Waves in a random medium: Endpoint Strichartz estimates and number estimates

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Abstract

In this article we reconsider the problem of the propagation of waves in a random medium in a kinetic regime. The final aim of this program would be the understanding of the conditions which allow to derive a kinetic or radiative transfer equation. Although it is not reached for the moment, accurate optimal and somehow surprising number estimates in the Fock space setting, which happen to be left invariant by the dynamics. Keel and Tao endpoint Strichartz estimates play a crucial role after being combined with Cauchy-Kowalevski type argument. Although the whole article is focussed on the simplest case of Schrödinger waves in a gaussian random potential of which the translation into a QFT problem is straightforward, several intermediate results are written in a general setting in order to be applied to other similar problems.

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1 Introduction

The asymptotic analysis or random homogenization of wave propagation in a random medium, in a kinetic or diffusive regime has motivated several works in the recent decades. It is not our purpose here to give an exhaustive list but we think essentially of two different approaches: the one initiated by G. Papanicolaou and coauthors (see e.g. [FGPS, Pap, RPK]) with a rather complete review by J. Garnier in [Gar] and the one proposed by L. Erdös, H.T. Yau and later with M. Salmhofer in [ErYa][EYS1][EYS2]. Those two approaches formulate their results in terms of a kinetic (or diffusive) evolution equation for some weak limit of scaled Wigner functions. The main difference between the two approaches can be summarized as follows : The first approach presented in [Gar] modeled on the problem of randomly layered media (see [FGPS]) focusses on space-time wave functions, by solving a space-time PDE (it can be a Schrödinger or a wave equation) with random coefficients but with a smooth and essentially deterministic right-hand side. With

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very strong assumptions on the right-hand side of the equation, essentially deterministic and smooth, a kinetic equation is written for the distributional weak limit of the Wigner function associated with the space-time wave function. The work of [ErYa][EYS1][EYS2] is concerned with Cauchy problems, at the quantum level for the Schrödinger equation and semiclassically at a classical level for a linear Boltzmann equation in [ErYa] or a heat equation in [EYS1][EYS2]. The strategy of this second approach consists after writing a Dyson expansion (the iteration of Duhamel's formula), in making an accurate combinatorial analysis of Feynman diagrams which label all the random interaction terms of the expanded Dyson series. This Dyson expansion technique was actually already used for a similar problem by H. Spohn in [Spo]. The final step which gives the asymptotic behaviour of the Wigner transform, essentially relies on the accurate control and expression of the remaining terms of the series by using stationary phase asymptotic expressions for the many oscillating integrals. The results of this second approach always require strong assumptions on the initial data at the initial time $t = 0$ and prove weak convergence results at the macroscopic time $t \neq 0$.

The main difficulty in this problem is concerned with the control of recollisions and especially the proof that the asymptotic evolution is Markovian, or given by some semigroup associated to a kinetic of heat equation, although the multiple scattering process of waves could destroy this markovian aspect. Depending on the asymptotic regime, the effective asymptotic evolution could be affected by some memory or non local in time effect. In the considered asymptotic problems, it must be checked that those memory effects vanish asymptotically. In the approach reviewed in [Gar] which is concerned with rather general random fields, this is proved by estimating higher moments. In the approach of [ErYa] the combinatorial accurate analysis of Feynman diagrams, is reminiscent of the accurate control of recollision terms by G. Gallavotti in [Gal] for the classical Lorentz gas problem (Wind tree model). Both approaches bring accurate information about a difficult problem in slightly different frameworks and with various range of applications.

However those results remain unsatifactory from the mathematical point of view and for the following reason: The dynamics of (quantum) waves is given by a semigroup (actually a unitary group when there is no dissipation) and the asymptotic kinetic or diffusive limits are also given by well defined (semi)-groups. In the Cauchy problem approach, one does not yet understand the dynamically stable class of initial data which makes the derivation of a classical kinetic or heat equation possible. Actually the results of [ErYa][EYS1][EYS2] are themselves puzzling because with very specific initial data at time $t = 0$, they prove the asymptotic expected behaviour at the macroscopic time $t \neq 0$. But this means that the time evolved quantum state at the macroscopic time $t/2 \neq 0$, enters in the class of admissible initial data for which the asymptotic evolution can be proved for a nonzero time interval (at the macroscopic scale). Such initial data do not enter in the very specfic class considered at time $t = 0$. In the space time approach reviewed in [Gar] the strong assumptions on the right-hand side compared with the weak convergence results of the wave function, have been considered in a negative way. Actually what is called "statistical stability" is shown to fail with rough data (see [Bal]). But no positive answer for a general class of random right-hand side seems to emerge. Although the two approaches are about slightly different problems, they seem related at least for some basic random processes on which we will focus in this article.
Our hope is that such an analysis about the propagation of random waves in a ran-

dom medium should lead to results relying on dynamically stable hypotheses. We are led in this direction by the strategy followed by the second author with Z. Ammari in [AmNi1][AmNi2] where they managed to give a general and robust class of initial data, dynamically stable, such that the quantum mean field dynamics can be followed.
About this very technical question a first attempt was tried by the first author in [Bre]. The idea was to exploit the link between gaussian random fields (and possibly other fields like the poissonian random fields) with quantum field theory. It rapidly appears that the asymptotic problem, of waves in a random medium in a gaussian random field in the kinetic regime, cannot be thought as an infinite semiclassical problem like the bosonic mean field problem. It has some similarities but the strength of the free wave propagator and the translation invariance lead to non quadratic and non "semiclassical" Wick quantized operators. For this reason the coherent state method presented in [Bre] led to an accurate Ansatz, only for $O(h^{1/2})$ macroscopic times, where $h > 0$ is the chosen small parameter, and the derivation of a linear Boltzmann equation was possible only by forcing the markovian nature of the asymptotic evolution by reinitializing on some intermediate time scale the random potential. It was not at all satisfactory. Actually the number estimates that we prove in this article confirm that a coherent state approach cannot work for those problems.

Another issue of this problem is the good understanding of the dispersive properties of the free wave propagator with the asymptotic behaviour of waves in a random medium. The different behaviours expected in small dimension, $d \leq 2$ for the Schrödinger equation in the kinetic regime compared to $d \geq 3$, are closely linked with the time integrability of the dispersion relation ($L^1 - L^\infty$ estimates). In the community of nonlinear PDE's, Strichartz estimates are known to be more robust and effective than the pointwise in time $L^1 - L^\infty$ estimate. With the endpoint Strichartz estimates proved by Keel and Tao in [KeTa], those inequalities are now well adapted for linear critical problems. This article shows that they actually lead to very accurate and somehow surprising "number estimates" with some non trivial consequences.

Before giving the outline of this text, let us point out some limitations and features of the present analysis:

- We are not yet able to derive a full kinetic equation, except if one makes some connection with the existing results of [ErYa]. The class of good initial data for which an asymptotic equation can be written is not yet identified.
- We work essentially with the Schrödinger equation in the presence of a gaussian random potential in the kinetic regime, as what we think to be the simplest, and richest model problem from the point of view of available structures.
- Once the two previous points are made clear, the interested reader will realize that several argument, especially the one making use of Strichartz estimates, have been written in a sufficiently general framework in order to be transposed in another framework.
- Some results like the possibility to define Wigner measures for all times, the localization in energy of the propagation phenomena, the class of potential corre-

sponding to the scale invariant potential for Strichartz estimates, definitely bring a partial but accurate information.

Our main results are about accurate number estimates, stated in Proposition 4.5 in a rather general abstract setting and in Theorem 5.1 for the case of our model problem of the Schrödinger equation with a gaussian translation invariant potential in the kinetic regime and dimension $d \geq 3$.

Outline of the article;

a) In Section 2 the link between gaussian Hilbert spaces and the bosonic Fock space is recalled and the equations in which we are interested are explicitely written.

b) In Section 3 the translation invariance is used in order to make appear in a crucial way the center of mass variable, with respect to the position of the field variable. The expression of the creation and annihiliation operators are given explicitely in the center of mass and relative variables and finally L^p-estimates are carefully checked for those creation and annihilation operators under the suitable assumptions on the potential.

c) Section 4 reviews the known results about endpoint Strichartz estimates, and gives consequences in connection with the L^p-estimate in the center of mass given in Section 3. Then a rather general fixed point is proved which combines endpoint Strichartz estimates with an adaptation of Cauchy-Kowalevski techniques.

d) In Section 5, the general assumptions of Section 4 are checked in the framework of the Schrödinger equation with a gaussian random field in the kinetic regime and ambient dimension $d \geq 3$.

e) Consequences and a priori information, for the asymptotic evolution of Wigner functions are given in Section 6, withouth computing them.

f) Finally various approximation or stability results are deduced as consequences of the general estimates proved in Sections 4, 5 and 6.

Before starting, be aware of the following assumed framework and conventions:

All our Hilbert spaces, real or complex, are separable. All measures are assumed sigma-finite. On a set $\mathscr{X}$ endowed with a sigma-set, a generic sigma-finite measure will be denoted $\mathbf{dx}$, while the normal calligraphy dx will be reserved for the Lebesgue measure on $\mathscr{X} = \mathbb{R}^d$. When $(\mathscr{X}, \mathbf{dx})$ and $(\mathscr{Y}, \mathbf{dy})$ are two sigma-finite measured spaces, the notation $L^p_x L^q_y$, $1 \leq p, q \leq +\infty$, is used for $L^p(\mathscr{X}, \mathbf{dx}; L^q(\mathscr{Y}, \mathbf{dy}))$. However a more general version of $L^p_x L^q_y$ will be introduced in Subsection 3.2.

2 Random fields and Fock space

2.1 Gaussian Hilbert space and random fields

Let $\mathscr{G}$ be the stochastic gaussian measure (see e.g. [Jan]) on the Lebesgue measured space $(\mathbb{R}^d, \mathscr{L}, dy)$. This defines a real Hilbert gaussian space indexed by $L^2(\mathbb{R}^d, dy; \mathbb{R})$ which is generated, as a Hilbert space, by the centered real gaussian variables $X_A \sim N(0, |A|)$, with A measurable set of $\mathbb{R}^d$ and $|A| = \int_A dy$. By Minlös theorem (see [Sim]) the space

$L^2(\Omega,\mathcal{G};\mathbb{R})$ which contains powers of those gaussian processes can be realized with $\Omega = \mathcal{S}'(\mathbb{R}^d, dy;\mathbb{R})$.
Complex valued elements $F \in L^2(\Omega,\mathcal{G};\mathbb{C})$ are written $F = \mathrm{Re}\,F + i\,\mathrm{Im}\,F$, $\mathrm{Re}\,F, \mathrm{Im}\,F \in L^2(\Omega,\mathcal{G};\mathbb{R})$ handled by the $\mathbb{R}$-linearity of the decomposition.
Once the complexification is fixed in this order (see [Jan] for an accurate description of various complex structures of gaussian measures), the chaos decomposition of elements in $F \in L^2(\Omega,\mathcal{G};\mathbb{C})$ can be written

$$F(\omega) = \bigoplus_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} F_n(y_1,\dots,y_n) : X_{y_1}\cdots X_{y_n} : \, dy_1\cdots dy_n, \tag{1}$$

where

- $F_n(y_{\sigma(1)},\dots,y_{\sigma(n)}) = F_n(y_1,\dots,y_n)$ for all $\sigma \in \mathfrak{S}_n$ and complex valued functions are treated by the $\mathbb{R}$-linearity of the decomposition $F_n = \mathrm{Re}(F_n) + i\,\mathrm{Im}\,F_n$;
- the above symmetry can be written $F_n = S_n F_n$ where S_n is the symmetrizing orthogonal projection on $L^2(\mathbb{R}^{dn}, dy_1\cdots dy_n;\mathbb{C})$ given by

$$(S_n F_n)(y_1,\dots,y_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} F_n(y_{\sigma(1)},\dots,y_{\sigma(n)}); \tag{2}$$

- the family $(X_y)_{y\in\mathbb{R}^d}$ is made of jointly gaussian real centered random fields such that $\mathbb{E}(X_y X_{y'}) = \delta(y-y')$, which actually means

$$\mathbb{E}[(\int_{\mathbb{R}^d} f(y)X_y\, dy)(\int_{\mathbb{R}^d} g(y')X_{y'}dy')] = \int_{\mathbb{R}^d} f(y)g(y)\, dy$$

  for all $f, g \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$;[1]
- products or Wick products of singular random variables X_{y_j}, $j = 1\dots J$, must be considered in their weak formulation as well;
- $: Y_1\cdots Y_n :$ stands for the Wick product of the random variables $Y_1,\dots,Y_n$;
- with the assumed symmetry of the F_n components,

$$\mathbb{E}(|F|^2) = \int_\Omega |F(\omega)|^2\, d\mathcal{G}(\omega) = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^{dn}} |F_n(y_1,\dots,y_n)|^2\, dy_1\cdots dy_n = \sum_{n=0}^{\infty} n!\|F_n\|^2_{L^2}. \tag{3}$$

A field is a random function of $x \in \mathbb{R}^d$ and we shall consider $F : \mathbb{R}^d \times \Omega \to \mathbb{C}$. A real gaussian centered translation invariant field can be written

$$\mathcal{V}(x,\omega) = \int_{\mathbb{R}^d} V(y-x)X_y\, dy.$$

An element $F \in L^2(\mathbb{R}^d_x \times \Omega, dx\otimes\mathcal{G};\mathbb{C})$ has the chaos decomposition

$$F(x,\omega) = \bigoplus_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} \tilde{F}_n(x,y_1,\dots,y_n) : X_{y_1}\cdots X_{y_n} : \, dy_1\cdots dy_n \tag{4}$$

$$= \bigoplus_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} F_n(x,y_1-x,\dots,y_n-x) : X_{y_1}\cdots X_{y_n} : \, dy_1\cdots dy_n \tag{5}$$

[1] We follow the general probabilistic convention which omits the ω argument with $X_y = X_y(\omega)$ e.g. in formula (1).

where $F_n(x,y_1,\dots,y_n)=\tilde F_n(x,y_1+x,\dots,y_n+x)$ shares the same symmetry in $(y_1,\dots,y_n)$ as $\tilde F_n$ and

$$\|F\|^2_{L^2(\mathbb{R}^d_x\times\Omega)}=\int_{\mathbb{R}^d}\mathbb{E}(|F(x,\cdot)|^2)\,dx=\sum_{n=0}^{\infty}n!\|\tilde F_n\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^{dn})}=\sum_{n=0}^{\infty}n!\|F_n\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^{dn})}\qquad(6)$$

Assumptions on the real potential function V will be specified later but we can already compute the product $\mathcal{V}(x,\omega)F(x,\omega)$ by making use of Wick formula (see e.g. [Jan]-Theorem 3.15)

$$X_y:X_{y_1}\cdots X_{y_n}:=:X_yX_{y_1}\cdots X_{y_n}:+\sum_{j=1}^{n}\delta(y-y_j):X_{y_1}\cdots X_{y_{j-1}}\underbrace{X_{y_j}}_{\text{removed}}X_{y_{j+1}}\cdots X_{y_n}:$$

which leads to the chaos decomposition of $\mathcal{V}(x,\omega)F(x,\omega)$ as

$$\begin{aligned}&\int_{\mathbb{R}^{d(n+1)}}\frac{1}{(n+1)!}\sum_{\sigma\in\mathfrak{S}_{n+1}}V(y_{\sigma(n+1)}-x)F_n(x,y_{\sigma(1)}-x,\dots,y_{\sigma(n)}-x)\;:X_{y_1}\cdots X_{y_{n+1}}:\,dy_1\cdots dy_{n+1}\\&+\int_{\mathbb{R}^{d(n-1)}}n\left[\int_{\mathbb{R}^d}V(y)F_n(x,y,y_1-x,\dots,y_{n-1}-x)\,dy\right]\;:X_{y_1}\cdots X_{y_{n-1}}:\,dy_1\cdots dy_{n-1}.\qquad(7)\end{aligned}$$

2.2 The Fock space presentation

The chaos decomposition (1) provides the isomorphism between $L^2(\Omega,\mathcal{G};\mathbb{C})$ and the bosonic Fock space

$$\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C}))=\bigoplus_{n=0}^{\infty}(L^2(\mathbb{R}^d,dy;\mathbb{C}))^{\odot n}$$

where for a (real or complex) Hilbert space $\mathfrak{h}$, $\mathfrak{h}^{\odot n}$ is the symmetric Hilbert completed tensor product, equal to $\mathbb{C}$ (or $\mathbb{R}$) for $n=0$, endowed with the norm such that

$$\|\varphi^{\otimes n}\|_{\mathfrak{h}^{\odot n}}=\|\varphi\|^n_{\mathfrak{h}}\quad,\quad\|f_n\|_{L^2(\mathbb{R}^d,dy;\mathbb{C})^{\odot n}}=\|f_n\|_{L^2(\mathbb{R}^{dn},dy_1\cdots dy_n;\mathbb{C})}.\qquad(8)$$

The above direct sum is also the Hilbert completed direct sum. Note that the Fock space norm (8) differs from the $\mathfrak{h}^{\odot n}$-norm chosen in [Jan] in adequation with Wick products by a factor $\sqrt{n!}$. The unitary operator from $L^2(\Omega,\mathcal{G};\mathbb{C})$ to $\Gamma(L^2(\mathbb{R}^d_y,dy;\mathbb{C}))$ is thus given by

$$F\mapsto\bigoplus_{n=0}^{\infty}f_n\quad,\quad f_n=\sqrt{n!}F_n,$$

since

$$\|F\|^2_{L^2(\Omega,\mathcal{G};\mathbb{C})}=\sum_{n=0}^{\infty}n!\|F_n\|^2_{L^2(\mathbb{R}^{dn},dy_1\cdots dy_n;\mathbb{C})}=\sum_{n=0}^{\infty}\|f_n\|^2_{L^2(\mathbb{R}^d,dy;\mathbb{C})^{\odot n}}.$$

The Fock space $\Gamma(\mathfrak{h})$ is endowed with densely defined Wick-quantized operators. For a monomial symbol $b(z)=\langle z^{\otimes q},\tilde b z^{\otimes p}\rangle$ with $\tilde b\in\mathscr{L}(\mathfrak{h}^{\otimes p};\mathfrak{h}^{\otimes q})$, the Wick quantization b^{Wick} is defined on $\bigoplus_{n\in\mathbb{N}}^{\mathrm{alg}}\mathfrak{h}^{\odot n}$ by

$$b^{\mathrm{Wick}}f_{n+p}=\frac{\sqrt{(n+p)!(n+q)!}}{n!}S_{n+q}(\tilde b\otimes\mathrm{Id}^{\otimes n})f_{n+p}$$

where $S_m:\mathfrak{h}^{\otimes m}\to\mathfrak{h}^{\odot m}$ is the symmetrizing orthogonal projection given by

$$S_m(g_1\otimes\cdots\otimes g_m)=\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}g_{\sigma(1)}\otimes\cdots\otimes g_{\sigma(m)}\qquad(9)$$

already introduced in (2).
Basic examples in our case $\mathfrak{h} = L^2(\mathbb{R}^d, dy; \mathbb{C})$ are given by

$$a(g) = (\langle g, z\rangle)^{\mathrm{Wick}}, \quad a(g)f_n(y_1,\dots,y_{n-1}) = \sqrt{n}\int_{\mathbb{R}^d} \overline{g(y)} f_n(y_1,\dots,y_{n-1},y)\, dy,$$

$$a^*(f) = (\langle z, f\rangle)^{\mathrm{Wick}}, \quad a^*(f)f_n(y_1,\dots,y_{n+1}) = \frac{\sqrt{n+1}}{(n+1)!}\sum_{\sigma\in\mathfrak{S}_{n+1}} f(y_{\sigma(1)}) f_n(y_{\sigma(2)},\dots,y_{\sigma(n+1)}),$$

$$\phi(V) = (\sqrt{2}\,\mathrm{Re}\,\langle V, z\rangle)^{\mathrm{Wick}}, \quad \phi(V) = \frac{1}{\sqrt{2}}[a(V) + a^*(V)],$$

$$d\Gamma(A) = (\langle z, Az\rangle)^{\mathrm{Wick}}, \quad d\Gamma(A) = \sum_{k=0}^{n-1} \mathrm{Id}^{\otimes k} \otimes A \otimes \mathrm{Id}^{\otimes n-1-k}.$$

with

$$[a(g), a^*(f)] = a(g)a^*(f) - a^*(f)a(g) = \langle g, f\rangle \mathrm{Id},$$

Remember also that more generally, if $(A, D(A))$ generates a strongly continuous semigroup of contractions e^{tA}, $t \geq 0$, then $\Gamma(e^{tA})f_n = [e^{tA}]^{\otimes n} f_n$ defines a strongly continuous semigroup of contractions $\Gamma(e^{tA})$ on $\Gamma(\mathfrak{h})$ with generator denoted by $(d\Gamma(A), D(d\Gamma(A)))$, which extends the above definition of $d\Gamma(A)$. In particular this makes sense for $A = -iB$ with $(B, D(B))$ self-adjoint on $\mathfrak{h}$ and $(d\Gamma(B), D(d\Gamma(B)))$ is a self-adjoint operator on $\Gamma(\mathfrak{h})$ when $(B, D(B))$ is self-adjoint on $\mathfrak{h}$.
According to (5)(6), random $L^2(\mathbb{R}^d, dx; \mathbb{C})$ functions $F(x,\omega)$ can be written as elements f of $L^2(\mathbb{R}^d, dx; \mathbb{C}) \otimes \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))$,

$$\begin{aligned} & F(x,\omega) \mapsto f(x, \cdot - x) = \bigoplus_{n\in\mathbb{N}} f_n(x, y_1 - x, \dots, y_n - x) \\ \text{with} \quad & f_n \in L^2_{\mathrm{sym}}(\mathbb{R}^d_x \times \mathbb{R}^{dn}, dx dy_1 \cdots dy_n; \mathbb{C}), \\ & \|F\|^2_{L^2(\mathbb{R}^d\times\Omega, dx\otimes\mathcal{G})} = \sum_{n=0}^{\infty} \|f_n\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^{dn}, dx dy_1\cdots dy_n)}, \end{aligned}$$

and where L^2_{sym} refers to the exchange symmetry in the y-variables.
When $V \in L^2(\mathbb{R}^d, dy; \mathbb{R})$ and $\mathcal{V}(x,\omega) = \int_{\mathbb{R}^d} V(y-x) X_y\, dy$, the Wick product formula (7) for $\mathcal{V}(x,\omega)F(x,\omega)$ is transformed into

$$\mathcal{V}(x,\omega)F(x,\omega) \mapsto [a(V) + a^*(V)]f(x, \cdot - x) = [\sqrt{2}\phi(V)f](x, \cdot - x). \tag{10}$$

With the notation $D_y = \frac{1}{i}\partial_y = \begin{pmatrix} \frac{1}{i}\partial_{y^1} \\ \vdots \\ \frac{1}{i}\partial_{y^d} \end{pmatrix}$ the operator $(x\cdot D_y, D(x\cdot D_y))$, with $x\cdot D_y = \sum_{k=1}^d x^k D_{y^k}$, is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^d, dy; \mathbb{C})$ for all $x \in \mathbb{R}^d$. This defines a strongly continuous unitary representation of the additive group $(\mathbb{R}^d_x, +)$ on $L^2(\mathbb{R}^d_x) \otimes \Gamma(L^2(\mathbb{R}^d_y))$ given by

$$e^{-ix\cdot d\Gamma(D_y)}\Big(\bigoplus_{n=0}^{\infty} f_n(x, y_1, \dots, y_n)\Big) = \bigoplus_{n=0}^{\infty} f_n(x, y_1 - x, \dots, y_n - x).$$

Therefore the above unitary correspondence $F(x,\omega) \mapsto f(x, \cdot - x)$ gives a unitary correspondence

$$F \in L^2(\mathbb{R}^d \times \Omega, dx \otimes \mathcal{G}; \mathbb{C}) \mapsto f \in L^2(\mathbb{R}^d, dx; \mathbb{C}) \otimes \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C})), \tag{11}$$

while (10) becomes for $V \in L^2(\mathbb{R}^d, dy; \mathbb{R})$

$$\mathcal{V}F \mapsto \Big[\sqrt{2}\Phi(V)f\Big]. \tag{12}$$

We now translate a general pseudo-differential operator in the x-variable, $a^{\mathrm{Weyl}}(x,D_x)\otimes \mathrm{Id}_{L^2(\Omega,\mathcal{G};\mathbb{C})}$ under the above transformation (11).
When $\mathfrak{h}$ is a complex Hilbert space, we recall that $L^2(\mathbb{R}^d,dx;\mathbb{C})\otimes\mathfrak{h}$ equals $L^2(\mathbb{R}^d,dx;\mathfrak{h})$ and

- the Fourier transform, with the normalization

$$Fu(\xi)=\int_{\mathbb{R}^d} e^{-i\xi\cdot x}u(x)\,dx,\quad F^{-1}v(x)=\int_{\mathbb{R}^d} e^{ix\cdot\xi}v(\xi)\,\frac{d\xi}{(2\pi)^d},$$

  is unitary from $L^2(\mathbb{R}^d,dx;\mathfrak{h})$ to $L^2(\mathbb{R}^d,\frac{d\xi}{(2\pi)^d};\mathfrak{h})$;

- $\mathscr{S}(\mathbb{R}^d;\mathfrak{h})$, $\mathscr{S}'(\mathbb{R}^d;\mathfrak{h})$ and the Fourier transform have the same properties as in the scalar case $\mathfrak{h}=\mathbb{C}$.

Be aware that the behavior of the Fourier transform when $\mathfrak{h}$ is a general Banach space is more tricky according to [Pee]. So when $\mathfrak{h}$ is a Hilbert space, we consider pseudo-differential operators in the x-variable of the form $a^{\mathrm{Weyl}}(x,D_x)=a^{\mathrm{Weyl}}(x,D_x)\otimes\mathrm{Id}_{\mathfrak{h}}$ for a symbol $a\in\mathscr{S}'(\mathbb{R}^{2d}_{x,\xi};\mathbb{C})$ given by its Schwartz' kernel

$$[a^{\mathrm{Weyl}}(x,D_x)](x,y)=\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a\Big(\frac{x+y}{2},\xi\Big)\,\frac{d\xi}{(2\pi)^d}.$$

When $\mathfrak{h}=\mathbb{C}$, $a^{\mathrm{Weyl}}(x,D_x)$ is a continuous endomorphism of $\mathscr{S}(\mathbb{R}^d_x;\mathbb{C})$ and $\mathscr{S}'(\mathbb{R}^d;\mathbb{C})$ with the formal adjoint $\overline{a}^{\mathrm{Weyl}}(x,D_x)$ and the alternative representations:

- When $v,u\in\mathscr{S}(\mathbb{R}^d;\mathbb{C})$,

$$\langle v\,,a^{\mathrm{Weyl}}(x,D_x)u\rangle=\int_{\mathbb{R}^{2d}} a(x,\xi)W[v,u](x,\xi)\,\frac{dx\,d\xi}{(2\pi)^d}$$

  where $W[v,u]$ is the Wigner function of the pair $[v,u]$ (or the Weyl symbol of $|u\rangle\langle v|$), given by

$$W[v,u](x,\xi)=\int_{\mathbb{R}^d} e^{i\xi\cdot s}\,u(x+\frac{s}{2})\,\overline{v}(x-\frac{s}{2})\,ds,$$

  and which belongs to $\mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$.

- By setting $[\![P,X]\!]=p_\xi\cdot x-p_x\cdot\xi$ for $P=(p_x,p_\xi)$, $X=(x,\xi)$ in $\mathbb{R}^{2d}=T^*\mathbb{R}^d$, and

$$\mathscr{F}a(P)=\int_{\mathbb{R}^{2d}} e^{i[\![P,X]\!]}a(X)\,\frac{dX}{(2\pi)^d}$$

  we have $a=\mathscr{F}(\mathscr{F}a)$ in $\mathscr{S}'(\mathbb{R}^{2d})$. When $\mathscr{F}a\in L^1(\mathbb{R}^{2d};\mathbb{C})$,

$$a^{\mathrm{Weyl}}(x,D_x)=\int_{\mathbb{R}^{2d}}\mathscr{F}a(P)\,\tau_P\,\frac{dP}{(2\pi)^d},$$

  where $\tau_P=e^{i(p_\xi\cdot x-p_x\cdot D_x)}=[e^{i(p_\xi\cdot x-p_x\cdot\xi)}]^{\mathrm{Weyl}}(x,D_x)$ is the unitary phase translation

$$\tau_P\,u\,(x)=e^{ip_\xi\cdot(x-p_x/2)}u(x-p_x).$$

  In particular, the above integral is a $\mathscr{L}(L^2(\mathbb{R}^d,dx;\mathbb{C}))$-integral when $\mathscr{F}a\in L^1(\mathbb{R}^{2d},dP;\mathbb{C})$ and a fortiori when $a\in\mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$.

