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OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

Abstract. We continue in this paper our study of the notion of radial uniformly upper semicontinuous functional that we developed in a previous paper (see [AHM14]) in the context of relaxation. We consider here the framework of Γ-convergence. We present general radial extension results with respect to Γ-convergence and give some applications to Γ-convergence and homogenization of integral functionals with constraints.

Contents

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1. Introduction

Let X be a vector space and let $\|\cdot\|$ be a norm on X. In [AHM14, Theorem 3.1] we proved that if $F : X \to [0,\infty]$ is ru-usc[1] on a subset D of the effective domain $\mathrm{dom}(F)$ of F, i.e. there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}} \sup_{u\in D} \frac{F(tu) - F(u)}{a + F(u)} \leqslant 0,$$

and if D is strongly star-shaped, i.e.

$$t\overline{D} \subset D \text{ for all } t \in]0,1[$$

with $\overline{D}$ denoting the closure of D with respect to $\|\cdot\|$, then the lsc[2] envelope $\overline{F+\chi_D}$ of $F+\chi_D$, where χ_D denotes the indicator function of D, is given by the sum of the radial extension of F and the indicator function of $\overline{D}$ whenever lsc envelope of $F+\chi_D$ is equal to F on D, i.e.

$$\overline{F+\chi_D} = F \text{ on } D \text{ implies } \overline{F+\chi_D} = \widehat{F} + \chi_{\overline{D}},$$

where $\widehat{F} : X \to [0,\infty]$ is defined by

$$\widehat{F}(u) := \underline{\lim_{t\to 1^-}} F(tu).$$

In this paper, we extend this result to the framework of Γ-convergence (see Theorem 2.7, Corollary 2.8 and Theorem 2.9).

In the spirit of De Giorgi (see [DG79, §4]), our motivation comes from the problem of finding an integral representation of the Γ-limit of a family of integral functionals subjected to constraints. A typical example is given by integral functionals $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ defined by

$$F_\varepsilon(u) := \int_\Omega f_\varepsilon(x, \nabla u(x))dx,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $p > N$, $\varepsilon > 0$ and $f_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ is a Borel measurable function, not necessarily convex, having g-growth, i.e. there exist $\alpha, \beta > 0$ and a Borel measurable function $g : \mathbb{M} \to [0,\infty]$, which do not depend on ε, such that for every $(x,\xi) \in \Omega \times \mathbb{M}$,

$$\alpha g(\xi) \leqslant f_\varepsilon(x,\xi) \leqslant \beta(1 + g(\xi))$$

with $\mathbb{M}$ denoting the space of $m \times N$ matrices. Thus, for each $\varepsilon > 0$, the effective domain $\mathrm{dom}(F_\varepsilon)$ of F_ε is equal to the effective domain $\mathrm{dom}(G)$ of $G : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ defined by

$$G(u) := \int_\Omega g(\nabla u(x))dx.$$

This means that integral functionals F_ε are subjected to constraints on gradients represented by the set $\mathrm{dom}(G)$. Under some requirements (see §3.2), it can be proved (see Theorem 3.8) that $\{F_\varepsilon\}_\varepsilon$ Γ-converges, with respect to the L^p-norm, on $\mathrm{dom}(G)$, i.e.

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u) \text{ for all } u \in \mathrm{dom}(G). \tag{1.1}$$

[1]The abbreviation ru-usc means radially uniformly upper semicontinuous.

[2]The abbreviation lsc means lower semicontinuous.

The difficulty is then to extend (1.1) to the whole space $W^{1,p}(\Omega;\mathbb{R}^m)$. This can be achieved by developing extension theorems with respect to the concept of ru-usc functional (see, in particular, Theorem 2.7).

To our knowledge, the concept of ru-usc finds its origin in [CDA02, Condition (10.1.13), pp. 213] in connection with relaxation problems with constraints. Later, this concept was proved very useful for relaxation problems in the vectorial case with bounded and convex constraints see [AH10]. Then, it was used to study homogenization and relaxation problems with constraints (see for instance [AHM11, AHM12, AHMZ15]). Ru-usc seems to be a key concept to deal with certain constrained variational problems (see Sect. 3). However, of course, this concept does not allow to handle all types of constraints as for example obstacle constraints.

The plan of the paper is as follows. In §2.1 we recall the definition of a ru-usc functionals (see Definition 2.1) and the one of a family of ru-usc functionals (see Definition 2.4). An important point is the fact that ru-usc is stable with respect to Γ-convergence (see Theorem 2.6). The definition of ru-usc functions together with the link with the one of ru-usc functionals is recalled in §3.1 (see Definitions 3.1 and 3.2). Roughly, functionals with ru-usc integrands are ru-usc (see Proposition 3.3). The main results of the paper are stated and proved in §2.2 (see Theorem 2.7, Corollary 2.8 and Theorem 2.9), and applications of these results to Γ-convergence and homogenization of integral functionals subjected to constraints are given in Section 3 (see Corollaries 3.10, 3.12 and 3.13). In the Applications, we need a partial Γ-convergence theorem (see Theorem 3.8) which was established in [AHM21, Theorem 4.3]. For the convenience of the reader, the proof of Theorem 3.8 is given in Appendix A, and the auxiliary results that are used in its proof are stated (without the proofs) in §B.1 and §B.2. For the application to homogenization we need a subadditive theorem which is stated (without proof) in §B.3.

Throughout the paper, we will use the following notation and terminology.

- Given a vector space X and a norm $\|\cdot\|$ on X, the closure (resp. interior) with respect to $\|\cdot\|$ of a set $A\subset X$ is denoted by $\overline{A}$ (resp. int$A$) and its boundary by ∂A.
- By the effective domain of a function $F : X \to [0,\infty]$ we mean $\mathrm{dom}(F)$ given by $\mathrm{dom}(F) := \{u\in X : F(u) < \infty\}$.
- A set $D\subset X$ such that $t\overline{D}\subset D$ for all $t\in]0,1[$ is said to be strongly star-shaped. When $t\overline{D}\subset \mathrm{int}D$ for all $t\in]0,1[$, we say that D is super-strongly star-shaped.
- For $D\subset X$ we denote the indicator function of D by χ_D, i.e.

$$\chi_D(x) = \begin{cases} 0 & \text{if } x\in D \\ \infty & \text{if } x\in X\backslash D. \end{cases}$$

- For $A\subset\mathbb{R}^N$, the diameter of A (resp. the distance from a point $x\in\mathbb{R}^N$ to the subset A) is defined by $\mathrm{diam}(A) := \sup_{x,y\in A}|x-y|$ (resp. $\mathrm{dist}(x,A) := \inf_{y\in A}|x-y|$).
- The symbol $⨍$ stands for the mean-value integral with respect to the Lebesgue measure $\mathscr{L}^N$ on $\mathbb{R}^N$, i.e. $⨍_Q = \frac{1}{\mathscr{L}^N(Q)}\int_Q$.

2. Main results

In what follows, X is a vector space and $\|\cdot\|$ is a norm on X. (In the applications $(X, \|\cdot\|)$ will be $W^{1,p}$ with $p > 1$ endowed with the L^p-norm.)

2.1. **Ru-usc functionals.** We begin by recalling the concept of ru-usc functional.

Definition 2.1. We say that $F : X \to [0, \infty]$ is ru-usc if there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}}\, \Delta_F^a(t) \leqslant 0$$

with $\Delta_F^a : [0, 1] \to]-\infty, \infty]$ defined by

$$\Delta_F^a(t) := \sup_{u\in \mathrm{dom}(F)} \frac{F(tu) - F(u)}{a + F(u)},$$

where $\mathrm{dom}(F)$ denotes the effective domain of F.

Let $\widehat{F} : X \to [0, \infty]$ be defined by

$$\widehat{F}(u) := \underline{\lim_{t\to 1^-}}\, F(tu).$$

Usually, $\widehat{F}$ is called the radial extension of F.

Remark 2.2. If F is ru-usc then $\widehat{F} \leqslant F$. Indeed, for every $u \in \mathrm{dom}(F)$, $F(tu) \leqslant F(u) + \Delta_F^a(t)(a+F(u))$ for all $t \in [0, 1]$. As $\overline{\lim}_{t\to 1^-} \Delta_F^a(t) \leqslant 0$ it follows that $\widehat{F}(u) = \underline{\lim}_{t\to 1^-} F(tu) \leqslant F(u) + \overline{\lim}_{t\to 1^-} \Delta_F^a(t)(a + F(u)) \leqslant F(u)$ for all $u \in \mathrm{dom}(F)$.

The interest of Definition 2.1 comes from the following theorem. (For a proof we refer to [AHM11, Theorem 3.5] and also [AHM12, §4.2])

Theorem 2.3. *If $F : X \to [0, \infty]$ is ru-usc and if $\mathrm{dom}(F)$ is super-strongly star-shaped, i.e.*

$$t\overline{\mathrm{dom}(F)} \subset \mathrm{intdom}(F) \text{ for all } t \in]0, 1[, \tag{2.1}$$

then:

(a) $\widehat{F}$ *is ru-usc*;
(b) $\widehat{F}(u) = \lim_{t\to 1^-} F(tu)$ *for all* $u \in X$.

If moreover, F is lower semicontinuous (lsc) with respect to $\|\cdot\|$ on $\mathrm{intdom}(F)$ then:

(c) $$\widehat{F}(u) = \begin{cases} F(u) & \text{if } u \in \mathrm{intdom}(F) \\ \lim\limits_{t\to 1^-} F(tu) & \text{if } u \in \partial\mathrm{dom}(F) \\ \infty & \text{otherwise;} \end{cases}$$

(d) $\widehat{F}$ *is the lsc envelope of* F.

The following definition generalizes Definition 2.1 to the case of a family of functionals.

Definition 2.4. For each $\varepsilon > 0$, let $F_\varepsilon : X \to [0, \infty]$. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc if there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}} \sup_{\varepsilon>0} \Delta_{F_\varepsilon}^a(t) \leqslant 0.$$

(When $F_\varepsilon = F$ for all $\varepsilon > 0$, we recover Definition 2.1.)

The interest of Definition 2.4 comes from the following theorem which says that ru-usc is conserved under Γ-convergence. We begin by recalling the definition of Γ-convergence (see [DM93, BD98, Bra06] for more details).

Definition 2.5. For each $\varepsilon > 0$, let $F_\varepsilon : X \to [0,\infty]$ and let $\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon : X \to [0,\infty]$ and $\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon : X \to [0,\infty]$ be respectively defined by:

$$\Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(u) := \inf\left\{\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(u_\varepsilon) : u_\varepsilon \overset{\|\cdot\|}{\to} u\right\};$$
$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u) := \inf\left\{\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u_\varepsilon) : u_\varepsilon \overset{\|\cdot\|}{\to} u\right\}.$$

Let $F : X \to [0,\infty]$. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to F, and we write

$$F = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon,$$

if the following two inequalities hold:

$$F \leqslant \Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} F_\varepsilon;$$
$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon \leqslant F.$$

The following theorem asserts that ru-usc is stable with respect to Γ-convergence.

Theorem 2.6. *Let $F : X \to [0,\infty]$ and, for each $\varepsilon > 0$, let $F_\varepsilon : X \to [0,\infty]$. If $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc and if $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to F then F is ru-usc. If moreover* (2.1) *holds then $\widehat{F} = F$.*

Proof of Theorem 2.6. Fix any $u \in \mathrm{dom}(F)$. Since $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to F, there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset X$ with $u_\varepsilon \in \mathrm{dom}(F_\varepsilon)$ such that:

$$u_\varepsilon \overset{\|\cdot\|}{\to} u; \tag{2.2}$$
$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(u). \tag{2.3}$$

Fix any $t \in [0,1]$. For every $\varepsilon > 0$, we have

$$\begin{aligned} F_\varepsilon(tu_\varepsilon) &\leqslant \Delta^a_{F_\varepsilon}(t)\big(a + F_\varepsilon(u_\varepsilon)\big) + F_\varepsilon(u_\varepsilon) \\ &\leqslant \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t)\big(a + F_\varepsilon(u_\varepsilon)\big) + F_\varepsilon(u_\varepsilon). \end{aligned} \tag{2.4}$$

From (2.2) we see that $tu_\varepsilon \overset{\|\cdot\|}{\to} tu$, and so, since $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to F,

$$F(tu) \leqslant \underline{\lim_{\varepsilon\to 0}} F_\varepsilon(tu_\varepsilon). \tag{2.5}$$

Letting $\varepsilon \to 0$ in (2.4) and using (2.3) and (2.5) we deduce that

$$F(tu) \leqslant \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t)\big(a + F(u)\big) + F(u).$$

Hence, for every $u \in \mathrm{dom}(F)$,

$$\frac{F(tu) - F(u)}{a + F(u)} \leqslant \sup_{\varepsilon>0} \Delta^a_{F_\varepsilon}(t).$$

Consequently, for every $t \in [0,1]$,

$$\sup_{u\in \mathrm{dom}(F)} \frac{F(tu)-F(u)}{a+F(u)} \leqslant \sup_{\varepsilon>0} \Delta^a_{F_\varepsilon}(t), \text{ i.e. } \Delta^a_F(t) \leqslant \sup_{\varepsilon>0} \Delta^a_{F_\varepsilon}(t).$$

As $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc we have $\overline{\lim}_{t\to 1^-} \sup_{\varepsilon>0} \Delta^a_{F_\varepsilon}(t) \leqslant 0$ and so $\overline{\lim}_{t\to 1^-} \Delta^a_F(t) \leqslant 0$, which proves that F is ru-usc.
Since $F = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$, F is lsc with respect to $\|\cdot\|$. Hence, if moreover (2.1) holds then $F = \widehat{F}$ by Theorem 2.3(d). ■

2.2. **Radial extension theorems with respect to Γ-convergence.** The following theorem is the first main result of the paper.

