on the learning attack by Ducas and Yu [24]. Data from our experiments can be found in Figure 2.

The attack is clearly less efficient than with signed noises, and the noise level seems to have more impact. However, while it seems that the attack stabilizes quickly for $\Delta(\mathbf{B}) = 1$ it is unclear how the different curves would evolve for larger sample sizes and noise levels.

Interestingly, one can remark that different curves never cross each other in this negative setting. This tends to show that the attack does not become more efficient as the dimension grows as it is the case in the signed setting.



(a) $\Delta(\mathbf{B}) = 1$ (b) $\Delta(\mathbf{B}) = D/2$

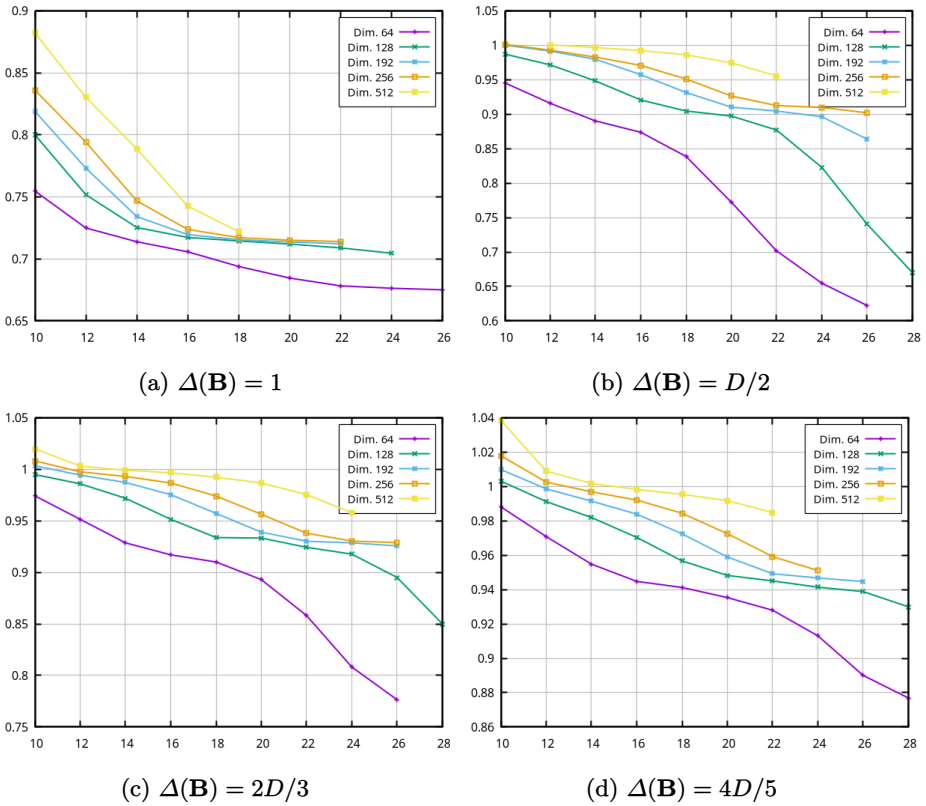
(c) $\Delta(\mathbf{B}) = 2D/3$

Fig. 1: Experimental measures of $r(n, N)$ for smaller reduction radius and different noise levels, when the noise is negative.

5.3 Negative DRS

In this section, we consider a modified version of the DRS scheme, tweaked to use negative noise matrices. Since this is the only major modification, we will not describe the different algorithms in detail nor will we prove correctness. In the following we focus on analysing its security. The basic strategies for the key recovery problem are the same as for DRE so we refer to Section 4.4. Thus, we only consider more advanced techniques through statistical analysis. Since the dominance level $\Delta(\mathbf{B})$ is an important parameter, we *do not fix it* at first and discuss its impact on security. Thus, we consider that $\Delta(\mathbf{B}) = \alpha \cdot D$ for some $\alpha \in]0, 1[$.