With those two remarks, for a general $a \in \mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$ the integral

$$a^{\mathrm{Weyl}}(x,D_x) = \int_{\mathbb{R}^{2d}} \mathscr{F}a(P)\, \underbrace{e^{i(p_\xi\cdot x - p_x\cdot D_x)}}_{=\tau_P} \frac{dP}{(2\pi)^d}$$

can be interpreted as the weak limit

$$a^{\mathrm{Weyl}}(x,D_x) = \operatorname*{w-lim}_{n\to\infty} \int_{\mathbb{R}^{2d}} \mathscr{F}a_n(P)\, e^{i(p_\xi\cdot x - p_x\cdot D_x)} \frac{dP}{(2\pi)^d},$$

where $a_n \in \mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$ is any approximation of $a \in \mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$.
While considering the $a^{\mathrm{Weyl}}(x,D_x)\otimes \mathrm{Id}_{\mathfrak{h}}$, the same construction makes sense after noticing that for $u,v \in \mathscr{S}(\mathbb{R}^d;\mathfrak{h})$, the Wigner transform $W[v,u]$ belongs to $\mathscr{S}(\mathbb{R}^{2d}_{x,\xi};\mathscr{L}^1(\mathfrak{h}))$[2] and

$$\langle v\,, a^{\mathrm{Weyl}}(x,D_x)u\rangle = \mathrm{Tr}\left[[a^{\mathrm{Weyl}}(x,D_x)\otimes \mathrm{Id}_{\mathfrak{h}}]\,|u\rangle\langle v|\right] = \int_{\mathbb{R}^{2d}} a(x,\xi)\mathrm{Tr}[W[v,u]](x,\xi)\,\frac{dxd\xi}{(2\pi)^d}.$$

We apply this with $\mathfrak{h} = L^2(\Omega,\mathscr{G};\mathbb{C})$ and $\mathfrak{h} = \Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C}))$: We start from

$$a^{\mathrm{Weyl}}(x,D_x) = a^{\mathrm{Weyl}}(x,D_x)\otimes \mathrm{Id}_{L^2(\Omega,\mathscr{G};\mathbb{C})} = \operatorname*{w-lim}_{n\to\infty} \int_{\mathbb{R}^{2d}} \mathscr{F}a_n(P) e^{i(p_\xi\cdot x - p_x\cdot D_x)} \frac{dP}{(2\pi)^d},$$

the correspondance

$$a^{\mathrm{Weyl}}(x,D_x)F \mapsto e^{ix\cdot d\Gamma(D_y)} a^{\mathrm{Weyl}}(x,D_x) e^{-ix\cdot d\Gamma(D_y)} f,$$

and

$$e^{ix\cdot\lambda} e^{i(p_\xi\cdot x - p_x\cdot D_x)}(e^{-ix\cdot\lambda}\times) = e^{i(p_x\cdot x - p_x\cdot(D_x-\lambda))} \quad \text{for all } \lambda\in\mathbb{R}^d$$

which gives by the functional calculus, the equality of unitary operators

$$e^{ix\cdot d\Gamma(D_y)} e^{i(p_\xi\cdot x - p_x\cdot D_x)}(e^{-ix\cdot d\Gamma(D_y)}) = e^{i(p_x\cdot x - p_x\cdot(D_x - d\Gamma(D_y)))}.$$

We deduce that for $a \in \mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$, $a^{\mathrm{Weyl}}(x,D_x)F \in \mathscr{S}'(\mathbb{R}^d_x;L^2(\Omega,\mathscr{G};\mathbb{C}))$ is transformed into

$$a^{\mathrm{Weyl}}(x,D_x)F \mapsto a^{\mathrm{Weyl}}(x,D_x - d\Gamma(D_y))f \in \mathscr{S}'(\mathbb{R}^d;\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C}))). \tag{13}$$

with

$$a^{\mathrm{Weyl}}(x,D_x - d\Gamma(D_y)) = \operatorname*{w-lim}_{n\to\infty} \int_{\mathbb{R}^{2d}} \mathscr{F}a_n(P) e^{i(p_\xi\cdot x - p_x\cdot(D_x - d\Gamma(D_y)))} \frac{dP}{(2\pi)^d}.$$

Let us continue by applying the Fourier transform in the x-variable with

$$F_x u(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\, dx \quad , \quad F_x^{-1}u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} u(\xi)\,\frac{d\xi}{(2\pi)^d}$$

and set for $f \in \mathscr{S}'(\mathbb{R}^d_x;\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C})))$

$$\hat f = F_x f \in \mathscr{S}'(\mathbb{R}^d;\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C}))).$$

With

$$F_x\, a^{\mathrm{Weyl}}(x,D_x) F_x^{-1} = a^{\mathrm{Weyl}}(-D_\xi,\xi)$$

[2] $\mathscr{L}^p(\mathfrak{h})$ denotes the Schatten space of compact operators for $1 \le p \le +\infty$.

where the functional calculus leads to $F_x\, a^{\mathrm{Weyl}}(x, D_x - d\Gamma(D_y))F_x^{-1} = a^{\mathrm{Weyl}}(-D_\xi, \xi - d\Gamma(D_y))$, we obtain the unitary correspondence

$$F \in L^2(\mathbb{R}^d \times \Omega, dx \otimes \mathscr{G}; \mathbb{C}) \mapsto \hat{f} = F_x f \in L^2(\mathbb{R}^d, \frac{d\xi}{(2\pi)^d}; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))), \tag{14}$$

$$\text{with} \qquad F(x,\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} \frac{1}{\sqrt{n!}} f_n(x, y_1 - x, \ldots, y_n - x) : X_{y_1} \cdots X_{y_n} : \, dy_1 \cdots dy_n \,, \tag{15}$$

and where (12) and (13) become

$$\mathscr{V} F \mapsto \sqrt{2}\phi(V)\hat{f}\,, \tag{16}$$

$$\underbrace{a^{\mathrm{Weyl}}(x, D_x)F}_{\in \mathscr{S}'(\mathbb{R}^d_\xi; L^2(\Omega, \mathscr{G}; \mathbb{C}))} \mapsto \underbrace{a^{\mathrm{Weyl}}(-D_\xi, \xi - d\Gamma(D_y))\hat{f}}_{\in \mathscr{S}'(\mathbb{R}^d_\xi; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C})))}\,. \tag{17}$$

From this point of view, the Fock space and functional analysis presentation is simpler than sticking with the usual chaos decomposition (4) where Fourier transforms and pseudo-differential operators do not seem to have simple probabilistic interpretation.

Remark 2.1. *As a final remark, all the above constructions can be tensorized with an additional separable Hilbert space* $\mathfrak{h}' = L^2(Z, \mathbf{dz}; \mathbb{C})$.

2.3 Our problem

We aim at studying the stochastic partial differential equation

$$\begin{cases} i\partial_t F = -\Delta_x F + \sqrt{h}\mathscr{V} F\,, \\ F(t=0) = F_0\,, \end{cases} \tag{18}$$

where

- $\mathscr{V}$ is the translation invariant gaussian random field

$$\mathscr{V}(x,\omega) = \int_{\mathbb{R}^d} V(y-x) X_y \, dy\,,$$

  with $V \in L^2(\mathbb{R}^d; \mathbb{R})$;

- the solution $F(t,x,\omega,z)$ is seeked in $\mathscr{C}^0(\mathbb{R}; L^2(\mathbb{R}^d \times \Omega \times Z, dx \otimes \mathscr{G} \otimes \mathbf{dz}; \mathbb{C}))$;

- $h > 0$ is a small parameter which will tend to 0.

In particular we will consider the asymptotic behavior of quantities

$$\langle F(\frac{t}{h}), a^{\mathrm{Weyl}}(hx, D_x)F(\frac{t}{h})\rangle_{L^2(\mathbb{R}^d \times \Omega \times Z)} = \int_Z \mathbb{E}\left[\langle F(\frac{t}{h}, z), a^{\mathrm{Weyl}}(hx, D_x)F(\frac{t}{h}, z)\rangle_{L^2(\mathbb{R}^d, dx)}\right] \mathbf{dz}(z) \tag{19}$$

for $a \in S(1, dx^2 + d\xi^2)$ and $t \in [0,T]$. Remember that the symbol class $S(1, dx^2 + d\xi^2)$ is the set of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{2d}$ with all derivatives bounded on $\mathbb{R}^{2d}$.
Note that the variable $z \in Z$ does not appear in the equation. The dynamics is thus well defined when it is defined for $Z = \{z_0\}$ and $\mathbf{dz} = \delta_{z_0}$. A sufficient condition was provided in [Bre] by making use of Nelson commutator method.

Lemma 2.2. ***Proposition 4.4 in [Bre]:*** *Assume $V \in H^2(\mathbb{R}^d;\mathbb{R})$ then the operator $-\Delta_x + \sqrt{h}V$ is essentially self-adjoint on $\bigoplus_{n\in\mathbb{N}}^{\rm alg} \mathscr{S}(\mathbb{R}^d_x;(L^2(\mathbb{R}^d,dy;\mathbb{C}))^{\odot n})$ which is a dense subset of $L^2(\mathbb{R}^d_x,dx;L^2(\Omega,\mathscr{G};\mathbb{C})) = L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C})$ by (4).*

Remark 2.3. *A side corollary of our analysis says that the dynamics is well defined under the assumption $V \in L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})$ with $r'_\sigma = \frac{2d}{d+2}$ in dimension $d \geq 3$, See Subsection 7.4 at the end of the article.*

Lemma 2.2 provides a natural self-adjoint realization of $-\Delta_x + \sqrt{h}V$ in $\mathfrak{h} = L^2(\mathbb{R}^d\times\Omega\times Z, dx\otimes\mathscr{G}\otimes\mathbf{dz};\mathbb{C})$ and any initial datum $F_0 \in \mathfrak{h}$ defines a unique solution $F \in \mathscr{C}^0(\mathbb{R};\mathfrak{h})$.
There are various reasons for introducing an additional variable $z \in Z$, and this trick will be used repeatedly. One of them is the following: Starting with $Z = \{z_0\}$ and $\mathbf{dz} = \delta_{z_0}$, one may consider instead of $F(\frac{t}{h}) = U_V(\frac{t}{h})F_0$ with $U_V(t) = e^{-it(-\Delta_x+\sqrt{h}V)}$, the evolution of a state

$$\varrho(\frac{t}{h}) = U_V(\frac{t}{h})\varrho_0 U_V^*(\frac{t}{h})$$

with $\varrho_0 \in \mathscr{L}^1(L^2(\mathbb{R}^d\times\Omega;\mathbb{C}))$, $\varrho_0 \geq 0$, $\mathrm{Tr}[\varrho_0] = 1$ possibly replacing $\|F_0\|_{L^2} = 1$. By writing $\varrho_0 = \varrho_0^{1/2}\varrho_0^{1/2}$ one gets

$$\varrho(\frac{t}{h}) = [U_V(\frac{t}{h})\varrho_0^{1/2}][U_V(\frac{t}{h})\varrho_0^{1/2}]^*$$

where $F(t) = U_V(t)\varrho_0^{1/2}$ is the solution to (18) in

$$\mathscr{L}^2(L^2(\mathbb{R}^d\times\Omega,dx\times\mathscr{G};\mathbb{C})) \simeq L^2(\mathbb{R}^d\times\Omega\times Z, dx\otimes\mathscr{G}\otimes\mathbf{dz})$$
$$\text{with}\quad Z = \mathbb{R}^d\otimes\Omega,\quad \mathbf{dz} = dx\otimes\mathscr{G},$$

while the trace to be computed at time $\frac{t}{h}$ equals

$$\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x)\varrho(\frac{t}{h})\right] = \int_Z \mathbb{E}\left[\langle F(\frac{t}{h},z), a^{\mathrm{Weyl}}(hx,D_x)F(\frac{t}{h},z)\rangle_{L^2(\mathbb{R}^d,dx)}\right]\mathbf{dz}(z).$$

Thus considering the evolution of non negative trace class operators instead of projectors on wave functions, becomes the same problem by introducing the suitable additional parameter $z \in Z$.
The unitary correspondence (14)(15), with (16)(17) and Remark 2.1, transforms the dynamics (18) into

$$\begin{cases} i\partial_t\hat{f} = (\xi - d\Gamma(D_y))^2\hat{f} + \sqrt{2h}\phi(V)\hat{f}, \\ \hat{f}(t=0) = \hat{f}_0, \end{cases} \tag{20}$$

and the quantity (19) into

$$\langle\hat{f}(\frac{t}{h}), a^{\mathrm{Weyl}}(-hD_\xi,\xi - d\Gamma(D_y))\hat{f}(\frac{t}{h})\rangle_{L^2(\mathbb{R}^d\times\Omega\times Z,\frac{d\xi}{(2\pi)^d}\otimes\mathscr{G}\otimes\mathbf{dz})}. \tag{21}$$

We will see that the variable $\xi \in \mathbb{R}^d$ and even some part Y' of the variable $Y = (y_1,\dots,y_n)$, when the total number is fixed to n, can be taken as another parameter like $z \in Z$ for some points of the analysis. This leads to a parameter z'-dependent, $z' = (\xi, Y', z) \in \mathbb{R}^d\times\mathbb{R}^{dn'}\times Z$, analysis in $L^2(\mathbb{R}^{d(n-n')},dY'')$. Those parameters appear in Section 3 by introducing the center of mass $Y'' = y_G = \frac{y_1+\cdots+y_n}{n}$ and the relative coordinates $y'_j = y_j - y_G$, a general functional framework for parameter dependent Strichartz estimates and their consequences are presented in Section 4 and finally those are detailled in Section 5 for (20).

3 The Fock space and the center of mass

According to (20) our stochastic dynamics has been translated in a parameter dependent dynamics in the Fock space. We shall consider an additional unitary transform using the center of mass and the relative variables

$$y_G^n = \frac{y_1 + \cdots + y_n}{n}, \quad y_j' = y_j - y_G^n$$

in the n-particles sector, $n \geq 1$. It trivializes the free dynamics when $\mathcal{V} \equiv 0$ or $V \equiv 0$. The expression of the interaction term $\sqrt{2h}\phi(V)$ becomes more tricky but various general estimates are given here.

3.1 The unitary transform associated with the center of mass

We shall use the following notations for $n \geq 1$:

- A generic element of $\mathbb{R}^{dn}$ will be written

$$Y_n = (y_1, \ldots, y_n) \quad \text{with} \quad |Y_n|^2 = \sum_{j=1}^n |y_j|^2. \tag{22}$$

- The center of mass of $Y_n \in \mathbb{R}^{dn}$ will be written

$$y_G = y_G^n = \frac{y_1 + \cdots + y_n}{n} \tag{23}$$

  and the relative coordinates $y_j' = y_j - y_G^n$ will be gathered into

$$Y_n' = (y_1', \ldots, y_n') = (y_1 - y_G^n, \ldots, y_n - y_G^n). \tag{24}$$

  The vector Y_n' actually belongs to the subspace $\mathscr{R}^n = \left\{ Y_n \in \mathbb{R}^{dn}, \sum_{j=1}^n y_j = 0 \right\}$ and we recall

$$|Y_n|^2 = n|y_G^n|^2 + \left|Y_n'\right|^2 = n|y_G^n|^2 + \sum_{j=1}^n |y_j'|^2. \tag{25}$$

With those notations the map $\mathbb{R}^{dn} \ni Y_n \mapsto (y_G^n, Y_n') \in \mathbb{R}^d \times \mathscr{R}^n \subset \mathbb{R}^d \times \mathbb{R}^{dn}$ is a measurable map and the image measure of the Lebesgue measure $|dY_n| = \prod_{j=1}^n |dy_j|$ is nothing but

$$dy_G \otimes d\mu_n(Y_n') = dy_G \otimes [n^d dy_1 \cdots dy_n \delta_0(y_1 + \cdots + y_n)]. \tag{26}$$

For $n \geq 2$ we can write $d\mu_n(Y_n') = n^d \prod_{j \neq j_0} dy_j'$ for any fixed $j_0 \in \{1, \ldots, n\}$ by taking the linear coordinates $(y_j')_{j \neq j_0}$ on $\mathscr{R}^n$ where $y_{j_0}' = -\sum_{j \neq j_0} y_j'$. For $n = 1$, $\mathscr{R}^1 = \{0\}$ and integrating with respect to $Y_1' = y_1' \in \mathscr{R}^1$ is nothing but the evaluation at $y_1' = 0$.

Definition 3.1. *On $\sqcup_{n=1}^\infty \mathbb{R}^{dn}$ the measure μ carried by $\mathscr{R} = \sqcup_{n=1}^\infty \mathscr{R}^n$ is defined by*

$$\forall g_n \in \mathscr{C}_c^0(\mathbb{R}^{dn}), \quad \int_{\mathscr{R}^n} g_n(Y')\, d\mu_n(Y') = \int_{\mathbb{R}^{dn}} g_n(y_1, \ldots, y_n) \delta_0(y_1 + \cdots + y_n) n^d \, dy_1 \cdots dy_n$$
$$\overset{n \geq 2}{=} \int_{\mathbb{R}^{d(n-1)}} g_n(y_1', \ldots, y_{n-1}', -\sum_{j=1}^{n-1} y_j') n^d \, dy_1' \cdots dy_{n-1}'.$$

For $1 \leq p < +\infty$, the space $L^p(\mathscr{R}, d\mu)$ is the direct sum $\bigoplus_{n=1}^\infty L^p(\mathscr{R}^n, d\mu_n)$ completed with respect to the norm $\| \bigoplus_{n=1}^\infty g_n \|_{L^p} = \left(\sum_{n=1}^\infty \| g_n \|_{L^p(\mathscr{R}^n, d\mu_n)}^p \right)^{1/p}$. The closed subspace of symmetric functions, $g_n(y_{\sigma(1)}', \ldots, y_{\sigma(n)}') = g_n(y_1', \ldots, y_n')$ for all $\sigma \in \mathfrak{S}_n$ and for all $n \geq 1$, is then denoted by $L^p_{\mathrm{sym}}(\mathscr{R}, d\mu(Y'))$.

For $g_n \in L^2(\mathbb{R}^{dn} \times Z, dY^n \otimes \mathbf{dz}; \mathbb{C})$, $n \geq 1$, the function

$$g_{G,n}(y_G, Y'_n, z) = U_G g_n(y_G, Y'_n, z) = g_n(y_G + Y'_n, z) \tag{27}$$

belongs to $L^2(\mathbb{R}^d \times \mathscr{R}^n \times Z, dy_G \otimes d\mu_n \otimes \mathbf{dz}; \mathbb{C})$ with

$$\|U_G g_n\|_{L^2(\mathbb{R}^d \times \mathscr{R}^n \times Z, dy_G \otimes d\mu_n \otimes \mathbf{dz})} = \|g_n\|_{L^2(\mathbb{R}^{dn} \times Z, dY_n \otimes \mathbf{dz})}$$

$$\text{and} \qquad g_n(Y_n, z) = (U_G^{-1} g_{G,n})(Y_n, z) = g_{G,n}(y_G^n, Y_n - y_G^n, z).$$

Additionally $U_G : L^2(\mathbb{R}^d, dy)^{\odot n} \mapsto L^2(\mathbb{R}^d, dy_G; L^2_{\mathrm{sym}}(\mathscr{R}^n, d\mu_n)) = L^2_{\mathrm{sym}}(\mathscr{R}^n, d\mu_n; L^2(\mathbb{R}^d, dy_G))$ is unitary and the same result holds for the parameter $z \in Z$ version.

Proposition 3.2. *The map U_G extended by $U_G g_0(z) = g_0(z)$ for $n = 0$, defines a unitary map*

$$U_G : L^2(Z, dz; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))) \to L^2(Z; \mathbb{C}) \oplus L^2_{\mathrm{sym}}(Z \times \mathscr{R}, dz \otimes d\mu; L^2(\mathbb{R}^d, dy_G; \mathbb{C})). \tag{28}$$

When $d\Gamma_G(A) = U_G[d\Gamma(A) \otimes \mathrm{Id}_{L^2(Z,dz)}]U_G^{-1}$ for a self-adjoint operator $(A, D(A))$ in $L^2(\mathbb{R}^d_y, dy)$, the case $A = D_y$ gives

$$d\Gamma_G(D_y) = U_G\, d\Gamma(D_y) U_G^{-1} = D_{y_G}. \tag{29}$$

For any bounded measurable function ϕ on $\mathscr{R} \times Z$ the multiplication by $\phi(Y', z)\big|_{\mathscr{R}^n} = \phi_n(Y'_n, z)$ for $n \geq 1$, while $\phi_0 : Z \to \mathbb{C}$, commutes with $d\Gamma_G(D_y) = D_{y_G}$ according to

$$\forall t \in \mathbb{R}^d, \forall u \in L^2(Z, \mathbf{dz}; \mathbb{C}) \oplus L^2_{\mathrm{sym}}(Z \times \mathscr{R}, \mathbf{dz} \otimes d\mu; L^2(\mathbb{R}^d, dy_G, \mathbb{C})) \quad e^{it \cdot D_{y_G}}(\phi u) = \phi(e^{it \cdot D_{y_G}} u).$$

A particular case is $\phi_n(Y'_n, z) = \varphi(n)$ for a bounded function $\varphi : \mathbb{N} \to \mathbb{C}$.

Proof. The unitarity of U_G comes at once from (27) and the componentwise unitarity already checked. For $d\Gamma_G(D_y) = D_{y_G}$, simply write

$$\partial_{y_G} g_{G,n}(y_G, Y'_n) = \partial_{y_G} g_n(y_G + y'_1, \ldots, y_G + y'_n) = \sum_{j=1}^n (\partial_{y_j} g_n)(y_G + y'_1, \ldots, y_G + y'_n).$$

The commutation statement comes from the separation of variables, y_G and (Y', z). □

Introducing the center of mass thus simplifies the free transport part of (20). It is not so for the interaction term $\sqrt{2h}\phi(V) = \sqrt{h}[a(V) + a^*(V)]$. An explicit and useful expression is nevertheless possible for

$$a_G(V) = U_G\, a(V) U_G^{-1} \quad \text{and} \quad a_G^*(V) = U_G\, a^*(V) U_G^{-1}. \tag{30}$$

Proposition 3.3. *The operator $a_G(V)$ and $a_G^*(V)$ for $V \in L^2(\mathbb{R}^d, dy; \mathbb{C})$ have the following action on $f_{G,n} \in L^2_{\mathrm{sym}}(\mathscr{R}^n \times Z, d\mu_n \otimes \mathbf{dz}; L^2(\mathbb{R}^d, dy_G; \mathbb{C}))$ for $n \geq 1$ and $f_{G,0} \in L^2(Z, \mathbf{dz}; \mathbb{C})$ where we omit the transparent variable $z \in Z$:*

$$a_G(V) f_{G,0} = 0, \quad [a_G(V) f_{G,1}] = \int_{\mathbb{R}^d} \overline{V(y_1)} f_{G,1}(y_1)\, dy_1, \tag{31}$$

$$\forall n > 1, \ [a_G(V) f_{G,n}](y_G, Y'_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \overline{V(y_G + y_n)} f_{G,n}(y_G + \frac{y_n}{n}, Y_n - \frac{y_n}{n})\, dy_n,$$

$$\text{with} \qquad Y_n = (y'_1, \ldots, y'_{n-1}, y_n) \in \mathbb{R}^{dn}, \quad Y'_{n-1} \in \mathscr{R}^{n-1}, \quad Y_n - \frac{y_n}{n} \in \mathscr{R}^n, \tag{32}$$

$$a_G^*(V) f_{G,0}(y_G) = V(y_G) f_{G,0}, \tag{33}$$

$$\forall n > 0, \qquad a_G^*(V) f_{G,n}(y_G, Y'_{n+1}) = \sqrt{n+1} S_{n+1}[V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n})],$$

$$\text{with} \qquad Y_n = (y'_1, \ldots, y'_n) \in \mathbb{R}^{dn}, \quad Y'_{n+1} \in \mathscr{R}^{n+1}, \quad Y_n + \frac{y'_{n+1}}{n} \in \mathscr{R}^n,$$

$$\text{and} \qquad S_{n+1}[v(y'_{n+1}) u(y'_1, \ldots, y'_{n+1})] = \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} v(y'_{\sigma(n+1)}) u(y'_{\sigma(1)}, \ldots, y'_{\sigma(n+1)}). \tag{34}$$

Proof. Write for $n>1$,

$$\begin{aligned}[a_G(V)f_{G,n}](y_G^{n-1},Y'_{n-1}) &= [a(V)U_G^{-1}f_{G,n}](Y'_{n-1}+y_G^{n-1})\\ &= \sqrt{n}\int_{\mathbb{R}^d}\overline{V(\tilde{y}_n)}[U_G^{-1}f_{G,n}](Y'_{n-1}+y_G^{n-1},\tilde{y}_n)\,d\tilde{y}_n\,.\end{aligned}$$

By setting $\tilde{y}_n = y_G^{n-1}+y_n$ the formula $(U_G^{-1}g_{G,n})(\cdot) = g_{G,n}(y_G^n,\cdot - y_G^n)$ with

$$y_G^n = \frac{y_1+\cdots+y_{n-1}+\tilde{y}_n}{n} = \frac{n-1}{n}y_G^{n-1}+\frac{\tilde{y}_n}{n} = y_G^{n-1}+\frac{y_n}{n}$$

leads to

$$\begin{aligned}[a_G(V)f_{G,n}](y_G^{n-1},Y'_{n-1}) &= \sqrt{n}\int_{\mathbb{R}^d}\overline{V(y_G^{n-1}+y_n)}f_{G,n}(y_G^{n-1}+\frac{y_n}{n},Y'_{n-1}-\frac{y_n}{n},y_n-\frac{y_n}{n})\,dy_n\\ &= \sqrt{n}\int_{\mathbb{R}^d}\overline{V(y_G^{n-1}+y_n)}f_{G,n}(y_G^{n-1}+\frac{y_n}{n},Y_n-\frac{y_n}{n})\,dy_n\end{aligned}$$

with $Y_n=(y'_1,\dots,y'_{n-1},y_n)$.
The computation of $a_G^*(V)f_{G,n}$ is done by duality:

$$\begin{aligned}\langle a_G^*(V)f_{G,n-1},g_{G,n}\rangle &= \langle f_{G,n-1},a_G(V)g_{G,n}\rangle\\ &= \int_{\mathbb{R}^d\times\mathscr{R}^{n-1}}\overline{f_{G,n-1}(y_G^{n-1},Y'_{n-1})}\times\\ &\quad\left[\sqrt{n}\int_{\mathbb{R}^d}\overline{V(y_G^{n-1}+y_n)}g_{G,n}(y_G^{n-1}+\frac{y_n}{n},Y_n-\frac{y_n}{n})\,dy_n\right]\,dy_G^{n-1}d\mu_{n-1}(Y'_{n-1}).\end{aligned}$$

Remember $Y_n=(y'_1,\dots,y'_{n-1},y_n)$ and $\tilde{Y}'_n = Y_n-\frac{y_n}{n}\in\mathscr{R}^n$. The change of variables

$$\tilde{Y}'_n = Y_n-\frac{y_n}{n},\quad y_G^n = y_G^{n-1}+\frac{y_n}{n},\quad y_G^{n-1}=y_G^n-\frac{\tilde{y}_n}{n-1},\quad Y'_{n-1}=\tilde{Y}_{n-1}+\frac{y_n}{n}=\tilde{Y}_{n-1}+\frac{\tilde{y}_n}{n-1},$$

with

$$\begin{aligned}dy_n dy_G^{n-1}d\mu^{n-1}(Y'_{n-1}) &= dy_G^{n-1}\delta_0(y'_1+\cdots+y'_{n-1})(n-1)^d dy'_1\cdots dy'_{n-1}\\ &= dy_G^n\frac{n^d}{(n-1)^d}(n-1)^d\delta_0(\tilde{y}'_1+\cdots+\tilde{y}'_n)d\tilde{y}'_1\cdots d\tilde{y}'_n\\ &= dy_G^n d\mu_n(\tilde{Y}'_n),\end{aligned}$$

gives

$$\begin{aligned}\langle a_G^*(V)f_{G,n-1},g_{G,n}\rangle = \sqrt{n}\int_{\mathbb{R}^d\times\mathscr{R}^n}&\overline{V(y_G^n+\tilde{y}'_n)f_{G,n-1}(y_G^n-\frac{\tilde{y}_n}{n-1},\tilde{Y}'_{n-1}+\frac{\tilde{y}'_n}{n-1})}\times\\ &g_{G,n}(y_G^n,\tilde{Y}'_n)\,dy_G^n d\mu_n(\tilde{Y}'_n).\end{aligned}$$

Replacing n by $n+1$, while remembering that $a_G^*(V)f_{G,n}$ is symmetric in the variables $(y'_1,\dots,y'_{n+1})$ yields

$$[a_G^*(V)f_{G,n}](y_G,Y'_{n+1}) = \sqrt{n+1}S_{n+1}[V(y_G+y'_{n+1})f_{G,n}(y_G-\frac{y'_{n+1}}{n},Y_n+\frac{y'_{n+1}}{n})]$$

with $Y_n=(y'_1,\dots,y'_n)$. □

3.2 General $L^p_x L^q_y$ spaces

When $(\mathscr{X},\mathbf{dx})$ and $(\mathscr{Y},\mathbf{dy})$ are sigma-finite measured spaces $L^p_x L^q_y$, $1\le p,q\le+\infty$, denotes the space $L^p_x L^q_y = L^p(\mathscr{X},\mathbf{dx};L^q(\mathscr{Y},\mathbf{dy}))$. This shortened notation is especially useful when estimates are written in those spaces, like in Strichartz estimates (see Section 4). However the final space of the unitary map U_G in (28) shows already that the product space $\mathscr{X}\times\mathscr{Y}$ is too restrictive. Below is a convenient generalization.

Definition 3.4. *Let $(\mathscr{X}_n,\mathbf{dx_n})_{n\in\mathscr{N}}$ and $(\mathscr{Y}_n,\mathbf{dy_n})_{n\in\mathscr{N}}$ be at most countable families ($\mathscr{N}\subset\mathbb{N}$) of sigma-finite measured spaces. Let $\mathscr{X}=\sqcup_{n\in\mathscr{N}}\mathscr{X}_n$ and $\mathscr{Y}=\sqcup_{n\in\mathscr{N}}\mathscr{Y}_n$ be endowed with the measures $\mathbf{dx}=\Sigma_{n\in\mathscr{N}}\mathbf{dx_n}$ and $\mathbf{dy}=\Sigma_{n\in\mathscr{N}}\mathbf{dy_n}$. In this framework, the space $L^p_x L^q_y$, $1\le p,q\le+\infty$, will denote the closed subspace of $L^p(\mathscr{X},\mathbf{dx};L^q(\mathscr{Y},\mathbf{dy}))$ given by*

$$L^p_x L^q_y = \left\{ f\in L^p(\mathscr{X},\mathbf{dx};L^q(\mathscr{Y},\mathbf{dy})),\quad f(x,y)=\sum_{n\in\mathscr{N}} 1_{\mathscr{X}_n}(x)1_{\mathscr{Y}_n}(y)f(x,y)\quad \text{a.e.}\right\}.$$

The above definition is coherent with the specific product case, which is the particular case $\mathscr{N}=\{0\}$. The differences will be clear from the different frameworks when $(\mathscr{X}_n,\mathbf{dx_n})_{n\in\mathscr{N}}$ and $(\mathscr{Y}_n,\mathbf{dy_n})_{n\in\mathscr{N}}$ will be specified.
The two following properties of the product case are still valid in this extended framework:

- The dual of $L^p_x L^q_y$, $1\le q,p<+\infty$ is $L^{p'}_x L^{q'}_y$ with $\frac{1}{q'}+\frac{1}{q}=1$ and $\frac{1}{p'}+\frac{1}{p}=1$.
- Minkowski's inequality says

$$\|f\|_{L^p_x L^q_y}\le\|f\|_{L^q_y L^p_x}\quad\text{for}\quad 1\le q\le p\le+\infty. \tag{35}$$

Below are examples, associated with the decomposition associated with the introduction of the center of mass (23) and the relative coordinates (24), where those notations will be used

- $\mathscr{N}=\{n\}$, $n\ge1$, $\mathscr{X}_n=\mathscr{R}_n\times Z'$, $\mathbf{dx_n}=d\mu_n\otimes\mathbf{dz'}$, $\mathscr{Y}_n=\mathbb{R}^d$, $\mathbf{dy_n}=dy_G$ and
$$L^p_{(Y'_n,z')}L^q_{y_G}=L^p_{x_n}L^q_{y_G}=L^p(\mathscr{R}_n\times Z',d\mu_n\otimes\mathbf{dz'};L^q(\mathbb{R}^d,dy_G)).$$
The notation $L^p_{(Y'_n,z'),\mathrm{sym}}L^q_{y_G}$ will stand for the closed subspace of functions which are symmetric with respect to the variables $Y'_n\in\mathscr{R}_n$.