Theorem 2.7. *Let $F : X \to [0,\infty]$ and, for each $\varepsilon > 0$, let $F_\varepsilon : X \to [0,\infty]$. Let $D, E \subset X$ be such that*:

$$tE \subset D \text{ for all } t \in]0,1[; \tag{2.6}$$

$$\mathrm{dom}\left(\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon\right) \subset E. \tag{2.7}$$

If $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc and if

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u) \text{ for all } u \in D, \tag{2.8}$$

then

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = \widehat{F} + \chi_E.$$

Proof of Theorem 2.7. First of all, taking (2.7) into account, if $u \notin E$ then

$$\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) = \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) = \infty.$$

So, it is sufficient to prove that for every $u \in E$,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \widehat{F}(u). \tag{2.9}$$

Fix $u \in E$. By (2.6), for any $t \in]0,1[$, we have $tu \in D$. From (2.8) it follows that

$$F(tu) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(tu) = \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu) \text{ for all } t \in]0,1[.$$

Hence, since the Γ-limit sup is lower semicontinuous with respect to $\|\cdot\|$ and $tu \overset{\|\cdot\|}{\to} u$ as $t \to 1^-$,

$$\widehat{F}(u) = \underline{\lim}_{t\to 1^-} F(tu) = \underline{\lim}_{t\to 1^-} \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu) \geqslant \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u). \tag{2.10}$$

Let $\{u_\varepsilon\}_{\varepsilon>0} \subset X$ be such that $u_\varepsilon \in \mathrm{dom}(F_\varepsilon)$ for all $\varepsilon > 0$ and:

$$u_\varepsilon \overset{\|\cdot\|}{\to} u; \tag{2.11}$$

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) < \infty. \tag{2.12}$$

For any $t \in]0,1[$, from (2.11) we see that $tu_\varepsilon \overset{\|\cdot\|}{\to} tu$ with $tu \in D$ by (2.6). Hence, by using (2.8),

$$\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu_\varepsilon) \geqslant \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu) = F(tu) \text{ for all } t \in]0,1[.$$

It follows that

$$\underline{\lim}_{t\to 1^-} \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu_\varepsilon) \geqslant \widehat{F}(u). \tag{2.13}$$

On the other hand, for every $t \in]0,1[$ and every $\varepsilon > 0$, we have

$$\begin{aligned} F_\varepsilon(tu_\varepsilon) &\leqslant \big(1 + \Delta^a_{F_\varepsilon}(t)\big) F_\varepsilon(u_\varepsilon) + a\Delta^a_{F_\varepsilon}(t) \\ &\leqslant \left(1 + \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t)\right) F_\varepsilon(u_\varepsilon) + a_\varepsilon \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t), \end{aligned}$$

and so, by letting $\varepsilon \to 0$ and by using (2.12), we get

$$\begin{aligned} \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu_\varepsilon) &\leqslant \left(1 + \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t)\right) \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) + a \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t) \\ &= \left(1 + \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t)\right) \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) + a \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t) \text{ for all } t \in]0,1[. \end{aligned}$$

As $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $a > 0$, i.e. $\overline{\lim}_{t\to 1^-} \sup_{\varepsilon'>0} \Delta^a_{F_{\varepsilon'}}(t) \leqslant 0$, letting $t \to 1^-$ we deduce that

$$\underline{\lim}_{t\to 1^-} \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(tu_\varepsilon) \leqslant \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u), \tag{2.14}$$

and (2.9) follows by combining (2.10), (2.13) and (2.14). ■

If $\mathrm{dom}(F_\varepsilon) \subset D$ for all $\varepsilon > 0$ then $\mathrm{dom}\,(\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon) \subset \overline{D}$ and so, as a direct consequence of Theorem 2.7, we have the following result which roughly asserts that the Γ-limit of a ru-usc family of functionals having effective domains included in a same strongly star-shaped set D can be computed from its Γ-limit on D as the sum of the radial extension of $\Gamma\text{-}\lim|_D$ and the indicator function of $\overline{D}$.

Corollary 2.8. *Let $F : X \to [0,\infty]$ and, for each $\varepsilon > 0$, let $F_\varepsilon : X \to [0,\infty]$. Let $D \subset X$ be a strongly star-shaped set, i.e. $t\overline{D} \subset D$ for all $t \in]0,1[$, such that $\mathrm{dom}(F_\varepsilon) \subset D$ for all $\varepsilon > 0$. If $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc and if $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u)$ for all $u \in D$, then $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = \widehat{F} + \chi_{\overline{D}}$.*

Here is the second main result of the paper. Roughly, this theorem asserts that Γ-convergence is stable under super-strongly star-shaped constraints.

Theorem 2.9. *Let $F : X \to [0,\infty]$ and, for each $\varepsilon > 0$, let $F_\varepsilon : X \to [0,\infty]$. Let $D \subset X$ be such that*

$$D \text{ is super-strongly star-shaped, i.e. } t\overline{D} \subset \mathrm{int}D \text{ for all } t \in]0,1[. \tag{2.15}$$

If $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc and if

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = F,$$

then

$$\Gamma\text{-}\lim_{\varepsilon\to 0} (F_\varepsilon + \chi_D) = F + \chi_{\overline{D}}.$$

Proof of Theorem 2.9. First of all, if $u \notin \overline{D}$ then

$$\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)=\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)=\infty.$$

So, it is sufficient to prove that $\{F_\varepsilon+\chi_D\}_{\varepsilon>0}$ Γ-converges to F on $\overline{D}$. As $F_\varepsilon \leqslant F_\varepsilon+\chi_D$ for all $\varepsilon>0$, we have

$$F=\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon \leqslant \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)\leqslant \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right). \tag{2.16}$$

Fix any $u\in X$. Let $\{u_\varepsilon\}_{\varepsilon>0}\subset X$ be such that

$$u_\varepsilon \overset{\|\cdot\|}{\to} u \text{ and } F(u)=\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(u)=\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon).$$

Then, for every $u\in X$,

$$\begin{aligned}\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)\leqslant \overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u_\varepsilon) &= \overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon(u_\varepsilon)+\chi_D(u_\varepsilon)\right)\\ &= \lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)+\overline{\lim}_{\varepsilon\to 0}\chi_D(u_\varepsilon)\\ &= F(u)+\overline{\lim}_{\varepsilon\to 0}\chi_D(u_\varepsilon)\\ &\leqslant F(u)+\chi_{\mathrm{int}D}(u).\end{aligned}$$

Taking (2.16) into account, we see that

$$F\leqslant \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)\leqslant \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)\leqslant F+\chi_{\mathrm{int}D},$$

which implies that $\{F_\varepsilon+\chi_D\}_{\varepsilon>0}$ Γ-converges to F on $\mathrm{int}D$. Now, fix any $u\in\overline{D}$. By (2.15) we have $tu\in\mathrm{int}D$ for all $t\in]0,1[$, and so

$$\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(tu)=\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(tu)=F(tu) \text{ for all } t\in]0,1[.$$

It follows that

$$\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)\leqslant \underline{\lim}_{t\to 1^-}\,\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(tu)\leqslant \underline{\lim}_{t\to 1^-}F(tu)=\widehat{F}(u).$$

Since $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc and $\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon=F$, from Theorem 2.6 we can asset that F is ru-usc, and so $\widehat{F}\leqslant F$ (see Remark 2.2). Consequently

$$\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)\leqslant F(u) \text{ for all } u\in\overline{D}. \tag{2.17}$$

From (2.16) and (2.17) we conclude that

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\left(F_\varepsilon+\chi_D\right)(u)=F(u) \text{ for all } u\in\overline{D},$$

and the proof is complete. ■

3. Applications to Γ-convergence and homogenization with constraints

Let $m,N\geqslant 1$ be two integers, let $p>1$ be a real number, let $\Omega\subset\mathbb{R}^N$ be a bounded open set and let $\mathbb{M}$ denote the space of $m\times N$ matrices. In order to deal with integral functionals of the calculus of variations, in what follows we consider $X=W^{1,p}(\Omega;\mathbb{R}^m)$ and $\|\cdot\|=\|\cdot\|_{L^p}$.

3.1. **Ru-usc functions.** As for functionals (see Definitions 2.1 and 2.4) we can define the concept of ru-usc for functions and families of functions from $\Omega \times \mathbb{M}$ to $[0,\infty]$ (with Ω which can be unbounded).

Definition 3.1. Let $f : \Omega \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function. We say that f is ru-usc if there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}}\, \delta_f^a(t) \leqslant 0,$$

where $\delta_f^a : [0,1] \to]-\infty,\infty]$ is given by

$$\delta_f^a(t) := \sup_{x\in\Omega} \sup_{\xi\in \mathrm{dom}(f)} \frac{f(x,t\xi) - f(x,\xi)}{a + f(x,\xi)}.$$

The following definition generalizes Definition 3.1 to the case of a family of functions.

Definition 3.2. For each $\varepsilon > 0$, let $f_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function. We say that $\{f_\varepsilon\}_{\varepsilon>0}$ is ru-usc if there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}} \sup_{\varepsilon>0} \delta_{f_\varepsilon}^a(t) \leqslant 0,$$

where, for each $\varepsilon > 0$, $\delta_{f_\varepsilon}^a : [0,1] \to]-\infty,\infty]$ is given by

$$\delta_{f_\varepsilon}^a(t) := \sup_{x\in\Omega} \sup_{\xi\in \mathrm{dom}(f_\varepsilon)} \frac{f_\varepsilon(x,t\xi) - f_\varepsilon(x,\xi)}{a + f_\varepsilon(x,\xi)}. \tag{3.1}$$

The following result makes clear the link between ru-usc families of integrands and ru-usc families of functionals. (For a proof we refer to [AHM21, Proposition 2.23].)

Proposition 3.3. *For each $\varepsilon > 0$, let $f_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function and let $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by $F_\varepsilon(u) := \int_\Omega f_\varepsilon(x,\nabla u(x))dx$. If $\{f_\varepsilon\}_{\varepsilon>0}$ is ru-usc then $\{F_\varepsilon\}_{\varepsilon>0}$ is ru-usc.*

Roughly, the following result, which is useful for dealing with homogenization, asserts that a ru-usc 1-periodic function generates a family of ru-usc functions. (For a proof we refer to [AHM21, Lemma 2.24].)

Proposition 3.4. *Let $f : \mathbb{R}^N \times \mathbb{M} \to [0,\infty]$ be Borel measurable function such that $f(\cdot,\xi)$ is 1-periodic for all $\xi \in \mathbb{M}$, i.e. for every $(x,z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $f(x+z,\xi) = f(x,\xi)$, and, for each $\varepsilon > 0$, let $f_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ be defined by*

$$f_\varepsilon(x,\xi) := f\left(\frac{x}{\varepsilon},\xi\right).$$

If f is ru-usc then $\{f_\varepsilon\}_{\varepsilon>0}$ is ru-usc.

3.2. **Γ-convergence and homogenization.** We begin by establishing a partial Γ-convergence theorem.

3.2.1. *Partial* Γ*-convergence theorem.* Let $g : \mathbb{M} \to [0,\infty]$ be a Borel measurable function. In what follows, we consider the following conditions:

(C_1) g is p-coercive i.e. there exists $c > 0$ such that for every $\xi \in \mathbb{M}$,
$$g(\xi) \geqslant c|\xi|^p;$$
(C_2) there exists $\gamma > 0$ such that for every $t \in]0,1[$ and every $\xi, \zeta \in \mathbb{M}$,
$$g(t\xi + (1-t)\zeta) \leqslant \gamma(1 + g(\xi) + g(\zeta));$$
(C_3) $0 \in \mathrm{intdom}(g)$;
(C_4) the functional $G : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by
$$G(u) := \int_\Omega g(\nabla u(x))dx$$
is L^p-lsc;
(C_5) $t\mathrm{dom}(\overline{G}) \subset \mathrm{dom}(G)$ for all $t \in]0,1[$, where $\overline{G} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ denotes the L^p-lsc envelope of G, i.e. for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$,
$$\overline{G}(u) := \inf\left\{\varliminf_{\varepsilon\to 0} G(u_\varepsilon) : u_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u\right\}.$$

Remark 3.5. (i) The condition (C_2) implies that $\mathrm{dom}(g)$ is convex. When g is convex, (C_2) can be dropped.

(ii) Since a convex function is continuous in the interior of its effective domain, if g is convex and if (C_3) holds then g is bounded at the neighborhood of the null matrix, i.e. there exists $r > 0$ such that
$$\sup_{|\xi|\leqslant r} g(\xi) < \infty.$$
More generally, it is proved in [AHMZ15, Lemma 4.1] that such a boundlessness condition holds if (C_2)–(C_3) are satisfied.