Original attack. The first statistical attack from Nguyen and Regev [49] and its improvements [22,40] assume at some point that signatures are of the form $\mathbf{s} = [s_1, \ldots, s_n] \cdot \mathbf{B}$ where the coordinates s_i are independent one to each other. There is no evidence that this condition is satisfied by DRS signatures. However, their distribution may be close to this ideal setting to the point where one can



(a) $\Delta(\mathbf{B}) = 1$ (b) $\Delta(\mathbf{B}) = D/2$

(c) $\Delta(\mathbf{B}) = 2D/3$ (d) $\Delta(\mathbf{B}) = 4D/5$

Fig. 2: Experimental measures of $r(n, N)$ for smaller reduction radius and different noise levels, for negative noises.

still apply the gradient descent with success. Moreover, remark that we know broad directions for the secret vectors $\mathbf{B}_i$. Thus, as mentioned by Nguyen and Regev for the GGH scheme in [49], one can start the descent with well-chosen initial vectors instead of drawing them uniformly on the unitary sphere. However, our experiments show that this strategy is asymptotically unsuccessful. Indeed, if $\mathbf{s}$ is a vector recovered by a descent, our experiments show that its distance to the secret key $\min_{i \in [\![1,n]\!]} \|\mathbf{s} - \mathbf{B}_i\|_2$ is typically around $n/2$, see Table 1. Thus, the best strategy remains the BDD-uSVP attack on $D \cdot \mathbf{e}_i$.

n \ N	10	11	12	13	14	15	16
37	19.23	19.34	19.24	19.22	19.10	19.19	19.15
71	38.04	36.80	36.28	36.19	36.51	36.26	36.36
211	208.91	120.25	109.56	106.95	106.67	106.42	106.21

Table 1: Minimal distance between the secret key and vectors recovered by 16 descents for several dimensions n and sample sizes N.

Learning attack from Ducas and Yu. [24] The data gathered by our experiments tend to show that the learning attack from Ducas and Yu is mitigated and that we can asymptotically assume that the key recovery can be done by replacing the target vector $D \cdot \mathbf{e}_i$ by a vector $\mathbf{t}_i$ such that $\|\mathbf{t}_i - \mathbf{B}_i\|_2 \geqslant r \cdot \|D \cdot e_i - \mathbf{B}_i\|_2$, where $r \in]0.7, 0.95[$ depending on the noise level. For example when $\alpha \approx 0$, Figure 1a shows that we can assume that $r \geqslant 0.7$ while a lower noise such as $\alpha \approx 0.8$ could reach $r \geqslant 0.9$. Without further thinking, one could jump on the counter-intuitive conclusion that a more orthogonal basis provides better security. However, one should not forget that the attack deeply relies on the *distance between the target and the secret basis*, i.e. the term $\|\mathbf{B}_i - D \cdot \mathbf{e}_i\|$ in Equation (5). Thus, one can get a larger r factor but a smaller term $r \cdot \|\mathbf{B}_i - D \cdot \mathbf{e}_i\|$. For example, as soon as $\Delta(\mathbf{B}) \geqslant D/2$ this norm *will be lower* than the one obtained for $\Delta(\mathbf{B}) = 1$ and taking the attack into account. Thus, choosing $\Delta(\mathbf{B}) = 1$ seems to be the safer option. Equation (5) is then replaced by

$$\sqrt{\beta/(n+1)} \cdot r(n, N) \cdot \|\mathbf{B}_i - D \cdot \mathbf{e}_i\| \approx \delta_\beta^{2\beta-n-1} \cdot D^{n/(n+1)}, \tag{7}$$

which gives $0.7 \cdot \sqrt{(1-\alpha) \cdot \beta} \approx \delta_\beta^{2\beta-n-1} \cdot n^{n/(n+1)}$.

This ensures an asymptotical security, even though the corresponding β for a given dimension would be significantly lower than for other schemes such as SQUIRRELS [28] or HAWK [25] for example.

Extending the learning attack. However one can wonder whether signing with very close vector reveals other information, such as (potentially approximate) Voronoi cells. This lead us to consider the setting of the Closest Vector Problem with Preprocessing (CVPP).

Assume that a learning attack *à la* Nguyen and Regev [49,22,40] allows us to recover vectors from a hidden parallelotop close to the Voronoi cell. First one may wonder if this structure is complex enough to hide the secret basis. Indeed, diagonally dominant matrices have a strong structure allowing for an efficient CVP solver. We first verified that a trivial attack through enumeration could not be done. To this end we computed the set of relevant Voronoi vectors for diagonally dominant matrices of small degree. The average size of such sets for a selected range of dimensions are gathered in Table 2.

n	$D - \Delta(\mathbf{B}) = 2$	$D - \Delta(\mathbf{B}) = 3$	$D - \Delta(\mathbf{B}) = 5$
6	122.6	118.0	72.6
8	500.8	497.6	430.4
10	2011.2	1975.6	1657.4
16	130458.	130319.2	125477.2
20	2094084.	2089558.6	2080385.4

Table 2: Average number of Voronoi relevant vectors for diagonally dominant lattices.