- $\mathscr{N}=\{0,1\}$ with
$$\mathscr{X}_0=Z',\quad \mathscr{X}_1=\mathscr{R}\times Z'=(\sqcup_{n=1}^\infty\mathscr{R}_n)\times Z',\quad \mathbf{dx_0}=\mathbf{dz'},\quad \mathbf{dx_1}=d\mu\otimes\mathbf{dz'},$$
$$\mathscr{Y}_0=\{0\},\quad \mathscr{Y}_1=\mathbb{R}^d,\quad \mathbf{dy_0}=\delta_0,\quad \mathbf{dy_1}=dy_G,$$
where
$$L^p_{(Y',z')}L^q_{y_G}=L^p(Z',\mathbf{dz'})\oplus L^p(\mathscr{R}\times Z',d\mu\otimes\mathbf{dz'};L^q(\mathbb{R}^d,dy_G)).$$
With the same convention as above for $L^p_{(Y',z'),\mathrm{sym}}L^q_{y_G}$, which refers to the symmetry for the $Y'\in\mathscr{R}$ variable, the formula (28) becomes
$$U_G:L^2(Z',\mathbf{dz'};\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C})))\to L^2_{(Y',z'),\mathrm{sym}}L^2_{y_G}.$$
The general spaces $L^2_{(Y',z'),\mathrm{sym}}L^p_{y_G}$, $1\le p\le+\infty$, will be especially useful after Section 4.

- The previous example can be written with $\mathscr{N}=\mathbb{N}$ and
$$\mathscr{X}_0=Z',\quad \mathscr{X}_{n>0}=\mathscr{R}_n\times Z',\quad \mathbf{dx_0}=\mathbf{dz'},\quad \mathbf{dx_{n>0}}=d\mu_n\otimes\mathbf{dz'},$$
$$\mathscr{Y}_0=\{0\},\quad \mathscr{Y}_{n>0}=\mathbb{R}^d,\quad \mathbf{dy_0}=\delta_0,\quad \mathbf{dy_{n>0}}=dy_G.$$

3.3 $L^p_{y_G}$-Estimates for $a_G(V)$ and $a^*_G(V)$

General L^p-esptimates, or more precisely $L^2_{(Y',z),\mathrm{sym}}L^p_{y_G}$-estimates, are proved in this paragraph for the operators $a_G(V)$ and $a^*_G(V)$. The use of the center of mass and the $L^p_{y_G}$ spaces, will be extremely useful for the application of Strichartz estimates in Section 4. Let us start with a simple application of Young's inequality.

Lemma 3.5. *For any $q',p'\in[1,2]$ such that $q'\le p'$, let $r'\in[1,2]$ be defined by $\frac{1}{r'}=\frac{1}{2}+\frac{1}{q'}-\frac{1}{p'}$. The inequality*

$$\|V(y_G+y')\varphi(y_G)\|_{L^2_{y'}L^{q'}_{y_G}}\le\|V\|_{L^{r'}}\|\varphi\|_{L^{p'}},$$

holds for all $V\in L^{r'}(\mathbb{R}^d,dy;\mathbb{C})$ and all $\varphi\in L^{p'}(\mathbb{R}^d,dy;\mathbb{C})$.

Proof. The conditions $\frac{1}{r'}+\frac{1}{p'}=\frac{1}{2}+\frac{1}{q'}$, $1\le q'\le p'\le 2$, ensure

$$\frac{1}{r'}=\frac{1}{2}+\frac{1}{q'}-\frac{1}{p'}\in[\frac{1}{2},1]\quad\text{and}\quad r'\in[1,2].$$

Young's inequality with $\frac{1}{\tilde r}+\frac{1}{\tilde p}=\frac{1}{\frac{2}{q'}}+1$ and $\tilde r,\tilde p,\frac{2}{q'}\ge 1$ yields

$$\begin{aligned}\|V(y_G+y')\varphi(y_G)\|_{L^2_{y'}L^{q'}_{y_G}}\le\||V|(y_G-y')|\varphi|(y_G)\|_{L^2_{y'}L^{q'}_{y_G}}&=\||V(-\cdot)|^{q'}*|\varphi|^{q'}\|^{1/q'}_{L^{2/q'}}\\&\le\||V|^{q'}\|^{1/q'}_{L^{\tilde r}}\||\varphi|^{q'}\|^{1/q'}_{L^{\tilde p}}.\end{aligned}$$

By taking $\tilde p=\frac{p'}{q'}\in[1,2]$ and $r'=\tilde r q'$ we obtain

$$\|V(y_G+y')\varphi(y_G)\|_{L^2_{y'}L^{q'}_{y_G}}\le\|V\|_{L^{r'}}\|\varphi\|_{L^{p'}}$$

□

The first result concerns the action of $a_G(V)$ and $a^*_G(V)$ on a fixed finite particles sector.

Proposition 3.6. *For any $p',q'\in[1,2]$ such that $q'\le p'$, $2\le p\le q\le+\infty$, let $r'\in[1,2]$ be defined by $\frac{1}{r'}=\frac{1}{2}+\frac{1}{q'}-\frac{1}{p'}$ like in Lemma 3.5. For any $V\in L^{q'}(\mathbb{R}^d,dy;\mathbb{C})\cap L^{r'}(\mathbb{R}^d,dy;\mathbb{C})$, the creation and annihilation operators satisfy the following estimates:*

$$\forall f_{G,0}\in L^2_z,\ \|a^*_G(V)f_{G,0}\|_{L^2_zL^{q'}_{y_G}}\le\|V\|_{L^{q'}}\|f_{G,0}\|_{L^2_z},\tag{36}$$

$$\forall n>0,\forall f_{G,n}\in L^2_{(Y'_n,z),\mathrm{sym}}L^{p'}_{y_G},\ \|a^*_G(V)f_{G,n}\|_{L^2_{(Y'_{n+1},z)}L^{q'}_{y_G}}\le\|V\|_{L^{r'}}\sqrt{n+1}\|f_{G,n}\|_{L^2_{(Y'_n,z)}L^{p'}_{y_G}},\tag{37}$$

$$\forall f_{G,1}\in L^2_zL^q_{y_G},\ \|a_G(V)f_{G,1}\|_{L^2_z}\le\|V\|_{L^{q'}}\|f_{G,1}\|_{L^2_zL^q_{y_G}},\tag{38}$$

$$\forall n>1,\forall f_{G,n}\in L^2_{(Y'_n,z),\mathrm{sym}}L^q_{y_G},\ \|a_G(V)f_{G,n}\|_{L^2_{(Y'_{n-1},n)}L^p_{y_G}}\le\|V\|_{L^{r'}}\sqrt{n}\|f_{G,n}\|_{L^2_{(Y'_n,z)}L^q_{y_G}}.\tag{39}$$

A notable case is when $q'=r'$ and $p'=p=2$.

Proof. The variable $z\in Z$ is actually a parameter which can be forgotten because our estimates are uniform w.r.t. $z\in Z$.
For (36) it suffices to notice $[a^*_G(V)f_{G,0}](y_G)=f_{G,0}\times V(y_G)$.
The estimate of $a^*_G(V)f_{G,n}$ for $n>0$ relies on Lemma 3.5. We start from the expression (34)

$$(a_G(V)^*f_{G,n})(y_G,Y'_{n+1})=\sqrt{n+1}\,S_{n+1}V(y_G+y'_{n+1})f_{G,n}(y_G-\frac{y'_{n+1}}{n},Y_n+\frac{y'_{n+1}}{n})$$

with $Y'_{n+1} = (y'_1,\dots,y'_n,y'_{n+1}) \in \mathscr{R}^{n+1}$, $Y_n = (y'_1,\dots,y'_n) \in \mathbb{R}^{dn}$, $Y_n + \frac{y'_{n+1}}{n} \in \mathscr{R}^n$. The symmetrization S_{n+1} simply takes the average of $n+1$-terms which have all the same form as

$$\sqrt{n+1}V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n}),$$

after circular permutation of the variables y'_j which does not change the $L^2_{Y'_{n+1}} L^{q'}_{y_G}$-norm. We can therefore forget the symmetrization S_{n+1} for proving the upper bound (37). When $n > 1$ integrations must be performed with respect to the independent variables $(y'_2,\dots,y'_n) \in \mathbb{R}^{d(n-1)}$. Remember that $(y'_2,\dots,y'_n,y'_{n+1})$ are coordinates on $\mathscr{R}^{n+1}$ such that $y'_1 = -y'_2 \cdots - y'_n - y'_{n+1}$, $d\mu_{n+1}(Y'_{n+1}) = (n+1)^d dy'_2 \cdots dy'_{n+1}$ and that the quantity

$$\left\| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n}) \right\|_{L^2_{Y'_{n+1}} L^{q'}_{y_G}}$$

equals

$$(n+1)^{d/2} \| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n})) \|_{L^2(\mathbb{R}^d, dy'_{n+1}; L^2(\mathbb{R}^{d(n-1)}, dy'_2 \cdots dy'_n; L^{q'}_{y_G}))}.$$

When $y'_{n+1} \in \mathbb{R}^d$ is fixed, setting $y'_1 = -\sum_{j=2}^n (y'_h + \frac{y'_{n+1}}{n})$ and $Y'_n = Y_n + \frac{y'_{n+1}}{n}$, provides the coordinates $(y'_2 + \frac{y'_{n+1}}{n}, \dots, y'_n + \frac{y'_{n+1}}{n})$ on $\mathscr{R}^n$ with $d\mu_n(Y'_n) = n^{d/2} dy'_2 \cdots dy'_n$, and then $\| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n}) \|_{L^2(\mathbb{R}^{d(n-1)}, dy'_2 \cdots dy'_n; L^{q'}_{y_G})}$ equals

$$n^{-d/2} \| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y'_n) \|_{L^2_{Y'_n} L^{q'}_{y_G}} = n^{-d/2} \| V(\tilde{y}_G + \frac{n+1}{n} y'_{n+1}) f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^2(Y'_n L^{q'}_{\tilde{y}_G})}.$$

We deduce

$$\begin{aligned}
&\left\| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n}) \right\|_{L^2_{Y'_{n+1}} L^{q'}_{y_G}} \\
&\quad \le \frac{(n+1)^{d/2}}{n^{d/2}} \left\| \| V(\tilde{y}_G + \frac{n+1}{n} y'_{n+1}) f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^2_{Y'_n} L^{q'}_{\tilde{y}_G}} \right\|_{L^2(\mathbb{R}^d, dy'_{n+1})} \\
&\quad \le \left\| \| V(\tilde{y}_G + y') f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^2_{Y'_n} L^{q'}_{\tilde{y}_G}} \right\|_{L^2(\mathbb{R}^d, dy')} \\
&\quad \le \left\| \| V(\tilde{y}_G + y') f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^2(\mathbb{R}^d, dy'; L^{q'}_{\tilde{y}_G})} \right\|_{L^2_{Y'_n}},
\end{aligned}$$

after using the change of variable $y' = \frac{n+1}{n} y'_{n+1}$ in $\mathbb{R}^d$ for the third line and $L^2_{y'} L^2_{Y'_n} = L^2_{Y'_n} L^2_{y'}$ for the last one. We now use Lemma 3.5 with

$$\| V(\tilde{y}_G + y') f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^2(\mathbb{R}^d, dy'; L^{q'}_{\tilde{y}_G})} \le \| V \|_{L^{r'}} \| f_{G,n}(\tilde{y}_G, Y'_n) \|_{L^{p'}_{\tilde{y}_G}}$$

for almost all $Y'_n \in \mathscr{R}^n$ and $1 \le q' \le p' \le 2$, $\frac{1}{r'} = \frac{1}{2} + \frac{1}{q'} - \frac{1}{p'}$. Integrating w.r.t. $Y'_n \in \mathscr{R}^n$ gives

$$\left\| V(y_G + y'_{n+1}) f_{G,n}(y_G - \frac{y'_{n+1}}{n}, Y_n + \frac{y'_{n+1}}{n}) \right\|_{L^2_{Y'_{n+1}} L^{q'}_{y_G}} \le \| V \|_{L^{r'}} \| f_{G,n} \|_{L^2_{Y'_n} L^{p'}_{y_G}}.$$

By multiplying by $\sqrt{n+1}$ and with the symmetrization S_{n+1}, we have proved (37).
The estimates with $a_G(V)$ follow by duality using

$$\| a_G(V) f_{G,n+1} \|_{L^2_{Y'_{n+1}} L^p_{y_G}} = \sup_{\| f_{G,n} \|_{L^2_{Y'_n} L^{p'}_{y_G}} = 1} | \langle a_G(V) f_{G,n+1}, f_{G,n} \rangle |.$$

Indeed, if $\|f_{G,n}\|_{L^2_{Y'_n}L^{p'}_{y_G}}=1$, then

$$\begin{aligned}|\langle a_G(V)f_{G,n+1},f_{G,n}\rangle| &= |\langle f_{G,n+1},a_G^*(V)f_{G,n}\rangle|\\ &\le \|f_{G,n+1}\|_{L^2_{Y'_{n+1}}L^q_{y_G}}\|a_G^*(V)f_{G,n}\|_{L^2_{Y'_n}L^{q'}_{y_G}}\\ &\le \begin{cases}\|V\|_{L^{q'}}\|f_{G,1}\|_{L^q_{y_G}} & \text{when } n=0,\\ \|V\|_{L^{r'}}\sqrt{n+1}\|f_{G,n+1}\|_{L^2_{Y'_{n+1}}L^q_{y_G}}, & \text{when } n>0,\end{cases}\end{aligned}$$

which implies the bounds (38) and (39). □

Remark 3.7. *Instead of Young's inequality one could use the more general Brascamp-Lieb inequality (see[BrLi][Lie]). This would not change the result (up to multiplicative constants). One may wonder whether it is possible to improve Lebesgue's exponent, in particular the integrability by reaching exponents $p<2$ in (39) by strengthening the assumptions on V. Actually it is not. Take $V\in\mathscr{S}(\mathbb{R}^d)$ and $\varphi\in L^2(\mathbb{R}^d,dy;\mathbb{C})$, then $a(V)\varphi^{\otimes n}=\sqrt{n}\langle V,\varphi\rangle\varphi^{\otimes n-1}$ and $a_G(V)U_G^{-1}(\varphi^{\otimes n})$ cannot be put in $L^2_{z,Y'_{n-1}}L^p_{y_G}$ with $p<2$ in general.*

Proposition 3.8. *Take $\alpha,\alpha'\in\mathbb{R}$, $\alpha<\alpha'$ and for $1\le q'\le p'\le 2$, $2\le p\le q\le+\infty$, and let $r'\in[1,2]$ be defined by $\frac{1}{r'}=\frac12+\frac{1}{q'}-\frac{1}{p'}$. For any $V\in L^{r'}(\mathbb{R}^d)\cap L^{q'}(\mathbb{R}^d)$, the following estimates hold*

$$\forall f\in e^{-\alpha' N}L^2_{z,Y',\mathrm{sym}}L^{p'}_{y_G},\ \|e^{\alpha N}a_G^*(V)f\|_{L^2_{z,Y'}L^{q'}_{y_G}}\le\frac{\max(\|V\|_{L^{r'}},\|V\|_{L^{q'}})e^{\alpha'}}{2\sqrt{\alpha'-\alpha}}\|e^{\alpha' N}f\|_{L^2_{z,Y'}L^{p'}_{y_G}},\tag{40}$$

$$\forall f\in e^{-\alpha' N}L^2_{z,Y',\mathrm{sym}}L^{q}_{y_G},\ \|e^{\alpha N}a_G(V)f\|_{L^2_{z,Y'}L^{p}_{y_G}}\le\frac{\max(\|V\|_{L^{r'}},\|V\|_{L^{q'}})e^{-\alpha}}{2\sqrt{\alpha'-\alpha}}\|e^{\alpha' N}f\|_{L^2_{z,Y'}L^{q}_{y_G}}.\tag{41}$$

Again, a notable case is when $q'=r'$ and $p=p'=2$.

Proof. By writing

$$e^{\alpha N}a_G^*(V)e^{-\alpha' N}(\bigoplus_{n=0}^{\infty}f_{G,n})=\bigoplus_{n=0}^{\infty}e^{\alpha(n+1)-\alpha' n}a_G^*(V)f_{G,n},$$

$$\text{and}\qquad e^{\alpha N}a_G(V)e^{-\alpha' N}(\bigoplus_{n=0}^{\infty}f_{G,n})=\bigoplus_{n=1}^{\infty}e^{\alpha(n-1)-\alpha' n}a_G(V)f_{G,n},$$

Proposition 3.6 tells us that it suffices to bound

$$\sup_{n\in\mathbb{N}}\sqrt{n+1}e^{-(\alpha'-\alpha)(n+1)}e^{\alpha'}\le\frac{e^{\alpha'}}{\sqrt{2e}\sqrt{\alpha'-\alpha}}\le\frac{e^{\alpha'}}{2\sqrt{\alpha'-\alpha}},$$

$$\text{and}\qquad \sup_{n\in\mathbb{N}}\sqrt{n}e^{-(\alpha'-\alpha)n}e^{-\alpha}\le\frac{e^{-\alpha}}{\sqrt{2e}\sqrt{\alpha'-\alpha}}\le\frac{e^{-\alpha}}{2\sqrt{\alpha'-\alpha}}.$$

□

4 Strichartz estimates in the center of mass variable

Here we review the celebrated results of Keel and Tao in [KeTa] and adapt them to our framework. We shall use like those authors the short notations

- $a(z) \lesssim b(z)$ for the uniform inequality

$$\forall z \in Z\,, \quad a(z) \leq C b(z)\,,$$

where C is a constant which depends only on the following data: the dimension d or the free one particle evolution on $\mathbb{R}^d$;

- for $1 \leq p, q \leq +\infty$, various uses of the general notation $L^p_x L^q_y$ introduced in Definition 3.4 will be specified;

- except in specified cases, L^p_x is used for $2 \leq p \leq +\infty$ while $L^{p'}_x$ is used for $1 \leq p' \leq 2$.

4.1 Endpoint Strichartz estimates

Keel and Tao's results about endpoint Strichartz estimates (see [KeTa]) written with uniform inequalities, obviously induce a parameter dependent version which will be needed. They start with a time-dependent operator $U(t) : \mathfrak{h}_{\rm in} \to L^2_x = L^2(X, \mathbf{dx}; \mathbb{C})$ where $t \in \mathbb{R}$ and $\mathfrak{h}_{\rm in}$ is a (separable) Hilbert space of initial data. We rather consider a parameter dependent operator $U(t, z_1) : \mathfrak{h}_{\rm in} \to L^2_x$ defined for $(t, z_1) \in \mathbb{R} \times Z_1$ such that

$$\|U(t, z_1) f\|_{L^2_x} \lesssim \|f\|_{\mathfrak{h}_{\rm in}}\,, \tag{42}$$

$$\|U(t, z_1) U^*(s, z) g\|_{L^\infty_x} \lesssim \frac{\|g\|_{L^1_x}}{|t-s|^\sigma} \quad \text{for all } t \neq s\,, \tag{43}$$

while $U^*(t, z_1)$ may be defined only on a dense set of L^1_x.
On the measured space $(Z_1, \mathbf{dz_1})$ the map $(t, z_1) \mapsto U(t, z_1) f \in L^2_x$ is assumed measurable for all $f \in \mathfrak{h}_{\rm in}$ and $U(t) : L^w(Z_1, \mathbf{dz_1}; \mathfrak{h}_{\rm in}) \to L^w_{z_1} L^2_x$, where $L^w_{z_1} L^2_x = L^w(Z_1, \mathbf{dz_1}; L^2(X, \mathbf{dx}))$ here, is defined by pointwise multiplication $(U(t) f)(z_1) = U(t, z_1) f(z_1)$.
The set of sharp σ-admissible space-time exponents is given by

$$q, r \geq 2 \quad \frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}\,,$$

and the dual exponents are denoted by q', r', $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{r} + \frac{1}{r'} = 1$ with $1 \leq q', r' \leq 2$, $\frac{1}{q'} + \frac{\sigma}{r'} = \frac{\sigma+2}{2}$.
We will consider cases where $\sigma > 1$ and the endpoint Strichartz estimates for $P = (2, \frac{2\sigma}{\sigma-1})$ holds true. The results for sharp σ-admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ are:

- the homogeneous estimate

$$\|U(t) f\|_{L^w_{z_1} L^q_t L^r_x} \lesssim \|f\|_{L^w(Z_1, \mathbf{dz_1}; \mathfrak{h}_{\rm in})}\,; \tag{44}$$

- the inhomogeneous estimate

$$\| \int U(s)^* F(s)\, ds \|_{L^w(Z_1, dz_1; \mathfrak{h}_{\rm in})} \lesssim \|F\|_{L^w_{z_1} L^{\tilde{q}'}_t L^{\tilde{r}'}_x}\,; \tag{45}$$

- the retarded estimate

$$\| \int_{s<t} U(t) U(s)^* F(s)\, ds \|_{L^w_{z_1} L^q_t L^r_x} \lesssim \|F\|_{L^w_{z_1} L^{\tilde{q}'}_t L^{\tilde{r}'}_x}\,, \tag{46}$$

where $s < t$ can be replaced by $s > t$.

Keel and Tao's results are written in [KeTa] with $Z_1=\{z_0\}$ and $\mathbf{dz_1}=\delta_{z_0}$, but the uniform inequalities with respect to $z_1\in Z_1$ can be integrated afterwards for data in $L^w_{z_1}$.
By requiring $\sigma>1$, the endpoint estimate allows to take $q=\tilde{q}'=2$ with the endpoint exponents $r_\sigma=\frac{2\sigma}{\sigma-1}$ and $r'_\sigma=\frac{2\sigma}{\sigma+1}$. This is a very convenient framework for fixed point and bootstrap method in our linear setting.

Below are the typical inequalities which will be used. In our applications like in Subsection 3.3, the vaccuum sector plays a separate role and it is convenient to use the general Definition 3.4 for $L^w_z L^q_x$

$$\begin{aligned}&\mathscr{N}=\{0,1\}\quad,\quad Z=Z_0\sqcup Z_1\\ \text{and}\qquad&\mathscr{X}_0=\{0\}\quad,\quad\mathscr{X}_1=X\quad,\quad\mathbf{dx_0}=\delta_0\quad,\quad\mathbf{dx_1}=\mathbf{dx}.\end{aligned}$$

In particular the spaces $L^2_z L^q_x$ for $1\le q\le\infty$ equal

$$L^2_z L^q_x=L^2(Z_0,\mathbf{dz_0})\oplus L^2(Z_1,\mathbf{dz_1};L^2(X,\mathbf{dx}))=\underbrace{L^2_{z_0}}_{\text{vacuum}}\oplus L^2_{z_1}L^2_x. \tag{47}$$

At this level the action of the dynamics $U(t)U(s)^*$ is considered only on the $L^w_{z_1}L^q_x$ component.

Proposition 4.1. *Consider $L^2_z L^q_x=L^2_{z_0}\oplus L^2_{z_1}L^2_x$ like in (47) and according to Definition 3.4. Assume that there is a dense Banach space D in $L^2_{z_1,x}=L^2_{z_1}L^2_x$ such that $D\subset L^2_{z_1}L^{r_\sigma}_x$ and $U(t)U(s)^*u\in L^2_{z_1}L^{r_\sigma}_x$ is measurable with respect to $t,s\in\mathbb{R}$ for all $u\in D$ with the uniform estimate $\|U(t)U(s)^*u\|_{L^2_{z_1}L^{r_\sigma}_x}\lesssim\|u\|_D$ for almost all $t,s\in\mathbb{R}$.*

Assume that the bounded operator $B^*_{t,s}:L^2_z L^2_x\to L^2_{z_1}L^{r'_\sigma}_x$ and its adjoint $B_{t,s}:L^2_{z_1}L^{r_\sigma}_x\to L^2_z L^2_x$ are strongly measurable with respect to $(t,s)\in[0,T]\times[0,T]$ with the assumption

$$\sup_{t\in[0,T]}\int_0^T\|B^*_{t,s}\|^2\,ds<+\infty,\qquad\|B^*_{t,s}\|=\|B^*_{t,s}\|_{L^2_{z_1}L^{r'_\sigma}_x\leftarrow L^2_z L^2_x}, \tag{48}$$

resp.

$$\sup_{s\in[0,T]}\int_0^T\|B_{t,s}\|^2dt<+\infty,\qquad\|B_{t,s}\|=\|B_{t,s}\|_{L^2_z L^2_x\leftarrow L^2_{z_1}L^{r_\sigma}_x}. \tag{49}$$

The operator A^*_T (resp. A_T) defined by

$$[A^*_T f](t)=1_{Z_1}(z)\int_0^T U(t)U(s)^*B^*_{t,s}f(s)\,ds, \tag{50}$$

resp.

$$[A_T f](t)=\int_0^T B_{t,s}U(t)U(s)^*1_{Z_1}(z)f(s)\,ds, \tag{51}$$

acts continuously on $L^\infty([0,T];L^2_z L^2_x)$ (resp. extends as a continuous operator on $L^1([0,T];L^2_z L^2_x)$) with

$$\mathrm{Ran}\,A^*_T\subset L^\infty([0,T];L^2_{z_1,x}),\quad\mathrm{Ker}(A_T)\supset L^1([0,T];L^2_{z_0}), \tag{52}$$

$$\|(A^*_T)^n\|_{\mathscr{L}(L^\infty([0,T];L^2_z L^2_x))}\lesssim\left(\sup_{t_{n+1}\in[0,T]}\int_{[0,T]^n}\|B^*_{t_{n+1},t_n}\|^2\ldots\|B^*_{t_2,t_1}\|^2\,dt_1\cdots dt_n\right)^{1/2}, \tag{53}$$

resp.

$$\|(A_T)^n\|_{\mathscr{L}(L^1([0,T];L^2_z L^2_x))}\lesssim\left(\sup_{t_0\in[0,T]}\int_{[0,T]^n}\|B_{t_n,t_{n-1}}\|^2\ldots\|B_{t_1,t_0}\|^2\,dt_1\cdots dt_n\right)^{1/2}, \tag{54}$$

for all non zero $n\in\mathbb{N}$.
When $B^\sharp_{t,s}=B^\sharp_{t,s}1_{s<t}$ or $B^\sharp_{t,s}=B^\sharp_{t,s}1_{s>t}$ ($B^\sharp=B^*$ resp. $B^\sharp=B$), the domain of integration $[0,T]^n$ can be replaced by the corresponding n-dimensional simplex $0<t_1<\ldots<t_n<T$ or $T>t_1\ldots>t_n>0$.

Remark 4.2. *The dense subspace D is introduced in order to get a dense domain of $L^1([0,T];L^2_{z_1,x})$ where A_T is well defined by its integral formula. The extension to the whole space $L^1([0,T];L^2_{z_1,x})$ is proved by using the fact that $L^\infty([0,T];L^2_{z_1,x})$ is the dual of $L^1([0,T];L^2_{z_1,x})$ and it cannot be done in the other way.*
Examples where the dense subset D is easy to construct are when $L^2(X,\mathbf{dx};\mathbb{C}) = L^2(\mathbb{R}^d,dx;\mathbb{C})$ and $U(t)U(s)^* : H^\mu(\mathbb{R}^d;\mathbb{C}) \to H^\mu(\mathbb{R}^d;\mathbb{C})$ are measurable and uniformly bounded w.r.t. $t,s\in\mathbb{R}$ for some $\mu > d/2$. In this simple case, the set D can be $L^2(Z_1,\mathbf{dz_1};H^\mu(\mathbb{R}^d;\mathbb{C}))$ with $\mu > \frac{d}{2}$.

Proof. Let us start with A_T^*. When $f\in L^\infty([0,T],dt;L^2_zL^2_x)$ the function $1_{[0,T]}f$ belongs to $L^2_zL^2_{t,x}$ and, for almost all $t_0\in[0,T]$, the function $(z,s,x)\mapsto B^*_{t_0,s}1_{[0,T]}(s)f(s)$ belongs to $L^2_{z_1}L^2_sL^{r'_\sigma}_x$. The inhomogeneous endpoint Strichartz estimate implies for almost all $t_0\in[0,T]$

$$\|A_T^*f(t_0)\|^2_{L^2_zL^2_x} \lesssim \int_0^T \|B^*_{t_0,s}f(s)\|^2_{L^2_{z_1}L^{r'_\sigma}_x}\,ds \lesssim \left(\int_0^T \|B^*_{t_0,s}\|^2\,ds\right)\|f\|^2_{L^\infty([0,T];L^2_zL^2_x)}. \tag{55}$$

This proves firstly that A_T^* acts continuously on $L^\infty([0,T];L^2_zL^2_x)$. The property $\mathrm{Ran}\,A_T^*\subset L^\infty([0,T];L^2_{z_1,x})$ comes from the assumption $B^*_{t,s}:L^2_zL^2_x\to L^2_{z_1}L^{r'_\sigma}_x$ and the redundant multiplication by $1_{Z_1}(z)$ in (50). Secondly iterating (55) with $(t_0,s)=(t_{n+1},t_n)$ leads to (53).
Consider now A_Tf when $f = 1_{Z_1}(z)f\in L^1([0,T];L^2_{z_1,x})$. For f in the dense subspace $L^1([0,T];D)$ of $L^1([0,T];L^2_{z_1,x})$, our assumptions ensure that A_Tf belongs to $L^\infty([0,T];L^2_zL^2_x)\subset L^1([0,T];L^2_zL^2_x)$ with

$$\|A_Tf\|_{L^1([0,T];L^2_zL^2_x)} \lesssim C_T\|f\|_{L^1([0,T];D)}.$$

With

$$\int_0^T \langle v(t),A_Tf(t)\rangle\,dt = \int_0^T \langle 1_{Z_1}(z)\int_0^T U(s)U^*(t)B^*_{t,s}v(t)\,dt, f(s)\rangle\,ds = \int_0^T \langle(\tilde A_T^*v)(s),f(s)\rangle\,ds,$$

where $B^*_{t,s}$ has simply been replaced by $B^*_{s,t}$ in $\tilde A_T^*v(t) = 1_{Z_1}(z)\int_0^T U(t)U(s)^*B^*_{s,t}v(s)\,ds$, we obtain

$$\forall v\in L^\infty([0,T];L^2_zL^2_x),\quad |\langle v, A_Tf\rangle| \lesssim \left(\int_0^T \|B_{s,t}\|^2\,ds\right)^{1/2}\|v\|_{L^\infty([0,T];L^2_zL^2_x)}\|f\|_{L^1([0,T];L^2_{z_1,x})},$$

while $L^\infty([0,T];L^2_zL^2_x) = (L^1([0,T];L^2_zL^2_x))'$.
This proves that A_T extends as a continuous operator $L^1([0,T];L^2_{z_1,x})\to L^1([0,T];L^2_zL^2_x)$ and the formula contains the extension by 0 on $L^1([0,T];L^2_{z_0})$, with $L^1([0,T];L^2_zL^2_x) = L^1([0,T];L^2_{z_0})\oplus L^1([0,T];L^2_{z_1,x})$. Its adjoint is $\tilde A_T^*: L^\infty([0,T];L^2_zL^2_x)\to L^\infty([0,T];L^2_zL^2_x)$. The estimate (53) for $\tilde A_T^*$ with $(\|B^*_{t,s}\|,t_k)$ replaced by $(\|B^*_{s,t}\| = \|B_{s,t}\|, t_{n+1-k})$ yields (54). □

Note that when $B^\sharp_{t,s} = B^\sharp_{t,s}1_{t>s}$ or $B^\sharp_{t,s} = B^\sharp_{t,s}1_{t<s}$ with $\|B^\sharp_{t,s}\|\le\beta$, the upper bounds of (53) and (54) are below

$$\left(\frac{(\beta^2T)^n}{n!}\right)^{1/2} \lesssim \left(\frac{e\beta^2T}{n}\right)^{n/2}.$$

This gives a hint of times scales with respect to β, e.g. when $\beta^2T\le C$ here, where iterative methods lead to convergent series or the associated fixed point methods can be used. We will use some refined versions of the scaling rule $\beta^2T\le C$. Although the L^p_t spaces estimates are written with $p=+\infty$ and $p=1$, this scaling really relies on the endpoint Strichartz estimate with $p=2$.
We complete our general corollaries of endpoint Strichartz estimates with a result which combines the action of operators like $B_{t,s}$ and $B^*_{t,s}$ in Proposition 4.1.