(iii) Under (C_2) it is clear that $\mathrm{dom}(G)$ is convex. Hence, if moreover (C_3) holds then $t\mathrm{dom}(G) \subset \mathrm{dom}(G)$ for all $t \in]0,1[$, and so, under (C_2)–(C_3), if (C_4) is satisfied then also is (C_5).

For each $\varepsilon > 0$, let $f_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function. In what follows, we consider the following conditions:

(C_6) there exist $\alpha, \beta > 0$ such that for every $\varepsilon > 0$ and every $(x,\xi) \in \Omega \times \mathbb{M}$,
$$\alpha g(\xi) \leqslant f_\varepsilon(x,\xi) \leqslant \beta(1 + g(\xi));$$
(C_7) $\{f_\varepsilon\}_{\varepsilon>0}$ is ru-usc;
(C_8) for every $x \in O$ and every $\xi \in \mathbb{G}$,
$$\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x,\xi) \geqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x,\xi),$$
where $\mathscr{H}^\rho[f_\varepsilon] : O \times \mathbb{M} \to [0,\infty]$ is defined by
$$\mathscr{H}^\rho[f_\varepsilon](x,\xi) := \inf\left\{\fint_{Q_\rho(x)} f_\varepsilon(y,\xi + \nabla v(y))dy : v \in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\} \tag{3.2}$$

with $Q_\rho(x) := x +]-\frac{\rho}{2}, \frac{\rho}{2}[^N$.

Remark 3.6. (i) Under (C_2) and (C_6) we have $\mathrm{dom}(f_\varepsilon) = \mathrm{dom}(g)$ for all $\varepsilon > 0$, and so $\mathrm{dom}(f_\varepsilon)$ is convex for all $\varepsilon > 0$.

(ii) In the periodic case, i.e. $f_\varepsilon(x,\xi) = f(\frac{x}{\varepsilon}, \xi)$ with $f : \mathbb{R}^N \times \mathbb{M} \to [0,\infty]$ a 1-periodic function with respect to the first variable, (C_8) can be proved by using Akcoglu-Krengel's subadditive theorem (see Theorem B.5).

For each $\varepsilon > 0$, let $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$F_\varepsilon(u) := \int_\Omega f_\varepsilon(x, \nabla u(x))dx.$$

Remark 3.7. (i) If (C_2) and (C_6) are satisfied then $\mathrm{dom}(F_\varepsilon) = \mathrm{dom}(G)$ for all $\varepsilon > 0$.

(ii) If (C_4) holds then $\mathrm{dom}\left(\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon\right) = \mathrm{dom}\left(\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon\right) = \mathrm{dom}(G)$. Otherwise, we have $\mathrm{dom}\left(\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon\right) = \mathrm{dom}\left(\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon\right) = \mathrm{dom}(\overline{G})$.

The following theorem establishes the Γ-convergence of $\{F_\varepsilon\}_{\varepsilon>0}$ on $\mathrm{dom}(G)$. It plays a central role for establishing Γ-convergence and homogenization results on the whole space $W^{1,p}(\Omega;\mathbb{R}^m)$.

Theorem 3.8. *Assume that $p > N$. If (C_1)–(C_3) and (C_6)–(C_7) hold then for every $u \in \mathrm{dom}(G)$,*

$$\begin{cases} \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) \geqslant \displaystyle\int_\Omega \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \underline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x))dx \\ \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) = \displaystyle\int_\Omega \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x))dx. \end{cases} \tag{3.3}$$

If moreover (C_8) is satisfied then for every $u \in \mathrm{dom}(G)$,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x))dx. \tag{3.4}$$

Theorem 3.8) was established in [AHM21, Theorem 4.3]. For the convenience of the reader, we give a proof in Appendix A.

3.2.2. Γ-*convergence.* From now on, let $F : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$F(u) := \int_\Omega \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x))dx. \tag{3.5}$$

By adding (C_4) to the hypotheses of Theorem 3.8 we obtain the following result. (Note that this result does not need any radial extension theorem.)

Corollary 3.9. *Assume that $p > N$. If (C_1)–(C_4) and (C_6)–(C_8) hold then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = F + \chi_{\mathrm{dom}(G)}.$$

Proof of Corollary 3.9. From (C_6) we see that for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$,

$$\alpha\overline{G}(u) \leqslant \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u) \leqslant \beta(1 + \overline{G}(u)),$$

hence $\mathrm{dom}(\underline{\lim}_{\varepsilon\to 0} F_\varepsilon) = \mathrm{dom}(G)$ by using (C_4), and the proof is complete. ■

By adding (C_5) to the hypotheses of Theorem 3.8, from Theorem 2.7 we deduce the following result.

Corollary 3.10. *Assume that $p > N$. If (C_1)–(C_3), (C_5) and (C_6)–(C_8) hold then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = \widehat{F} + \chi_{\mathrm{dom}(\overline{G})}. \tag{3.6}$$

Proof of Corollary 3.10. From (C_6) we have $\mathrm{dom}(\underline{\lim}_{\varepsilon\to 0} F_\varepsilon) = \mathrm{dom}(\overline{G})$. Hence, taking Theorem 3.8 into account and according to (C_5), (3.6) follows by applying Theorem 2.7 with $D = \mathrm{dom}(G)$ and $E = \mathrm{dom}(\overline{G})$. ■

Let $\Phi : W^{1,p}(\Omega;\mathbb{R}^m) \to]-\infty,\infty]$ be such that:

(C_9) Φ is L^p-continuous;
(C_{10}) for every $t \in]0,1[$, $\Phi(tu) < \Phi(u)$.

Let $\theta \in \mathbb{R}$ and let $D \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be given by

$$D := \Big\{ u \in W^{1,p}(\Omega;\mathbb{R}^m) : \Phi(u) < \theta \Big\}.$$

Lemma 3.11. *The set D is L^p-closed and super-strongly star-shaped.*

Proof of Lemma 3.11. First of all, from (C_9) we can assert that D is L^p-open, i.e. $D = \mathrm{int}D$. Let $t \in]0,1[$ and let $u \in \overline{D}$. By using again (C_9) we have $\Phi(u) \leqslant \theta$. But, by (C_{10}), $\Phi(tu) < \Phi(u)$ and so $\Phi(tu) < \theta$. Hence $tu \in D$ and the proof is complete. ■

Taking Lemma 3.11 into account, as a direct consequence of Corollary 3.9 (resp. Corollary 3.10) and Theorem 2.9 we deduce Corollary 3.12(i) (resp. Corollary 3.12(ii)) below.

Corollary 3.12. *Assume that $p > N$.*

(i) *If (C_1)–(C_4) and (C_6)–(C_{10}) hold then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}(F_\varepsilon + \chi_D) = F + \chi_{\mathrm{dom}(G)} + \chi_{\overline{D}}.$$

(ii) *If (C_1)–(C_3), (C_5) and (C_6)–(C_{10}) hold then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}(F_\varepsilon + \chi_D) = \widehat{F} + \chi_{\mathrm{dom}(\overline{G})} + \chi_{\overline{D}}.$$

3.2.3. *Homogenization.* Let $f : \mathbb{R}^N \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function satisfying the following conditions:

(C_6^H) there exist $\alpha, \beta > 0$ such that for every $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}$,

$$\alpha g(\xi) \leqslant f(x,\xi) \leqslant \beta(1 + g(\xi));$$

(C_7^H) f is ru-usc;
(C_8^H) f is 1-periodic with respect to its first variable, i.e. for every $\xi \in \mathbb{M}$,

$$f(x+z,\xi) = f(x,\xi)$$

for all $(x,z) \in \mathbb{R}^N \times \mathbb{Z}^N$.

For each $\varepsilon > 0$, we consider $f_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ be defined by

$$f_\varepsilon(x, \xi) := f\left(\frac{x}{\varepsilon}, \xi\right). \tag{3.7}$$

Then, for every $\varepsilon > 0$, $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, \infty]$ is given by

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

Let $F_{\rm hom} : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, \infty]$ be defined by

$$F_{\rm hom}(u) := \int_\Omega \widehat{f}_{\rm hom}(\nabla u(x)) dx$$

with $\widehat{f}_{\rm hom} : \mathbb{M} \to [0, \infty]$ given by

$$\widehat{f}_{\rm hom}(\xi) := \varliminf_{t \to 1^-} f_{\rm hom}(t\xi),$$

where $f_{\rm hom} : \mathbb{M} \to [0, \infty]$ is defined by

$$f_{\rm hom}(\xi) := \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \inf \left\{ \int_{]0,k[^N} f(y, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}(]0, k[^N; \mathbb{R}^m) \right\}.$$

As a consequence of Corollaries 3.9, 3.10 and 3.12 we have the following homogenization results.

Corollary 3.13. *Assume that $p > N$.*

(i) *If* (C_1)–(C_4) *and* $(\mathrm{C}_6^{\mathrm{H}})$–$(\mathrm{C}_8^{\mathrm{H}})$ *hold then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon = F_{\rm hom} + \chi_{{\rm dom}(G)}.$$

(ii) *If* (C_1)–(C_3), (C_5) *and* $(\mathrm{C}_6^{\mathrm{H}})$–$(\mathrm{C}_8^{\mathrm{H}})$ *hold then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon = \widehat{F}_{\rm hom} + \chi_{{\rm dom}(\overline{G})}.$$

(iii) *If* (C_1)–(C_4), $(\mathrm{C}_6^{\mathrm{H}})$–$(\mathrm{C}_8^{\mathrm{H}})$ *and* (C_9)–(C_{10}) *hold then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} (F_\varepsilon + \chi_D) = F_{\rm hom} + \chi_{{\rm dom}(G)} + \chi_{\overline{D}}.$$

(iv) *If* (C_1)–(C_3), (C_5), $(\mathrm{C}_6^{\mathrm{H}})$–$(\mathrm{C}_8^{\mathrm{H}})$ *and* (C_9)–(C_{10}) *hold then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} (F_\varepsilon + \chi_D) = \widehat{F}_{\rm hom} + \chi_{{\rm dom}(\overline{G})} + \chi_{\overline{D}}.$$

Proof of Corollary 3.13. First of all, taking (3.7) into account, it is easy to see that (C_6) can be deduced from $(\mathrm{C}_6^{\mathrm{H}})$. On the hand, from Proposition 3.4 we see that $(\mathrm{C}_7^{\mathrm{H}})$ implies (C_7). Let $x \in \Omega$, let $\xi \in \mathbb{G}$ and let $\mathcal{S}^\xi : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0, \infty]$ defined by

$$\mathcal{S}^\xi(A) := \inf \left\{ \int_A f(x, \xi + \nabla \varphi(x)) dx : \varphi \in W_0^{1,p}(A; \mathbb{R}^m) \right\},$$

where $\mathcal{O}_{\rm b}(\mathbb{R}^N)$ denotes the class of all bounded open subsets of $\mathbb{R}^N$. It is clear that $\mathcal{S}^\xi$ is subadditive. Then, by using $(\mathrm{C}_8^{\mathrm{H}})$ we see that $\mathcal{S}^\xi$ is $\mathbb{Z}^N$-invariant. Finally, from the

right inequality in (C_6^H) we deduce that for every $A \in \mathcal{O}_b(\mathbb{R}^N)$, $\mathcal{S}^\xi(A) \leqslant C_\xi \mathscr{L}^N(A)$ with $C_\xi := \beta(1+g(\xi)) < \infty$ because $\xi \in \mathbb{G}$. From Theorem B.5 it follows that for every $\rho > 0$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}^\xi\left(\frac{1}{\varepsilon}Q_\rho(x)\right)}{\mathscr{L}^N\left(\frac{1}{\varepsilon}Q_\rho(x)\right)} = \inf_{k\geqslant 1} \frac{\mathcal{S}^\xi(]0,k[^N)}{k^N},$$

i.e.

$$\lim_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x,\xi) = f_{\rm hom}(\xi).$$

We are thus proved that

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x,\xi) = f_{\rm hom}(\xi) \text{ for all } x\in\Omega \text{ and all } \xi\in\mathbb{G}, \tag{3.8}$$

which implies (C_8). Let $F : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by (3.5). By (3.8) we have

$$F(u) = F_{\rm hom}(u) \text{ for all } u \in \mathrm{dom}(G),$$

and (i) and (iii) follow from Corollaries 3.9 and 3.12(i) respectively. When (C_5) holds, by using (3.8), we have

$$\widehat{F}(u) = \widehat{F}_{\rm hom}(u) \text{ for all } u \in \mathrm{dom}(\overline{G}),$$

which gives (ii) and (iv) by applying Corollaries 3.10 and 3.12(ii) respectively. ■

Appendix A. Proof of the partial Γ-convergence theorem

Here we give a proof of Theorem 3.8.