It evolves exponentially as expected. Thus, one cannot just go through these sets to recover the secret basis even if given the Voronoi cell for free.

Then we considered the possibility that the recovered vector could help in solving CVP_γ more efficiently, to the point where one could forge a signature in polynomial time. As mentioned earlier, this setting is close to the one of CVPP algorithms. We established in Appendix A that the average approximation factor reached by Algorithm 1, both for signed or negative noises, is a small constant. Following [21] the query phase for solving such an instance of the Approximate Closest Vector Problem with Preprocessing (CVPP_γ) is exponential for arbitrary lattices. Note that the size of the preprocessed list of lattice vectors should be (at least) subexponential as well and requires to compute the shortest vectors (up to some approximation factor) of the lattice, among which are the vectors of the secret basis. Thus, one would certainly recover the secret basis as a byproduct of the query phase. Thus, we deem that forging a signature using (approximate) Voronoi cells or classical algorithms solving the CVPP_γ [21] is as hard as recovering the secret key.

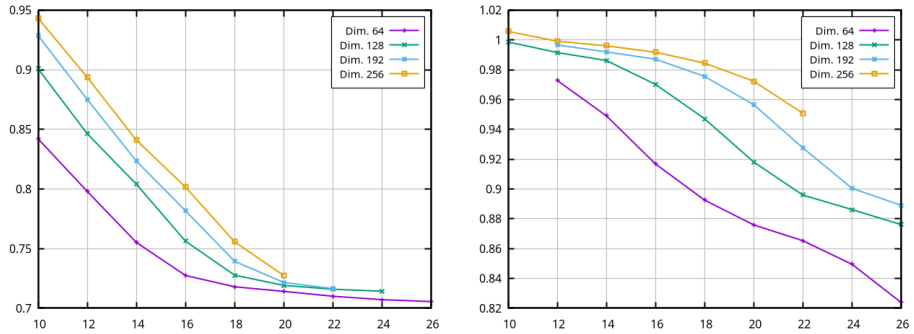
Conclusion. From our study, we deem that the DRS scheme should be asymptotically secure, with the most secure noise level being $D - 1$ or equivalently $\delta(\mathbf{B}) = 1$. We also deem diagonally dominant matrices with negative noise an interesting direction to pursue. Indeed, they offer good decoding properties which seem to mitigate the statistical attack from Ducas and Yu [24].

5.4 One mask to loose them all ?

In this section, we explore a counter-measure to statistical attacks linking signatures to the secret key. We use a mask to "drown" the information leaked by the signature, to the cost of increasing them. More formally, if $\mathbf{m}$ is a vector to sign and $\mathtt{Sign}$ is our base signing algorithm, our masked signing algorithm will be $\mathtt{MaskedSign} : \mathbf{m} \mapsto \mathtt{Sign}(\mathbf{m} + \mathbf{e}) - \mathbf{e}$, where $\mathbf{e}$ is drawn following a *public* distribution which is *independent* of the secret basis. Then the produced signature $\mathbf{s} = \mathtt{MaskedSign}(\mathbf{m})$ is still congruent to $\mathbf{m}$ module the lattice $\mathcal{L}(\mathbf{B})$. The idea is that $\mathbf{s}$ will lie in a space still dependent of the basis but where the geometry of latter is hidden by the distribution added by $\mathbf{e}$.

In order to fit the setting of DRS, we choose the mask to be drawn from a uniform distribution over the integral hypercube $[\![-c_{\mathsf{mask}} \cdot D, c_{\mathsf{mask}} \cdot D]\!]^n$ for some fixed $c_{\mathsf{mask}} \in \mathbb{N}$. In terms of security, adding such a mask does not improve the attacks on the secret key. However, forging a signature becomes easier since signatures have larger norms.

One can find experimental results for the learning attack from Ducas and Yu in Figure 3. First we consider the impact of a small mask fixing $c_{\mathsf{mask}} = 1$ for $\Delta(\mathbf{B}) \in \{1, D/2\}$ then we focus on $\Delta(\mathbf{B}) = 1$ with larger masks.



(a) $\Delta(\mathbf{B}) = 1$ (b) $\Delta(\mathbf{B}) = D/2$

Fig. 3: Experimental measures of $r(n, N)$ for different noise levels, with negative noise and a mask s.t. $c_{\mathsf{mask}} = 1$.