Proposition 4.3. *Let $\mathscr{I},\mathscr{J}$ be at most countable families of disjoint finite intervals, and set $UI=\sqcup_{I\in\mathscr{I}}I$ and $UJ=\sqcup_{I\in\mathscr{J}}J$. For a given $\varphi_\infty\in L^\infty(UJ;L^2_zL^2_x)$ consider*

$$\varphi_{1,I}(t)=1_I(t)\sum_{J\in\mathscr{J}}\int_0^t B_{1,IJ}U(t)U(s)^*B^*_{2,IJ}(s)\varphi_{\infty,J}(s)\,ds$$

$$\textit{with}\qquad \varphi_{\infty,J}(s)=\varphi_\infty(s)1_J(s),$$

$$\textit{and}\qquad \|B_{1,IJ}\|_{L^2_zL^2_x\leftarrow L^2_{z_1}L^{r_\sigma}_x}\leq\beta_{1,IJ}\quad,\quad\sup_{s\in J}\|B^*_{2,IJ}(s)\|_{L^2_{z_1}L^{r'_\sigma}_x\leftarrow L^2_zL^2_x}\leq\beta_{2,IJ},$$

where $B_{1,IJ}:L^2_{z_1}L^{r_\sigma}_x\to L^2_zL^2_x$ does not depend on $(t,s)\in I\times J$ while $B^*_{2,IJ}(s):L^2_zL^2_x\to L^2_{z_1}L^{r'_\sigma}_x$ does not depend on the time variable $t\in I$ and is strongly measurable with respect to $s\in J$. Then the function $\varphi_1=\sum_{I\in\mathscr{I}}\varphi_{1,I}$ belongs to $L^1(UI,dt;L^2_zL^2_x)$ with

$$\|\varphi_1\|_{L^1(UI,dt;L^2_zL^2_x)}\lesssim\left[\sum_{\substack{I\in\mathscr{I},J\in\mathscr{J}\\ \inf J<\sup I}}|I|^{1/2}\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\right]\|\varphi_\infty\|_{L^\infty(UJ,dt;L^2_zL^2_x)},$$

as soon as $\left[\sum_{I\in\mathscr{I},J\in\mathscr{J}}1_{]0,+\infty[}(\sup I-\inf J)|I|^{1/2}\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\right]<+\infty$.

Proof. Every term of $\varphi_{1,I}$ can be written

$$\psi_{IJ}(t)=B_{1,IJ}\int_0^t U(t)U(s)^*\phi_{2,IJ}(s)\,ds$$

where $\phi_{2,IJ}=B^*_{2,IJ}(\cdot)\varphi_{\infty,J}(\cdot)\in L^2(\mathbb{R};L^2_{z_1}L^{r'_\sigma}_x)$ satisfies

$$\phi_{2,IJ}=0\quad\text{if }\inf J\geq\sup I,$$

$$\text{and}\qquad\|\phi_{2,IJ}\|_{L^2(\mathbb{R},dt;L^2_{z_1}L^{r'_\sigma}_x)}\leq|J|^{1/2}\beta_{2,IJ}\|\varphi_{\infty,J}\|_{L^\infty(J,dt;L^2_zL^2_x)}\leq|J|^{1/2}\beta_{2,IJ}\|\varphi_\infty\|_{L^\infty(UJ,dt;L^2_zL^2_x)}.$$

The retarded endpoint Strichartz estimate with $\|B_{1,IJ}\|_{L^2_zL^2_x\leftarrow L^2_zL^{r_\sigma}_x}\leq\beta_{1,IJ}$ implies

$$\|\psi_{IJ}\|_{L^2(I,dt;L^2_zL^2_x)}\lesssim1_{]0,+\infty[}(\sup I-\inf J)\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\|\varphi_\infty\|_{L^\infty(UJ,dt;L^2_zL^2_x)}$$

and therefore

$$\|\psi_{IJ}\|_{L^1(I,dt;L^2_zL^2_x)}\lesssim\left[1_{]0,+\infty[}(\sup I-\inf J)|I|^{1/2}\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\right]\|\varphi_\infty\|_{L^\infty(UJ,dt;L^2_zL^2_x)}.$$

The finiteness of $\sum_{I\in\mathscr{I},J\in\mathscr{J}}\left[1_{]0,+\infty[}(\sup I-\inf J)|I|^{1/2}\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\right]$ ensures that $\varphi_{1,I}=\sum_{J\in\mathscr{J}}\psi_{IJ}$ belongs to $L^1(I,dt;L^2_zL^2_x)$ and finally

$$\begin{aligned}\|\varphi_1\|_{L^1(UI,dt;L^2_zL^2_x)}&=\sum_{I\in\mathscr{I}}\|\varphi_{1,I}\|_{L^1(I,dt;L^2_zL^2_x)}\\&\lesssim\sum_{I\in\mathscr{I},J\in\mathscr{J}}\left[1_{]0,+\infty[}(\sup I-\inf J)|I|^{1/2}\beta_{1,IJ}\beta_{2,IJ}|J|^{1/2}\right]\|\varphi_\infty\|_{L^\infty(UJ,dt;L^2_zL^2_x)}.\end{aligned}$$

□

4.2 Fixed point in weighted spaces

In this section, we apply the general framework of Strichartz estimates for evolution equations in the spaces

$$F^2=L^2(Z',\mathbf{dz}';\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C})))=L^2(Z',\mathbf{dz}';\mathbb{C})\oplus L^2_{\mathrm{sym}}(\mathscr{R}\times Z',d\mu\otimes\mathbf{dz}';L^2(\mathbb{R}^d,dy_G;\mathbb{C})).$$

The measured space of parameters $(Z', \mathbf{dz}')$ will be specified later and by following the notations of Definition 3.4 and (47) for the application of Strichartz estimates, we write

$$\begin{aligned} &Z_0 = Z' \quad , \quad Z_1 = \mathscr{R} \times Z' \quad , \quad \mathbf{dz_0} = \mathbf{dz}' \quad , \quad \mathbf{dz_1} = \mu \otimes \mathbf{dz}' \\ &X_0 = \{0\} \quad , \quad X_1 = \mathbb{R}^d \quad , \quad \mathbf{dx_0} = \delta_0 \quad , \quad \mathbf{dx} = dy_G\,, \\ &F_2 = L^2_{z,\mathrm{sym}} L^2_{y_G} = L^2_{z_0} \oplus L^2_{z_1,\mathrm{sym}} L^2_{y_G} = L^2_{z_0} \oplus L^2_{(Y',z'),\mathrm{sym}} L^2_{y_G}\,, \end{aligned} \tag{56}$$

where the second variable $x \in \mathscr{X} = X_0 \sqcup X_1$ has been replaced by y_G in order to recall its link with the center of mass on the non vacuum sector.
We will use the $L^p_{y_G}$,$1 \le p \le +\infty$, version

$$L^2_{z,\mathrm{sym}} L^p_{y_G} = L^2_{z_0} \oplus L^2_{(Y',z'),\mathrm{sym}} L^p_{y_G} \qquad \text{with } z_1 = (Y', z')\,.$$

In all the above identities the subscript $_{\mathrm{sym}}$ refers to the symmetry for the relative variable $Y' \in \mathscr{R}$. Because the symmetry is preserved by all our defined operators, this subscript will be forgotten when we write estimates.
Only the useful conditions on the "free dynamics" $U(t)$, or more precisely $U(t)U(s)^* : F^2 \to F^2$ will be specified. Those will be checked for our model later in Section 5. The free dynamics or more precisely $U(t)U(s)^* : F^2 \to F^2$ is assumed to preserve the number of particles

$$[U(t)U(s)^*, N] = 0$$

with the following decomposition:

$$U(t)U(s)^* = (K_0(t,z')\overline{K_0(s,z')}\times_{z'}) \oplus (U_1(t,Y',z')U_1^*(s,Y',z')\times_{(Y',z')}) \tag{57}$$

$$\text{in} \qquad F^2 = \underbrace{L^2(Z', \mathbf{dz}'; \mathbb{C})}_{=L^2_{z_0}\ \text{(vacuum)}} \oplus \underbrace{L^2_{\mathrm{sym}}(\mathscr{R} \times Z', d\mu \otimes \mathbf{dz}'; L^2(\mathbb{R}^d, dy_G; \mathbb{C}))}_{=L^2_{(Y',z'),\mathrm{sym}} L^2_{y_G}}, \tag{58}$$

where $\times_{z'}$ or $\times_{(Y',z')}$ stands for the pointwise multiplication. So $U_1(t,Y',z')U_1^*(s,Y',z')$ is a one particle operator acting in the y_G-variable, parametrized by $z_1 = (Y',z')$ and we add the following conditions which make the results of Subsection 4.1 relevant:

- The measured space $(X_1, \mathbf{dx_1})$ is nothing but $(\mathbb{R}^d, dy_G)$ in the center of mass variable and the $z_1 = (Y',z')$-dependent one particle operators $U_1(t,z_1) : \mathfrak{h}_{\mathrm{in}} \to L^2(\mathbb{R}^d, dy_G; \mathbb{C})$ and its adjoint are assumed to satisfy the estimate (42)(43) with $\sigma > 1$. Remember $r'_\sigma = \frac{2\sigma}{\sigma+1}$ and $r_\sigma = \frac{2\sigma}{\sigma-1}$.
- The additional assumption of Proposition 4.1 concerned with the dense subset D is also assumed for $U_1(t,z_1)$.
- The vacuum component K_0 belongs to $L^\infty(\mathbb{R} \times Z', dt \otimes \mathbf{dz}'; \mathbb{C})$.

The interaction terms will be

$$B^*_{t,s} = c_1(t,s)e^{\alpha(t,s)N}\sqrt{h}a^*_G(V_1)e^{-\alpha'(t,s)N} \quad \text{and} \quad B_{t,s} = c_2(t,s)\sqrt{h}e^{\alpha(t,s)N}a_G(V_2)e^{-\alpha'(t,s)N}$$

with $V_1, V_2 \in L^{r'_\sigma}(\mathbb{R}^d, dy; \mathbb{C})$ (complex valued V are allowed here) and where c_1, c_2, α and α' are real measurable functions of $(t,s) \in [0,T]^2$ with $\alpha - \alpha' < 0$. Those will be specified further and we shall check the estimates (48)(49). Because $Z_0 = Z'$ corresponds to the vacuum sector, $N = 0$, on which $a_G(V)$ vanishes while the range of $a_G(V)^*$ lies in the non vacuum sector $N \ge 1$, the range $B^*_{t,s}$ lies naturally in $L^2_{z_1} L^{r'_\sigma}_{y_G}$, $z_1 = (Y',z')$, once the proper

estimates are checked while it adjoints $B_{t,s}$ sends $L^2_{z_1}L^{r_\sigma}_{y_G}$ into $L^2_zL^2_{y_G}$ and is naturally extended by 0 on the vacuum sector $L^2_{z_0}$.
We will consider the following system

$$u^h_\infty(t) = -i\int_0^t U(t)U^*(s)\left(\sqrt{h}a^*_G(V_1)u^h_\infty(s) + \sqrt{h}u^h_2(s) + u^h_1(s)\right)ds + f^h_\infty(t) \tag{59}$$

$$u^h_2(t) = -i\int_0^t a_G(V_2)U(t)U(s)^*\sqrt{h}u^h_2(s)\,ds + f^h_2(t), \tag{60}$$

$$u^h_1(t) = -i\int_0^t a_G(V_2)U(t)U(s)^*\left(ha^*_G(V_1)u^h_\infty(s) + \sqrt{h}u^h_1(s)\right)ds + f^h_1(t). \tag{61}$$

written shortly as

$$\forall q\in\{\infty,2,1\},\quad u^h_q = \sum_{p\in\{\infty,2,1\}} L_{qp}(u^h_p) + f^h_q \tag{62}$$

or

$$\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1\end{pmatrix} = L\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1\end{pmatrix} + \begin{pmatrix} f^h_\infty \\ f^h_2 \\ f^h_1\end{pmatrix},\quad L = \begin{pmatrix} L_{\infty\infty} & L_{\infty 2} & L_{\infty 1} \\ 0 & L_{22} & 0 \\ L_{1\infty} & 0 & L_{11}\end{pmatrix}. \tag{63}$$

This system will be studied in spaces with the number weight $e^{\alpha N}$ and we will use the following functional spaces.

Definition 4.4. *For $T>0$, $h\in]0,h_0[$, I^h_T denotes the interval $I^h_T =]-T/h, T/h[$.*
Fix $\alpha_0,\alpha_1\in\mathbb{R}$, $\alpha_0<\alpha_1$ and set $M_{\alpha 01} = \frac{\max(e^{\alpha_1},e^{-\alpha_0})}{2}\geq 1/2$.
Assume $V_1,V_2\in L^{r'_\sigma}(\mathbb{R}^d,dy;\mathbb{C})$ with $\max(\|V_1\|_{L^{r'_\sigma}},\|V_2\|_{L^{r'_\sigma}}) < C_V$.
For a parameter $\gamma>0$ and $\alpha\in[\alpha_0,\alpha_1[$ set

$$T_\alpha = \gamma(\alpha_1-\alpha).$$

The space $\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma}$ is the set of $(e^{-\alpha_0 N}L^2_{z,\mathrm{sym}}L^2_{y_G})^3$-valued measurable functions $I^h_{T_{\alpha_0}}\ni t\mapsto \begin{pmatrix} u_\infty(t)\\ u_2(t)\\ u_1(t)\end{pmatrix}$ such that for all α in $[\alpha_0,\alpha_1[$,

$$\begin{aligned} |t|^{-1/2}u_\infty &\in L^\infty(I^h_{T_\alpha},dt;e^{-\alpha N}L^2_zL^2_{y_G}),\\ u_2 &\in L^2_{\mathrm{loc}}(I^h_{T_\alpha},dt;e^{-\alpha N}L^2_zL^2_{y_G}),\\ |t|^{-1/2}u_1 &\in L^1_{\mathrm{loc}}(I^h_{T_\alpha},dt;e^{-\alpha N}L^2_zL^2_{y_G}).\end{aligned}$$

and $M(u_\infty,u_2,u_1)<+\infty$ with

$$M(u_\infty,u_2,u_1) = M_\infty(u_\infty) + M_2(u_2) + M_1(u_1), \tag{64}$$

$$M_\infty(u_\infty) = \sup_{\alpha_0\leq\alpha<\alpha_1}\left\|\left(\frac{T_\alpha-|ht|}{|ht|}\right)^{1/2}e^{\alpha N}u_\infty\right\|_{L^\infty(I^h_{T_\alpha};L^2_zL^2_{y_G})}, \tag{65}$$

$$M_2(u_2) = \frac{1}{M_{\alpha 01}C_V\gamma^{1/2}}\sup_{\substack{\alpha_0\leq\alpha<\alpha_1\\ \tau\in[0,T_\alpha[}}\sqrt{T_\alpha-\tau}\left\|e^{\alpha N}u_2\right\|_{L^2(I^h_\tau;L^2_zL^2_{y_G})}, \tag{66}$$

$$M_1(u_1) = \frac{1}{M_{\alpha 01}C_V\gamma^{1/2}}\sup_{\substack{\alpha_0\leq\alpha<\alpha_1\\ \tau\in[0,T_\alpha[}}\sqrt{T_\alpha-\tau}\left\|\frac{e^{\alpha N}u_1}{\sqrt{|ht|}}\right\|_{L^1(I^h_\tau;L^2_zL^2_{y_G})}. \tag{67}$$

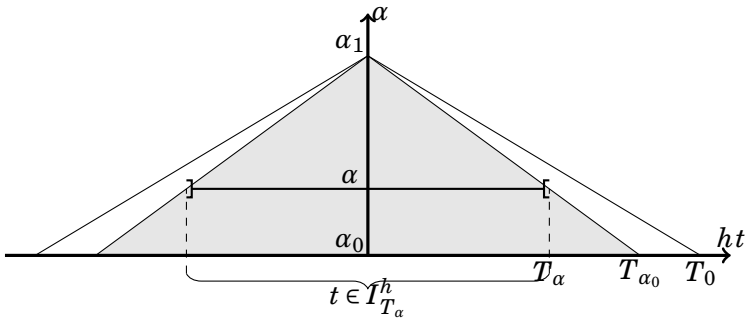

Figure 1: The time interval $I^h_{T_\alpha} = \left]-\frac{\gamma(\alpha_1-\alpha)}{h}, \frac{\gamma(\alpha_1-\alpha)}{h}\right[$ according to α.

Endowed with the norm $M(u_\infty,u_2,u_1)$, $\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma}$ is a Banach space for all $h\in]0,h_0[$. The α-dependent time domain $I^h_{T_\alpha}$ where weighted L^∞_t, L^2_t and L^1_t norms are evaluated is illustrated in Figure 1.
The constants $C_V>0$ and $M_{\alpha 01}=\max(e^{\alpha_1},e^{-\alpha_0})/2\geq 1/2$ were chosen so that Proposition 3.8 applied with $q'=r'_\sigma$ and $p'=2$, gives

$$\|e^{\alpha N}a^*_G(V)e^{-\alpha' N}\varphi\|_{L^2_{z_1}L^{r'_\sigma}_{y_G}}\leq\frac{C_V e^{\alpha'}}{2\sqrt{\alpha'-\alpha}}\|\varphi\|_{L^2_z L^2_{y_G}}\leq\frac{M_{\alpha 01}C_V}{\sqrt{\alpha'-\alpha}}\|\varphi\|_{L^2_z L^2_{y_G}}$$
$$\|e^{\alpha N}a_G(V)e^{-\alpha' N}\varphi\|_{L^2_z L^2_{y_G}}\leq\frac{C_V e^{-\alpha}}{2\sqrt{\alpha'-\alpha}}\|\varphi\|_{L^2_{z_1}L^{r_\sigma}_{y_G}}\leq\frac{M_{\alpha 01}C_V}{\sqrt{\alpha'-\alpha}}\|\varphi\|_{L^2_z L^{r_\sigma}_{y_G}},$$

for all $\alpha,\alpha'\in[\alpha_0,\alpha_1[$, $\alpha<\alpha'$.
Finally the normalization of (66) and (67) was chosen in order to make the contraction statement simple.

Proposition 4.5. *Assume that the free dynamics $U_1(t,z_1):\mathfrak{h}_{in}\to L^2(\mathbb{R}^d,dy_G;\mathbb{C})$ satisfies (42)(43) (uniformly w.r.t. $z\in Z$) with $\sigma>1$ and the additional existence of the dense subset D assumed in Proposition 4.1.*
Let $h_0>0$, $\alpha_0,\alpha_1\in\mathbb{R}$, $\alpha_0<\alpha_1$ and $V_1,V_2\in L^{r'_\sigma}(\mathbb{R}^d,dy;\mathbb{C})$ be fixed. The positive constants $M_{\alpha 01},C_V$, the space $\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma}$ and its norm M are the ones of Definition 4.4. By choosing the parameter $\gamma>0$ small enough the linear operator L given by (63) is a contraction of the Banach space $(\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma},M)$ for all $h\in]0,h_0[$ and the system (63), explicitely written (59)(60)(61), admits a unique solution for any $(f^h_\infty,f^h_2,f^h_1)\in\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma}$.
More precisely there exists a constant $C_{d,U}$ determined by the dimension d and the free dynamics U, given by the pair K_0 and U_1, such that

$$\forall h\in]0,h_0[,\quad \|L\|_{\mathscr{L}(\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma})}\leq C_{d,U}M_{\alpha 01}C_V\gamma^{1/2}.$$

Taking e.g. $\gamma=\frac{1}{2C^2_{d,U}M^2_{\alpha 01}C^2_V}$ ensures $\|L\|_{\mathscr{L}(\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma})}\leq\frac{1}{2}$ so that the solution to (63) satisfies

$$M(u^h_\infty,u^h_2,u^h_1)\leq 2M(f^h_\infty,f^h_2,f^h_1).$$

Proof. The non-vanishing entries of $L\begin{pmatrix}u_\infty\\u_2\\u_1\end{pmatrix}$, namely

$$L_{\infty\infty}(u_\infty)\quad,\quad L_{\infty 2}(u_2)\quad,\quad L_{\infty 1}(u_1)\quad,\quad L_{22}(u_2)\quad,\quad L_{11}(u_1)\quad\text{and}\quad L_{1\infty}(u_\infty)$$

will be considered separately in this order of increasing difficulty. Additionally the symmetry $t \mapsto -t$ allows us to restrict the analysis to $t \geq 0$, that is $t \in [0, \frac{T_\alpha}{h}[$ for $\alpha \in [\alpha_0, \alpha_1[$. Accordingly I_T^h is, in this proof, the restricted interval $[0, \frac{T}{h}[$.

We use like in Section 4 the symbol $\lesssim$ for inequalities with constants which depend only on the dimension d and the free dynamics U.

$\mathbf{L_{\infty\infty}(u_\infty)}$: For this term and up to the square root and the parameter $h \in]0, h_0[$, we follow exactly the method of [Nir] for Cauchy-Kowalevski theorem. Write for $t \in]0, T_\alpha/h[$, $ht \in]0, T_\alpha[$, $\alpha < \alpha_1 - \frac{ht}{\gamma}$, and

$$\left(\frac{T_\alpha - ht}{ht}\right)^{1/2} e^{\alpha N} L_{\infty\infty}(u_\infty)(t) = -i \int_0^{T_\alpha/h} U(t)U(s)^* B_{t,s}^* \left(\frac{T_{\alpha_s} - hs}{hs}\right)^{1/2} e^{\alpha_s N} u_\infty(s)\, ds$$

with

$$B_{t,s}^* = 1_{s<t} \left(\frac{T_\alpha - ht}{ht}\right)^{1/2} e^{\alpha N} \sqrt{h} a_G^*(V) e^{-\alpha_s N} \left(\frac{hs}{T_{\alpha_s} - hs}\right)^{1/2}, \tag{68}$$

and $\alpha < \alpha_s < \alpha_1 - \frac{hs}{\gamma}$. Hence $hs < T_{\alpha_s}$ and

$$\left(\frac{T_{\alpha_s} - hs}{hs}\right)^{1/2} \|e^{\alpha_s N} u_\infty(s)\|_{L_z^2 L^2 y_G} \leq M_\infty(u_\infty) \tag{69}$$

while $\alpha < \alpha_s$ implies that $\|B_{t,s}^*\| = \|B_{t,s}^*\|_{L_z^2 L_{y_G}^{r'_\sigma} \leftarrow L_z^2 L_{y_G}^2}$ satisfies

$$\|B_{t,s}^*\|^2 \leq h 1_{s<t} \frac{M_{\alpha 01}^2 C_V^2}{(\alpha_s - \alpha)} \frac{(T_\alpha - ht)(hs)}{ht(T_{\alpha_s} - hs)} = h 1_{s'<t'} \frac{M_{\alpha 01}^2 C_V^2}{(\alpha_{s'/h} - \alpha)} \frac{(T_\alpha - t')s'}{t'(T_{\alpha_{s'/h}} - s')},$$

by setting $s' = hs$, $t' = ht$. By choosing

$$\alpha_s = \frac{\alpha_1 + \alpha - hs/\gamma}{2} = \frac{\alpha_1 + \alpha - s'/\gamma}{2},$$

we obtain

$$\gamma(\alpha_s - \alpha) = \frac{\gamma(\alpha_1 - \alpha) - s'}{2} = \frac{T_\alpha - s'}{2},$$
$$T_{\alpha_{s'/h}} = \gamma(\alpha_1 - \alpha_{s'/h}) = \frac{\gamma(\alpha_1 - \alpha) + s'}{2} \quad , \quad T_{\alpha_{s'/h}} - s' = \frac{T_\alpha - s'}{2},$$

and

$$\frac{(T_\alpha - t')s'}{(\alpha_{s'/h} - \alpha)t'(T_{\alpha_{s'/h}} - s')} = 4\gamma \frac{(T_\alpha - t)s'}{t'(T_\alpha - s')^2}.$$

This yields

$$\int_0^{T_\alpha/h} \|B_{t,s}^*\|^2\, ds \leq 4\gamma M_{\alpha 01}^2 C_V^2 \frac{T_\alpha - t'}{t'} \int_0^{t'} \frac{s'}{(T_\alpha - s')^2}\, ds' \leq 4\gamma M_{\alpha 01}^2 C_V^2. \tag{70}$$

The inequalities (69) and (70) combined with the inequality (53) with $n = 1$ of Proposition 4.1 imply

$$\left\| \left(\frac{T_\alpha - ht}{ht}\right)^{1/2} e^{\alpha N} L_{\infty\infty}(u_\infty) \right\|_{L^\infty([0, T_\alpha/h]; L_z^2 L_{y_G}^2)} \lesssim 2\gamma^{1/2} M_{\alpha 01} C_V M_\infty(u_\infty). \tag{71}$$

$\mathbf{L_{\infty 2}(u_2)}$: The Cauchy-Schwarz inequality applied to

$$\left(\frac{T_\alpha - ht}{ht}\right)^{1/2} e^{\alpha N} L_{\infty 2}(u_2)(t) = -i\sqrt{T_\alpha - ht} \frac{1}{\sqrt{t}} \int_0^t U(t)U(s)^* e^{\alpha N} u_2(s)\, ds,$$

imply

$$\begin{aligned}\left\|\left(\frac{T_\alpha-ht}{ht}\right)^{1/2}e^{\alpha N}L_{\infty 2}(u_2)(t)\right\|_{L^2_zL^2_{y_G}} &\le \sqrt{T_\alpha-ht}\,\frac{1}{\sqrt{t}}\|e^{\alpha N}u_2(s)\|_{L^2_zL^1([0,t];L^2_{y_G})}\\ &\le \sqrt{T_\alpha-ht}\|e^{\alpha N}u_2(s)\|_{L^2([0,t];L^2_zL^2_{y_G})}\\ &\le \sup_{\tau\in]0,T_\alpha[}\sqrt{T_\alpha-\tau}\|e^{\alpha N}u_2(s)\|_{L^2([0,\tau/h];L^2_zL^2_{y_G})}.\end{aligned}$$

Taking the supremum over $\alpha\in[\alpha_0,\alpha_1[$ yields

$$M_\infty(L_{\infty 2}(u_2))\lesssim M_{\alpha 01}C_V\gamma^{1/2}M_2(u_2). \tag{72}$$

$\mathbf{L_{\infty 1}(u_1)}$: The expression

$$\left(\frac{T_\alpha-ht}{ht}\right)^{1/2}e^{\alpha N}L_{\infty 1}(u_1)(t)=-i\sqrt{T_\alpha-ht}\int_0^t\frac{\sqrt{hs}}{\sqrt{ht}}\left[U(t)U(s)^*e^{\alpha N}\frac{1}{\sqrt{hs}}u_1(s)\right]ds,$$

gives

$$\begin{aligned}\left\|\left(\frac{T_\alpha-ht}{ht}\right)^{1/2}e^{\alpha N}L_{\infty 1}(u_1)(t)\right\|_{L^2_zL^2_{y_G}} &\le \sqrt{T_\alpha-ht}\|e^{\alpha N}\frac{u_1(s)}{\sqrt{hs}}\|_{L^1([0,t];L^2_zL^2_{y_G})}\\ &\le \sup_{\tau\in]0,T_\alpha[}\sqrt{T_\alpha-\tau}\|\frac{u_1(s)}{\sqrt{hs}}\|_{L^1([0,\frac{\tau}{h}];L^2_zL^2_{y_G})}\\ &\le M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1)\end{aligned}$$

and

$$\|\left(\frac{T_\alpha-ht}{ht}\right)^{1/2}e^{\alpha N}L_{\infty 1}(u_1)\|_{L^\infty([0,\frac{T_\alpha}{h}];L^2_zL^2_{y_G})}\le M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1). \tag{73}$$

The entries $L_{22}(u_2)$, $L_{11}(u_1)$ and finally $L_{1\infty}(u_\infty)$ require some additional techniques. The proof, done in several steps for each of them, relies on a dyadic partition of the interval $[0,T_\alpha[$ around T_α. In the two cases of $L_{22}(u_2)$ and $L_{11}(u_1)$, the norms $M_2(\varphi)$ and $M_1(\varphi)$ are transformed into equivalent norms corresponding to this dyadic partition, the proof being given in Lemma 4.6 below. Finally the entry $L_{1\infty}(u_\infty)$ is treated via dyadic partitions around T_α and 0 and happens to be a direct application of Proposition 4.3.

Splitting the interval $[0,T[$. Fix $\alpha\in[\alpha_0,\alpha_1[$ and therefore $T=T_\alpha$. The intervals J_T^n are defined for $n\in\mathbb{N}$ by

$$\begin{aligned}&J_T^n=T+2^{-n}[-T,-T/2[=[(1-2^{-n})T,(1-2^{-n-1})T[,\\ &J_T^{\le n_0}=\bigcup_{n\le n_0}J_T^n \text{ for } n_0\in\mathbb{N},\end{aligned}$$

so that $[0,T]=\cup_{n\in\mathbb{N}}J_T^n=J_T^{\le n_0}\cup(\cup_{n>n_0}J_T^n)$, see Figure 2.
With the exponents

$$\alpha_0'=\frac{\alpha_1+6\alpha}{7}\quad\text{and}\quad\alpha_n'=\frac{\alpha_1+(2^{n+2}-1)\alpha}{2^{n+2}}\text{ for } n\ge 1$$

we note that

$$J^{\le 2}_{T_{\alpha_0'}}=\frac{7}{8}T_{\alpha_0'}=\frac{7}{8}\frac{6}{7}T_\alpha=\frac{3}{4}T_\alpha=J^{\le 1}_{T_\alpha},$$

$$\text{and for } n\ge 1\qquad T_{\alpha_n'}=T_\alpha-\frac{1}{2}\frac{T_\alpha}{2^{n+1}}=(1-2^{-n-2})T_\alpha.$$

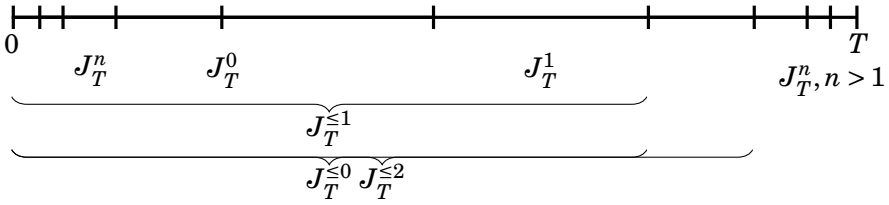

Figure 2: The time intervals J_T^n, $n \in \mathbb{N}$, with length $\frac{T}{2^{n+1}}$

By taking $\delta_n = \frac{T_\alpha}{2^{n+2}}$ and $2\delta_n = \frac{T_\alpha}{2^{n+1}}$ for $n > 1$, we obtain in particular

$$J_{T_\alpha}^n = [T_\alpha - 4\delta_n, T_\alpha - 2\delta_n[= [T_{\alpha'_n} - 3\delta_n, T_{\alpha'_n} - \delta_n[\quad \text{with } \delta_n \leq \frac{T_{\alpha'_n}}{12} \ (n > 1)$$

as summarized in Figure 3.