Proof of Theorem 3.8. In what follows, $\mathcal{O}(\Omega)$ denotes the class of open subsets of Ω. Let $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega) \to [0,\infty]$ be defined by

$$F_\varepsilon(u,A) = \int_A f_\varepsilon(x,\nabla u(x))dx$$

and let $\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon, \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)\to[0,\infty]$ be given by:

$$\Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(u,A) := \inf\left\{\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(u_\varepsilon,A) : u_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u\right\};$$
$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u,A) := \inf\left\{\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u_\varepsilon,A) : u_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u\right\}.$$

(For $A=\Omega$, $F_\varepsilon(u,\Omega) = F_\varepsilon(u)$, $\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,\Omega) = \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u)$ and $\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,\Omega) = \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u)$.) The proof is divided into six steps.

Step 1: integral representation of $\Gamma\text{-}\underline{\lim}$ and $\Gamma\text{-}\overline{\lim}$. For each $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ we consider the set functions $\mathcal{S}_u^-, \mathcal{S}_u^+ : \mathcal{O}(\Omega)\to[0,\infty]$ given by:

$$\mathcal{S}_u^-(A) := \Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(u,A);$$
$$\mathcal{S}_u^+(A) := \Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u,A).$$

Step 1 consists of proving the following lemma.

Lemma A.1. *Assume that $p > N$. If (C_1)–(C_3) and (C_6)–(C_7) hold then for every $u \in \mathrm{dom}(G)$ and every $A \in \mathcal{O}(\Omega)$,*

$$\mathcal{S}_u^-(A) = \int_A \lambda_u^-(x)dx;$$
$$\mathcal{S}_u^+(A) = \int_A \lambda_u^+(x)dx,$$

where $\lambda_u^-, \lambda_u^+ \in L^1(\Omega)$ are given by:

$$\lambda_u^-(x) = \lim_{\rho\to 0} \frac{\mathcal{S}_u^-(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))};$$
$$\lambda_u^+(x) = \lim_{\rho\to 0} \frac{\mathcal{S}_u^+(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}.$$

Proof of Lemma A.1. Fix $u \in \mathrm{dom}(G)$. Using the right inequality in (C_6) we see that for every $A \in \mathcal{O}(\Omega)$,

$$\mathcal{S}_u^-(A) \leqslant \beta\left(\mathscr{L}^N(A) + \int_A g(\nabla u(x))dx\right); \tag{A.1}$$
$$\mathcal{S}_u^+(A) \leqslant \beta\left(\mathscr{L}^N(A) + \int_A g(\nabla u(x))dx\right).$$

Thus, the condition (iv) of Lemma B.1 is satisfied with $\nu = \beta[1 + g(\nabla u(\cdot))]\mathscr{L}^N$ (which is absolutely continuous with respect to $\mathscr{L}^N$). On the other hand, it is easily seen that the conditions (i) and (ii) of Lemma B.1 are satisfied. Hence, the proof is completed if we prove the condition (iii) of Lemma B.1, i.e.

$$\mathcal{S}_u^-(A \cup B) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B) \text{ for all } A, B \in \mathcal{O}(\Omega); \tag{A.2}$$
$$\mathcal{S}_u^+(A \cup B) \leqslant \mathcal{S}_u^+(A) + \mathcal{S}_u^+(B) \text{ for all } A, B \in \mathcal{O}(\Omega). \tag{A.3}$$

Indeed, by Lemma B.1, the set function $\mathcal{S}_u^-$ (resp. $\mathcal{S}_u^+$) can be uniquely extended to a finite positive Radon measure which is absolutely continuous with respect to $\mathscr{L}^N$, and the theorem follows by using Radon-Nikodym's theorem and then Lebesgue's differentiation theorem.

Remark A.2. Lemma A.1 shows that $\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,\cdot)$ and $\Gamma\text{-}\overline{\lim}_{t\to\infty} F_\varepsilon(u,\cdot)$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to $\mathscr{L}^N$.

Substep 1-1: an auxiliary result for proving Lemma A.1. To show (A.2) (resp. (A.3)) we need the following lemma.

Lemma A.3. *Assume that $p > N$ and (C_1)–(C_3) and (C_6)–(C_7) hold. If $U, V, Z, T \in \mathcal{O}(\Omega)$ are such that $\overline{Z} \subset U$ and $T \subset V$, then:*

$$\mathcal{S}_u^-(Z \cup T) \leqslant \mathcal{S}_u^-(U) + \mathcal{S}_u^-(V); \tag{A.4}$$
$$\mathcal{S}_u^+(Z \cup T) \leqslant \mathcal{S}_u^+(U) + \mathcal{S}_u^+(V). \tag{A.5}$$

Proof of Lemma A.3. As the proofs of (A.4) and (A.5) are the same, we only give the proof of (A.4). Let $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ be two sequences in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that:

$$u_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u; \tag{A.6}$$

$$v_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u; \tag{A.7}$$

$$\lim_{\varepsilon\to 0}\int_U f_\varepsilon(x,\nabla u_\varepsilon(x))dx = \mathcal{S}_u^-(U) < \infty; \tag{A.8}$$

$$\lim_{\varepsilon\to 0}\int_V f_\varepsilon(x,\nabla v_\varepsilon(x))dx = \mathcal{S}_u^-(V) < \infty. \tag{A.9}$$

By (C_1) and (C_6) we have $\sup_{\varepsilon>0}\|\nabla u_\varepsilon\|_{L^p(U)} < \infty$ and $\sup_{\varepsilon>0}\|\nabla v_\varepsilon\|_{L^p(V)} < \infty$. Taking (A.6) and (A.7) into account, as $p > N$, up to a subsequence, we have:

$$u_\varepsilon \xrightarrow{L^\infty(U)} u; \tag{A.10}$$

$$v_\varepsilon \xrightarrow{L^\infty(V)} u. \tag{A.11}$$

Fix $\delta \in]0, \mathrm{dist}(Z,\partial U)[$ with $\partial U := \overline{U}\backslash U$, fix any $q \geqslant 1$ and consider $W_i^-, W_i^+ \subset \Omega$ given by:

$$W_i^- := \left\{x \in \Omega : \mathrm{dist}(x,Z) \leqslant \tfrac{\delta}{3} + \tfrac{(i-1)\delta}{3q}\right\};$$
$$W_i^+ := \left\{x \in \Omega : \tfrac{\delta}{3} + \tfrac{i\delta}{3q} \leqslant \mathrm{dist}(x,Z)\right\},$$

where $i \in \{1,\cdots,q\}$. For every $i \in \{1,\cdots,q\}$ there exists a cut-off function $\varphi_i \in C^\infty(\Omega)$ for the pair (W_i^+, W_i^-), i.e. $\varphi_i(x) \in [0,1]$ for all $x \in \Omega$, $\varphi_i(x) = 0$ for all $x \in W_i^+$ and $\varphi_i(x) = 1$ for all $x \in W_i^-$. Fix any $\varepsilon > 0$ and define $w_\varepsilon^i \in W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$w_\varepsilon^i := \varphi_i u_\varepsilon + (1-\varphi_i)v_\varepsilon. \tag{A.12}$$

Fix any $t \in]0,1[$. Setting $W_i := \Omega\backslash(W_i^- \cup W_i^+)$ we have

$$\nabla(tw_\varepsilon^i) = t\nabla w_\varepsilon^i = \begin{cases} t\nabla u_\varepsilon & \text{in } W_i^- \\ (1-t)\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - v_\varepsilon) + t\big(\varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla v_\varepsilon\big) & \text{in } W_i \\ t\nabla v_t & \text{in } W_i^+. \end{cases}$$

Noticing that $Z \cup T = ((Z\cup T)\cap W_i^-) \cup (W \cap W_i) \cup (T \cap W_i^+)$ with $(Z\cup T)\cap W_i^- \subset U$, $T \cap W_i^+ \subset V$ and $W := T \cap \{x \in U : \frac{\delta}{3} < \mathrm{dist}(x,Z) < \frac{2\delta}{3}\}$ we deduce that for every $i \in \{1,\cdots,q\}$,

$$\begin{aligned}\int_{Z\cup T} f_\varepsilon(x,t\nabla w_\varepsilon^i)dx \;\leqslant\; & \int_U f_\varepsilon(x,t\nabla u_\varepsilon)dx + \int_V f_\varepsilon(x,t\nabla v_\varepsilon)dx \\ & + \int_{W\cap W_i} f_\varepsilon(x,t\nabla w_\varepsilon^i)dx.\end{aligned} \tag{A.13}$$

Fix any $i \in \{1, \cdots, q\}$. From (C_2) and (C_6) we see that

$$\begin{aligned}\int_{W\cap W_i} f_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta\mathscr{L}^N(W\cap W_i) + \beta\int_{W\cap W_i} g(t\nabla w_\varepsilon^i)dx\\ &\leqslant \beta(1+\gamma)\mathscr{L}^N(W\cap W_i)\\ &\quad +\beta\gamma\int_{W\cap W_i} g(\varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla v_\varepsilon)dx\\ &\quad +\beta\gamma\int_{W\cap W_i} g\left(\frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx,\end{aligned}$$

and so, by using again (C_2) and (C_6),

$$\begin{aligned}\int_{W\cap W_i} f_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta(1+\gamma+\gamma^2)\mathscr{L}^N(W\cap W_i)\\ &\quad +\frac{\beta\gamma^2}{\alpha}\left(\int_{W\cap W_i} f_\varepsilon(x,\nabla u_\varepsilon)dx + \int_{W\cap W_i} f_\varepsilon(x,\nabla v_\varepsilon)dx\right)\\ &\quad +\beta\gamma\int_{W\cap W_i} g\left(\frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx. \qquad (A.14)\end{aligned}$$

On the other hand, we have

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x)-v_\varepsilon(x))\right| \leqslant \left|\frac{t}{1-t}\right|\|\nabla\varphi_i\|_{L^\infty}\|u_\varepsilon - v_\varepsilon\|_{L^\infty(U\cap V)}$$

for $\mathscr{L}_N$-a.a. $x \in W\cap W_i \subset U\cap V$. But, by (C_2)–(C_3) (see Remark 3.5(ii)), there exists $r>0$ such that

$$\theta := \sup_{|\xi|\leqslant r} g(\xi) < \infty, \qquad (A.15)$$

and $\lim_{\varepsilon\to 0}\|u_\varepsilon - v_\varepsilon\|_{L^\infty(U\cap V)} = 0$ by (A.10) and (A.11), hence for each $t \in]0,1[$ and each $i \in \{1,\cdots,q\}$ there exists $\varepsilon_{t,i} > 0$ such that

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x)-v_\varepsilon(x))\right| \leqslant r$$

for $\mathscr{L}_N$-a.a. $x \in W\cap W_i$ and all $\varepsilon \in]0,\varepsilon_{t,i}]$. Consequently, for every $i \in \{1,\cdots,q\}$,

$$\int_{W\cap W_i} g\left(\frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx \leqslant \theta\mathscr{L}^N(W\cap W_i) \qquad (A.16)$$

for all $t \in]0,1[$ and all $\varepsilon \in]0,\overline{\varepsilon}_{t,q}]$ with $\overline{\varepsilon}_{t,q} := \min\{\varepsilon_{t,i} : i \in \{1,\cdots,q\}\}$. Moreover, we have:

$$\int_U f_\varepsilon(x,t\nabla u_\varepsilon)dx \leqslant \left(1+\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\right)\int_U f_\varepsilon(x,\nabla u_\varepsilon)dx + a\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\mathscr{L}^N(U); \qquad (A.17)$$

$$\int_V f_\varepsilon(x,t\nabla v_\varepsilon)dx \leqslant \left(1+\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\right)\int_V f_\varepsilon(x,\nabla v_\varepsilon)dx + a\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\mathscr{L}^N(V), \qquad (A.18)$$

where $a>0$ is given by (C_7) and $\delta^a_{f_{\varepsilon'}} : [0,1] \to]-\infty,\infty]$ is defined by (3.1). Taking (A.16) into account and substituting (A.14), (A.17) and (A.18) into (A.13) and then averaging these

inequalities, it follows that for every $t\in]0,1[$, every $q\geqslant 1$ and every $\varepsilon\in]0,\overline{\varepsilon}_{t,q}]$, there exists $i_{\varepsilon,t,q}\in\{1,\cdots,q\}$ such that

$$\begin{aligned}\int_{Z\cup T} f_\varepsilon(x,\nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx \;\leqslant\; & \left(1+\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\right)\int_U f_\varepsilon(x,\nabla u_\varepsilon)dx + a\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\mathscr{L}^N(U)\\ &+\left(1+\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\right)\int_V f_\varepsilon(x,\nabla v_\varepsilon)dx + a\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\mathscr{L}^N(V)\\ &+\frac{C}{q}\left(\mathscr{L}^N(\Omega)+\int_U f_\varepsilon(x,\nabla u_\varepsilon)dx+\int_V f_\varepsilon(x,\nabla v_\varepsilon)dx\right)\end{aligned}$$

with $C=\max\left\{\beta(1+\gamma+\gamma^2)+\theta,\frac{\beta\gamma^2}{\alpha}\right\}$. Thus, letting $\varepsilon\to 0$, $q\to\infty$ and $t\to 1^-$ and using (C_7), i.e. $\overline{\lim}_{t\to 1^-}\sup_{\varepsilon'>0}\delta^a_{f_{\varepsilon'}}(t)\leqslant 0$, (A.8) and (A.9), we get

$$\overline{\lim_{t\to 1^-}}\;\overline{\lim_{q\to\infty}}\;\overline{\lim_{\varepsilon\to 0}}\int_{Z\cup T} f_\varepsilon(x,\nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx\leqslant \mathcal{S}_u^-(U)+\mathcal{S}_u^-(V). \tag{A.19}$$