One can observe from Figure 3 that the impact of a small mask seems to be limited. Indeed, the factors $r(n, N)$ for $\Delta(\mathbf{B}) = 1$ are larger for very small sample sizes but the asymptotic value seems to be unchanged an reached for small sample sizes as well. The situation is mitigated for $\Delta(\mathbf{B}) = D/2$, as the attack is clearly less efficient with the mask. However there is no definitive sign of convergence, so the asymptotical value could soon be reached as well.

From Figure 4 it seems that the sample size N for which the limit value of r is reached increases for larger masks. However the difference with the original signature procedure is rather small.

Finally our study seems to indicate that adding a mask is not an efficient counter-measure against Ducas and Yu's attack [24], especially considering that the scheme would be slower and weaker against signature forgery.

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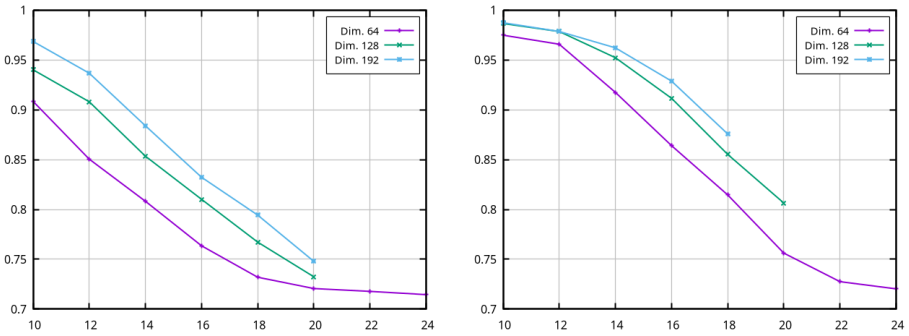

(a) $c_{\mathsf{mask}} = 2$ (b) $c_{\mathsf{mask}} = 5$

Fig. 4: Experimental measures of $r(n, N)$ for $\Delta(\mathbf{B}) = 1$, with negative noise and a mask s.t. $c_{\mathsf{mask}} \in \{2, 5\}$.

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A Average quality of reduction

Average quality of CVP We evaluated experimentally the quality of the approximation factor obtained by Algorithm 1 as a CVP_γ solver for small dimensions. To this end we used the CVP solver from FPYLLL [62], called with the method `CVP.closest_vector(L, t)`, where `L` is the lattice and `t` is the target vector. From our computations, for a fixed dominance level $\Delta(\mathbf{B})$, the average approximation factor reached by Algorithm 1 is smaller than a constant, seemingly decreasing with respect to the dimension.

Note that since one is able to recover $\mathbf{B}$ from its HNF in exponential time, this indicates that approximating the CVPP within a small constant factor should be solvable in polynomial time for diagonally dominant matrices. This contrasts with the situation over general lattices [3].

	$\Delta(\mathbf{B})$ \ n	10	15	20	25	30	35	40	45	50	55	60
`PSW`	1	2.91	2.86	2.79	2.73	2.67	2.61	2.61	2.56	2.56	2.51	2.50
	$D/2$	1.55	1.50	1.50	1.37	1.38	1.41	1.43	1.36	1.38	1.36	1.38
`neg-PSW`	1	1.44	1.24	1.26	1.21	1.22	1.16	1.18	1.15	1.15	1.13	1.14
	$D/2$	1.066	1.022	1.028	1.015	1.021	1.012	1.018	1.010	1.011	1.010	1.009

Table 3: Average approximation factor reached by `PSW` and `neg-PSW` for small dimensions and $\Delta(\mathbf{B}) \in \{1, D/2\}$.

B Short vectors and reduction algorithms for Column Diagonally Dominant matrices

In this section we consider Column Diagonally Dominant matrices. A c.d.d matrices can be simply defined as the transpose matrix of a Diagonally Dominant matrix (Definition 9). Conceqeuntly, we will note $\Delta^T(\mathbf{B}) = \Delta(\mathbf{B}^T)$.

The overall methodology used in this subsection is very similar to the previous one. Again, the results proven in this subsection can be grouped in the following theorem.