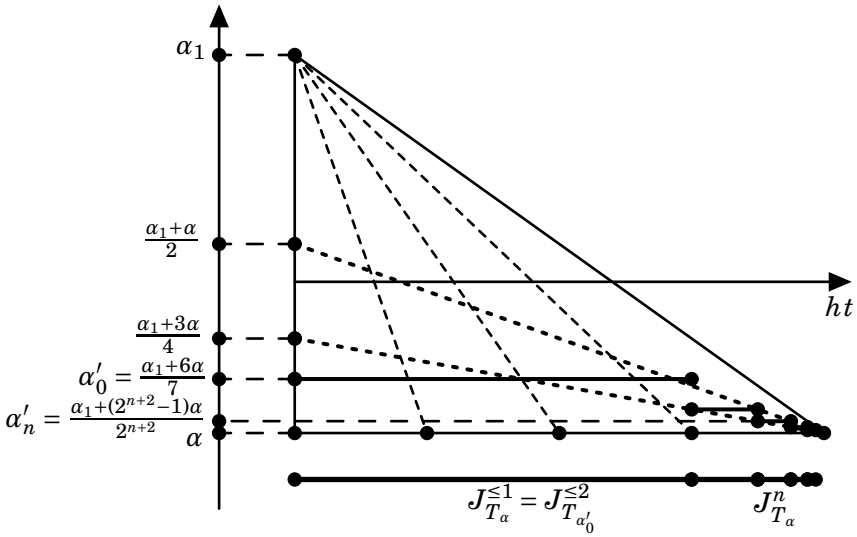

Figure 3: The exponent α'_0 is determined by $\frac{7}{8}T_{\alpha'_0} = \frac{3}{4}T_\alpha$ while for $n > 1$, α'_n is determined by $T_{\alpha'_n} = (1 - \frac{2^{-n-1}}{2})T_\alpha = (1 - 2^{-n-2})T_\alpha$.

The equivalence of norms

$$\kappa_2^{-1} N_{2,1}(\varphi) \leq N_{2,i}(\varphi) \leq \kappa_2 N_{2,1}(\varphi), \quad 2 \leq i \leq 4, \tag{74}$$

for some universal constant $\kappa_2 > 1$ is proved in Lemma 4.6 for

$$N_{2,1}(\varphi) = \sup_{\tau \in [0,T[} \sqrt{T-\tau}\, \|\varphi\|_{L^2([0,\frac{\tau}{h}];L_z^2 L_{y_G}^2)}, \tag{75}$$

$$N_{2,2}(\varphi) = \sqrt{T}\, \|\varphi\|_{L^2(h^{-1}J_T^{\leq 1};L_z^2 L_{y_G}^2)} + \sup_{\delta \in]0,T/8]} \sqrt{\delta}\, \|\varphi\|_{L^2(h^{-1}[T-2\delta,T-\delta];L_z^2 L_{y_G}^2)}, \tag{76}$$

$$N_{2,3}(\varphi) = \sqrt{T} \sup_{n \in \mathbb{N}} 2^{-n/2}\, \|\varphi\|_{L^2(h^{-1}J_T^n;L_z^2 L_{y_G}^2)}, \tag{77}$$

$$N_{2,4}(\varphi) = \sqrt{T}\, \|\varphi\|_{L^2(h^{-1}J_T^{\leq 2};L_z^2 L_{y_G}^2)} + \sup_{\delta \in]0,T/12]} \sqrt{\delta}\, \|\varphi\|_{L^2(h^{-1}[T-3\delta,T-\delta];L_z^2 L_{y_G}^2)}. \tag{78}$$

$\mathbf{L_{22}(u_2)}$: For $\alpha \in [\alpha_0, \alpha_1[$, we seek an upper bound of $N_{2,1}(\varphi)$ (with $T = T_\alpha$) for

$$\varphi(t) = e^{\alpha N} L_{22}(u_2)(t) = -i \int_0^t e^{\alpha N} \sqrt{h}\, a_G(V_2) U(t) U(s)^* u_2(s)\, ds.$$

By the equivalence of norms $N_{2,1}$ and $N_{2,3}$ this is the same as finding an upper bound for

$$\sqrt{T_\alpha}\,2^{-n/2}\|\varphi\|_{L^2(h^{-1}J^n_{T_\alpha};L^2_zL^2_{y_G})}$$

uniformly in both $\alpha\in[\alpha_0,\alpha_1[$ and $n\geq 0$, or equivalently for

$$\sqrt{T_\alpha}\|\varphi\|_{L^2(h^{-1}J^{\leq 1}_{T_\alpha};L^2_zL^2_{y_G})}\quad\text{and}\quad\sqrt{T_\alpha}\,2^{-n/2}\|\varphi\|_{L^2(h^{-1}J^n_{T_\alpha};L^2_zL^2_{y_G})}\ (n>1),$$

with the same uniformity.
For $t\in h^{-1}J^{\leq 1}_{T_\alpha}$ we write

$$\sqrt{T_\alpha}\varphi(t)=-i\sqrt{T_\alpha}e^{\alpha N}a_G(V)e^{-\alpha'_0N}\int_{s<t}U(t)U(s)^*\sqrt{h}\,w_1(s)\,ds$$

with

$$w_1(s)=e^{\alpha'_0N}1_{h^{-1}J^{\leq 1}_{T_\alpha}}(s)u_2(s)=e^{\alpha'_0N}1_{h^{-1}J^{\leq 2}_{T_{\alpha'_0}}}(s)u_2(s).$$

Then Proposition 3.8, the retarded Strichartz estimate (46) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}\sqrt{T_\alpha}\|\varphi\|_{L^2(h^{-1}J^{\leq 1}_{T_\alpha};L^2_zL^2_{y_G})}&\lesssim\sqrt{T_\alpha}\frac{C_VM_{\alpha 01}}{\sqrt{\alpha'_0-\alpha}}\|\int_{s<t}U(t)U(s)^*\sqrt{h}w_1(s)\,ds\|_{L^2_zL^2_t(h^{-1}J^{\leq 1}_{T_\alpha};L^{r\sigma}_{y_G})}\\&\lesssim C_VM_{\alpha 01}\sqrt{\gamma}\|\sqrt{h}w_1\|_{L^2_zL^1(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_{y_G})}\\&\lesssim C_VM_{\alpha 01}\sqrt{\gamma}\sqrt{T_{\alpha'_0}}\|w_1\|_{L^2_zL^2_t(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_{y_G})}\\&\lesssim C_VM_{\alpha 01}\sqrt{\gamma}\sqrt{T_{\alpha'_0}}\|e^{\alpha'_0}u_2\|_{L^2_t(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_zL^2_{y_G})}.\end{aligned}$$

The equivalence between the norms $N_{2,1}$ and $N_{2,4}$ implies

$$\sqrt{T_\alpha}\|\varphi\|_{L^2(h^{-1}J^{\leq 1}_{T_\alpha};L^2_zL^2_{y_G})}\lesssim C_V^2M_{\alpha 01}^2\gamma M_2(u_2).\tag{79}$$

For $t\in h^{-1}J^n_{T_\alpha}$, $n>1$, write

$$\begin{aligned}\sqrt{T_\alpha}\,2^{-n/2}\varphi(t)=&-i\sqrt{T_\alpha}2^{-n/2}e^{\alpha N}a_G(V)e^{-\alpha'_0N}\int_{s<t}U(t)U(s)^*\sqrt{h}\,w_1(s)\,ds\\&-i\sqrt{T_\alpha}2^{-n/2}\underbrace{\sum_{m=2}^n e^{\alpha N}a_G(V_2)e^{-\alpha'_mN}\int_{s<t}U(t)U(s)^*\sqrt{h}\,w_m(s)\,ds}_{=\tilde\varphi}\end{aligned}$$

with for $m\geq 2$

$$w_m(s)=1_{h^{-1}J^m_{T_\alpha}}(s)e^{\alpha'_mN}u_2(s)=1_{h^{-1}\left[T_{\alpha'_m}-3\delta_m,T_{\alpha'_m}-\delta_m\right]}(s)e^{\alpha'_mN}u_2(s).$$

The first term is actually estimate as we did for (79) with the additional factor $2^{-n/2}\leq 1$. It suffices to consider the application of Proposition 3.8, the retarded Strichartz estimate (46) and the Cauchy-Schwarz inequality to

$$\begin{aligned}&\sqrt{T_\alpha}\,2^{-n/2}\|\tilde\varphi\|_{L^2(h^{-1}J^n_{T_\alpha};L^2_zL^2_{y_G})}\\&\lesssim\sqrt{T_\alpha}\,2^{-n/2}\sum_{m=2}^n\frac{C_V\,M_{\alpha 01}}{\sqrt{\alpha'_m-\alpha}}\left\|\int_{s<t}U(t)U(s)^*\sqrt{h}w_m(s)\,ds\right\|_{L^2_zL^2_t(h^{-1}J^n_{T_\alpha};L^{r\sigma}_{y_G})}\\&\lesssim C_V\,M_{\alpha 01}\,\sqrt{\gamma}\,2^{-n/2}\sum_{m=2}^n 2^{m/2}\left\|\sqrt{h}w_m\right\|_{L^2_zL^1_t(h^{-1}J^m_{T_\alpha};L^2_{y_G})}\\&\lesssim C_V\,M_{\alpha 01}\sqrt{\gamma}\,2^{-n/2}\sum_{m=2}^n\sqrt{T_\alpha}\,\|w_m\|_{L^2_zL^2_t(h^{-1}J^m_{T_\alpha};L^2_{y_G})}.\end{aligned}\tag{80}$$

Thanks to the equivalence of the norms $N_{2,1}$ and $N_{2,4}$ (with $T = T_{\alpha'_m}$), we obtain for $m \geq 2$

$$\sqrt{T_\alpha}\,\|w_m\|_{L^2_z L^2_t(h^{-1}J^m_{T_\alpha};L^2_{y_G})} = 2^{\frac{m+2}{2}}\sqrt{\delta_m}\left\|e^{\alpha'_m N}u_2(s)\right\|_{L^2_t(h^{-1}\left[T_{\alpha'_m}-3\delta_m,T_{\alpha'_m}-\delta_m\right];L^2_z L^2_{y_G})}$$
$$\lesssim 2^{m/2}C_V\,M_{\alpha 01}\sqrt{\gamma}M_2(u_2). \tag{81}$$

Putting together (80) and (81) gives

$$\sqrt{T_\alpha}\,2^{-n/2}\|\varphi\|_{L^2(h^{-1}J^n_{T_\alpha};L^2_z L^2_{y_G})} \lesssim 2^{-n/2}\sum_{m=0}^{n} 2^{m/2}C_V^2\,M_{\alpha 01}^2\,\gamma\,M_2(u_2)$$
$$\lesssim C_V^2\,M_{\alpha 01}^2\,\gamma\,M_2(u_2)$$

which, combined with (79) and the normalization of $M_2(L_{22}(u_2))$, yields

$$M_2(L_{22}(u_2)) \lesssim C_V\,M_{\alpha 01}\sqrt{\gamma}\,M_2(u_2). \tag{82}$$

The estimate of $L_{11}(u_1)$ starts with the same decomposition of the interval $[0,T/h]$ with the norms

$$N_{1,1}(\varphi) = \sup_{\tau\in[0,T[}\sqrt{T-\tau}\left\|\frac{\varphi(t)}{\sqrt{ht}}\right\|_{L^1([0,\frac{\tau}{h}];L^2_z L^2_{y_G})}, \tag{83}$$

$$N_{1,2}(\varphi) = \left\|\left(\frac{T}{ht}\right)^{1/2}\varphi\right\|_{L^1(h^{-1}J^{\leq 1}_T;L^2_z L^2_{y_G})} + \sup_{\delta\in]0,T/8]}\left(\frac{\delta}{T}\right)^{1/2}\|\varphi\|_{L^1(h^{-1}[T-2\delta,T-\delta];L^2_z L^2_{y_G})}, \tag{84}$$

$$N_{1,3}(\varphi) = \left\|\left(\frac{T}{ht}\right)^{1/2}\varphi\right\|_{L^1(h^{-1}J^{\leq 1}_T;L^2_z L^2_{y_G})} + \sup_{n>1}2^{-n/2}\|\varphi\|_{L^1(h^{-1}J^n_T;L^2_z L^2_{y_G})}, \tag{85}$$

$$N_{1,4}(\varphi) = \left\|\left(\frac{T}{ht}\right)^{1/2}\varphi\right\|_{L^1(h^{-1}J^{\leq 2}_T;L^2_z L^2_{y_G})} + \sup_{\delta\in]0,T/12]}\left(\frac{\delta}{T}\right)^{1/2}\|\varphi\|_{L^1(h^{-1}[T-3\delta,T-\delta];L^2_z L^2_{y_G})}. \tag{86}$$

Those norms are equivalent according to

$$\kappa_1^{-1}N_{1,1}(\varphi) \leq N_{1,i}(\varphi) \leq \kappa_1 N_{1,1}(\varphi) \quad ,2\leq i\leq 4 \tag{87}$$

with a universal constant $\kappa_1 > 1$. See Lemma 4.6 for the proof.

$L_{11}(u_1)$-Step 1, Decomposition of $L_{11}(u_1)$: For $\alpha\in[\alpha_0,\alpha_1[$, we seek an upper bound of $N_{1,1}(\varphi)$ for

$$\varphi(t) = e^{\alpha N}L_{11}(u_1)(t) = -i\int_0^t e^{\alpha N}\sqrt{h}a_G(V_2)U(t)U(s)^*u_1(s)\,ds.$$

By the equivalence of norms $N_{1,1}$ and $N_{1,3}$ this is the same as finding a uniform upper bound for

$$\left\|\left(\frac{T_\alpha}{ht}\right)^{1/2}\varphi\right\|_{L^1(h^{-1}J^{\leq 1}_{T_\alpha};L^2_z L^2_{y_G})} \quad\text{and}\quad 2^{-n/2}\|\varphi\|_{L^1(h^{-1}J^n_{T_\alpha};L^2_z L^2_{y_G})} \text{ for } n>1.$$

Setting $\psi_1(t) = \left(\frac{T_\alpha}{ht}\right)^{1/2}1_{h^{-1}J^{\leq 1}_{T_\alpha}}(t)\varphi(t)$ and, for $n>1$, $\psi_n(t) = 2^{-n/2}1_{h^{-1}J^n_{T_\alpha}}(t)\varphi(t)$ gives

$$\psi_1(t) = -i\int_0^t\left(\frac{T_\alpha}{ht}\right)^{1/2}e^{\alpha N}\sqrt{h}a_G(V_2)U(t)U(s)^*1_{h^{-1}J^{\leq 1}_{T_\alpha}}(s)u_1(s)\,ds, \quad t\in h^{-1}J^{\leq 1}_{T_\alpha},$$

and, for $n>1$,

$$\psi_n(t) = -i\int_0^t 2^{-n/2} e^{\alpha N}\sqrt{h}a_G(V_2)U(t)U(s)^* 1_{h^{-1}J^{\leq 1}_{T_\alpha}}(s)u_1(s)\,ds$$
$$-i\sum_{1<m\leq n}\int_0^t \sqrt{h}2^{-n/2}e^{\alpha N}a_G(V_2)U(t)U(s)^* 1_{h^{-1}J^m_{T_\alpha}}(s)u_1(s)\,ds, \quad t\in h^{-1}J^n_{T_\alpha}.$$

This allows to rewrite the above decomposition as

$$\psi_1(t) \overset{t\leq \frac{3T_\alpha}{4h}}{=} -i\int_0^{\frac{3T_\alpha}{4h}} \underbrace{1_{[0,t]}(s)\left(\frac{T_\alpha}{T_{\alpha'_0}}\right)^{1/2}\left(\frac{hs}{ht}\right)^{1/2} e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_0 N}}_{B_{11}(t,s)} U(t)U(s)^* w_1(s)\,ds,$$

and, for $n>1$,

$$\psi_n(t) \overset{t\in h^{-1}J^n_{T_\alpha}}{=} -i\int_0^{\frac{T_\alpha}{h}} \underbrace{1_{[0,\frac{3T_\alpha}{4h}]}(s)2^{-n/2}e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_0 N}\left(\frac{hs}{T_{\alpha'_0}}\right)^{1/2}}_{B_{n1}(t,s)} U(t)U(s)^* w_1(s)\,ds$$
$$-i\sum_{m=2}^n\int_0^{\frac{T_\alpha}{h}} \underbrace{1_{[0,t]\cap h^{-1}J^m_{T_\alpha}}(s)2^{-(n-m)/2}e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_m N}}_{B_{nm}(t,s)} U(t)U(s)^* w_m(s)\,ds,$$

$$\text{with}\quad w_1(s) = 1_{h^{-1}J^{\leq 2}_{T_{\alpha'_0}}}(s)\left(\frac{T_{\alpha'_0}}{hs}\right)^{1/2} e^{\alpha'_0 N}u_1(s)$$
$$\text{and}\quad w_m(s) \overset{m>1}{=} 2^{-m/2}1_{h^{-1}J^m_{T_\alpha}}(s)e^{\alpha'_m N}u_1(s) = 2^{-m/2}1_{\left[\frac{T_{\alpha'_m}-3\delta_m}{h},\frac{T_{\alpha'_m}-\delta_m}{h}\right]}(s)e^{\alpha'_m N}u_1(s).$$

Proposition 4.1 tells us

$$\|\psi_1\|_{L^1(h^{-1}J^{\leq 1}_{T_\alpha};L^2_z L^2_{y_G})} \lesssim \left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_0^{\frac{3T_\alpha}{4h}}\|B_{11}(t,s)\|^2\,dt\right)^{1/2}\|w_1\|_{L^1(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_z L^2_{y_G})},$$

and, for $n>1$,

$$\|\psi_n\|_{L^1(h^{-1}J^n_{T_\alpha};L^2_z L^2_{y_G})} \lesssim \left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_{h^{-1}J^n_{T_\alpha}}\|B_{n1}(t,s)\|^2\,dt\right)^{1/2}\|w_1\|_{L^1(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_z L^2_{y_G})}$$
$$+\sum_{m=2}^n\left(\sup_{s\in h^{-1}J^m_{T_\alpha}}\int_{h^{-1}J^n_{T_\alpha}}\|B_{nm}(t,s)\|^2\,dt\right)^{1/2}\|w_m\|_{L^1([\frac{T_{\alpha'_m}-3\delta_m}{h},\frac{T_{\alpha'_m}-\delta_m}{h}];L^2_z L^2_{y_G})}.$$

From the comparison between the norms $N_{1,1}$ and $N_{1,4}$ we know

$$\|w_1\|_{L^1(h^{-1}J^{\leq 2}_{T_{\alpha'_0}};L^2_z L^2_{y_G})} \lesssim \sup_{\tau\in[0,T_{\alpha'_0}[}\sqrt{T_{\alpha'_0}-\tau}\left\|\frac{e^{\alpha'_0 N}u_1}{\sqrt{ht}}\right\|_{L^1([0,\frac{\tau}{h}];L^2_z L^2_{y_G})}$$
$$\lesssim M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1),$$

while for $m>1$,

$$\|w_m\|_{L^1([\frac{T_{\alpha'_m}-3\delta_m}{h},\frac{T_{\alpha'_m}-\delta_m}{h}];L^2_z L^2_{y_G})} \lesssim \left(\frac{T_{\alpha'_m}}{\delta_m}\right)^{1/2}2^{-m/2}\sup_{\tau\in[0,T_{\alpha'_m}[}\sqrt{T_{\alpha'_m}-\tau}\left\|\frac{e^{\alpha'_m N}u_1}{\sqrt{ht}}\right\|_{L^1([0,\frac{\tau}{h}];L^2_z L^2_{y_G})}$$
$$\lesssim \left(\frac{T_\alpha}{T_\alpha 2^{-m-2}}\right)^{1/2}2^{-m/2}M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1)$$
$$\lesssim M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1).$$

We have proved

$$\frac{\sup_{\tau\in[0,T_\alpha[}\sqrt{T_\alpha-\tau}\left\|\frac{e^{\alpha N}L_{11}(u_1)}{\sqrt{ht}}\right\|_{L^1([0,\frac{\tau}{h}];L^2_zL^2_{y_G})}}{M_{\alpha 01}C_V\gamma^{1/2}M_1(u_1)}$$
$$\lesssim\left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_0^{\frac{3T_\alpha}{4h}}\|B_{11}(t,s)\|^2\,dt\right)^{1/2}+\sup_{n\geq 1}\left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_{h^{-1}J^n_{T_\alpha}}\|B_{n1}(t,s)\|^2\,dt\right)^{1/2}$$
$$+\sup_{n>1}\sum_{m=2}^{n}\left(\sup_{s\in h^{-1}J^m_{T_\alpha}}\int_{h^{-1}J^n_{T_\alpha}}\|B_{nm}(t,s)\|^2\,dt\right)^{1/2}. \quad (88)$$

It remains to estimate every term of the above right-hand side.

$\mathbf{L_{11}(u_1)}$-Step 2, Estimate for B_{11}: The expression

$$B_{11}(t,s)=1_{[0,t]}(s)\left(\frac{T_\alpha}{T_{\alpha'_0}}\right)^{1/2}\left(\frac{hs}{ht}\right)^{1/2}e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_0N}$$

implies, with $T_\alpha=\gamma(\alpha_1-\alpha)=7\gamma(\alpha'_0-\alpha)$,

$$\|B_{11}(t,s)\|^2\leq\frac{T_\alpha}{T_{\alpha'_0}}\frac{7\gamma}{T_\alpha}M^2_{\alpha 01}C^2_Vh1_{[0,t]}(s)\frac{hs}{ht}\leq 7\gamma M^2_{\alpha 01}C^2_V\frac{4hs}{3T_\alpha}\frac{1_{[0,t]}(s)}{t}.$$

We obtain

$$\int_0^{\frac{3T_\alpha}{4h}}\|B_{11}(t,s)\|^2\,dt\leq 7\gamma M^2_{\alpha 01}C^2_V\frac{4hs}{3T_\alpha}\ln(\frac{3T_\alpha}{4hs}).$$

and

$$\left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_0^{\frac{3T_\alpha}{4h}}\|B_{11}(t,s)\|^2\,dt\right)^{1/2}\lesssim\gamma^{1/2}M_{\alpha 01}C_V. \quad (89)$$

$\mathbf{L_{11}(u_1)}$-Step 3, Estimate for B_{n1}, $n>1$: From

$$B_{n1}(t,s)=1_{[0,\frac{3T_\alpha}{4h}]}(s)2^{-n/2}e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_0N}\left(\frac{hs}{T_{\alpha'_0}}\right)^{1/2}$$

we deduce with $\alpha'_0-\alpha=\frac{\alpha_1-\alpha}{7}=\frac{T_\alpha}{7\gamma}$ and $T_{\alpha'_0}=\frac{6T_\alpha}{7}$,

$$\|B_{n1}(t,s)\|^2\leq 1_{[0,\frac{3T_\alpha}{4h}]}(s)2^{-n}\frac{hM^2_{\alpha 01}C^2_V}{(\alpha'_0-\alpha)}\left(\frac{3T_\alpha/4}{T_{\alpha'_0}}\right)\leq 1_{[0,\frac{3T_\alpha}{4h}]}(s)\frac{2^{-n}7\gamma h}{T_\alpha}M^2_{\alpha 01}C^2_V.$$

With $J^n_{T_\alpha}=[(1-2^{-n})T_\alpha,(1-2^{-n-1})T_\alpha[$ for $n>1$ we obtain

$$\int_{h^{-1}J^n_{T_\alpha}}\|B_{n0}(t,s)\|^2\,dt\leq 2^{-n-1}T_\alpha\times\frac{2^{-n}7\gamma}{T_\alpha}M^2_{\alpha 01}C^2_V\leq\frac{7\gamma}{4}M^2_{\alpha 01}C^2_V.$$

and

$$\sup_{n>1}\left(\sup_{s\in[0,\frac{3T_\alpha}{4h}]}\int_{h^{-1}J^n_{T_\alpha}}\|B_{n0}(t,s)\|^2\,dt\right)^{1/2}\lesssim\gamma^{1/2}M_{\alpha 01}C_V. \quad (90)$$

$\mathbf{L_{11}(u_1)}$-Step 4, Estimate for the B_{nm}'s, $n,m>1$: From

$$B_{nm}(t,s)=1_{[0,t]\cap h^{-1}J^m_{T_\alpha}}(s)2^{-(n-m)/2}e^{\alpha N}\sqrt{h}a_G(V_2)e^{-\alpha'_mN}$$

and $\alpha'_m - \alpha = 2^{-(m+2)}(\alpha_1 - \alpha) = \frac{2^{-(m+2)}T_\alpha}{\gamma}$, we deduce

$$\|B_{nm}(t,s)\|^2 \le 1_{[0,t]\cap h^{-1}J^m_{T_\alpha}}(s) 2^{-(n-m)} \frac{h 2^{m+2}\gamma}{T_\alpha} M^2_{\alpha 01} C_V^2 .$$

Using again that the length of $J^n_{T_\alpha}$ is $2^{-(n+1)}T_\alpha$, we get

$$\sup_{s\in h^{-1}J^m_{T_\alpha}} \int_{h^{-1}J^n_{T_\alpha}} \|B_{nm}(t,s)\|^2 \, dt \le 2\gamma 2^{-2(n-m)} M^2_{\alpha 01} C_V^2$$

and

$$\sup_{n\ge 1} \sum_{m=1}^{n} \left(\sup_{s\in h^{-1}J^m_{T_\alpha}} \int_{h^{-1}J^n_{T_\alpha}} \|B_{nm}(t,s)\|^2 \, dt \right)^{1/2} \lesssim \gamma^{1/2} M_{\alpha 01} C_V . \tag{91}$$

$\mathbf{L_{1\infty}(u_\infty)}$-Step 1, Decomposition of $\mathbf{L_{1\infty}(u_\infty)}$:
Compared with the decomposition of $\mathbf{L_{22}(u_2)}$ and $\mathbf{L_{11}(u_1)}$, an additional dyadic decomposition has to be done around 0 in order to absorb the weight $\frac{1}{\sqrt{ht}}$ properly and to use Proposition 4.3. Decompose now $[0,T] = \cup_{n\in\mathbb{Z}} J^n_T$ where J^0_T is now the interval $[T/4, T/2[$ and $J^n_T = 2^n J^0_T$ for $n < 0$, according to figure 4. In particular, the interval previously denoted by J^0_T is now $J^{\le 0}_T$ while $J^{\le n_0}_T$ is not changed for $n_0 > 0$.

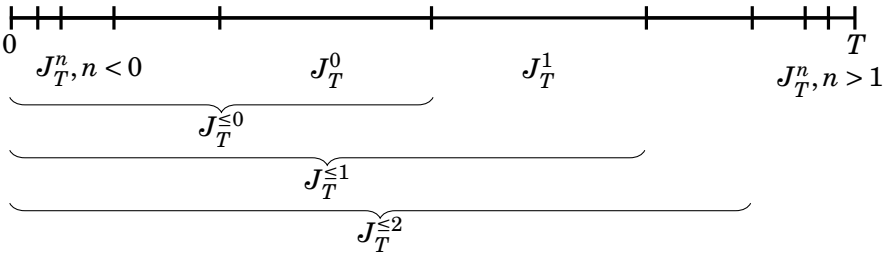

Figure 4: The time intervals J^n_T, $n \in \mathbb{Z}$.