On the other hand, taking (A.12) into account and using (A.6) and (A.7) we see that

$$\lim_{t\to 1^-}\lim_{q\to\infty}\lim_{\varepsilon\to 0}\|tw_\varepsilon^{i_{\varepsilon,t,q}}-u\|_{L^p}=0.$$

By diagonalization, there exist increasing mappings $\varepsilon\mapsto t_\varepsilon$ and $\varepsilon\mapsto q_\varepsilon$ with $t_\varepsilon\to 1^-$ and $q_\varepsilon\to\infty$ such that:

$$\begin{aligned}&\underline{\lim_{\varepsilon\to 0}}\int_{Z\cup T} f_\varepsilon(x,\nabla\hat{w}_\varepsilon)dx\leqslant \overline{\lim_{t\to 1^-}}\;\overline{\lim_{q\to\infty}}\;\overline{\lim_{\varepsilon\to 0}}\int_{Z\cup T} f_\varepsilon(x,\nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx;\\ &\lim_{\varepsilon\to 0}\|\hat{w}_\varepsilon-u\|_{L^p}=0,\end{aligned}$$

where $\hat{w}_\varepsilon:=t_\varepsilon w_\varepsilon^{i_{\varepsilon,t_\varepsilon,q_\varepsilon}}$. Hence

$$\mathcal{S}_u^-(Z\cup T)\leqslant \overline{\lim_{t\to 1^-}}\;\overline{\lim_{q\to\infty}}\;\overline{\lim_{\varepsilon\to 0}}\int_{Z\cup T} f_\varepsilon(x,\nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx,$$

and (A.4) follows from (A.19). ■

Substep 1-2: end of the proof of Lemma A.1. We now prove (A.2). Fix $A,B\in\mathcal{O}(\Omega)$ such that $\mathcal{S}_u^-(A)<\infty$ and $\mathcal{S}_u^-(B)<\infty$. Then, by (A.1), $\int_{A\cup B} g(\nabla u(x))dx<\infty$. Fix any $\eta>0$ and consider $C_0,D_0\in\mathcal{O}(\Omega)$ such that $\overline{C}_0\subset A$, $\overline{D}_0\subset B$ and

$$\beta\left(\mathscr{L}^N(E)+\int_E g(\nabla u(x))dx\right)<\eta$$

with $E:=A\cup B\backslash\overline{C_0\cup D_0}$. Then $\mathcal{S}_u^-(E)\leqslant\eta$ by (A.1). Let $\hat{C},\hat{D}\in\mathcal{O}(\Omega)$ be such that $\overline{C}_0\subset C$, $\overline{C}\subset\hat{C}$, $\overline{\hat{C}}\subset A$, $\overline{D}_0\subset D$, $\overline{D}\subset\hat{D}$ and $\overline{\hat{D}}\subset B$. Applying Lemma A.3 with $U=\hat{C}\cup\hat{D}$, $V=T=E$ and $Z=C\cup D$ (resp. $U=A$, $V=B$, $Z=\hat{C}$ and $T=\hat{D}$) we obtain

$$\mathcal{S}_u^-(A\cup B)\leqslant\mathcal{S}_u^-(\hat{C}\cup\hat{D})+\eta\ \big(\text{resp. }\mathcal{S}_u^-(\hat{C}\cup\hat{D})\leqslant\mathcal{S}_u^-(A)+\mathcal{S}_u^-(B)\big),$$

i.e. $\mathcal{S}_u^-(A\cup B)\leqslant\mathcal{S}_u^-(A)+\mathcal{S}_u^-(B)+\eta$, and (A.2) follows by letting $\eta\to 0$. ■

Step 2: other formulas for Γ-$\underline{\lim}$ and Γ-$\overline{\lim}$. Let $F_0^-, F_{0,}^+ : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)\to[0,\infty]$ be defined by:

$$F_0^-(u,A) := \inf\left\{\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) : W_0^{1,p}(\Omega;\mathbb{R}^m)\ni u_\varepsilon - u \xrightarrow{\|\cdot\|_{L^p}} 0\right\};$$
$$F_0^+(u,A) := \inf\left\{\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) : W_0^{1,p}(\Omega;\mathbb{R}^m)\ni u_\varepsilon - u \xrightarrow{\|\cdot\|_{L^p}} 0\right\}.$$

Since $W_0^{1,p}(\Omega;\mathbb{R}^m)\subset W^{1,p}(\Omega;\mathbb{R}^m)$, for every $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and every $A\in\mathcal{O}(\Omega)$, we have:

$$\mathcal{S}_u^-(A) \leqslant F_0^-(u,A); \tag{A.20}$$
$$\mathcal{S}_u^+(A) \leqslant F_0^+(u,A). \tag{A.21}$$

On the other hand, we have the following lemma.

Lemma A.4. *Assume that $p>N$ and (C_1)–(C_3) and (C_6)–(C_7) hold. Then, for every $u\in\mathrm{dom}(G)$, every $A\in\mathcal{O}(\Omega)$ and every $t\in]0,1[$, we have*:

$$F_0^-(tu,A) \leqslant \left(1+\sup_{\varepsilon>0}\delta^a_{f_\varepsilon}(t)\right)\mathcal{S}_u^-(A) + a\sup_{\varepsilon>0}\delta^a_{f_\varepsilon}(t)\mathscr{L}^N(A); \tag{A.22}$$
$$F_0^+(tu,A) \leqslant \left(1+\sup_{\varepsilon>0}\delta^a_{f_\varepsilon}(t)\right)\mathcal{S}_u^+(A) + a\sup_{\varepsilon>0}\delta^a_{f_\varepsilon}(t)\mathscr{L}^N(A), \tag{A.23}$$

where $a>0$ is given by (C_7) and $\delta^a_{f_{\varepsilon'}}:[0,1]\to]-\infty,\infty]$ is defined by (3.1). As a consequence of (A.20)-(A.22) and (A.21)-(A.23), for every $u\in\mathrm{dom}(G)$ and every $A\in\mathcal{O}(\Omega)$, we have:

$$\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A) = \lim_{t\to 1^-} F_0^-(tu,A); \tag{A.24}$$
$$\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A) = \lim_{t\to 1^-} F_0^+(tu,A).$$

Proof of Lemma A.4. Fix $u\in\mathrm{dom}(G)$ and $A\in\mathcal{O}(\Omega)$. As the proofs of (A.22) and (A.23) are the same, we only prove (A.22). Let $\{u_\varepsilon\}_{\varepsilon>0}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ be such that:

$$u_\varepsilon \xrightarrow{\|\cdot\|_{L^p}} u; \tag{A.25}$$
$$\lim_{\varepsilon\to 0}\int_A f_\varepsilon(x,\nabla u_\varepsilon(x))dx = \mathcal{S}_u^-(A) < \infty. \tag{A.26}$$

From (C_1) and (A.26) we see that $\sup_{\varepsilon>0}\|\nabla u_\varepsilon\|_{L^p}<\infty$. As $p>N$, taking (A.25) into account, we can assert, up to a subsequence, that

$$u_\varepsilon \xrightarrow{\|\cdot\|_{L^\infty}} u. \tag{A.27}$$

Fix $\delta>0$ and set $A_\delta := \{x\in A : \mathrm{dist}(x,\partial A)>\delta\}$ with $\partial A := \overline{A}\backslash A$. Fix any $\varepsilon>0$ and any $q\geqslant 1$ and consider $W_i^-, W_i^+\subset\Omega$ given by:

$$W_i^- := \left\{x\in\Omega : \mathrm{dist}(x,A_\delta)\leqslant \tfrac{\delta}{3}+\tfrac{(i-1)\delta}{3q}\right\};$$
$$W_i^+ := \left\{x\in\Omega : \tfrac{\delta}{3}+\tfrac{i\delta}{3q}\leqslant \mathrm{dist}(x,A_\delta)\right\},$$

where $i \in \{1, \cdots, q\}$. (Note that $W_i^- \subset A$.) For every $i \in \{1, \cdots, q\}$ there exists a cut-off function $\varphi_i \in C^\infty(\Omega)$ for the pair (W_i^+, W_i^-), i.e. $\varphi_i(x) \in [0,1]$ for all $x \in \Omega$, $\varphi_i(x) = 0$ for all $x \in W_i^+$ and $\varphi_i(x) = 1$ for all $x \in W_i^-$. Define $w_\varepsilon^i : \Omega \to \mathbb{R}^m$ by

$$w_\varepsilon^i := \varphi_i u_\varepsilon + (1 - \varphi_i)u. \tag{A.28}$$

Then $w_\varepsilon^i - u \in W_0^{1,p}(A; \mathbb{R}^m)$. Fix any $t \in]0,1[$. Setting $W_i := X \backslash (W_i^- \cup W_i^+) \subset A$, we have

$$\nabla(tw_\varepsilon^i) = t\nabla w_\varepsilon^i = \begin{cases} t\nabla u_\varepsilon & \text{in } W_i^- \\ (1-t)\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u) + t\left(\varphi_i \nabla u_\varepsilon + (1-\varphi_i)\nabla u\right) & \text{in } W_i \\ t\nabla u & \text{in } W_i^+. \end{cases}$$

Fix any $i \in \{1, \cdots, q\}$. Noticing that $A = W_i^- \cup W_i \cup (A \cap W_i^+)$, we deduce that

$$\int_A f_\varepsilon(x, t\nabla w_\varepsilon^i)dx \leqslant \int_A f_\varepsilon(x, t\nabla u_\varepsilon)dx + \int_{A\cap W_i^+} f_\varepsilon(x, t\nabla u)dx + \int_{W_i} f_\varepsilon(x, t\nabla w_\varepsilon^i)dx. \tag{A.29}$$

From the right inequality in (C_6) and the inequality (C_2), we see that ...

$$\begin{aligned} \int_{W_i} f_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta\mathscr{L}^N(W_i) + \beta\int_{W_i} g(t\nabla w_\varepsilon^i)dx \\ &\leqslant \beta(1+\gamma)\mathscr{L}^N(W_i) + \beta\gamma\int_{W_i} g(\varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla u)dx \\ &\quad + \beta\gamma\int_{W_i} g\left(\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u)\right)dx, \end{aligned} \tag{A.30}$$

and by using again (C_2) and the left inequality in (C_6) we obtain

$$\begin{aligned} \int_{W_i} f_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta(1+\gamma+\gamma^2)\mathscr{L}^N(W_i) + \frac{\beta\gamma^2}{\alpha}\left(\int_{W_i} f_\varepsilon(x, \nabla u_\varepsilon)dx + \int_{W_i} f_\varepsilon(x, \nabla u)dx\right) \\ &\quad + \beta\gamma\int_{W_i} g\left(\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u)\right)dx. \end{aligned} \tag{A.31}$$

On the other hand, for $\mathscr{L}^N$-a.e. $x \in \Omega$, we have

$$\left|\frac{t}{1-t}\nabla\varphi_i(x) \otimes (u_\varepsilon(x) - u(x))\right| \leqslant \left|\frac{t}{1-t}\right| \|\nabla\varphi_i\|_{L^\infty}\|u_\varepsilon - u\|_{L^\infty}.$$

But $\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{L^\infty} = 0$ by (A.27), hence for each $i \in \{1, \cdots, q\}$ there exists $\varepsilon_i > 0$ such that for $\mathscr{L}^N$-a.e. $x \in \Omega$ and every $\varepsilon \in]0, \varepsilon_i]$,

$$\left|\frac{t}{1-t}\nabla\varphi_i(x) \otimes (u_\varepsilon(x) - u(x))\right| \leqslant r$$

with $r > 0$ given by (C_2)–(C_3) (see Remark 3.5(ii)). Hence, for every $i \in \{1, \cdots, q\}$,

$$\int_{W_i} g\left(\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u)\right)dx \leqslant \theta\mathscr{L}^N(W_i) \tag{A.32}$$

for all $\varepsilon \in]0, \overline{\varepsilon}_q]$ with $\overline{\varepsilon}_q = \min\{\varepsilon_i : i \in \{1, \cdots, q\}\}$, where θ is defined by (A.15). Moreover, we have:

$$\begin{aligned}\int_A f_\varepsilon(x, t\nabla u_\varepsilon)dx &\leqslant \left(1 + \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t)\right) \int_A f_\varepsilon(x, \nabla u_\varepsilon)dx \\ &\quad + a \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t) \mathscr{L}^N(A); \qquad (A.33)\\ \int_{A\cap W_i^+} f_\varepsilon(x, t\nabla u)dx &\leqslant \left(1 + \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t)\right) \int_{A\cap W_i^+} f_\varepsilon(x, \nabla u)dx \\ &\quad + a \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t) \mathscr{L}^N(A \cap W_i^+). \qquad (A.34)\end{aligned}$$