Theorem 3. *Consider $\mathbf{B} \in \mathbb{Z}^n$ a c.d.d. matrix and $\mathcal{L} = \mathcal{L}(\mathbf{B})$. Then $\lambda_1(\mathcal{L}) \geqslant \Delta^T(\mathbf{B})$ and there is an algorithm, `RSR` (Alg. 6), running within a polynomial amount of arithmetic operations such that*

$$\forall \mathbf{v} \in \mathrm{span}(\mathcal{L}), \mathtt{RSR}(v) \equiv v \bmod \mathcal{L}, \|\mathtt{RSR}(\mathbf{v})\|_\infty \leqslant D - \frac{\Delta^T(\mathbf{B})}{2}.$$

Consequently one has $\mu^{(\infty)}(\mathcal{L}) \leqslant D - \frac{\Delta^T(\mathbf{B})}{2}$.

As done previously, the proof of this theorem will be done in two steps: bounding the minimal size of the shortest vector, then bounding the maximal convergence radius of a reduction algorithm. Note that the acronym `RSR` stands for `RepeatedSingleReduce`.

B.1 Specific notations

We will use the following objects and notations.

- For $I \subset [\![1, n]\!]$, we denote by $\mathbf{B}_I \in M_{|I|,|I|}(\mathbb{Z})$ the submatrix of $\mathbf{B}$ composed of the rows and columns of indexes in I. Naturally, if $\mathbf{B}$ is a r.d.d/c.d.d matrix, so is $\mathbf{B}_I$.

- $S_\infty(l)$ is the set of positions i given $l \in \mathbb{Z}^n$ such that $|l_i| = \|l\|_\infty$
- $\mathcal{B}(I, \mathbf{B}) = \min\left\{\max_{j\in I}\{|(l\cdot\mathbf{B})_j| \mid \|l\|_\infty = 1, S_\infty(l) = I\}\right\}$ given any set of indexes I.
  It is simply $\min\{\|l\cdot\mathbf{B}_I\|_\infty \mid l \in \{-1,1\}^{|I|}\}$.
  We denote $\mathcal{B}(I,\mathbf{B})$ by $\mathcal{B}_I$ when $\mathbf{B}$ is implied, and stress that $\mathcal{B}_I \neq \lambda_1(\mathbf{B})$.

B.2 Short vectors

First let us study the norm of a shortest vector.

Lemma 5 (Minimal largest value of non-zero combinations). *Consider $k \in \mathbb{Z}^n\backslash\{0\}$, $j \in [\![1,n]\!]$ such that $|k_j| = \|k\|_\infty$, $\mathbf{B}$ be a c.d.d matrix, and $\mathbf{v} = k\cdot\mathbf{B}$. Then one has $|\mathbf{v}_j| \geqslant \|k\|_\infty \cdot \delta_j(\mathbf{B}^T)$.*

Proof. Without any loss of generality we can assume $\mathbf{v}_i \geq 0$ and $k_j > 0$. Then

$$|\mathbf{v}_i| = \left|\sum_{i=1}^n k_i\mathbf{B}_{i,j}\right| \geqslant k_j D - \sum_{\substack{i=1\\i\neq j}}^n |k_i\mathbf{B}_{i,j}| \geqslant k_j(D - \sum_{\substack{i=1\\i\neq j}}^n |\mathbf{B}_{i,j}|) = k_j\delta_j(\mathbf{B}^T).$$

□

This directly implies that $\lambda_1^{(\infty)}(\mathcal{L}(\mathbf{B})) \geqslant \Delta^T(\mathbf{B})$. Let us show some additional results on c.d.d. matrices.

Lemma 6 (Submatrix bound on non-zero combinations). *Consider $\mathbf{B}$ a c.d.d. matrix, $k \in \mathbb{Z}^n$, $I = S_\infty(k)$ and $\mathbf{v} = k\cdot\mathbf{B}$. Then there is $j \in I$ such that $|\mathbf{v}_j| \geqslant \mathcal{B}(I,\mathbf{B})$.*

Proof. If $k \in \{-\|k\|_\infty, 0, \|k\|_\infty\}^n$, then there is $j \in S_\infty(k)$ such that $|\mathbf{v}_j| \geqslant \|k\|_\infty \times \mathcal{B}(S_\infty(k),\mathbf{B})$. If $\exists j_1, |k_{j_1}| \notin \{0, \|k\|_\infty\}$ with $k_{j_1} \neq 0$, one can pick j_1 such that $|k_{j_1}| \geqslant |k_j|$ for all $j \notin S_\infty(k)$. Consider the vectors k' and k'' such that $k = k' + k''$ and

$$k'_j = \begin{cases} \text{sign}(k_j)\cdot(|k|_\infty - |k_{j_1}|), & \text{if } j \in I \\ 0, & \text{otherwise.}\end{cases}$$