We seek an upper bound of $N_{1,1}(\varphi)$ for

$$\varphi(t) = e^{\alpha N} L_{1\infty}(u_\infty)(t) = -h \int_0^t e^{\alpha N} a_G(V_2) U(t) U(s)^* a_G(V_1)^* u_\infty(s) \, ds .$$

By the equivalence of norms $N_{1,1}$ and $N_{1,3}$ this is equivalent to proving a uniform upper bound for

$$\left\| \left(\frac{T_\alpha}{ht}\right)^{1/2} \varphi \right\|_{L^1(h^{-1}J^{\le 1}_{T_\alpha}; L^2_z L^2_{y_G})} \quad \text{and} \quad 2^{-n/2} \|\varphi\|_{L^1(h^{-1}J^n_{T_\alpha}; L^2_z L^2_{y_G})} \text{ for } n > 1 .$$

But the dyadic decomposition around 0 says

$$\left\| \left(\frac{T_\alpha}{ht}\right)^{1/2} \varphi \right\|_{L^1(h^{-1}J^{\le 1}_{T_\alpha}; L^2_z L^2_{y_G})} \le 2 \sum_{n\le 1} \|2^{-\frac{n+1}{2}} \varphi\|_{L^1(h^{-1}J^n_{T_\alpha}; L^2_z L^2_{y_G})} = 2 \| \sum_{n\le 1} 2^{-\frac{n+1}{2}} 1_{h^{-1}J^n_{T_\alpha}}(t) \varphi \|_{L^1(h^{-1}J^{\le 1}_{T_\alpha}; L^2_z L^2_{y_G})} .$$

$\mathbf{L_{1\infty}(u_\infty)}$-Step 2, Estimate on $\mathbf{h^{-1}J^{n\le 1}_{T_\alpha}}$:
We write $\varphi_1 = \sum_{n\le 1} 2^{-\frac{n+1}{2}} 1_{h^{-1}J^n_{T_\alpha}}(t) e^{\alpha N} L_{1\infty}(u_\infty) = \sum_{n\le 1} \varphi_{1,n}(t)$ where

$$\begin{aligned}
\varphi_{1,n}(t) = & -h \sum_{m=-\infty}^{1} 2^{-\frac{n+1}{2}} 1_{h^{-1}J^n_{T_\alpha}}(t) \times \\
& \int_0^t e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_0}{2} N} U(t) U(s)^* e^{\frac{\alpha+\alpha'_0}{2} N} a^*_G(V_1) e^{-\alpha'_0 N} 1_{h^{-1}J^m_{T_\alpha}}(s) e^{\alpha'_0 N} u^h_\infty(s) \, ds \\
= & -h \sum_{m=-\infty}^{1} 1_{h^{-1}J^n_{T_\alpha}} \int_0^t B_{1n} U(t) U(s)^* B^*_{2m}(s) \varphi_{\infty,m}(s) \, ds
\end{aligned}$$

with

$$B_{1n} = 2^{-\frac{n+1}{2}} e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_0}{2}N} \quad ,$$
$$\|B_{1n}\|_{L^2_z L^2_{y_G} \leftarrow L^2_z L^{r_\sigma}_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_2\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_0-\alpha}} 2^{-n/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha_1}^{1/2}} 2^{-n/2},$$
$$B^*_{2m} = e^{\frac{\alpha+\alpha'_0}{2}N} a^*_G(V_1) e^{-\alpha'_0 N} 1_{h^{-1}J^m_{T_\alpha}}(s) \frac{\sqrt{hs}}{\sqrt{T_\alpha - hs}} \quad ,$$
$$\|B^*_{2m}\|_{L^2_z L^{r'_\sigma}_{y_G} \leftarrow L^2_z L^2_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_1\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_0-\alpha}} 2^{m/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha}^{1/2}} 2^{m/2},$$
$$\varphi_{\infty,m}(s) = 1_{h^{-1}J^m_{T_\alpha}}(s)\varphi_\infty(s) \quad , \quad \varphi_\infty(s) = e^{\alpha'_0 N}\frac{\sqrt{T_\alpha - hs}}{\sqrt{hs}} u_\infty(s).$$

By noticing

$$|h^{-1}J^n_{T_\alpha}| \leq T_\alpha h^{-1} 2^n$$

the upper bound of Proposition 4.3 gives

$$\begin{aligned}\|\varphi_1\|_{L^1(h^{-1}J^{\leq 1}_{T_\alpha}; L^2_z L^2_{y_G})} &\lesssim \left[\sum_{-\infty\leq m\leq n\leq 1} 2^{n/2}(M_{\alpha 01}C_V\gamma^{1/2}2^{-n/2})(M_{\alpha 01}C_V\gamma^{1/2}2^{m/2})\right]\|\varphi_\infty\|_{L^\infty(h^{-1}J^{\leq 1}_{T_\alpha}; L^2_z L^2_{y_G})} \\ &\lesssim M^2_{\alpha 01}C^2_V\gamma M_\infty(u_\infty).\end{aligned}$$

We proved

$$\|\frac{1}{\sqrt{ht}} e^{\alpha N} L_{1\infty}(u_\infty)\|_{L^1(h^{-1}J^{\leq 1}_{T_\alpha}; L^2_z L^2_{y_G})} \lesssim M_{\alpha 01}C^2_V\gamma M_\infty(u_\infty). \tag{92}$$

$L_{1\infty}(u_\infty)$-Step 3, Estimate on $h^{-1}J^n_{T_\alpha}$, $n>1$:
Write $\varphi_1(t) = 2^{-n/2} 1_{h^{-1}J^n_{T_\alpha}}(t) e^{\alpha N} L_{1\infty}(u_\infty)$, where

$$\begin{aligned}\varphi_1(t) = &-h\sum_{m=-\infty}^{1} 2^{-\frac{n}{2}} 1_{h^{-1}J^n_{T_\alpha}}(t)\times \\ &\int_0^t e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_0}{2}N} U(t)U(s)^* e^{\frac{\alpha+\alpha'_0}{2}N} a^*_G(V_1) e^{-\alpha'_0 N} 1_{h^{-1}J^m_{T_\alpha}}(s) e^{\alpha'_0 N} u^h_\infty(s)\, ds \\ &-h\sum_{m=2}^{n} 2^{-\frac{n}{2}} 1_{h^{-1}J^n_{T_\alpha}}(t)\times \\ &\int_0^t e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_m}{2}N} U(t)U(s)^* e^{\frac{\alpha+\alpha'_m}{2}N} a^*_G(V_1) e^{-\alpha'_m N} 1_{h^{-1}J^m_{T_\alpha}}(s) e^{\alpha'_m N} u^h_\infty(s)\, ds \\ = &-h\sum_{m=-\infty}^{1} 1_{h^{-1}J^n_{T_\alpha}}(t) \times \int_0^t B_{1n}U(t)U(s)^* B^*_{2m}(s)\varphi_{\infty,m}(s)\, ds \\ &-h\sum_{m=2}^{n} 1_{h^{-1}J^n_{T_\alpha}}(t) \times \int_0^t B_{1nm}U(t)U(s)^* B^*_{2nm}(s)\varphi_{\infty,m}(s)\, ds.\end{aligned}$$

The family $\mathscr{I}$ of Proposition 4.3 is made here of the single interval $h^{-1}J^n_{T_\alpha}$ while the family $\mathscr{J} = \left\{h^{-1}J^m_{T_\alpha}, m\leq n\right\}$ is splitted in two parts $m\leq 1$ and $2\leq m\leq n$. In the last two

lines the notations correspond to

$$B_{1n} = 2^{-\frac{n}{2}} e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_0}{2}N} \quad ,$$

$$\|B_{1n}\|_{L^2_z L^2_{y_G} \leftarrow L^2_z L^{r_\sigma}_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_2\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_0-\alpha}} 2^{-n/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha_1}^{1/2}} 2^{-n/2},$$

$m \leq 1$ $$B^*_{2m} = e^{\frac{\alpha+\alpha'_0}{2}N} a^*_G(V_1) e^{-\alpha'_0 N} 1_{h^{-1}J^m_{T_\alpha}}(s) \frac{\sqrt{hs}}{\sqrt{T_\alpha - hs}} \quad ,$$

$m \leq 1$ $$\|B^*_{2m}\|_{L^2_z L^{r'_\sigma}_{y_G} \leftarrow L^2_z L^2_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_1\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_0-\alpha}} 2^{m/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha}^{1/2}} 2^{m/2},$$

$m \geq 2$ $$B_{1nm} = 2^{-\frac{n}{2}} e^{\alpha N} a_G(V_2) e^{-\frac{\alpha+\alpha'_m}{2}N} \quad ,$$

$m \geq 2$ $$\|B_{1nm}\|_{L^2_z L^2_{y_G} \leftarrow L^2_z L^{r_\sigma}_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_2\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_m-\alpha}} 2^{-n/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha_1}^{1/2}} 2^{m/2-n/2}$$

$m \geq 2$ $$B^*_{2nm} = e^{\frac{\alpha+\alpha'_m}{2}N} a^*_G(V_1) e^{-\alpha'_m N} 1_{h^{-1}J^m_{T_\alpha}}(s) \frac{\sqrt{hs}}{\sqrt{T_\alpha - hs}} \quad ,$$

$m \geq 2$ $$\|B^*_{2mn}\|_{L^2_z L^{r'_\sigma}_{y_G} \leftarrow L^2_z L^2_{y_G}} \lesssim \frac{M_{\alpha 01}\|V_1\|_{L^{r'_\sigma}}}{\sqrt{\alpha'_m-\alpha}} 2^{m/2} \lesssim \frac{M_{\alpha 01} C_V \gamma^{1/2}}{T_{\alpha}^{1/2}} 2^{m},$$

$$\varphi_m(s) = 1_{h^{-1}J^m_{T_\alpha}}(s)\varphi_\infty(s) \quad ,$$

$$\varphi_\infty(s) = 1_{h^{-1}J^{\leq 1}_{T_\alpha}}(s) e^{\alpha'_0 N} \frac{\sqrt{T_\alpha - hs}}{\sqrt{hs}} u_\infty(s) + \sum_{m=2}^{\infty} e^{\alpha'_m N} 1_{h^{-1}J^m_{T_\alpha}}(s) \frac{\sqrt{T_\alpha - hs}}{\sqrt{hs}} u_\infty(s).$$

The size of the intervals are estimated respectively by $|h^{-1}J^n_{T^\alpha}| \lesssim h^{-1}2^{-n}T_\alpha$ and

$$|h^{-1}J^m_{T_\alpha}| \lesssim h^{-1}2^m T_\alpha \quad \text{for } m \leq 1 \quad , \quad |h^{-1}J^m_{T_\alpha}| \lesssim h^{-1}2^{-m}T_\alpha \quad \text{for} \quad m \geq 2.$$

Proposition 4.3 gives

$$\begin{aligned}\frac{\|\varphi_1\|_{L^1(h^{-1}J^n_{T_\alpha};L^2_z L^2_{y_G})}}{\|\varphi_\infty\|_{L^\infty(h^{-1}J^{\leq n}_{T_\alpha};L^2_z L^2_{y_G})}} &\lesssim \left[\sum_{-\infty \leq m \leq 1} 2^{-n/2}(M_{\alpha 01}C_V\gamma^{1/2}2^{-n/2})(M_{\alpha 01}C_V\gamma^{1/2}2^{m/2})2^{m/2}\right] \\ &\quad + \left[\sum_{m=2}^{n} 2^{-n/2}(M_{\alpha 01}C_V\gamma^{1/2}2^{m/2-n/2})(M_{\alpha 01}C_V\gamma^{1/2}2^{m})2^{-m/2}\right] \\ &\lesssim M_{\alpha 01}C_V^2\gamma.\end{aligned}$$

With

$$\|\varphi_\infty\|_{L^\infty(h^{-1}J^{\leq n}_{T_\alpha};L^2_z L^2_{y_G})} \leq \|\varphi_\infty\|_{L^\infty([0,T_\alpha/h[;L^2_z L^2_{y_G})} \leq M_\infty(u_\infty),$$

we have proved

$$\sup_{n\geq 1} 2^{-n/2}\|e^{\alpha N} L_{1\infty}(u_\infty)\|_{L^1(h^{-1}J^n_{T_\alpha};L^2_z L^2_{y_G})} \lesssim M_{\alpha 01}C_V^2\gamma M_\infty(u_\infty). \tag{93}$$

Conclusion. From (71), (72) and (73) we deduce

$$M_\infty(L_{\infty\infty}(u_\infty) + L_{\infty 2}(u_2) + L_{\infty 1}(u_1)) \lesssim \gamma^{1/2} M_{\alpha 01} C_V M(u_\infty, u_2, u_1). \tag{94}$$

Combining (88), (89), (90), (91), and taking the supremum over $\alpha \in [\alpha_0, \alpha_1[$ yields

$$M_1(L_{11}(u_1)) \lesssim \gamma^{1/2} M_{\alpha 01} C_V M_1(u_1),$$

while (82) says

$$M_2(L_{22}(u_2)) \lesssim C_V M_{\alpha 01}\sqrt{\gamma} M_2(u_2).$$

Finally the upper bounds (92),(93) combined, firstly with the equivalence of norms N_{11} and N_{31}, and secondly the normalization of (67) of M_1 yields

$$M_1(L_{1\infty}(u_\infty)) \lesssim \gamma^{1/2} M_{\alpha 01} C_V M(u_\infty, u_2, u_1).$$

The sum of all those inequalities is

$$M(L(u_\infty, u_2, u_1)) \lesssim \gamma^{1/2} M_{\alpha 01} C_V M(u_\infty, u_2, u_1),$$

which means that there exists a constant $C_{d,U}$ determined by the dimension d and the free dynamics U such that

$$\|L\|_{\mathscr{L}(\mathscr{E}^h_{\alpha_0,\alpha_1,\gamma})} \le C_{d,U} M_{\alpha 01} C_V \gamma^{1/2}.$$

□

Lemma 4.6. *The norms $N_{p,1}, N_{p,2}, N_{p,3}, N_{p,4}$ defined in (75)(76)(77)(78) for $p = 2$ (resp. (83)(84)(85)(86) for $p = 1$) are equivalent according to (74) (resp. (87)).*

Proof. We forget the notation $L^2_z L^2_{y_G}$ because it is a time integration issue and it can be done with any Banach space valued functions.
With the Definition (77) of $N_{2,3}(\varphi)$, the equality

$$\|\varphi\|_{L^2(h^{-1}J_T^{\le 1})} = \left(\|\varphi\|^2_{L^2(h^{-1}J_T^0)} + \|\varphi\|^2_{L^2(h^{-1}J_T^1)}\right)^{1/2}$$

allows to replace $N_{2,3}(\varphi)$ by the equivalent norm

$$\sqrt{T}\|\varphi\|_{L^2(h^{-1}J_T^{\le 1})} + \sqrt{T}\sup_{n>1} 2^{-n/2}\|\varphi\|_{L^2(h^{-1}J_T^n)}$$

For $p = 1$, the inequality

$$\forall t \in [\frac{3T}{4h}, \frac{T}{h}[, \quad \frac{1}{T} \le \frac{1}{ht} \le \frac{4}{3T}$$

allows to replace the second term of the definitions (84) (85)(86) of $N_{1,2}$, $N_{1,3}$ and $N_{1,4}$, respectively by

$$\begin{aligned}
&\sup_{\delta\in]0,T/8]} \sqrt{\delta}\left\|\frac{\varphi}{\sqrt{ht}}\right\|_{L^1(h^{-1}[T-2\delta,T-\delta[)} \\
&\sup_{n>1}\sqrt{T}2^{-n/2}\left\|\frac{\varphi}{\sqrt{ht}}\right\|_{L^1(h^{-1}J_T^n)}, \\
&\sup_{\delta\in]0,T/12]} \sqrt{\delta}\left\|\frac{\varphi}{\sqrt{ht}}\right\|_{L^1(h^{-1}[T-3\delta,T-\delta[)}.
\end{aligned}$$

Additionally the $\sup_{\tau\in[0,T[}$ in the definitions (75)(83) can be replaced by $\sup_{\tau\in[3T/4,T[}$. We are thus led to compare the norms, for $p = 1, 2$,

$$\begin{aligned}
N_{p,1,T,h}(\varphi) &= \sup_{\tau\in[3T/4,T[} \sqrt{T-\tau}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p([0,\frac{\tau}{h}])}, \\
N_{p,2,T,h}(\varphi) &= \sqrt{T}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}J_T^{\le 1})} + \sup_{\delta\in]0,T/8]}\sqrt{\delta}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}[T-2\delta,T-\delta])}, \\
N_{p,3,T,h}(\varphi) &= \sqrt{T}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}J_T^{\le 1})} + \sqrt{T}\sup_{n>1}2^{-n/2}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}J_T^n)}, \\
N_{p,4,T,h}(\varphi) &= \sqrt{T}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}J_T^{\le 2})} + \sup_{\delta\in]0,T/12]}\sqrt{\delta}\left\|\frac{\varphi}{(ht)^{1/p-1/2}}\right\|_{L^p(h^{-1}[T-3\delta,T-\delta])}.
\end{aligned}$$

The elementary homogeneity of those expressions gives

$$N_{p,i,T,h}(\varphi) = \frac{T}{h^{1/p}} N_{p,i,1,1}(\tilde{\varphi}) \quad \text{with } \tilde{\varphi}(t) = \varphi(ht) \quad \text{for } p = 1,2 \quad \text{and } 1 \le i \le 4,$$

and it suffices to consider the case $T = h = 1$ while setting $\psi = \frac{\tilde{\varphi}}{t^{1/p-1/2}}$.
For $\tau \in [3/4, 1[$ the identity

$$\|\psi\|_{L^p([0,\tau])} = \left(\|\psi\|^p_{L^p(J_1^{\le 1})} + \|\psi\|^p_{L^p([3/4,\tau])} \right)^{1/p}$$

reduces the comparison of $N_{p,1,1,1}(\tilde{\varphi})$, $N_{p,2,1,1}(\tilde{\varphi})$ and $N_{p,3,1,1}(\tilde{\varphi})$ to the comparison of

$$\begin{aligned}
A_1(\psi) &= \sup_{\tau \in [3/4,1[} \sqrt{1-\tau}\, \|\psi\|_{L^p([3/4,\tau])}, \\
A_2(\psi) &= \sup_{\delta \in]0,1/8]} \sqrt{\delta}\, \|\psi\|_{L^p[1-2\delta,1-\delta])}, \\
A_3(\psi) &= \sup_{n>1} 2^{-n/2} \|\psi\|_{L^p(J_1^n)}.
\end{aligned}$$

Taking $\tau = 1-\delta$, $\delta \le 1/8$, in $A_1(\psi)$ and $\delta = 2^{-n-1}$, $n > 1$, in $A_2(\psi)$ gives

$$A_3(\psi) \le \sqrt{2} A_2(\psi) \le \sqrt{2} A_1(\psi).$$

For $\tau \in [3/4, 1[$ there exists $n_\tau > 1$ such that $\tau \in [1-2^{-n_\tau}, 1-2^{-n_\tau-1}[= J_1^{n_\tau}$ and

$$\|\psi\|^p_{L^p([3/4,\tau])} = \sum_{n=2}^{n_\tau} \|\psi\|^p_{L^p(J_1^n)} \le \sum_{n=2}^{n_\tau} 2^{np/2} (2^{-n/2} \|\psi\|_{L^p(J_1^n)})^p \le \frac{2^{p(n_\tau+1)/2}}{2^{p/2}-1} A_3(\psi)^p.$$

The inequality

$$(1-2^{-n_\tau}) \le \tau \quad \text{or} \quad \sqrt{1-\tau} \le 2^{-n_\tau/2},$$

while taking the supremum over $\tau \in [3/4, 1[$, implies

$$A_1(\psi) \le \frac{\sqrt{2}}{(2^{p/2}-1)^{1/p}} A_3(\psi).$$

We have proved the equivalence

$$\kappa_{p,1}^{-1} N_{p,1}(\varphi) \le N_{p,i}(\varphi) \le \kappa_{p,1} N_{p,1}(\varphi) \quad \text{for } p = 1,2,\ i = 2,3,$$

with a universal constant $\kappa_{p,1} > 1$.
It now suffices to compare $N_{p,2}$ and $N_{p,4}$ or equivalently $N_{p,2,1,1}(\tilde{\varphi})$ and $N_{p,4,1,1}(\tilde{\varphi})$ written with $\psi = \frac{\tilde{\varphi}}{t^{1/p-1/2}}$

$$\begin{aligned}
N_{p,2,1,1}(\tilde{\varphi}) &= \|\psi\|_{L^p(J_1^{\le 1})} + \sup_{\delta \in]0,1/8]} \sqrt{\delta} \|\psi\|_{L^p([1-2\delta,1-\delta])} =: B_2(\psi), \\
N_{p,4,1,1}(\tilde{\varphi}) &= \|\psi\|_{L^p(J_1^{\le 2})} + \sup_{\delta \in]0,1/12]} \sqrt{\delta} \|\psi\|_{L^p([1-3\delta,1-\delta])} =: B_4(\psi).
\end{aligned}$$

For the first terms of $B_2(\psi)$ and $B_4(\psi)$,

$$\|\psi\|^p_{L^p(J_1^{\le 1})} \le \|\psi\|^p_{L^p(J_1^{\le 2})} = \|\psi\|^p_{L^p(J_1^{\le 1})} + \|\psi\|^p_{L^p(J_1^2)}$$

gives

$$\|\psi\|_{L^p(J_1^{\le 1})} \le \|\psi\|_{L^p(J_1^{\le 2})} \le \|\psi\|_{L^p(J_1^{\le 1})} + \sup_{\delta \in]0,1/8]} \|\psi\|_{L^p([1-2\delta,1-\delta])}.$$

For the second terms of $B_2(\psi)$ and $B_4(\psi)$,

$$\sqrt{\delta}\|\psi\|_{L^p([1-3\delta,1-3\delta/2])} \le \sqrt{\delta}\|\psi\|_{L^p([1-3\delta,1-\delta])} \le \sqrt{\delta}\|\psi\|_{L^p([1-3\delta,1-3\delta/2])} + \sqrt{\delta}\|\psi\|_{L^p([1-2\delta,1-\delta])}$$

leads to

$$(2/3)^{1/2} \sup_{\delta\in]0,1/8]} \sqrt{\delta}\|\psi\|_{L^p([1-3\delta,1-\delta])} \le \sup_{\delta\in]0,1/12]} \sqrt{\delta}\|\psi\|_{L^p([1-2\delta,1-\delta])} \le 2 \sup_{\delta\in]0,1/8]} \sqrt{\delta}\|\psi\|_{L^p([1-3\delta,1-\delta])}.$$

Adding the two terms yields the equivalences

$$\begin{aligned} & \kappa_{p,2}^{-1} B_2(\psi) \le B_4(\psi) \le \kappa_{p,2} B_2(\psi) \\ \text{and} \quad & \kappa_{p,2}^{-1} N_{2,2}(\varphi) \le N_{2,4}(\varphi) \le \kappa_{p,2} N_{2,2}(\varphi) \end{aligned}$$

for a universal constant $\kappa_{p,2} > 1$. The proof ends by taking $\kappa_p = \kappa_{p,1}\kappa_{p,2} > 1$. □

5 Consequences of Strichartz estimates for our model problem

The general results of Section 4 are applied to our model problem presented in Subsection 2.3.

5.1 Validity of the general hypotheses and main result

Let us consider (20)

$$\begin{cases} i\partial_t \hat{f}^h = (\xi - d\Gamma(D_y))^2 \hat{f}^h + \sqrt{h}[a(V) + a^*(V)]\hat{f}, \\ \hat{f}^h(t=0) = \hat{f}_0^h, \end{cases} \tag{95}$$

where $\hat{f}^h(t) \in L^2(\mathbb{R}^d \times Z'', \frac{d\xi}{(2\pi)^d} \otimes \mathbf{dz}''; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C})))$, ξ is the Fourier variable of $x \in \mathbb{R}^d$ and $z'' \in Z''$ is a parameter, e.g. $L^2(Z'', \mathbf{dz}'') = L^2(\mathbb{R}^d, \frac{d\xi}{(2\pi)^d}; \mathbb{C}) \otimes \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))$ when we want to handle the evolution of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d, \frac{d\xi}{(2\pi)^d}; \mathbb{C}) \otimes \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))$ as described in the end of Section 2.3. Our complete parameter is thus

$$z' = (\xi, z') \in \mathbb{R}^d \times Z'' = Z'$$

and remember the writing introduced in Definition 3.4 and specified in (47) and (58)

$$L^2(\mathbb{R}^d \times Z'', \frac{d\xi}{(2\pi)^d} \otimes \mathbf{dz}; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))) = L^2_{z,\mathrm{sym}} L^2_{y_G} = \underbrace{L^2_{z_0}}_{\text{vacuum}} \oplus L^2_{z_1,\mathrm{sym}} L^2_{y_G}.$$

$$\text{with} \qquad Z_0 = Z' \quad , \quad Z_1 = \mathscr{R} \otimes Z',$$

where the subscript $_{\mathrm{sym}}$ refers to the symmetry for the relative coordinate variable $Y' \in \mathscr{R}$.

Using the center of mass variable (see Section 3) by setting $t \mapsto u_G^h(t) = U_G^{-1}\hat{f}^h(t)$, (95) becomes

$$\begin{cases} i\partial_t u_G^h = (\xi - D_{y_G})^2 u_G^h + \sqrt{h}[a_G^*(V) + a_G(V)]u_G^h, \\ u_G^h(t=0) = u_{G,0}^h. \end{cases} \tag{96}$$

In this context, the free dynamics $U(t)$ involved in (57) acts simply on $L^2_z L^2_{y_G}$ and equals

$$U(t) = K_0(t, z') \oplus U_1(t, Y', z') = (e^{-it|\xi|^2} \times_{z'}) \oplus (e^{-it(\xi - D_{y_G})^2} \times_{(Y', z')})$$

where we recall $z_0 = z' \in Z'$ and $z_1 = (Y', z') \in \mathscr{R} \times Z'$. Because $\|e^{\pm i\xi\cdot y}\varphi\|_{L^p_y} = \|\varphi\|_{L^p_y}$ for all $1 \le p \le +\infty$ and $e^{-itD_y^2} = e^{it\Delta_y}$ satisfies

$$\|e^{it\Delta_y} f\|_{L^2_y} \le \|f\|_{L^2_y},$$
$$\|e^{it\Delta_y}(e^{is\Delta_y})^* g\|_{L^\infty_y} = \|e^{i(t-s)\Delta_y} g\|_{L^\infty_y} \le \frac{\|g\|_{L^1_y}}{(4\pi)^{d/2}|t-s|^{d/2}} \quad t \ne s,$$

the assumption (42)(43) are satisfied for $U_1(t, z_1)$, $z_1 = (Y', z')$, as soon as $d \ge 3$ with $\sigma = \frac{d}{2} > 1$, uniformly with respect to $z_1 \in \mathscr{R} \times Z'$.
The dense subset D in $L^2_{z_1} L^2_{y_G}$ such that $D \subset L^2(Z_1, \mathbf{dz_1}; L^{r_\sigma}(\mathbb{R}^d, dy_G; \mathbb{C}))$, with $r^\sigma = \frac{2d}{d-2}$ and $d \ge 3$ here, is simply $D = L^2(Z_1, dz_1; H^\mu(\mathbb{R}^d))$ with $\mu > d/2$. Remember that the dense subset D was introduced in Proposition 4.1 for the dense *a priori* definition of the operator A_T on $L^1([0,T]; L^2_{z,x})$ (see Remark 4.2 and the proof of Proposition 4.1).
Below are reviewed assumptions on V:

1. If $V \in L^{\frac{2d}{d+2}}(\mathbb{R}^d, dy; \mathbb{R})$, the assumptions of Proposition 4.5 are satisfied with $C_V = 1 + \|V\|_{L^{\frac{2d}{d+2}}} > 0$ and $r'_\sigma = \frac{2\sigma}{\sigma+1} = \frac{2d}{d+2}$.
2. If $V \in H^2(\mathbb{R}^d; \mathbb{R})$ then (95) (or (96)) defines a unitary dynamics with a rather well understood domain of its generator in $L^2(\mathbb{R}^d \times Z'', \frac{d\xi}{(2\pi)^d} \otimes \mathbf{dz''}; \Gamma(L^2(\mathbb{R}^d, dy; \mathbb{C}))) \simeq L^2_z L^2_{y_G}$.

We will always assume $V \in L^{r'_\sigma}$ in the sequel, and depending on the statement we might assume that $V \in H^2$ or not.

If $V \in H^2(\mathbb{R}^d; \mathbb{R})$, the unique solution $t \mapsto u^h_G(t) = U_G^{-1}\hat{f}^h(t) \in \mathscr{C}^0(\mathbb{R}; L^2_z L^2_{y_G})$ to (96) satisfies

$$u^h_G(t) = U(t)u^h_{G,0} - i\int_0^t U(t)U(s)^*\sqrt{h}[a^*_G(V) + a_G(V)]u^h_G(s)\, ds, \tag{97}$$

and we will now seek for a solution of this equation using the fixed point method developed in Section 4.2, for $V \in L^{r'_\sigma}$ but not necessarily $V \in H^2(\mathbb{R}^d; \mathbb{R})$.

If $(u^h_\infty, u^h_2, u^h_1)$ solves

$$u^h_\infty(t) = -i\int_0^t U(t)U^*(s)\left(\sqrt{h}a^*_G(V)u^h_\infty(s) + \sqrt{h}u^h_2(s) + u^h_1(s)\right) ds + f^h_\infty(t), \tag{98}$$
$$u^h_2(t) = -i\int_0^t a_G(V)U(t)U(s)^*\sqrt{h}u^h_2(s)\, ds \qquad + f^h_2(t), \tag{99}$$
$$u^h_1(t) = -i\int_0^t a_G(V)U(t)U(s)^*\left(ha^*_G(V)u^h_\infty(s) + \sqrt{h}u^h_1(s)\right) ds, \tag{100}$$

with

$$f^h_\infty(t) = -i\int_0^t U(t)U(s)^* a^*_G(V)\sqrt{h}U(s)u^h_{G,0}\, ds, \tag{101}$$
$$f^h_2(t) = -i\, a_G(V)\int_0^t U(t)U(s)^* a^*_G(V)\sqrt{h}U(s)u^h_{G,0}\, ds + a_G(V)U(t)u^h_{G,0}, \tag{102}$$

written shortly as

$$\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} = L\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} + \begin{pmatrix} f^h_\infty \\ f^h_2 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_{\infty\infty} & L_{\infty 2} & L_{\infty 1} \\ 0 & L_{22} & 0 \\ L_{1\infty} & 0 & L_{11} \end{pmatrix}, \tag{103}$$

then $u^h_G(t) = u_\infty(t) + U(t)u^h_{G,0}$ will yield a solution to (97).

Actually, with $u^h_G(t) = u^h_\infty(t) + U(t)u^h_{G,0}$, applying $a_G(V)$ to (98) on the one hand, and summing $\sqrt{h}\times$(99) and (100) on the other hand yields $\sqrt{h}a_G(V)u^h_G = u^h_1 + \sqrt{h}u^h_2$, which inserted in (98) provides (97).

Theorem 5.1. *Assume $d \geq 3$ and $V \in L^{\frac{2d}{d+2}}(\mathbb{R}^d, dy; \mathbb{R})$ with*

$$\|V\|_{L^{\frac{2d}{d+2}}} < C_V .$$

Assume that there exists $\alpha_1 > 0$ and $C_{\alpha_1} > 0$ such that

$$\forall h \in]0, h_0[, \quad \|e^{\alpha_1 N} u^h_{G,0}\|_{L^2_z L^2_{y_G}} \leq C_{\alpha_1} .$$

There exists a constant $C_d > 0$ depending on the dimension $d \geq 3$, such that when $\gamma > 0$ is chosen such that

$$2\|L\|_{\mathscr{L}(\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma})} \leq C_d e^{\alpha_1} C_V \gamma^{1/2} \leq 1,$$

the function $u^h_G(t) = u^h_\infty(t) + U(t)u^h_{G,0}$ with $(u^h_\infty, u^h_2, u^h_1)$ the unique solution to (103) in $(\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma}, M)$ satisfies

$$\forall t \in I^h_{T_\alpha}, \quad \left\| e^{\alpha N}[u^h_G(t) - U(t)u^h_{G,0}] \right\|_{L^2_z L^2_{y_G}} \leq C_d C_V e^{\alpha_1} C_{\alpha_1} \gamma^{1/2} \frac{\sqrt{|ht|}}{\sqrt{T_\alpha - |ht|}}, \tag{104}$$

with

$$T_\alpha = \gamma(\alpha_1 - \alpha) \tag{105}$$

for all $\alpha \in [0, \alpha_1[$ and all $h \in]0, h_0[$.

If, moreover, $V \in H^2(\mathbb{R}^d; \mathbb{R})$, then u^h_G is the only solution to (96) in $\mathscr{C}^0(I^h_{T_0}; L^2_z L^2_{y_G})$.