Taking (A.32) into account and substituting (A.31), (A.33) and (A.34) into (A.29) and then averaging these inequalities, it follows that for every $q \geqslant 1$ and every $\varepsilon \in]0, \overline{\varepsilon}_{t,q}]$, there exists $i_{\varepsilon,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned}\int_A f_\varepsilon(x, \nabla(tw_t^{i_{\varepsilon,q}}))dx &\leqslant \left(1 + \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t)\right) \left(\int_A f_\varepsilon(x, \nabla u_\varepsilon)dx + \frac{1}{q}\int_A f_\varepsilon(x, \nabla u)dx\right) \\ &\quad + a \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t) \mathscr{L}^N(A) \left(1 + \frac{1}{q}\right) + \frac{C}{q} \mathscr{L}^N(A) \\ &\quad + \frac{C}{q} \left(\int_A f_\varepsilon(x, \nabla u_\varepsilon)dx + \int_A f_\varepsilon(x, \nabla u)dx\right)\end{aligned}$$

with $C = \max\left\{\beta(1 + \gamma + \gamma^2) + \theta, \frac{\beta\gamma^2}{\alpha}\right\}$. Since $u \in \mathrm{dom}(G)$, from (C_6) we see that $\overline{\lim}_{\varepsilon\to 0} \int_A f_\varepsilon(x, \nabla u)dx < \infty$. Thus, letting $\varepsilon \to 0$ and $q \to \infty$ and using (A.26), we get

$$\overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}} \int_A f_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,q}}))dx \leqslant \left(1 + \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t)\right) \mathcal{S}_u^-(A) + a \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t) \mathscr{L}^N(A). \qquad (A.35)$$

On the other hand, taking (A.28) into account and using (A.25) we see that

$$\lim_{q\to\infty} \lim_{\varepsilon\to 0} \|tw_\varepsilon^{i_{\varepsilon,q}} - tu\|_{L^p} = 0.$$

By diagonalization, there exists an increasing mapping $\varepsilon \mapsto q_\varepsilon$ with $q_\varepsilon \to \infty$ such that:

$$\begin{aligned}&\underline{\lim_{\varepsilon\to 0}} \int_A f_\varepsilon(x, \nabla\hat{w}_\varepsilon)dx \leqslant \overline{\lim_{\varepsilon\to 0}} \int_A f_\varepsilon(x, \nabla\hat{w}_\varepsilon)dx \leqslant \overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}} \int_A f_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,q}}))dx; \\ &\lim_{\varepsilon\to 0} \|\hat{w}_\varepsilon - u\|_{L^p} = 0,\end{aligned}$$

where $\hat{w}_\varepsilon := tw_\varepsilon^{i_{\varepsilon,q_\varepsilon}}$ is such that $\hat{w}_\varepsilon - tu \in W_0^{1,p}(A; \mathbb{R}^m)$. Hence

$$F_0^-(tu, A) \leqslant \overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}} \int_A f_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,q}}))dx,$$

and (A.22) follows from (A.35).
Since $\overline{\lim}_{t\to 1^-} \sup_{\varepsilon'>0} \delta^a_{f_{\varepsilon'}}(t) \leqslant 0$ by (C_7), from (A.22) we deduce that

$$\overline{\lim_{t\to 1^-}}\, F_0^-(tu, A) \leqslant \mathcal{S}_u^-(A).$$

Moreover, by (A.20) we have

$$\mathscr{S}_u^-(A) \leqslant \underline{\lim}_{t\to 1^-} \mathscr{S}_{tu}^-(A) \leqslant \underline{\lim}_{t\to 1^-} F_0^-(tu, A),$$

which gives (A.24). ■

Step 3: using the Vitali envelope. For each $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we consider the set functions $\mathscr{M}_u^-, \mathscr{M}_u^+ : \mathscr{O}(\Omega) \to [0,\infty]$ defined by:

$$\begin{aligned} \mathscr{M}_u^-(A) &:= \overline{\lim}_{t\to 1^-} \mathrm{m}_{tu}^-(A); \\ \mathscr{M}_u^+(A) &:= \overline{\lim}_{t\to 1^-} \mathrm{m}_{tu}^+(A). \end{aligned} \tag{A.36}$$

where, for each $z \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\mathrm{m}_z^-, \mathrm{m}_z^+ : \mathscr{O}(\Omega) \to [0,\infty]$ are given by:

$$\begin{aligned} \mathrm{m}_z^-(A) &:= \underline{\lim}_{\varepsilon\to 0} \inf\left\{F_\varepsilon(v,A) : v - z \in W_0^{1,p}(A;\mathbb{R}^m)\right\}; \\ \mathrm{m}_z^+(A) &:= \overline{\lim}_{\varepsilon\to 0} \inf\left\{F_\varepsilon(v,A) : v - z \in W_0^{1,p}(A;\mathbb{R}^m)\right\}. \end{aligned}$$

For each $\delta > 0$ and each $A \in \mathscr{O}(\Omega)$, we denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i) := x_i +]-\frac{\rho_i}{2}, \frac{\rho_i}{2}[^N\}_{i\in I}$ of disjoint open cubes of A with $x_i \in A$ and $\rho_i \in]0,\delta[$ and such that $\mathscr{L}^N(A \setminus \cup_{i\in I} Q_i) = 0$ by $\mathscr{V}_\delta(A)$, we consider $\mathscr{M}_{u,\delta}^+ : \mathscr{O}(\Omega) \to [0,\infty]$ given by

$$\mathscr{M}_{u,\delta}^+(A) := \inf\left\{\sum_{i\in I} \mathscr{M}_u^+(Q_i) : \{Q_i\}_{i\in I} \in \mathscr{V}_\delta(A)\right\},$$

and we define $\overline{\mathscr{M}}_u^+ : \mathscr{O}(\Omega) \to [0,\infty]$ by

$$\overline{\mathscr{M}}_u^+(A) := \sup_{\delta>0} \mathscr{M}_{u,\delta}^+(A) = \lim_{\delta\to 0} \mathscr{M}_{u,\delta}^+(A).$$

The set function $\overline{\mathscr{M}}_u^+$ is called the Vitali envelope of $\mathscr{M}_u^+$, see §**??**. Step 3 consists of proving the following lemma.

Lemma A.5. *Assume that $p > N$ and (C_1)–(C_3) and (C_6)–(C_7) hold. Then, for every $u \in \mathrm{dom}(G)$ and every $A \in \mathscr{O}(\Omega)$, we have*:

$$\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A) \geqslant \mathscr{M}_u^-(A); \tag{A.37}$$

$$\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A) = \overline{\mathscr{M}}_u^+(A). \tag{A.38}$$

Proof of Lemma A.5. Fix $u \in \mathrm{dom}(G)$. Given any $A \in \mathscr{O}(\Omega)$ it is easy to see that for every $t \in]0,1[$, we have:

$$\begin{aligned} \mathrm{m}_{tu}^-(A) &\leqslant F_0^-(tu,A); \\ \mathrm{m}_{tu}^+(A) &\leqslant F_0^+(tu,A). \end{aligned}$$

Hence, by Lemma A.4, we have:

$$\begin{aligned} \mathscr{M}_u^-(A) = \overline{\lim}_{t\to 1^-} \mathrm{m}_{tu}^-(A) &\leqslant \lim_{t\to 1^-} F_0^-(tu,A) = \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A); \\ \mathscr{M}_u^+(A) = \overline{\lim}_{t\to 1^-} \mathrm{m}_{tu}^+(A) &\leqslant \lim_{t\to 1^-} F_0^+(tu,A) = \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,A). \end{aligned}$$

Consequently

$$\overline{\mathscr{M}}^+_u(A) \leqslant \Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(u,A)$$

because in the proof of Lemma A.4 it is established that $\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,\cdot)$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to $\mathscr{L}^N$, see Remark A.2. Hence (A.37) holds and, to establish (A.38), it remains to prove that

$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(u,A) \leqslant \overline{\mathscr{M}}^+_u(A) \tag{A.39}$$

with $\overline{\mathscr{M}}^+_u(A) < \infty$. Fix any $\delta > 0$. By definition of $\mathscr{M}^+_{u,\delta}(A)$ there exists $\{Q_i\}_{i\in I} \in \mathscr{V}_\delta(A)$ such that

$$\sum_{i\in I} \mathscr{M}^+_u(Q_i) \leqslant \mathscr{M}^+_{u,\delta}(A) + \frac{\delta}{2}. \tag{A.40}$$

Fix any $\varepsilon > 0$ and define $\mathrm{m}^\varepsilon_u : \mathscr{O}(\Omega) \to [0,\infty]$ by

$$\mathrm{m}^\varepsilon_u(U) := \inf\Big\{F_\varepsilon(v,U) : v - u \in W^{1,p}_0(U;\mathbb{R}^m)\Big\}. \tag{A.41}$$

(Thus $\mathrm{m}^+_u(\cdot) = \overline{\lim}_{\varepsilon\to 0} \mathrm{m}^\varepsilon_u(\cdot)$.) Fix any $t \in]0,1[$. For each $i \in I$, by definition of $\mathrm{m}^\varepsilon_{tu}(Q_i)$ there exists $v^i_{\varepsilon,t} \in W^{1,p}(Q_i;\mathbb{R}^m)$ such that $v^i_{\varepsilon,t} - tu \in W^{1,p}_0(Q_i;\mathbb{R}^m)$ and

$$F_\varepsilon(v^i_{\varepsilon,t}, Q_i) \leqslant \mathrm{m}^\varepsilon_{tu}(Q_i) + \frac{\delta \mathscr{L}^N(Q_i)}{2\mathscr{L}^N(A)}. \tag{A.42}$$

Define $u^\delta_{\varepsilon,t} : \Omega \to \mathbb{R}^m$ by

$$u^\delta_{\varepsilon,t} := \begin{cases} tu & \text{in } \Omega\backslash A \\ v^i_{\varepsilon,t} & \text{in } Q_i. \end{cases}$$

Then $u^\delta_{\varepsilon,t} - tu \in W^{1,p}_0(A;\mathbb{R}^m)$. From (A.42) we see that

$$F_\varepsilon(u^\delta_{\varepsilon,t}, A) \leqslant \sum_{i\in I} \mathrm{m}^\varepsilon_{tu}(Q_i) + \frac{\delta}{2},$$

hence $\overline{\lim}_{t\to 1^-}\, \overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u^\delta_{\varepsilon,t}, A) \leqslant \mathscr{M}^+_{u,\delta}(A) + \varepsilon$ by using (A.40), and consequently

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u^\delta_{\varepsilon,t}, A) \leqslant \overline{\mathscr{M}}^+_u(A). \tag{A.43}$$

On the other hand, we have

$$\begin{aligned} \|u^\delta_{\varepsilon,t} - u\|^p_{L^p} &\leqslant 2^p\left(\|u^\delta_{\varepsilon,t} - tu\|^p_{L^p} + \|tu - u\|^p_{L^p}\right) \\ &= 2^p\left(\int_A |u^\delta_{\varepsilon,t} - tu|^p dx + (1-t)^p \|u\|^p_{L^p}\right) \\ &= 2^p\left(\sum_{i\in I}\int_{Q_i} |v^i_{\varepsilon,t} - tu|^p dx + (1-t)^p\|u\|^p_{L^p}\right). \end{aligned}$$

As $\operatorname{diam}(Q_i) \in]0,\delta[$ for all $i \in I$, from Poincaré's inequality we can assert that there exists $C > 0$ (which only depends on p) such that

$$\sum_{i\in I}\int_{Q_i} |v^i_{\varepsilon,t} - tu|^p dx \leqslant \delta^p C \sum_{i\in I}\int_{Q_i} |\nabla v^i_{\varepsilon,t} - t\nabla u|^p dx,$$

hence

$$\sum_{i\in I}\int_{Q_i} |v^i_{\varepsilon,t} - tu|^p dx \leqslant 2^p\delta^p C \left(\sum_{i\in I}\int_{Q_i} |\nabla v^i_{\varepsilon,t}|^p dx + t^p\int_A |\nabla u|^p dx\right),$$

and consequently

$$\|u^\delta_{\varepsilon,t} - u\|^p_{L^p} \leqslant 2^{2p}\delta^p C \left(\sum_{i\in I}\int_{Q_i} |\nabla v^i_{\varepsilon,t}|^p dx + t^p\int_A |\nabla u|^p dx\right) + 2^p(1-t)^p\|u\|^p_{L^p}. \tag{A.44}$$

Taking (C_1), the left inequality in (C_6), (A.40) and (A.42) into account, from (A.44) we deduce that

$$\overline{\lim_{t\to 1^-}}\,\overline{\lim_{\varepsilon\to 0}}\,\|u^\delta_{\varepsilon,t} - u\|^p_{L^p} \leqslant 2^{2p}\delta^p C \left(\frac{1}{\alpha c}(\mathscr{M}^+_{u,\delta}(A) + \delta) + \int_A |\nabla u|^p dx\right),$$

which gives

$$\lim_{\delta\to 0}\overline{\lim_{t\to 1^-}}\,\overline{\lim_{\varepsilon\to 0}}\,\|u^\delta_{\varepsilon,t} - u\|^p_{L^p} = 0 \tag{A.45}$$

because $\lim_{\varepsilon\to 0}\mathscr{M}^+_{u,\delta}(A) = \overline{\mathscr{M}}^+_u(A) < \infty$. According to (A.43) and (A.45), by diagonalization there exist mappings $\varepsilon \mapsto t_\varepsilon$ and $\varepsilon \to \delta_\varepsilon$, with $t_\varepsilon \to 1^-$ and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, such that:

$$\lim_{\varepsilon\to 0}\|w_\varepsilon - u\|^p_{L^p} = 0; \tag{A.46}$$

$$\overline{\lim_{\varepsilon\to 0}}\,F_\varepsilon(w_\varepsilon, A) \leqslant \overline{\mathscr{M}}^+_u(A) \tag{A.47}$$

with $w_\varepsilon := u^{\delta_\varepsilon}_{\varepsilon,t_\varepsilon}$. By (A.46) we have $\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}F_\varepsilon(u,A) \leqslant \overline{\lim}_{\varepsilon\to 0}F_\varepsilon(w_\varepsilon,A)$, and inequality (A.39) follows from (A.47). ■

Step 4: differentiation with respect to $\mathscr{L}^N$. Using Lemma A.1, Remark A.2 and Lemma A.5, it is easily seen that for every $u \in \operatorname{dom}(G)$ and every $A \in \mathcal{O}(\Omega)$,

$$\Gamma\text{-}\underline{\lim_{\varepsilon\to 0}}\,F_\varepsilon(u,A) \geqslant \int_A \overline{\lim_{\rho\to 0}}\frac{\mathscr{M}^-_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx = \int_A \overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to 1^-}}\,\frac{\mathrm{m}^-_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx; \tag{A.48}$$

$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\,F_\varepsilon(u,A) = \int_A \lim_{\rho\to 0}\frac{\overline{\mathscr{M}}^+_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx. \tag{A.49}$$

The goal of Step 4 is to apply Theorem B.3 (with $\mathcal{S} = \mathscr{M}^+_u$ where $u \in \operatorname{dom}(G)$) for proving the following lemma.