Therefore we also have

$$k''_j = \begin{cases} \text{sign}(k_j)\cdot|k_s|, & \text{if } j \in I \\ k_j, & \text{otherwise.}\end{cases}$$

Remark that for all $j \in S_\infty(k)$ we have $\text{sign}(k''_j) = \text{sign}(k'_j) = \text{sign}(k_i)$ and $|k''_j| = |k''|_\infty$. From what precedes we know that there is $j \in S_\infty(k)$ such that $|(k'\cdot\mathbf{B})_j| \geqslant \mathcal{B}(S_\infty(k),\mathbf{B})$. Moreover $S_\infty(k) \subset S_\infty(k'')$ and the signs are the same so $\text{sign}((k''\cdot\mathbf{B})_j) = \text{sign}((k'\cdot\mathbf{B})_j)$. Thus we obtain $|(k\cdot\mathbf{B})_j| \geqslant \mathcal{B}(S_\infty(k),\mathbf{B})$. □

This gives us the following theorem.

Theorem 4 (Bound by the minimal submatrix). *Let $\mathbf{B}$ be a c.d.d. matrix. Then $\lambda_1^{(\infty)}(\mathcal{L}(\mathbf{B})) \geqslant \min_{I\subseteq[\![1,n]\!]} \mathcal{B}_I$.*

B.3 Reduction algorithms for c.d.d. matrices

The previous reduction algorithm only concerned r.d.d matrices and are not guaranteed to terminate on c.d.d matrices. We will propose here a different algorithm relying on the c.d.d structure. Before we present the full algorithm, we first introduce the core part that we denote by `SingleReduce`. It is described in Algorithm 5.

Algorithm 5 `SingleReduce`

Require: $\mathbf{v} \in \mathbb{Z}^n$, $\mathbf{B}$ a c.d.d matrix, $R_i \geqslant D - \frac{\delta_i(\mathbf{B}^T)}{2}$.
Ensure: $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$ and $\|\mathbf{w}\|_\infty \leqslant \max(qR_i, \|\mathbf{v}\|_\infty - q\Delta^T(\mathbf{B}))$, where $q = \max\{t \in \mathbb{N}^* \mid \forall i \in [\![1,n]\!], \|\mathbf{v}\|_\infty - tR_i \geqslant t(\delta_i(\mathbf{B}^T))\}$

$w \leftarrow v$, $i \leftarrow 1$, $s \leftarrow [0, ..., 0] \in \{0,1\}^n$ {initialization vector, index, reduction status}
$q \leftarrow \max\{t \in \mathbb{N}^* \mid \forall i \in [\![1,i]\!], \|\mathbf{v}\|_\infty - tR_i \geqslant t(\delta_i(\mathbf{B}^T))\}$
while $\bigvee_{j=1}^n ((|\mathbf{w}_j| > qR_j) \wedge (s_j = 0))$ **do**
  if $|\mathbf{w}_i| > qR_i$ and $s_i = 0$ **then**
    $w \leftarrow w - q\frac{w_i}{|w_i|}\mathbf{B}_i$ {Reduce $|\mathbf{w}_i|$}
    $s_i \leftarrow 1$ {"Update" the reduction status of index i}
  end if
  $i \leftarrow (i \bmod n) + 1$ {Enforces i to be within $[1,n]$ and not $[0, n-1]$}
end while
return w

Lemma 7. *`SingleReduce` (Alg. 5) outputs $\mathbf{w} \in \mathbb{Z}^n$ verifying the following properties:*

1. $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(B)$.
2. $\forall i \in [\![1,n]\!], |\mathbf{v}_i| > qR_i \implies |\mathbf{w}_i| < |\mathbf{v}_i|$.
3. $\forall i \in [\![1,n]\!], |\mathbf{v}_i| \leqslant qR_i \implies |\mathbf{w}_i| \leqslant qR_i$.

Moreover the algorithm performs at most n additions on vectors.