Proof. We take $\alpha_0 = -\alpha_1$ where $\alpha_1 > 0$ is fixed. The constant $M_{\alpha 01}$ of Definition 4.4 is nothing but

$$M_{\alpha 01} = \frac{e^{\alpha_1}}{2}.$$

Accordingly to Definition 4.4, for a fixed $\gamma > 0$ the time scale T_α is given by $T_\alpha = \gamma(\alpha_1 - \alpha)$ for all $\alpha \in [-\alpha_1, \alpha_1[$. Proposition 4.5 tells us that the condition

$$C_{d,U} \frac{e^{\alpha_1}}{2} C_V \gamma^{1/2} \leq \frac{1}{2}$$

where $C_{d,U} = C_d$ is determined by the dimension $d \geq 3$ here, ensures that the operator L is a contraction in $\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma}$ for all $h \in]0, h_0[$:

$$\|L\|_{\mathscr{L}(\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma})} \leq \frac{1}{2}.$$

If $M(f^h_\infty, f^h_2, 0) < \infty$, then the system (103) admits a unique solution in $\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma}$ for all $h \in]0, h_0[$ with

$$M(u^h_\infty, u^h_2, u^h_1) \leq 2M(f^h_\infty, f^h_2, 0).$$

It remains to check two things:

- the right-hand side $(f^h_\infty, f^h_2, 0)$ given by (101)(102) belongs to $\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma}$ and to estimate $M(f^h_\infty, f^h_2, 0)$;
- the unique solution $(u^h_\infty, u^h_2, u^h_1)$ to (103) yields after setting $u^h_G(t) = u^h_\infty(t) + U(t)u^h_{G,0}$ the unique solution to (18) in $\mathscr{C}^0(]-\frac{T_0}{h}, \frac{T_0}{h}[; L^2_{z,y_G})$.

The first step is simpler than what we did for Proposition 4.5. Let us start with

$$\begin{aligned} e^{\alpha N} f^h_\infty(t) &= -i\int_0^t U(t)U(s)^* e^{\alpha N}\sqrt{h}a^*_G(V)e^{-\alpha_1 N}U(s)e^{\alpha_1 N}u^h_{G,0}\,ds \\ &= \int_0^t U(t)U(s)^* F(s)\,ds \end{aligned}$$

with $F(s) = -i1_{[0,t]}(s)e^{\alpha N}\sqrt{h}a^*_G(V)e^{-\alpha_1 N}U(s)e^{\alpha_1 N}u^h_{G,0}$. By Proposition 3.8 we know that

$$\|F\|_{L^2_s L^2_{z_1} L^{r'_\sigma}_{y_G}} \le \frac{C_V e^{\alpha_1}}{2\sqrt{\alpha_1-\alpha}}|ht|^{1/2}\|e^{\alpha_1 N}u^h_{G,0}\|_{L^2_z L^2_{y_G}} \le \frac{C_V e^{\alpha_1}\gamma^{1/2}}{\sqrt{T_\alpha}}C_{\alpha_1}|ht|^{1/2}.$$

A direct application of the retarded endpoint Strichartz estimate (46) yields

$$\left(\frac{T_\alpha - |ht|}{|ht|}\right)^{1/2}\|e^{\alpha N}f^h_\infty(t)\|_{L^2_z L^2_{y_G}} \lesssim C_V e^{\alpha_1}C_{\alpha_1}\gamma^{1/2}.$$

and by taking the supremum over $|ht| < T_\alpha$,

$$M(f^h_\infty, 0) \lesssim C_V e^{\alpha_1}C_{\alpha_1}\gamma^{1/2}. \tag{106}$$

For

$$f^h_{2,1}(t) = -i\,a_G(V)\int_0^t U(t)U(s)^* a^*_G(V)\sqrt{h}U(s)u^h_{G,0}\,ds,$$

the Proposition 3.8 and the retarded Strichartz estimate (44) give

$$\begin{aligned} &\sqrt{T_\alpha-\tau}\|e^{\alpha N}f^h_{2,1}\|_{L^2_t(I^h_\tau;L^2_z L^2_{y_G})} \\ &\quad\lesssim \sqrt{T_\alpha-\tau}\frac{C_V e^{\alpha_1}}{\sqrt{\alpha_1-\alpha}}\left\|\int_0^t U(t)U(s)^* e^{\frac{\alpha+\alpha_1}{2}N}a^*_G(V)\sqrt{h}U(s)u^h_{G,0}\,ds\right\|_{L^2_{z_1}L^2_t(I^h_\tau;L^{r_\sigma}_{y_G})} \\ &\quad\lesssim \sqrt{T_\alpha-\tau}\frac{C_V e^{\alpha_1}}{\sqrt{\alpha_1-\alpha}}\left\|e^{\frac{\alpha+\alpha_1}{2}N}a^*_G(V)\sqrt{h}U(s)u^h_{G,0}\right\|_{L^2_{z_1}L^2_s(I^h_\tau;L^{r'_\sigma}_{y_G})}, \end{aligned}$$

where here $r'_\sigma = \frac{2d}{d+2}$ and $r_\sigma = \frac{2d}{d-2}$.
Then using Proposition 3.8 again, the square integrability of 1 on I^h_τ and the boundedness of $U(s)$ in the L^2 norm,

$$\begin{aligned} \sqrt{T_\alpha-\tau}\|e^{\alpha N}f^h_{2,1}\|_{L^2_t(I^h_\tau;L^2_z L^2_{y_G})} &\lesssim \sqrt{T_\alpha-\tau}\frac{C_V^2 e^{2\alpha_1}}{\alpha_1-\alpha}\left\|e^{\alpha_1 N}\sqrt{h}U(s)u^h_{G,0}\right\|_{L^2_z L^2_s(I^h_\tau;L^2_{y_G})} \\ &\lesssim C_V^2 e^{2\alpha_1}\gamma\frac{\sqrt{T_\alpha-\tau}\sqrt{\tau}}{T_\alpha}\left\|e^{\alpha_1 N}U(s)u^h_{G,0}\right\|_{L^\infty_s(I^h_\tau;L^2_z L^2_{y_G})} \\ &\lesssim C_V^2 e^{2\alpha_1}\gamma\left\|e^{\alpha_1 N}u^h_{G,0}\right\|_{L^2_z L^2_{y_G}} \end{aligned}$$

By taking the supremum w.r.t. $\alpha \in [-\alpha_1, \alpha_1[$ and dividing by $C_V e^{\alpha_1}\gamma^{1/2}/2$ we obtain

$$M_2(f^h_{2,1}) \lesssim C_V e^{\alpha_1}C_{\alpha_1}\gamma^{1/2}. \tag{107}$$

It remains to control

$$f^h_{2,2}(t) = a_G(V)U(t)u^h_{G,0}.$$

For $-\alpha_1 \le \alpha < \alpha_1$ and $0 \le \tau < T_\alpha$, Proposition 3.8 and the homogeneous Strichartz estimate (44) yield

$$\begin{aligned} &\sqrt{T_\alpha-\tau}\|e^{\alpha N}a_G(V)U(t)u^h_{G,0}\|_{L^2(I^h_\tau;L^2_z L^2_{y_G})} \\ &\quad\lesssim \sqrt{T_\alpha-\tau}\frac{C_V\, e^{\alpha_1}}{\sqrt{\alpha_1-\alpha}}\|U(t)e^{\alpha_1 N}u^h_{G,0}\|_{L^2_{z_1}L^2(I^h_\tau;L^{r_\sigma}_{y_G})} \\ &\quad\lesssim C_V\, e^{\alpha_1}\sqrt{\gamma}\|e^{\alpha_1 N}u^h_{G,0}\|_{L^2_z L^2_{y_G}}. \end{aligned} \tag{108}$$

Taking the supremum over $\tau \in [0, T_\alpha[$, $\alpha \in [-\alpha, \alpha_1[$ and dividing by $C_V e^{\alpha_1} \gamma^{1/2}/2$ gives

$$M(0, f^h_{2,2}, 0) \lesssim C_{\alpha_1}$$

It can be improved by rewriting the system

$$\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} = (\mathrm{Id} - L)^{-1} \begin{pmatrix} f^h_\infty \\ f^h_{2,1} + f^h_{2,2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f^h_{2,2} \\ 0 \end{pmatrix} + (\mathrm{Id} - L)^{-1} \begin{pmatrix} f^h_\infty \\ f^h_{2,1} \\ 0 \end{pmatrix} + (\mathrm{Id} - L)^{-1} L \begin{pmatrix} 0 \\ f^h_{2,2} \\ 0 \end{pmatrix}$$

from which we deduce

$$M(u^h_\infty, u^h_2 - f^h_{2,2}, u^h_1) \lesssim C_V e^{\alpha_1} \sqrt{\gamma} \left[M(f^h_\infty, f^h_{2,1}, 0) + M(0, f^h_{2,2}, 0) \right].$$

The inequalities (106), (107) and (108) prove that $(f^h_\infty, f^h_2, 0) \in \mathscr{E}^h_{-\alpha_1, \alpha_1, \gamma}$ and thus

$$M(u^h_\infty, u^h_2 - f^h_{2,2}, u^h_1) \le 2M(f^h_\infty, f^h_2, 0) \lesssim C_V e^{\alpha_1} C_{\alpha_1} \gamma^{1/2}.$$

By possibly enlarging the constant $C_d > 0$, the above inequality becomes

$$M(u^h_\infty, u^h_2 - f^h_{2,2}, u^h_1) \le C_d C_V C_{\alpha_1} e^{\alpha_1} \gamma^{1/2}.$$

We have finished the proof as soon as we can identify

$$u^h_\infty(t) = u^h_G(t) - U(t) u^h_{G,0}$$

for $t \in I^h_{T_\alpha}$ and $\alpha \in [0, \alpha_1[$. For $t \in I^h_{T_0}$, the function $u^h_G(t) = u^h_\infty(t) + U(t) u^h_{G,0}$ belongs to $\mathscr{C}^0(I^h_{T_0}; L^2_z L^2_{y_G})$ and satisfies (97) which is equivalent to (96). By the existence and uniqueness for (97) or (96) in $\mathscr{C}^0(I^h_{T_0}; L^2_z L^2_{y_G})$ when $V \in H^2(\mathbb{R}^d; \mathbb{R})$, u^h_G is the unique solution to (97) or (96) in $\mathscr{C}^0(I^h_{T_0}; L^2_z L^2_{y_G})$. □

5.2 Consequences of Theorem 5.1

Let us work now with a general initial time t_0 , specified later, and consider (96)

$$\begin{cases} i\partial_t u^h_G = (\xi - D_{y_G})^2 u^h_G + \sqrt{h}[a^*_G(V) + a_G(V)] u^h_G, \\ u^h_G(t = t_0) = u^h_{G,t_0}, \end{cases} \tag{109}$$

with the solution $u^h_G(t) = u^h_G(t' + t_0) = U(t') u^h_{G,t_0} + u^h_\infty$ in the framework of Theorem 5.1. For simplicity and because we work definitely in the framework of (109) we use here $U(t)U(s)^* = U(t-s)$. Remember that $(u^h_\infty, u^h_2, u^h_1)$ solves

$$\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} = \underbrace{\begin{pmatrix} L_{\infty\infty} & L_{\infty 2} & L_{\infty 1} \\ 0 & L_{22} & 0 \\ L_{1\infty} & 0 & L_{11} \end{pmatrix}}_{=L} \begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} + \begin{pmatrix} f^h_\infty \\ f^h_2 \\ 0 \end{pmatrix} \tag{110}$$

with

$$\begin{aligned} L_{\infty\infty}(\varphi)(t') &= -i \int_0^{t'} U(t'-s) \sqrt{h} a^*_G(V) \varphi(s)\, ds, \\ L_{\infty 1}(\varphi)(t') &= -i \int_0^{t'} U(t'-s) \varphi(s)\, ds, \quad L_{\infty 2}(\varphi)(t') = \sqrt{h} L_{\infty 1}(\varphi)(t'), \\ L_{qq}(\varphi)(t') &= -i \int_0^{t'} a_G(V) U(t'-s) \sqrt{h} \varphi(s)\, ds, \quad q \in \{2, 1\}, \\ L_{1\infty}(\varphi)(t') &= -ih \int_0^{t'} a_G(V) U(t'-s) a^*_G(V) \varphi(s)\, ds, \end{aligned}$$

and

$$f^h_\infty(t') = -i\int_0^{t'} U(t'-s)a^*_G(V)\sqrt{h}U(s)u^h_{G,t_0}\,ds,$$
$$f^h_2(t') = \underbrace{-ia_G(V)\int_0^{t'} U(t'-s)a^*_G(V)\sqrt{h}U(s)u^h_{G,t_0}\,ds}_{f^h_{2,1}(t')} + \underbrace{a_G(V)U(t')u^h_{G,t_0}}_{f^h_{2,2}(t')}.$$

Theorem 5.1 provides a framework in which L is a contraction and we will use it twice while inverting

$$\begin{pmatrix} u^h_\infty \\ u^h_2 \\ u^h_1 \end{pmatrix} = (\mathrm{Id}-L)^{-1}\begin{pmatrix} f^h_\infty \\ f^h_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f^h_{2,2} \\ 0 \end{pmatrix} + (\mathrm{Id}-L)^{-1}\begin{pmatrix} f^h_\infty \\ f^h_{2,1} \\ 0 \end{pmatrix} + (\mathrm{Id}-L)^{-1}L\begin{pmatrix} 0 \\ f^h_{2,2} \\ 0 \end{pmatrix}$$

and then using the Neumann expansion $(\mathrm{Id}-L)^{-1} = \sum_{k=0}^\infty L^k$ for different values of t_0 and of the parameter γ in Theorem 5.1. The following result is an easy consequence of Theorem 5.1.

Proposition 5.2. *Assume that the initial datum $u^h_{G,0}$ for $t_0 = 0$ in (109) satisfies the uniform bound $\|e^{2\alpha_1 N}u^h_{G,0}\|_{L^2_z L^2_{y_G}} \le C_{\alpha_1}$ for all $h \in]0,h_0[$. Then there exists $\hat{T}_{\alpha_1} > 0$ and $\tilde{C}_{\alpha_1} > 0$, $\delta_{\alpha_1} > 0$, such that*

a) *The following weighted estimate*

$$\|e^{\alpha_1 N}u^h_G(t)\|_{L^2_z L^2_{y_G}} \le \tilde{C}_{\alpha_1}$$

holds true for all $t \in I^h_{\hat{T}_{\alpha_1}} =]-\frac{\hat{T}_{\alpha_1}}{h}, \frac{\hat{T}_{\alpha_1}}{h}[$ and all $h \in]0,h_0[$.

b) *For $t_0 \in I^h_{\hat{T}_{\alpha_1}}$, $u^h_G(t_0+\delta/h)$ admits in $e^{-\frac{\alpha_1}{2}N}L^2_z L^2_{y_G}$ the following asymptotic expansion in terms of $\delta \in [-\delta_1,\delta_1]$,*

$$u^h_G(t_0+\delta/h) = \underbrace{U(\delta/h)u^h_G(t_0)}_{\mathcal{O}(1)} \underbrace{-i\sqrt{h}\int_0^{\delta/h} U(\delta/h-s)[a^*_G(V)+a_G(V)]U(s)u^h_G(t_0)\,ds}_{\mathcal{O}(|\delta|^{1/2})}$$
$$\underbrace{-h\int_0^{\delta/h}\int_0^s U(\delta/h-s)[a^*_G(V)+a_G(V)]U(s-s')[a^*_G(V)+a_G(V)]U(s')u^h_G(t_0)\,ds'\,ds}_{\mathcal{O}(|\delta|)}$$
$$+\mathcal{O}(|\delta|^{3/2})$$

where $v(h,\delta) = \mathcal{O}(|\delta|^{k/2})$, $k = 0,1,2,3$, means $\|e^{\frac{\alpha_1}{2}N}v(h,\delta)\|_{L^2_z L^2_{y_G}} \le \tilde{C}_{\alpha_1}|\delta|^{k/2}$ uniformly with respect to $h \in]0,h_0[$ and $t_0 \in I^h_{\hat{T}_{\alpha_1}}$.

Proof. **a)** Fix $\alpha_1 > 0$ and apply Theorem 5.1 with α_1 replaced by $2\alpha_1$. There exists $\gamma = \gamma_1 > 0$, determined by α_1, $C_{12}(V)$ and the dimension $d \ge 3$, such that the operator L is a contraction in $\mathscr{E}^h_{-2\alpha_1,2\alpha_1,\gamma_1}$. The system (110) for $t_0 = 0$ admits a unique solution with the norm M in $\mathscr{E}^h_{-2\alpha_1,2\alpha_1,\gamma_1}$ estimated by

$$M(u^h_\infty,u^h_2,u^h_1) \lesssim C_{\alpha_1} \tag{111}$$

and the solution u_G^h to (18) equals

$$u_G^h(t) = U(t)u_{G,0}^h + u_\infty^h(t).$$

With $T_{\alpha_1} = \gamma_1(2\alpha_1 - \alpha_1) = \gamma_1\alpha_1$, the estimate (111) says in particular

$$\forall t \in I^h_{T_{\alpha_1}}, \quad \|e^{\alpha_1 N} u_\infty^h(t)\|_{L_z^2 L_{y_G}^2} \lesssim C_{\alpha_1} \frac{\sqrt{|ht|}}{\sqrt{T_{\alpha_1} - |ht|}}.$$

By taking $\hat{T}_{\alpha_1} = \frac{T_{\alpha_1}}{2}$ with $|ht| \le \frac{T_{\alpha_1}}{2}$ when $t \in I_{\hat{T}^h_{\alpha_1}}$ and with

$$\|e^{\alpha_1 N} U(t) u_{G,0}^h\|_{L_z^2 L_{y_G}^2} \le \|e^{\alpha_1} u_{G,0}^h\|_{L_z^2 L_{y_G}^2} \le C_{\alpha_1},$$

we finally obtain

$$\forall t \in I^h_{\hat{T}_{\alpha_1}}, \quad \|e^{\alpha_1 N} u_G^h(t)\|_{L_z^2 L_{y_G}^2} \lesssim \tilde{C}_{\alpha_1},$$

for $\tilde{C}_{\alpha_1}$ large enough.

b) With **a)** the initial datum $u^h_{G,t_0} = u_G^h(t_0)$ of (109) fulfils the assumptions of Theorem 4.1 after time translation $t' = t - t_0$ and where $t' \in I_T^h$ means $t \in t_0 + I_T^h$. For any $\gamma > 0$ small enough and by setting $T_\alpha = \gamma(\alpha_1 - \alpha)$ for $\alpha \in [0, \alpha_1]$ we know that the system (110) satisfies

$$\|L\|_{\mathscr{L}(\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma})} \lesssim \gamma^{1/2} \quad , \quad M(f_\infty^h, f_{2,1}^h, 0) \lesssim C_{\alpha_1}\gamma^{1/2} \quad , \quad M(0, f_{2,2}^h, 0) \lesssim C_{\alpha_1},$$

while $u_G^h(t' + t_0) = U(t')u_G^h(t_0) + u_\infty^h(t')$ for $t' \in I^h_{T_\alpha}$.
In particular

$$\begin{pmatrix} u_\infty^h \\ u_2^h \\ u_1^h \end{pmatrix} = \begin{pmatrix} 0 \\ f_{2,2}^h \\ 0 \end{pmatrix} + (\mathrm{Id} - L)^{-1} \begin{pmatrix} f_\infty^h \\ f_{2,1}^h \\ 0 \end{pmatrix} + (\mathrm{Id} - L)^{-1} L \begin{pmatrix} 0 \\ f_{2,2}^h \\ 0 \end{pmatrix}$$

leads to

$$\begin{pmatrix} u_\infty^h \\ u_2^h \\ u_1^h \end{pmatrix} = \begin{pmatrix} f_\infty^h + L_{\infty 2}(f_{2,2}^h) + L_{\infty\infty}(f_\infty^h + L_{\infty 2}(f_{2,2}^h)) + L_{\infty 2}(f_{2,1}^h + L_{22}(f_{2,2}^h)) \\ f_{2,2}^h + f_{2,1}^h + L_{22}(f_{2,2}^h) + L_{22}(f_{2,1}^h + L_{22}(f_{2,2}^h)) \\ L_{1\infty}(f_\infty^h + L_{\infty 2}(f_{2,2}^h)) \end{pmatrix} + \mathscr{O}(\gamma^{3/2})$$

in $\mathscr{E}^h_{-\alpha_1,\alpha_1,\gamma}$. By using the first line with $\alpha = \frac{\alpha_1}{2}$, and by setting

$$\begin{aligned} v^h(t') = U(t')u_G^h(t_0) + [f_\infty^h(t') + L_{\infty 2}(f_{2,2}^h)](t') \\ + L_{\infty\infty}[f_\infty^h + L_{\infty 2}(f_{2,2}^h)](t') + L_{\infty 2}[f_{2,1}^h + L_{22}(f_{2,2}^h)](t') \end{aligned}$$

the difference $u_G^h(t_0 + t') - v^h(t')$ satisfies satisfies

$$\forall t' \in I^h_{T_{\frac{\alpha_1}{2}}}, \quad \|e^{\frac{\alpha_1}{2}N}[u_G^h(t_0 + t') - v^h(t')]\|_{L_z^2 L_{y_G}^2} \lesssim \gamma^{3/2} \frac{\sqrt{|ht'|}}{\sqrt{T_0 - |ht'|}},$$

where $T_{\frac{\alpha_1}{2}} = \frac{\gamma\alpha_1}{2}$. For $\delta = \pm\frac{T_{\frac{\alpha_1}{2}}}{2} = \pm\frac{\gamma\alpha_1}{4}$ we obtain

$$\|e^{\frac{\alpha_1}{2}N}[u_G^h(t_0 + \delta/h) - v^h(\delta/h)]\|_{L_z^2 L_{y_G}^2} = \mathscr{O}(|\delta|^{3/2}).$$

It now suffices to specify all the terms of $v^h(\delta/h)$:

- The first one is nothing but $U(\delta/h)u_G^h(t_0)$ with an $\mathscr{O}(1)$-norm.

- The second term

$$f^h_\infty(\delta/h)+L_{\infty 2}(f^h_{2,2})(\delta/h)=-i\int_0^{\delta/h}U(\delta/h-s)\sqrt{h}[a^*_G(V)+a_G(V)]U(s)u^h_G(t_0)\,ds$$

  has an $\mathcal{O}(\delta^{1/2})$-norm.

- All the other terms have an $\mathcal{O}(\delta)$-norm and equal

$$\begin{aligned}
L_{\infty\infty}(f^h_\infty)(\delta/h)&=-h\int_0^{\delta/h}\int_0^s U(\delta/h-s)a^*_G(V)U(s-s')a^*_G(V)U(s')u^h_G(t_0)\,ds'ds\,,\\
L_{\infty\infty}(L_{\infty 2}(f^h_{2,2}))(\delta/h)&=-h\int_0^{\delta/h}\int_0^s U(\delta/h-s)a^*_G(V)U(s-s')a_G(V)U(s')u^h_G(t_0)\,ds'ds\,,\\
L_{\infty 2}(f^h_{2,1})(\delta/h)&=-h\int_0^{\delta/h}\int_0^s U(\delta/h-s)a_G(V)U(s-s')a^*_G(V)U(s')u^h_G(t_0)\,ds'ds\,,\\
L_{\infty 2}(L_{22}(f^h_{2,2}))(\delta/h)&=-h\int_0^{\delta/h}\int_0^s U(\delta/h-s)a_G(V)U(s-s')a_G(V)U(s')u^h_G(t_0)\,ds'ds\,.
\end{aligned}$$

This ends the proof. □

6 Semiclassical measures

We will check here that semiclassical (or Wigner) measures for our model problem presented in Section 2.3 can be defined simultaneously for all macroscopic times $t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[$.

6.1 Framework

Below are reviewed a few properties of semiclassical measures or Wigner measures. We refer the reader e.g. to [CdV][Ger][GMMP][HMR][LiPa][Sch] for various presentations of those now well known objects.

a) The Anti-Wick quantization on $\mathbb{R}^d$ is defined by

$$a^{\mathrm{A-Wick}}(hx,D_x)=\int_{T^*\mathbb{R}^d}a(X)\,|\varphi^h_X\rangle\langle\varphi^h_X|\,\frac{dX}{(2\pi h)^d}$$

is defined for any $a\in L^\infty(T^*\mathbb{R}^d,dx;\mathbb{C})$ with

$$\varphi^h_{X_0}(x)=\frac{h^{d/4}}{\pi^{d/4}}e^{i\xi_0\cdot(x-\frac{x_0}{2h})}e^{-\frac{h(x-\frac{x_0}{h})^2}{2}}\,,\quad X_0=(x_0,\xi_0)\in T^*\mathbb{R}^d\,.$$

It is a non negative quantization for which

$$(a\geq 0)\Rightarrow(a^{\mathrm{A-Wick}}(hx,D_x)\geq 0)\quad\text{and}\quad\|a^{\mathrm{A-Wick}}(hx,D_x)\|_{\mathscr{L}(L^2(\mathbb{R}^d,dx;\mathbb{C}))}\leq\|a\|_{L^\infty}\,.$$

A natural separable subspace of $L^\infty(T^*\mathbb{R}^d;\mathbb{C})$ is

$$\mathscr{C}^0_0(T^*\mathbb{R}^d;\mathbb{C})=\left\{a\in\mathscr{C}^0(T^*\mathbb{R}^d;\mathbb{C}),\lim_{X\to\infty}a(X)=0\right\}$$

$$\text{resp.}\qquad\mathscr{C}^0(T^*\mathbb{R}^d\sqcup\{\infty\};\mathbb{C})=\mathscr{C}^0_0(T^*\mathbb{R}^d;\mathbb{C})\oplus\mathbb{C}\,1=\left\{a\in\mathscr{C}^0(T^*\mathbb{R}^d;\mathbb{C}),\lim_{X\to\infty}a(X)\in\mathbb{C}\right\},$$

endowed with the $\mathscr{C}^0$ norm, of which the dual is the space $\mathscr{M}_b(T^*\mathbb{R}^d;\mathbb{C})$ (resp. $\mathscr{M}_b(T^*\mathbb{R}^d\sqcup\{\infty\};\mathbb{C})$) of bounded Radon measures on $T^*\mathbb{R}^d$ (resp. $T^*\mathbb{R}^d\sqcup\{\infty\}$).

b) For a bounded family $(\varrho_h)_{h\in]0,h_0[}$ of normal states $\varrho_h \in \mathscr{L}^1(L^2(\mathbb{R}^d,dx;\mathbb{C}))$, $\varrho_h \geq 0$, $\mathrm{Tr}[\varrho_h]=1$, the set of semiclassical measures on $T^*\mathbb{R}^d$ (resp. $T^*\mathbb{R}^d \sqcup \{\infty\}$) is defined as the weak* limit point in $\mathscr{M}_b(T^*\mathbb{R}^d;\mathbb{R}_+)$ (resp. $\mathscr{M}_b(T^*\mathbb{R}^d \sqcup \{\infty\};\mathbb{R}_+)$) of $\frac{\sigma^{\mathrm{Wick}}(\varrho_h)}{(2\pi h)^d}$ with

$$\sigma^{\mathrm{Wick}}(\varrho_h)(X) = \langle \varphi_X^h, \varrho_h \varphi_X^h \rangle_{L^2(\mathbb{R}^d)} = \mathrm{Tr}\left[\varrho_h |\varphi_X^h\rangle\langle\varphi_X^h|\right].$$

This is extended by linearity for any bounded family $(\varrho_h)_{h\in]0,h_0[}$ in $\mathscr{L}^1(L^2(\mathbb{R}^d,dx;\mathbb{C}))$. The set of semiclassical measures is denoted by

$$\mathscr{M}(\varrho_h, h\in]0,h_0[),$$

and when h is restricted to a set $\mathscr{E}\subset]0,h_0[$, $0\in\overline{\mathscr{E}}$, we use

$$\mathscr{M}(\varrho_h, h\in\mathscr{E}).$$

After recalling

$$\int_{T^*\mathbb{R}^d} a(X)\sigma^{\mathrm{Wick}}(\varrho_h)(X)\frac{dX}{(2\pi h)^d} = \mathrm{Tr}\left[a^{\mathrm{A-Wick}}(hx,D_x)\varrho_h\right],$$

semiclassical measures $\mu\in\mathscr{M}(\varrho_h,h\in]0,h_0[)$ are characterized by the existence of a sequence $(h_k)_{k\in\mathbb{N}^*}$, $h_k\in\mathscr{E}$ such that

$$\begin{aligned}
&\lim_{k\to\infty} h_k = 0,\\
&\lim_{k\to\infty} \mathrm{Tr}\left[a^{\mathrm{A-Wick}}(h_k x,D_x)\varrho_{h_k}\right] = \int_{T^*\mathbb{R}^d} a(X)\,d\mu(X), \quad \forall a\in\mathscr{D},\\
&\lim_{k\to\infty} \mathrm{Tr}\left[\varrho_{h_k}\right] = \mu(T^*\mathbb{R}^d\sqcup\{\infty\}) = \mu(T^*\mathbb{R}^d)+\mu(\infty),
\end{aligned}$$

where $\mathscr{D}$ is any dense set of $\mathscr{C}_0^0(T^*\mathbb{R}^d;\mathbb{C})$.

d) After choosing $\mathscr{D}=\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$ and by recalling $\|a^{\mathrm{A-Wick}}(hx,D_x)-a^{\mathrm{Weyl}}(hx,D_x)\| = \mathscr{O}(h)$, for any $a\in S(1,dx^2+d\xi^2)\supset\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$, semiclassical measures are characterized by

$$\forall a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C}), \quad \lim_{k\to\infty}\mathrm{Tr}\left[a^{\mathrm{Weyl}}(h_k x,D_x)\varrho_{h_k}\right] = \int_{T^*\mathbb{R}^d} a(X)\,d\mu(X),$$

or

$$\forall P\in T^*\mathbb{R}^d, \quad \lim_{k\to\infty}\mathrm{Tr}\left[\tau_P^{h_k}\varrho_{h_k}\right] = \int_{T^*\mathbb{R}^d} e^{i(p_\xi\cdot x - p_x\cdot\xi)}\,d\mu(x,\xi),$$

with

$$\tau_P^h = (e^{i(p_\xi\cdot x-p_x\cdot\xi)})^{\mathrm{Weyl}}(hx,D_x) = e^{i(p_\xi\cdot(hx)-p_x\cdot D_x)}, \quad P=(p_x,p_\xi).$$

The compactification $T^*\mathbb{R}^d\sqcup\{\infty\}$ is just a way to count the mass of $(\varrho_{h_k})_{k\in\mathbb{N}^*}$ which is not caught by the compactly supported obervables $a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$.

e) Semiclassical measures are transformed by the dual action of semiclassical Fourier integral operator on $a^{\mathrm{Weyl}}(hx,D_x)$, $a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$.

f) When $\mathscr{M}(\varrho_{h,1},h\in\mathscr{E})=\{\mu_1\}$ and $\mathscr{M}(\varrho_{h,2},h\in\mathscr{E})=\{\mu_2\}$ the total variation between μ_1 and μ_2 is estimated by

$$|\mu_2-\mu_1|(\underbrace{T^*\mathbb{R}^d}_{\text{or } T^*\mathbb{R}^d\sqcup\{\infty\}}) \leq 4\liminf_{h\to 0}\|\varrho_{h,1}-\varrho_{h,2}\|_{\mathscr{L}^1}.$$

g) When (Λ, d_Λ) is a metric space and $(\varrho_h(\lambda))_{h\in]0,h_0[,\lambda\in\Lambda}$ is a bounded family in $\mathscr{L}^1(L^2(\mathbb{R}^d,dx;\mathbb{C}))$, semiclassical measures can be defined simultaneously for all $\lambda\in\Lambda$, if for any sequence $(h_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} h_n = 0^+$, there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that

$$\forall \lambda\in\Lambda, \exists \mu_\lambda \in \mathscr{M}_b(T^*\mathbb{R}^d \sqcup \{\infty\}),$$
$$\lim_{k\to\infty} \mathrm{Tr}\left[a^{\mathrm{A-Wick}}(h_{n_k}x,D_x)\varrho_{h_{n_k}}(\lambda)\right] = \int_{T^*\mathbb{R}^d\sqcup\{\infty\}} a(X)\,d\mu_\lambda(X).$$

By assuming (Λ, d_Λ) separable, sufficient conditions for this property are either

- For all given $a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$, $\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x)\varrho_h(\lambda)\right]$ is an equicontinuous family of continuous functions from Λ to $\mathbb{C}$, or
- The map $(P,\lambda)\mapsto \mathrm{Tr}\left[\tau_P^h\varrho_h(\lambda)\right]$ is an equicontinuous family of continuous functions from $T^*\mathbb{R}^d\times\Lambda$ to $\mathbb{C}$.