Lemma A.6. *Assume that $p > N$ and (C_1)–(C_3) and (C_6)–(C_7) hold. Then, for every $u \in \operatorname{dom}(G)$ and every $A \in \mathcal{O}(\Omega)$,*

$$\overline{\mathscr{M}}^+_u(A) = \int_A \lim_{\rho\to 0}\frac{\mathscr{M}^+_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx. \tag{A.50}$$

As a consequence, for every $u \in \mathrm{dom}(G)$ and every $A \in \mathcal{O}(\Omega)$, we have

$$\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u,A) = \int_A \lim_{\rho\to 0} \frac{\mathscr{M}_u^+(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx = \int_A \lim_{\rho\to 0} \overline{\lim_{t\to 1^-}} \frac{\mathrm{m}_{tu}^+(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx. \tag{A.51}$$

Proof of Lemma A.6. Fix $u \in \mathrm{dom}(G)$. The integral representation of $\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} F_\varepsilon(u,\cdot)$ in (A.51) follows from (A.50), (A.49) and the definition of $\mathscr{M}_u^+$ in (A.36). So, we only need to establish (A.50). For this, it is sufficient to prove that $\mathscr{M}_u^+$ is subadditive and there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to $\mathscr{L}^N$ such that

$$\mathscr{M}_u^+(A) \leqslant \nu(A) \tag{A.52}$$

for all $A \in \mathcal{O}(\Omega)$, and then to apply Theorem B.3. For each $\varepsilon > 0$ and each $t \in]0,1[$, from the definition of $\mathrm{m}_{tu}^\varepsilon$ in (A.41), it is easy to see that for every $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathscr{L}^N(A\backslash(B \cup C)) = 0$,

$$\mathrm{m}_{tu}^\varepsilon(A) \leqslant \mathrm{m}_{tu}^\varepsilon(B) + \mathrm{m}_{tu}^\varepsilon(C)$$

(with $\mathrm{m}_z^\varepsilon(\cdot)$ defined in (A.41)), and so

$$\overline{\lim_{t\to 1^-}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathrm{m}_{tu}^\varepsilon(A) \leqslant \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathrm{m}_{tu}^\varepsilon(B) + \overline{\lim_{\tau\to 1^-}}\, \overline{\lim_{t\to \infty}}\, \mathrm{m}_{tu}^\varepsilon(C),$$

i.e.

$$\mathscr{M}_u^+(A) \leqslant \mathscr{M}_u^+(B) + \mathscr{M}_u^+(C), \tag{A.53}$$

which shows the subadditivity of $\mathscr{M}_u^+$.

Remark A.7. As, in general, the $\underline{\lim}$ of the sum is not smaller than the sum of the $\underline{\lim}$, we cannot assert that (A.53) holds for $\mathscr{M}_u^-$ instead of $\mathscr{M}_u^+$ and so that $\mathscr{M}_u^-$ is subadditive.

On the other hand, given any $\varepsilon > 0$ and any $t \in]0,1[$, by using the right inequality in (C_6) we have

$$\mathrm{m}_{tu}^\varepsilon(A) \leqslant \beta\left(\mathscr{L}^N(A) + \int_A g(t\nabla u(x))dx\right).$$

But, from (C_2) we see that $g(t\nabla u(x)) \leqslant \gamma(1 + g(\nabla u(x)) + g(0))$ for $\mathscr{L}^N$-a.a. $x \in \Omega$, hence

$$\mathrm{m}_{tu}^\varepsilon(A) \leqslant \beta(1+\gamma+\gamma g(0))\mathscr{L}^N(A) + \beta\gamma\int_A g(\nabla u(x))dx.$$

Letting $\varepsilon \to 0$ and $t \to 1^-$ we conclude that

$$\mathscr{M}_u^+(A) \leqslant C\left(\mathscr{L}^N(A) + \int_A g(\nabla u(x))dx\right)$$

with $C := \beta(1+\gamma+\gamma g(0))$. Thus (A.52) holds with the Radon measure $\nu := C\big(1 + g(\nabla u(\cdot))\big)\mathscr{L}^N$ which is necessarily finite since $u \in \mathrm{dom}(G)$ and $g(0) < \infty$ by (C_3). ■

Step 5: establishing Γ-$\underline{\lim}$ and Γ-$\overline{\lim}$ formulas. According to (A.48) and (A.51), Γ-$\underline{\lim}$ and Γ-$\overline{\lim}$ formulas, see (3.3), will be established (see Substep 5-2) if we prove that for every

$u \in \mathrm{dom}(G)$ and $\mathscr{L}^N$-a.e. $x \in \Omega$, we have:

$$\underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \frac{\mathrm{m}^-_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \leqslant \overline{\lim}_{\rho\to 0} \frac{\mathscr{M}^-_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}; \tag{A.54}$$

$$\underline{\lim}_{t\to 1^-} \lim_{\rho\to 0} \frac{\mathrm{m}^+_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \lim_{\rho\to 0} \frac{\mathscr{M}^+_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}, \tag{A.55}$$

where $u_x(y) := u(x) + \nabla u(x)(y-x)$ and (A.55) is equivalent to:

$$\underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \frac{\mathrm{m}^+_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \leqslant \lim_{\rho\to 0} \frac{\mathscr{M}^+_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}; \tag{A.56}$$

$$\underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \frac{\mathrm{m}^+_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \geqslant \lim_{\rho\to 0} \frac{\mathscr{M}^+_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}. \tag{A.57}$$

Substep 5-1: proofs of (A.54), (A.56) and (A.57). As the proofs of (A.54), (A.56) and (A.57) use the same method, we only give the proof of (A.54). First of all, by diagonalization there exists a mapping $s \mapsto t_s$ with $t_s \to 1^-$ as $s \to 1^-$ such that:

$$\lim_{s\to 1^-} \frac{t_s}{s} = 1;$$
$$\overline{\lim}_{t\to 1^-} \overline{\lim}_{s\to 1^-} \Delta\left(\frac{t}{s}\right) \leqslant \overline{\lim}_{s\to 1^-} \Delta\left(\frac{t_s}{s}\right),$$

where $\Delta(\cdot) := \sup_{\varepsilon>0} \delta^a_{f_\varepsilon}(\cdot)$ with $a > 0$ given by (C_7) and $\delta^a_{f_\varepsilon}(\cdot)$ defined by (3.1). But, by (C_7), $\overline{\lim}_{r\to 1^-} \Delta(r) \leqslant 0$, hence

$$\overline{\lim}_{t\to 1^-} \overline{\lim}_{s\to 1^-} \Delta\left(\frac{t}{s}\right) \leqslant 0. \tag{A.58}$$

Fix any $\eta > 0$. For every $t \in]0,1[$ there exists $s_t \in]\tau, 1[$ such that for every $s \in [s_t, 1[$,

$$\Delta\left(\frac{t}{s}\right) \leqslant \overline{\lim}_{s\to 1^-} \Delta\left(\frac{t}{s}\right) + \frac{\eta}{2}. \tag{A.59}$$

In the same way, there exists $t_0 \in]0,1[$ such that for every $t \in [t_0, 1[$,

$$\overline{\lim}_{s\to 1^-} \Delta\left(\frac{t}{s}\right) \leqslant \overline{\lim}_{t\to 1^-} \overline{\lim}_{s\to 1^-} \Delta\left(\frac{t}{s}\right) + \frac{\eta}{2}, \tag{A.60}$$

and from (A.58), (A.59) and (A.60) we deduce that for every $t \in [t_0, 1[$ and every $s \in [s_t, 1[$,

$$\Delta\left(\frac{t}{s}\right) \leqslant \eta. \tag{A.61}$$

Fix $u \in \mathrm{dom}(G)$. Fix any $\varepsilon > 0$, any $\lambda \in]0,1[$, any $\rho > 0$, any $t \in [t_0, 1[$ and any $s \in [s_t, 1[$. By definition of $\mathrm{m}^\varepsilon_{su}(Q_{\lambda\rho}(x))$ in (A.41), there exists $w : \Omega \to \mathbb{R}^m$ such that $w - su \in W^{1,p}_0(Q_{\lambda\rho}(x); \mathbb{R}^m)$ and

$$\int_{Q_{\lambda\rho}(x)} f_\varepsilon(y, \nabla w(y))dy \leqslant \mathrm{m}^\varepsilon_{su}(Q_{\lambda\rho}(x)) + \eta \mathscr{L}^N(Q_{\lambda\rho}(x)). \tag{A.62}$$

Let $\varphi_i \in C^\infty(\Omega)$ be a cut-off function for the pair $(\Omega\backslash Q_\rho(x), \overline{Q}_{\lambda\rho}(x))$, i.e. $\varphi(x) \in [0,1]$ for all $x \in \Omega$, $\varphi(x) = 0$ for all $\Omega\backslash Q_\rho(x)$ and $\varphi(x) = 1$ for all $x \in \overline{Q}_{\lambda\rho}(x)$, such that

$$\|\nabla\varphi\|_{L^\infty} \leqslant \frac{4}{\rho(1-\lambda)}. \tag{A.63}$$

Let $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ be defined by

$$v := \varphi\frac{t}{s}u + (1-\varphi)\frac{t}{s}u_x = \varphi\frac{t}{s}(u-u_x) + \frac{t}{s}u_x.$$

Then $v - \frac{t}{s}u_x \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$ and

$$\nabla(sv) = \begin{cases} \nabla(tu) & \text{in } \overline{Q}_{\lambda\rho}(x) \\ (1-t)\frac{t}{1-t}\nabla\varphi\otimes(u-u_x) + t\big(\varphi\nabla u + (1-\varphi)\nabla u(x)\big) & \text{in } Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x). \end{cases}$$

As $\frac{t}{s}w - tu \in W_0^{1,p}(Q_{\lambda\rho}(x);\mathbb{R}^m)$ we have $sv + (\frac{t}{s}w - tu) - tu_x \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$, and so

$$\begin{aligned} \mathrm{m}^\varepsilon_{tu_x}(Q_\rho(x)) &\leqslant \int_{Q_\rho(x)} f_\varepsilon\left(y, \nabla(sv) + \nabla\Big(\frac{t}{s}w - tu\Big)\right)dy \\ &= \int_{Q_{\lambda\rho}(x)} f_\varepsilon\left(y, \frac{t}{s}\nabla w\right)dx + \int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} f_\varepsilon(y,\nabla(sv))dy. \end{aligned}$$

Taking (A.62), (C_2) and the right inequality in (C_6) into account, we deduce that

$$\begin{aligned} \frac{\mathrm{m}^\varepsilon_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} &\leqslant \left(1+\Delta\left(\frac{t}{s}\right)\right)\left(\frac{\mathrm{m}^\varepsilon_{su}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} + \eta\right) + \frac{a}{\lambda^N}\Delta\left(\frac{t}{s}\right) \\ &\quad + C\left(\frac{1}{\lambda^N} - 1\right)(1 + g(\nabla u(x)) + \frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g(\nabla u(y))dy \\ &\quad + \frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g\left(\frac{t}{1-t}\nabla\varphi\otimes(u-u_x)\right)dy \end{aligned}$$

with $C := \beta + \beta\gamma + \beta\gamma^2$. Thus, taking (A.61) into account, as $\mathscr{L}^N(Q_\rho(x)) \geqslant \mathscr{L}^N(Q_{\lambda\rho}(x))$, we get

$$\begin{aligned} \frac{\mathrm{m}^\varepsilon_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} &\leqslant (1+\eta)\left(\frac{\mathrm{m}^\varepsilon_{su}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} + \eta\right) + \frac{a}{\lambda^N}\eta + C\left(\frac{1}{\lambda^N} - 1\right)(1 + g(\nabla u(x)) \\ &\quad + \frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g\left(\frac{t}{1-t}\nabla\varphi\otimes(u-u_x)\right)dy \\ &\quad + \frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g(\nabla u(y))dy. \end{aligned} \tag{A.64}$$