Proof. First remark that we add or remove at most one time each row vector to the variable $\mathbf{w}$. This is ensured by the flag vector s. Therefore we add at most n vectors to $\mathbf{w}$. Write $\mathbf{v} = \mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(r)} = \mathbf{w}$ the two-by-two distinct values of the variable $\mathbf{w}$ with $r \leqslant n$. Similarly write $s^{(0)}, \ldots, s^{(r)}$ the different values taken by s. Fix some index $i \in [\![1,n]\!]$. First assume $s_i^{(r)} = 0$. Then we know that $|\mathbf{w}_i^{(r)}| \leqslant qR_i$ and w_i satisfies the claimed properties. Now assume $s_i^{(r)} = 1$. Let us denote by k_0 the integer such that $\mathbf{w}_i^{(k_0)} = \mathbf{w}_i^{(k_0-1)} \pm qD$. Without loss of generality we can assume $\mathbf{w}_i^{(0)} = \mathbf{v}_i \geqslant 0$. First we consider the case where $\mathbf{w}_i^{(0)} > qR_i$. Then for some $J \subset [\![1,n]\!] \setminus \{i\}$ we have

$$\mathbf{w}_i^{(k_0-1)} = \mathbf{w}_i^{(0)} + \sum_{j \in J} \pm q b_{j,i} \geqslant \mathbf{w}_i^{(0)} - q(D - \delta_i(\mathbf{B}^T)) > qR_i - q(D - \delta_i(\mathbf{B}^T)) \geqslant q\frac{\delta_i(\mathbf{B}^T)}{2} > 0$$

therefore $\mathbf{w}_i^{(k_0)} = \mathbf{w}_i^{(k_0-1)} - qD$. We can write

$$\mathbf{w}_i^{(n)} = \mathbf{w}_i^{(0)} - qD + \sum_{\substack{j \in [\![1,n]\!] \\ j \neq i}} \pm q b_{j,i} > qR_i - qD - q(D - \delta_i(\mathbf{B}^T) \geqslant -q(D - \frac{\delta_i(\mathbf{B}^T)}{2})$$

which ensures $|\mathbf{w}_i^{(n)}| < |\mathbf{w}_i^{(0)}|$. Now consider the case where $\mathbf{w}_i^{(0)} \leqslant qR_i$. From $D - \frac{\delta_i(B^T)}{2} > D - \delta_i(B^T)$ we deduce that $\mathbf{w}_i^{(k_0-1)} > 0$ and $\mathbf{w}_i^{(k_0)} = \mathbf{w}_i^{(k_0-1)} - qD$. With the same reasoning as before we can conclude $\mathbf{w}_i^{(n)} < \mathbf{w}_i^{(0)}$ and $\mathbf{w}_i^{(n)} > \mathbf{w}_i^{(k_0)} - qD - q(D - \delta_i(\mathbf{B})) > -q(D - \frac{\delta_i(\mathbf{B})}{2})$ which ensures $|w_i^{(n)}| \leqslant qR_i$. Finally we remark that the results obtained are independent of the choice of i. □

This building block naturally gives us the `RSR` reduction algorithm, which is guaranteed to finish given a c.d.d. lattice basis. Theoretically, there is no algorithm that can provide strictly better bounds on l_∞ for every single column diagonally dominant lattice: the covering radius cannot be lower than half the size of the shortest vector, and for $\Delta^T(\mathbf{B}) = D$ we do reach this extremity.

Algorithm 6 `RSR`

Require: $v \in \mathbb{Z}^n$, B a c.d.d matrix, $R_i \geqslant D - \frac{\delta_i(\mathbf{B})}{2}$.
Ensure: $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$ and $|\mathbf{w}_i| \leq R_i$.
1: $\mathbf{w} \leftarrow \mathbf{v}$
2: **while** $\bigvee_{j=1}^{n}(|\mathbf{w}_j| > R_j)$ **do**
3: $w \leftarrow$`SingleReduce`$(\mathbf{w}, \mathbf{B}, R)$.
4: **end while**
5: **return** w

Proposition 1. *Given a vector $\mathbf{v} \in \mathbb{Z}^n$, $R \in \mathbb{Z}^n$ such that $R_i \geqslant D - \frac{\delta_i(\mathbf{B})}{2}$ where $D, \delta_i(\mathbf{B})$ are associated to a c.d.d. matrix $\mathbf{B}$, `RSR` (Alg. 6) outputs $\mathbf{w} \in \mathbb{Z}^n$ verifying the following properties:*

1. $\mathbf{w} \equiv \mathbf{v} \bmod \mathcal{L}(\mathbf{B})$.
2. $\forall i \in [\![1, n]\!],\ |\mathbf{w}_i| \leqslant R_i$

Moreover the algorithm performs at most $n \left\lceil \log_b \frac{2\|\mathbf{v}\|_\infty}{2D + \Delta^T\mathbf{B}} \right\rceil + n$ additions on vectors, where $b = \frac{2D + \Delta^T\mathbf{B}}{2D - \Delta^T\mathbf{B}}$.