On the other hand, by (A.63), for $\mathscr{L}^N$-a.a. $y \in Q_\rho(x)\backslash Q_{\lambda\rho}(x)$, we have

$$\begin{aligned} \left|\frac{t}{1-t}\nabla\varphi(y)\otimes(u(y)-u_x(y))\right| &\leqslant \left|\frac{t}{1-t}\right|\|\nabla\varphi\|_{L^\infty}\|u-u_x\|_{L^\infty} \\ &\leqslant \frac{4t}{(1-t)(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty}. \end{aligned}$$

But, since $p > N$, $\lim_{\rho\to 0}\frac{1}{\rho}\|u-u_x\|_{L^\infty}=0$, hence there exists $\rho_0>0$ (which depends on t and λ) such that for $\mathscr{L}^N$-a.e. $y\in Q_\rho(x)\backslash Q_{\lambda\rho}(x)$ and every $\rho\in]0,\rho_0[$,

$$\left|\frac{t}{1-t}\nabla\varphi(y)\otimes(u(y)-u_x(y))\right|\leqslant r$$

with $r>0$ given by (C_2)–(C_3) (see Remark 3.5(ii)). Hence

$$\frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g\left(\frac{t}{1-t}\nabla\varphi\otimes(u-u_x)\right)dy\leqslant C\theta\left(\frac{1}{\lambda^N}-1\right), \tag{A.65}$$

where θ is defined by (A.15). Moreover, it easy to see that

$$\begin{aligned}\frac{C}{\lambda^N\rho^N}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} g(\nabla u(y))dy &\leqslant \frac{C}{\lambda^N}\fint_{Q_\rho(x)}\big|g(\nabla u(y))-g(\nabla u(x))\big|dy\\ &\quad +C\left(\frac{1}{\lambda^N}-1\right)g(\nabla u(x)).\end{aligned} \tag{A.66}$$

Taking (A.65) and (A.66) into account, from (A.64) we deduce that

$$\begin{aligned}\frac{\mathrm{m}^\varepsilon_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} &\leqslant (1+\eta)\left(\frac{\mathrm{m}^\varepsilon_{su}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))}+\eta\right)+\frac{a}{\lambda^N}\eta+C\left(\frac{1}{\lambda^N}-1\right)(1+2g(\nabla u(x))+\theta)\\ &\quad +\frac{C}{\lambda^N}\fint_{Q_\rho(x)}\big|g(\nabla u(y))-g(\nabla u(x))\big|dy.\end{aligned} \tag{A.67}$$

As $u\in\mathrm{dom}(G)$, i.e. $g(\nabla u(\cdot))\in L^1(\Omega)$, we can assert that

$$\lim_{\rho\to 0}\fint_{Q_\rho(x)}\big|g(\nabla u(y))-g(\nabla u(x))\big|dy=0. \tag{A.68}$$

Letting $\varepsilon\to 0$, $s\to 1^-$ and $\rho\to 0$ in (A.67) and using (A.68), we see that

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}^-_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} &\leqslant (1+\eta)\left(\overline{\lim_{\rho\to 0}}\frac{\mathscr{M}^-_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}+\eta\right)+\frac{a}{\lambda^N}\eta\\ &\quad +C\left(\frac{1}{\lambda^N}-1\right)(1+2g(\nabla u(x))+\theta).\end{aligned} \tag{A.69}$$

Letting $t\to 1^-$ and $\lambda\to 1^-$ in (A.69) we conclude that

$$\underline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}^-_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}\leqslant(1+\eta)\left(\lim_{\rho\to 0}\frac{\mathscr{M}^-_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}+\eta\right)+a\eta,$$

and (A.54) follows by letting $\eta\to 0$.

Substep 5-2: end of Step 5. Combining (A.48) with (A.54) and (A.51) with (A.55), for every $u\in\mathrm{dom}(G)$ and every $A\in\mathscr{O}(\Omega)$, we have:

$$\begin{cases}\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}F_\varepsilon(u,A)\geqslant\displaystyle\int_A\underline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}^-_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx\\ \Gamma\text{-}\overline{\lim}_{\varepsilon\to 0}F_\varepsilon(u,A)=\displaystyle\int_A\underline{\lim_{t\to 1^-}}\lim_{\rho\to 0}\frac{\mathrm{m}^+_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}dx.\end{cases} \tag{A.70}$$

On the other hand, it is easily seen that for $\mathscr{L}^N$-a.e. $x \in \Omega$, we have:

$$\underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \frac{\mathrm{m}^-_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \underline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x));$$
$$\underline{\lim}_{t\to 1^-} \lim_{\rho\to 0} \frac{\mathrm{m}^+_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \underline{\lim}_{t\to 1^-} \lim_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, t\nabla u(x)),$$

and (3.3) follows by taking $A = \Omega$ in (A.70).

Step 6: end of the proof. Let $u \in \mathrm{dom}(G)$. Then, $\nabla u(x) \in \mathbb{G}$ for $\mathscr{L}^N$-a.a. $x \in \Omega$. By using (C_8) it follows that $\mathscr{L}^N$-a.e. $x \in \Omega$,

$$\underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \underline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, \nabla u(x)) = \underline{\lim}_{t\to 1^-} \overline{\lim}_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho[f_\varepsilon](x, \nabla u(x)),$$

and (3.4) follows from (3.3). ■

Appendix B. Auxiliary results

B.1. **The De Giorgi-Letta lemma.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\mathscr{O}(\Omega)$ be the class of open subsets of Ω. The following result is due to De Giorgi and Letta (see [DGL77] and also [But89, Lemma 3.3.6 pp. 105]).

Lemma B.1. *Let $\mathcal{S} : \mathscr{O}(\Omega) \to [0, \infty]$ be an increasing set function, i.e. $\mathcal{S}(A) \leqslant \mathcal{S}(B)$ for all $A, B \in \mathscr{O}(\Omega)$ such $A \subset B$, satisfying the following four conditions*:

(i) $\mathcal{S}(\varnothing) = 0$;
(ii) *$\mathcal{S}$ is superadditive, i.e. $\mathcal{S}(A \cup B) \geqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathscr{O}(\Omega)$ such that $A \cap B = \varnothing$;*
(iii) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathscr{O}(\Omega)$;*
(iv) *there exists a finite Radon measure ν on Ω such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathscr{O}(\Omega)$.*

Then, $\mathcal{S}$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to ν.

B.2. **Integral representation of the Vitali envelope of a set function.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\mathscr{O}(\Omega)$ be the class of open subsets of Ω. For each $\delta > 0$ and each $A \in \mathscr{O}(\Omega)$, we denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i) := x_i +]-\frac{\rho_i}{2}, \frac{\rho_i}{2}[^N\}_{i\in I}$ of disjoint open cubes of A with $x_i \in A$ and $\rho_i \in]0, \delta[$ and such that $\mathscr{L}^N(A \setminus \cup_{i\in I} Q_i) = 0$ by $\mathscr{V}_\delta(A)$.

Definition B.2. Given $\mathcal{S} : \mathscr{O}(\Omega) \to [0, \infty]$, for each $\delta > 0$ we define $\mathcal{S}_\delta : \mathscr{O}(\Omega) \to [0, \infty]$ by

$$\mathcal{S}_\delta(A) := \inf \left\{ \sum_{i\in I} \mathcal{S}(Q_i) : \{Q_i\}_{i\in I} \in \mathscr{V}_\delta(A) \right\}.$$

By the Vitali envelope of $\mathcal{S}$ we call the set function $\overline{\mathcal{S}} : \mathscr{O}(\Omega) \to [-\infty, \infty]$ defined by

$$\overline{\mathcal{S}}(A) := \sup_{\delta>0} \mathcal{S}_\delta(A) = \lim_{\delta\to 0} \mathcal{S}_\delta(A).$$

The interest of Definition B.2 comes from the following integral representation result. (For a proof we refer to [AHM18, §3.3] or [AHCM17, §A.4].)

Theorem B.3. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0,\infty]$ be a set function satisfying the following two conditions*:

(i) *there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to $\mathscr{L}^N$ such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathcal{O}(\Omega)$;*
(ii) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$ for all $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathscr{L}^N(A \backslash (B \cup C)) = 0$.*

Then $\lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(\cdot))}{\mathscr{L}^N(Q_\rho(\cdot))} \in L^1(\Omega)$ and for every $A \in \mathcal{O}(\Omega)$, one has

$$\overline{\mathcal{S}}(A) = \int_A \lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} dx.$$

B.3. **A subadditive theorem.** Let $\mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$ be the class of all bounded open subsets of $\mathbb{R}^N$. We begin with the following definition.

Definition B.4. Let $\mathcal{S} : \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N) \to [0,\infty]$ be a set function.

(i) We say that $\mathcal{S}$ is subadditive if

$$\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$$

for all $A, B, C \in \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathscr{L}^N(A \backslash (B \cup C)) = 0$.
(ii) We say that $\mathcal{S}$ is $\mathbb{Z}^N$-invariant if

$$\mathcal{S}(A + z) = \mathcal{S}(A)$$

for all $A \in \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$ and all $z \in \mathbb{Z}^N$.

Let $\mathrm{Cub}(\mathbb{R}^N)$ be the class of all open cubes in $\mathbb{R}^N$. The following theorem is due to Akcoglu and Krengel (see [AK81] and also [LM02] and [AHM11, Theorem 3.11]).

Theorem B.5. *Let $\mathcal{S} : \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N) \to [0,\infty]$ be a subadditive and $\mathbb{Z}^N$-invariant set function for which there exists $C > 0$ such that for every $A \in \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$,*

$$\mathcal{S}(A) \leqslant C\mathscr{L}^N(A).$$

Then, for every $Q \in \mathrm{Cub}(\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\mathscr{L}^N\left(\frac{1}{\varepsilon}Q\right)} = \inf_{k\geqslant 1} \frac{\mathcal{S}(]0,k[^N)}{k^N}.$$

References

[AH10] Omar Anza Hafsa. On the integral representation of relaxed functionals with convex bounded constraints. *ESAIM Control Optim. Calc. Var.*, 16(1):37–57, 2010.

[AHCM17] Omar Anza Hafsa, Nicolas Clozeau, and Jean-Philippe Mandallena. Homogenization of nonconvex unbounded singular integrals. *Ann. Math. Blaise Pascal*, 24(2):135–193, 2017.

[AHM11] Omar Anza Hafsa and Jean-Philippe Mandallena. Homogenization of nonconvex integrals with convex growth. *J. Math. Pures Appl. (9)*, 96(2):167–189, 2011.

[AHM12] Omar Anza Hafsa and Jean-Philippe Mandallena. On the relaxation of unbounded multiple integrals. *arXiv:1207.2652*, 2012.

[AHM14] Omar Anza Hafsa and Jean-Philippe Mandallena. Radial representation of lower semicontinuous envelope. *Boll. Unione Mat. Ital.*, 7(1):1–18, 2014.

[AHM18] Omar Anza Hafsa and Jean-Philippe Mandallena. Relaxation of nonconvex unbounded integrals with general growth conditions in Cheeger-Sobolev spaces. *Bull. Sci. Math.*, 142:49–93, 2018.

[AHM21] Omar Anza Hafsa and Jean-Philippe Mandallena. Integral representation of unbounded variational functionals on sobolev spaces. *Ricerche di Matematica*, to appear, 2021.

[AHMZ15] Omar Anza Hafsa, Jean-Philippe Mandallena, and Hamdi Zorgati. Homogenization of unbounded integrals with quasiconvex growth. *Ann. Mat. Pura Appl. (4)*, 194(6):1619–1648, 2015.

[AK81] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.

[BD98] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998.

[Bra06] Andrea Braides. Chapter 2 a handbook of γ-convergence. volume 3 of *Handbook of Differential Equations: Stationary Partial Differential Equations*, pages 101–213. North-Holland, 2006.

[But89] Giuseppe Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations*, volume 207 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1989.

[CDA02] Luciano Carbone and Riccardo De Arcangelis. *Unbounded functionals in the calculus of variations*, volume 125 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002. Representation, relaxation, and homogenization.

[DG79] Ennio De Giorgi. Convergence problems for functionals and operators. In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pages 131–188. Pitagora, Bologna, 1979.

[DGL77] E. De Giorgi and G. Letta. Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(1):61–99, 1977.

[DM93] Gianni Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[LM02] Christian Licht and Gérard Michaille. Global-local subadditive ergodic theorems and application to homogenization in elasticity. *Ann. Math. Blaise Pascal*, 9(1):21–62, 2002.