Proof. Consider $\|\mathbf{v}\|_\infty$ such that there is no integer $t > 0$ such that $\|\mathbf{v}\|_\infty - tR_i \geqslant t\delta_i(\mathbf{B})$, i.e. $\|\mathbf{v}\|_\infty - R_i < \delta_i(\mathbf{B})$. Then a call to `SingleReduce` with $q = 1$ outputs $\mathbf{w}$ such that $\|\mathbf{w}_i\| \leqslant R_i$. Now consider $\|\mathbf{v}\|_\infty$ sufficiently large so that q exists. One call to `SingleReduce` outputs $\mathbf{w}$ such that $\|\mathbf{w}\|_\infty \leqslant \max\{qR_i, \|\mathbf{v}\|_\infty - q\Delta^T(\mathbf{B}))\} \leqslant \|v\|_\infty - q\Delta^T(B))$ by definition of q. Thus we get $\|\mathbf{w}\|_\infty \leqslant \|\mathbf{v}\|_\infty \cdot$

$(1-Q)$, where $Q = q\frac{\Delta^T(\mathbf{B})}{\|\mathbf{v}\|_\infty}$. Clearly $Q > 0$, and let us prove that $Q < 1$. By definition we have

$$\|\mathbf{v}\|_\infty - qR_i \geqslant q\delta_i(\mathbf{B}^T)) \implies \frac{q}{\|\mathbf{v}\|_\infty} \leqslant \frac{2D + \Delta^T(\mathbf{B})}{2}$$

which gives

$$Q \leqslant \frac{2\Delta^T(\mathbf{B})}{2D + \Delta^T(\mathbf{B})}.$$

Since $\Delta^T(\mathbf{B}) > 0$ one has $2D + \Delta^T(\mathbf{B}) > 2D$, which leads to $Q < 2D/2D = 1$.

Then, writing $a := 1 - \frac{2\Delta^T(\mathbf{B})}{2D+\Delta^T(\mathbf{B})} = \frac{2D-\Delta^T(\mathbf{B})}{2D+\Delta^T(\mathbf{B})}$ one has $0 < 1 - Q < a < 1$ and $\|\mathbf{w}\|_\infty \leqslant a \cdot \|\mathbf{v}\|_\infty$. Consequently, after i calls to `SingleReduce`, one has $\|\mathbf{w}\|_\infty \leqslant a^i \cdot \|\mathbf{v}\|_\infty$. Let us find i the number of calls to `SingleReduce` after which a single call to `SingleReduce` with $q = 1$ will output a well-reduced vector. This is ensured by

$$\begin{aligned}
\|\mathbf{w}\|_\infty \leqslant a^i \cdot \|\mathbf{v}\|_\infty < R + \Delta^T(\mathbf{B}) &\iff a^i \leqslant \frac{2D + \Delta^T(\mathbf{B})}{2\|\mathbf{v}\|_\infty} \\
&\iff i \geqslant \log_a \frac{2D + \Delta^T(\mathbf{B})}{2\|\mathbf{v}\|_\infty} \\
&\iff i = \left\lceil \log_a \frac{2D + \Delta^T(\mathbf{B})}{2\|\mathbf{v}\|_\infty} \right\rceil \\
&\iff i = \left\lceil \log_{1/a} \frac{2\|\mathbf{v}\|_\infty}{2D + \Delta^T(\mathbf{B})} \right\rceil.
\end{aligned}$$

Since each call to `SingleReduce` has at most n vector additions, we get the claimed worst-case cost. □

We want to stress this does not show the algorithm is practically efficient: `SingleReduce` might run a *quadratic* amount of absolute value comparisons on scalars in a single call. However, the reduction still runs a polynomial amount of vector operations in the dimension and in the entry size.

Comparison with Babai's Nearest Plane Unlike the r.d.d case, we do not have a measure of $\|b_i\|_1$. However, we estimate that it is possible in the case of c.d.d to have rows with very large noise, which might give $\|b_i\|_1 > 2D$ and thus a larger worst-case bound than a r.d.d for Babaï's nearest plane algorithm.