For the first characterization, apply a diagonal extraction process for a dense countable subset of (Λ,d_Λ) (and a dense countable subset of $\mathscr{C}_0^0(T^*\mathbb{R}^d)$ lying in $\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$) and then apply the various characterisations of elements of $\mathscr{M}(\varrho_h(\lambda), h\in\mathscr{E})$.

Like in our problem, semiclassical measures can be defined for bounded families $\varrho_h\in\mathscr{L}^1(L^2(\mathbb{R}^d\times Z',dx\otimes\mathbf{dz}';\mathbb{C}))$ after using observables $a^{\mathrm{Weyl}}(hx,D_x)\otimes\mathrm{Id}_{L^2_z}$.
When $(\varrho_h)_{h\in]0,h_0[}$ is a family of states, $\varrho_h\geq 0$ and $\mathrm{Tr}[\varrho_h]=1$, the relationship with the study of pure states can be done in two ways:

- Firstly by writing a general state as a convex combination of pure states, provided that this convex decomposition is explicit enough to follow the behaviour as $h\to 0^+$.
- Secondly by writing $\varrho_h=\varrho_h^{1/2}\varrho_h^{1/2}$ and taking $\Psi_h=\varrho_h^{1/2}\in\mathscr{L}^2(L^2(\mathbb{R}^d\times Z',dx\otimes\mathbf{dz}';\mathbb{C}))\sim L^2(\mathbb{R}^d\times Z'\times\hat{Z},dx\otimes\mathbf{dz}'\otimes\mathbf{d\hat{z}};\mathbb{C})$ where $\hat{Z}$ is another copy of $\mathbb{R}^d\times Z'$ with $\mathbf{d\hat{z}}=dx\otimes\mathbf{dz}'$. Then
$$\mathrm{Tr}\left[(a^{\mathrm{Weyl}}(hx,D_x)\otimes\mathrm{Id}_{L^2_{z'}})\varrho_h\right] = \langle\Psi_h,(a^{\mathrm{Weyl}}(hx,D_x)\otimes\mathrm{Id}_{L^2_{z',\hat{z}}})\Psi_h\rangle.$$

6.2 Equicontinuity

The following result, which is the first useful information about semiclassical measures, before computing them, comes from the equicontinuity directly deduced from Proposition 5.2. The unitary transforms introduced in Section 2.3 and Section 3 in order to transform (18) into (96) and $a^{\mathrm{Weyl}}(hx,D_x)\otimes\mathrm{Id}$ into $a^{\mathrm{Weyl}}(-hD_\xi,\xi-D_{y_G})$ are not recalled here and the results are directly formulated for the initial problem (18) and the semiclassical observables $a^{\mathrm{Weyl}}(hx,D_x)\otimes\mathrm{Id}$.

Proposition 6.1. *Assume*

$$V\in L^{r'_\sigma}(\mathbb{R}^d,dx;\mathbb{R})\cap H^2(\mathbb{R}^d;\mathbb{R}),\quad r'_\sigma=\frac{2d}{d+2},\quad d\geq 3,$$

and let $U_V(t)=e^{-it(-\Delta_x+\sqrt{h}V)}$ *like in Subsection 2.3.*
Assume that there exists $\alpha_1>0$ *such that* $\varrho_h(0)\in\mathscr{L}^1(L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C}))$, $\varrho_h(0)\geq 0$, $\mathrm{Tr}[\varrho_h(0)]=1$ *satisfies*

$$\exists C_{\alpha_1}>0,\ \forall h\in]0,h_0[,\quad \mathrm{Tr}\left[e^{\alpha_1 N}\varrho_h(0)e^{\alpha_1 N}\right]\leq C_{\alpha_1}.$$

Then there exists $\hat{T}_{\alpha_1}>0$ *such that elements of* $\mathscr{M}(\varrho_h(t),h\in]0,h_0[)$ *can be defined simultaneously for all macroscopic times* $t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[$ *when* $\varrho_h(t)=U_V(\frac{t}{h})\varrho_h(0)U_V^*(\frac{t}{h})$.

Proof. When $U(s)=e^{-is(-\Delta_x)}$ denotes the free unitary transform , the time evolved observable $U^*(\frac{s}{h})[a^{\mathrm{Weyl}}(hx,D_x)\otimes \mathrm{Id}_{L^2_\omega}]U(\frac{s}{h})$ equals exactly $a^{\mathrm{Weyl}}(hx,D_x,s)\otimes \mathrm{Id}_{L^2_\omega}$ with

$$a(x,\xi,s)=a(x+2\xi s,\xi).$$

It is clearly equicontinuous in $h\in]0,h_0[$ with respect to $s\in[-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}]$ in $\mathscr{L}(L^2_{x,\omega})$ for any given $a\in\mathscr{C}^\infty_0(T^*\mathbb{R}^d;\mathbb{C})$:

$$\|a^{\mathrm{Weyl}}(hx,D_x,s)-a^{\mathrm{Weyl}}(hx,D_x,0)\|_{\mathscr{L}(L^2_{x,\omega})}\leq C_a|s|.$$

We drop the tensorization with Id_{L^2}. With

$$\begin{aligned}&\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x)\varrho_h(t+\delta)\right]-\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x)\varrho_h(t)\right]\\&\quad=\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x,\delta)U^*(\frac{\delta}{h})U_{\mathscr{V}}(\frac{\delta}{h})\varrho_h(t)U^*_{\mathscr{V}}(\frac{\delta}{h})U(\frac{\delta}{h})\right]-\mathrm{Tr}\left[a^{\mathrm{Weyl}}(hx,D_x,0)\varrho_h(t)\right]\end{aligned}$$

it thus suffices to check, uniformly with respect to $(h,t)\in]0,h_0[\times]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[$, the estimate

$$\|U^*(\frac{\delta}{h})U_{\mathscr{V}}(\frac{\delta}{h})\varrho_h(t)U^*_{\mathscr{V}}(\frac{\delta}{h})U(\frac{\delta}{h})-\varrho_h(t)\|_{\mathscr{L}^1}=o_{\delta\to0}(1).\tag{112}$$

We now use the decomposition $\varrho_h(0)=\varrho_h(0)^{1/2}\varrho_h(0)^{1/2}$ and consider the evolution

$$U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}\in\mathscr{L}^2(L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C}))\sim L^2(\mathbb{R}^d\times\Omega\times\hat{Z},dx\otimes\mathscr{G}\otimes\mathbf{d\hat{z}};\mathbb{C})$$

with $\hat{Z}=\mathbb{R}^d\times\Omega$, $\mathbf{d\hat{z}}=dx\otimes\mathscr{G}$.
The estimate (112) is done as soon as

$$\|U^*(\frac{\delta}{h})U_{\mathscr{V}}(\frac{\delta}{h})[U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}]-[U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}]\|_{L^2_{x,\omega,\hat{z}}}=o_{\delta\to0}(1)$$

uniformly with respect to $(h,t)\in]0,h_0[\times]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[$.
This problem is now translated in a problem in

$$\underbrace{L^2(\mathbb{R}^d\times\hat{Z},\frac{d\xi}{(2\pi)^d}\otimes\mathbf{d\hat{z}};\mathbb{C})}_{\text{vacuum}}\oplus L^2_{\mathrm{sym}}(\mathbb{R}^d_{y_G}\times Z_1;dy_G\otimes\mathbf{dz_1};\mathbb{C})$$

by the unitary transform U_G associated with the center of mass y_G of Section 3, the translation invariance and its Fourier variable $\xi\in\mathbb{R}^d$ and the relative coordinates $Y'\in\mathscr{R}$. The variable $z_1\in Z_1$ is nothing but $z_1=(\xi,Y',\hat{z})\in\mathbb{R}^d\times\mathscr{R}\times\hat{Z}$ with $\mathbf{dz_1}=\frac{d\xi}{(2\pi)^d}\otimes\mu\otimes\mathbf{d\hat{z}}$. The subscript $_{\mathrm{sym}}$ refers to the symmetry in the variable $Y'\in\mathscr{R}$. All the assumptions of Theorem 5.1 have been checked in Section 5. In particular we can use Proposition 5.2-b) with

$$u^h_G(\frac{t}{h})=U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}\quad\text{and}\quad\frac{t}{h}\in I^h_{\hat{T}_{\alpha_1}}.$$

It says in particular

$$u^h_G(\frac{t}{h}+\frac{\delta}{h})=U(\frac{\delta}{h})u^h_G(\frac{t}{h})+\mathscr{O}(|\delta|^{1/2}),$$

uniformly with respect to $(h,\frac{t}{h})\in]0,h_0[\times I^h_{\hat{T}_{\alpha_1}}$, and therefore

$$\|U^*(\frac{\delta}{h})U_{\mathscr{V}}(\frac{\delta}{h})[U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}]-[U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}]\|_{L^2_{x,\omega,\hat{z}}}=\mathscr{O}_{\delta\to0}(|\delta|^{1/2})$$

uniformly with respect to $(h,t)\in]0,h_0[\times]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[$.
This ends the proof. □

7 Approximations

With our number estimates stated in Section 5, various approximations can be considered for the general class of initial data $(\varrho_h(0))_{h\in]0,h_0[}$, $\varrho_h(0)\in\mathscr{L}^1(L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C}))$, $\varrho_h(0)\geq 0$, $\mathrm{Tr}[\varrho_h(0)]=1$ under the sole additional assumption $\mathrm{Tr}[e^{\alpha_1 N}\varrho_h(0)e^{\alpha_1 N}]\leq C_{\alpha_1}$. Before computing the evolution of the semiclassical measures $(\mu_t)_{t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[}$ given by Proposition 6.1 (this will be done in a subsequent article), it provides useful a priori information for them.

7.1 Truncation with respect to the number operator N

For $\varepsilon>0$, let $\chi_\varepsilon:[0,+\infty)\to[0,1]$ be a decaying function such that

$$\forall k\in\mathbb{N},\forall\varepsilon\in]0,1[,\exists C_{k,\varepsilon}>0,\quad \sup_{s\in[0,+\infty)} s^k\chi_\varepsilon(s)\leq C_{k,\varepsilon}\,, \tag{113}$$

$$\forall\alpha_1>0,\exists C_{\alpha_1}>0,\forall\varepsilon\in]0,1[,\quad \sup_{s\in[0,+\infty)} e^{-\alpha_1 s}(1-\chi_\varepsilon(s))\leq C_{\alpha_1}\times\varepsilon\,. \tag{114}$$

Examples are

$$\chi_\varepsilon(s)=1_{[0,\varepsilon^{-1}]}(s)\quad\text{and}\quad\chi_\varepsilon(s)=e^{-\varepsilon s}\,.$$

Then the operators

$$a_{G,\varepsilon}(V)=\chi_\varepsilon(N)a_G(V)\chi_\varepsilon(N)\quad,\quad a^*_{G,\varepsilon}(V)=\chi_\varepsilon(N)a^*_G(V)\chi_\varepsilon(N)$$

are bounded operators on

$$F^2=L^2(Z',\mathbf{dz}';\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C})))=L^2_{z,\mathrm{sym}}=L^2_{z_0}\oplus L^2_{z_1,\mathrm{sym}}L^2_{y_G}$$

according to (56) and $\sqrt{h}(a_{G,\varepsilon}(V)+a^*_{G,\varepsilon}(V))$ is an $\mathscr{O}_\varepsilon(\sqrt{h})$ bounded self-adjoint perturbation of $(\xi-D_{y_G})^2$. Additionally for $\varepsilon>0$ the estimates of Proposition 3.6 hold true when $a_G(V)$ and $a^*_G(V)$ are replaced by $a_{G,\varepsilon}(V)$ and $a^*_{G,\varepsilon}(V)$. Actually, (39) with $n>1$ and (37) with $n>0$ become

$$\|a_{G,\varepsilon}(V)f_{G,n}\|_{L^2_{z',Y'_{n-1}}L^p_{y_G}}\leq\|V\|_{L^{r'}}\chi_\varepsilon(n-1)^2\sqrt{n}\|f_{G,n}\|_{L^2_{z',Y'_n}L^q_{y_G}}\leq C_\varepsilon\|V\|_{L^{r'}}\|f_{G,n}\|_{L^2_{z',Y'_n}L^q_{y_G}} \tag{115}$$

$$\|a^*_{G,\varepsilon}(V)f_{G,n}\|_{L^2_{z',Y'_{n+1}}L^{q'}_{y_G}}\leq\|V\|_{L^{r'}}\chi_\varepsilon(n)^2\sqrt{n+1}\|f_{G,n}\|_{L^2_{z',Y'_n}L^{p'}_{y_G}}\leq C_\varepsilon\|V\|_{L^{r'}}\|f_{G,n}\|_{L^2_{z',Y'_n}L^{p'}_{y_G}} \tag{116}$$

when $V\in L^{q'}(\mathbb{R}^d;\mathbb{C})\cap L^{r'}(\mathbb{R}^d;\mathbb{C})$, $\frac{1}{r'}=\frac{1}{2}+\frac{1}{q'}-\frac{1}{p'}$, $p',q'\in[1,2]$. All the analysis can thus be carried out with $a_G(V)$ and $a^*_G(V)$ replaced by $a_{G,\varepsilon}(V)$ and $a^*_{G,\varepsilon}(V)$, either with estimates which are uniform in $\varepsilon\in]0,1[$, or by replacing the N-dependent estimates by constants C_ε depending on $\varepsilon\in]0,1[$.
In particular the solution $v^h_{G,\varepsilon}$ to

$$\begin{cases} i\partial_t v^h_{G,\varepsilon}=(\xi-D_{y_G})^2v^h_{G,\varepsilon}+\sqrt{h}[a^*_{G,\varepsilon}(V)+a_{G,\varepsilon}(V)]v^h_{G,\varepsilon}\,,\\ v^h_{G,\varepsilon}(t=0)=v^h_{G,\varepsilon,0}=u^h_{G,0}\,,\end{cases} \tag{117}$$

satisfies the same properties as the solution u^h_G to (96) stated in Theorem 5.1 and Proposition 5.2, uniformly with respect to $\varepsilon\in]0,1[$.

Proposition 7.1. *Assume* $\|e^{2\alpha_1 N} u^h_{G,0}\|_{L^2_z L^2_{y_G}} \leq C_{\alpha_1}$ *for all* $h \in]0,h_0[$ *like in Proposition 5.2. There exists* $\hat{C}_{\alpha_1} > 0$ *and* $\hat{T}_{\alpha_1} > 0$ *such that the solutions* u^h_G *to (96) and* $v^h_{G,\varepsilon}$ *to (117) for* $\varepsilon \in]0,1[$*, satisfy*

$$\|u^h_G(t) - v^h_{G,\varepsilon}(t)\|_{L^2_z L^2_{y_G}} \leq \hat{C}_{\alpha_1}\varepsilon$$

for all $t \in I^h_{\hat{T}_\alpha} =]-\frac{\hat{T}_{\alpha_1}}{h}, \frac{\hat{T}_{\alpha_1}}{h}[$.
Additionally the statement ***b)*** *of Proposition 5.2 holds true when* u^h_G, $a_G(V)$, $a^*_G(V)$ *are replaced by* $v^h_{G,\varepsilon}$, $a_{G,\varepsilon}(V)$, $a^*_{G,\varepsilon}(V)$.

Proof. The statements **a)** and **b)** of Proposition 5.2 hold true uniformly with respect to $\varepsilon \in]0,1[$ for $v^h_{G,\varepsilon}$ as a consequence of the previous arguments.
In particular $v^h_{G,\varepsilon}(t) = U(\frac{t}{h})u^h_{G,0} + v^h_{\infty,\varepsilon}$ where $(v^h_{\infty,\varepsilon}, v^h_{2,\varepsilon}, v^h_{1,\varepsilon})$ solves the system

$$\begin{pmatrix} v^h_{\infty,\varepsilon} \\ v^h_{2,\varepsilon} \\ v^h_{1,\varepsilon} \end{pmatrix} = L_\varepsilon \begin{pmatrix} v^h_{\infty,\varepsilon} \\ v^h_{2,\varepsilon} \\ v^h_{1,\varepsilon} \end{pmatrix} + \begin{pmatrix} f^h_{\infty,\varepsilon} \\ f^h_{2,\varepsilon} \\ 0 \end{pmatrix}, \quad L_\varepsilon = \begin{pmatrix} L_{\infty\infty,\varepsilon} & L_{\infty 2,\varepsilon} & L_{\infty 1,\varepsilon} \\ 0 & L_{22,\varepsilon} & 0 \\ L_{1\infty,\varepsilon} & 0 & L_{11,\varepsilon} \end{pmatrix}, \tag{118}$$

with

$$f^h_\varepsilon(t) = -i\int_0^t U(t)U(s)^* a^*_{G,\varepsilon}(V)\sqrt{h}U(s)u^h_{G,0}\, ds, \tag{119}$$

$$f^h_{2,\varepsilon}(t) = -i\, a_{G,\varepsilon}(V)\int_0^t U(t)U(s)^* a^*_{G,\varepsilon}(V)\sqrt{h}U(s)u^h_{G,0}\, ds + a_{G,\varepsilon}(V)U(t)u^h_{G,0}, \tag{120}$$

and where the entries L_ε are the same as the ones of L with $a_G(V)$ and $a^*_G(V)$ replaced by $a_{G,\varepsilon}(V)$ and $a^*_{G,\varepsilon}(V)$. When $\chi_\varepsilon(s) = e^{-\varepsilon s}$, one recovers the system for u^h_G by taking $\varepsilon = 0$.
We start now with the equation for u^h_G

$$u^h_G(t) = U(\frac{t}{h})u^h_{G,0} - i\sqrt{h}\int_0^{\frac{t}{h}} U(t-s)[a_G(V) + a^*_G(V)]u^h_G(s)\, ds,$$

which implies

$$\begin{aligned}\chi_\varepsilon(N)u^h_G(t) = U(\frac{t}{h})\chi_\varepsilon(N)u^h_{G,0} - i\sqrt{h}\int_0^{\frac{t}{h}} U(\frac{t}{h}-s)\chi_\varepsilon(N)[a_G(V) + a^*_G(V)]\chi_\varepsilon(N)^2 u^h_G(s)\, ds \\ - i\sqrt{h}\chi_\varepsilon(N)\int_0^{\frac{t}{h}} U(\frac{t}{h}-s)[a_G(V) + a^*_G(V)](1-\chi^2_\varepsilon(N))u^h_G(s)\, ds.\end{aligned}$$

The function $w^h_{G,\varepsilon}(t) = \chi_\varepsilon(N)u^h_G(t)$ solves

$$w^h_{G,\varepsilon}(t) = U(\frac{t}{h})\chi_\varepsilon(N)u^h_{G,0} - i\sqrt{h}\int_0^{\frac{t}{h}} U(t-s)[a_{G,\varepsilon}(V) + a^*_{G,\varepsilon}(V)]w^h_{G,\varepsilon}(s)\, ds + g^h_{\infty,\varepsilon} \tag{121}$$

$$\text{with} \quad g^h_{\infty,\varepsilon} = -i\sqrt{h}\chi_\varepsilon(N)\int_0^{\frac{t}{h}} U(t-s)[a_G(V) + a^*_G(V)](1-\chi^2_\varepsilon(N))u^h_G(s)\, ds. \tag{122}$$

The system for $(w^h_{\infty,\varepsilon}, w^h_{2,\varepsilon}, w^h_{1,\varepsilon})$ after decomposing $w^h_{G,\varepsilon}(t) = U(\frac{t}{h})\chi_\varepsilon(N)u^h_{G,0} + w^h_{\infty,\varepsilon}(t)$ is

$$\begin{pmatrix} w^h_{\infty,\varepsilon} \\ w^h_{2,\varepsilon} \\ w^h_{1,\varepsilon} \end{pmatrix} = L_\varepsilon \begin{pmatrix} w^h_{\infty,\varepsilon} \\ w^h_{2,\varepsilon} \\ w^h_{1,\varepsilon} \end{pmatrix} + \begin{pmatrix} \tilde{f}^h_{\infty,\varepsilon} \\ \tilde{f}^h_{2,\varepsilon} \\ 0 \end{pmatrix} + \begin{pmatrix} g^h_{\infty,\varepsilon} \\ 0 \\ 0 \end{pmatrix},$$

where $\tilde f^h_{\infty,\varepsilon}$ and $\tilde f^h_{2,\varepsilon}$ have the same expressions as (119)(120) with $u^h_{G,0}$ replaced by $\chi_\varepsilon(N)u^h_{G,0}$.
By taking the difference with (118), and because $\|L_\varepsilon\|_{\mathscr{L}(\mathscr{E}_{0,\alpha_1,\gamma})} \le 1/2$ for $\gamma>0$ small enough, the proof is done as soon as the three norms

$$\|u^h_G(t)-\chi_\varepsilon(N)u^h_G(t)\|_{L^2_zL^2_{y_G}} \tag{123}$$
$$M(\tilde f^h_{\infty,\varepsilon}-f^h_{\infty,\varepsilon},\tilde f^h_{2,\varepsilon}-f^h_{2,\varepsilon},0) \tag{124}$$
$$M(g^h_{\infty,\varepsilon},0,0), \tag{125}$$

are bounded by $\hat C_{\alpha_1}\varepsilon$.
Because the time interval is restricted to $I^h_{\hat T_{\alpha_1}}$ with $\hat T_{\alpha_1}<T_{\alpha_1}$, the weight $\sqrt{T_{\alpha_1}-|ht|}$ or $\sqrt{T_{\alpha_1}-\tau}$ used in Definition 4.4 or in Proposition 4.5 can be forgotten now (simply multiply $f^h_{q,\varepsilon}$, $\tilde f^h_{q,\varepsilon}$, $q\in\{\infty,2\}$ and $g^h_{\infty,\varepsilon}$ by $1_{I_{\frac{\hat T_{\alpha_1}}{h}}}(t)$).
The estimate of (123) is obvious since

$$\|(1-\chi_\varepsilon(N))u^h_G(t)\|_{L^2_zL^2_{y_G}} \le \underbrace{\sup_{s\ge0}|(1-\chi_\varepsilon(s))e^{-\alpha_1 s}|}_{\mathscr{O}(\varepsilon)} \times \underbrace{\|e^{\alpha_1 N}u^h_G(t)\|_{L^2_zL^2_{y_G}}}_{\le\tilde C_{\alpha_1}}.$$

The estimate of (124) is very similar. Actually in the proof of Theorem 5.1 we checked $M(f^h_\infty,f^h_2,0)\lesssim\|e^{\alpha_1 N}u^h_{G,0}\|_{L^2_zL^2_{y_G}}$. It gives now

$$M(\tilde f^h_{\infty,\varepsilon}-f^h_{\infty,\varepsilon},\tilde f^h_{2,\varepsilon}-f^h_{2,\varepsilon},0)\lesssim\|e^{\alpha_1 N}(\chi_\varepsilon(N)-1)u^h_{G,0}\|_{L^2_zL^2_{y_G}}\le\hat C_{\alpha_1}\varepsilon.$$

For (125) let us first decompose $g^h_{\infty,\varepsilon}$ as

$$g^h_{\infty,\varepsilon}=g^h_{\infty,1,\varepsilon}+g^h_{\infty,2,\varepsilon}$$
$$\text{with}\quad g^h_{\infty,1,\varepsilon}=-i\sqrt{h}\chi_\varepsilon(N)\int_0^{\frac{t}{h}}U(t-s)a^*_G(V)(1-\chi^2_\varepsilon(N))u^h_G(s)\,ds$$
$$\text{and}\quad g^h_{\infty,2,\varepsilon}=-i\sqrt{h}\chi_\varepsilon(N)\int_0^{\frac{t}{h}}U(t-s)a_G(V)(1-\chi^2_\varepsilon(N))u^h_G(s)\,ds.$$

The estimate of $g^h_{\infty,1,\varepsilon}$ follows the method for the bound of $M(f^h_\infty,0,0)$ in the proof of Theorem 5.1, where we simply used the uniform bound in time for $\|U(s)e^{\alpha_1 N}u^h_{G,0}\|_{L^2_zL^2_{y_G}}$.
With

$$\sup_t\|(1-\chi^2_\varepsilon(N))u^h_G(t)\|_{L^2_zL^2_{y_G}}\le\underbrace{\sup_{s\ge0}|(1-\chi^2_\varepsilon(s))e^{-\alpha_1 s}|}_{\mathscr{O}(\varepsilon)}\times\underbrace{\|e^{\alpha_1 N}u^h_G(t)\|_{L^2_zL^2_{y_G}}}_{\le\tilde C_{\alpha_1}},$$

this gives

$$M(g^h_{\infty,1,\varepsilon},0,0)\le\hat C_{\alpha_1}\varepsilon.$$

For $g^h_{\infty,2,\varepsilon}$, remember firstly that the assumption is $\|e^{2\alpha_1 N}u^h_{G,0}\|_{L^2_zL^2_{y_G}}\le C_{\alpha_1}$ and by possibly reducing $\hat T_{\alpha_1}$, we may assume $\|e^{\frac{3\alpha_1}{2}N}u_{G,h}(t)\|_{L^2_zL^2_{y_G}}\le\tilde C_{\alpha_1}$. We now use the obvious relation $a_G(V)\phi(N)=\phi(N+1)a_G(V)$ and write

$$g^h_{\infty,2,\varepsilon}=-i\chi_\varepsilon(N)e^{-\frac{\alpha_1}{2}(N+1)}(1-\chi^2_\varepsilon(N+1))e^{\frac{\alpha_1}{2}(N+1)}\int_0^{\frac{t}{h}}U(t-s)\sqrt{h}a_G(V)u^h_G(s)\,ds.$$

Remember that the equivalent system (96) says $\sqrt{h}a_G(V)u^h_G(t)=u^h_1(t)+\sqrt{h}u^h_2(t)$ with $M(0,u^h_2,u^h_1)\lesssim C_{\alpha_1}$. The above equality becomes

$$g^h_{\infty,2,\varepsilon}(t)=\chi_\varepsilon(N)(1-\chi^2_\varepsilon(N+1))e^{-\frac{\alpha_1}{2}(N+1)}e^{\frac{\alpha_1}{2}(N+1)}[L_{\infty1}(u^h_1)+L_{\infty2}(u^h_2)].$$

The bounds for $L_{\infty 1}$ and $L_{\infty 2}$ in the Theorem 5.1, lead to

$$\||ht|^{-1/2}e^{\frac{\alpha_1}{2}(N+1)}[L_{\infty 1,\varepsilon}(u_1^h)+L_{\infty 2,\varepsilon}(u_2^h)](t)\|_{L^\infty(I^h_{\hat{T}_{\alpha_1}};L^2_z L^2_{y_G})}\lesssim C_{\alpha_1}.$$

With

$$\|\chi_\varepsilon(N)(1-\chi_\varepsilon^2(N+1))e^{-\frac{\alpha_1}{2}(N+1)}\|_{\mathscr{L}(L^2_z L^2_{y_G})}\leq \sup_{s\geq 0}|(1-\chi_\varepsilon^2(s))e^{-\frac{\alpha_1}{2}s}|=\mathscr{O}(\varepsilon),$$

this proves

$$M(g^h_{\infty,2,\varepsilon},0,0)\leq \hat{C}_{\alpha_1}\varepsilon.$$

□

Let us go back to our initial problem and let us compare the evolution of states for the dynamics $U(\frac{t}{h})=e^{-it(-\Delta_x+\sqrt{h}V)}$ for $\varepsilon=0$ and the case $\varepsilon>0$ where $\chi_\varepsilon(N)V\chi_\varepsilon(N)$ is a bounded self-adjoint perturbation of $-\Delta_x$. Set in particular

$$U_{V,\varepsilon}=e^{-it(-\Delta_x+\sqrt{h}V_\varepsilon)}\quad\text{with}\quad V_\varepsilon=\chi_\varepsilon(N)V\chi_\varepsilon(N). \tag{126}$$

Proposition 7.2. *Assume like in Proposition 6.1*

$$V\in L^{r'_\sigma}(\mathbb{R}^d,dx;\mathbb{R})\cap H^2(\mathbb{R}^d;\mathbb{R})\quad,\quad r'_\sigma=\frac{2d}{d+2}\quad,\quad d\geq 3,$$

and assume that there exists $\alpha_1>0$ such that $\varrho_h(0)\in\mathscr{L}^1(L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C}))$, $\varrho_h(0)\geq 0$, $\mathrm{Tr}[\varrho_h(0)]=1$ satisfies

$$\exists C_{\alpha_1}>0,\forall h\in]0,h_0[,\quad \mathrm{Tr}\left[e^{\alpha_1 N}\varrho_h(0)e^{\alpha_1 N}\right]\leq C_{\alpha_1}.$$

Call $\varrho_h(t)=U_V(\frac{t}{h})\varrho_h(0)U_V^*(\frac{t}{h})$ and $\varrho_{h,\varepsilon}(t)=U_{V,\varepsilon}(\frac{t}{h})\varrho_h(0)U_{V,\varepsilon}^*(\frac{t}{h})$. When the subset $\mathscr{E}\subset]0,h_0[$, $0\in\overline{\mathscr{E}}$, is chosen such that

$$\forall t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[,\quad \mathscr{M}(\varrho_h(t),h\in\mathscr{E})=\{\mu_t\}\quad\text{and}\quad \mathscr{M}(\varrho_{h,\varepsilon}(t),h\in\mathscr{E})=\{\mu_{t,\varepsilon}\}$$

Then the total variation of $\mu_t-\mu_{t,\varepsilon}$ is estimated by

$$\forall t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[,\quad |\mu_t-\mu_{t,\varepsilon}|(\underbrace{T^*\mathbb{R}^d}_{\text{or } T^*\mathbb{R}^d\sqcup\{\infty\}})\leq C'_{\alpha_1}\varepsilon,$$

for some constant $C'_{\alpha_1}>0$ determined by $\alpha_1>0$.

Proof. From

$$\begin{aligned}\varrho_h(t)-\varrho_{h,\varepsilon}(t)=&\left[U_V(\frac{t}{h})\varrho_h(0)^{1/2}-U_{V,\varepsilon}(\frac{t}{h})\varrho_h(0)^{1/2}\right]\varrho_h(0)^{1/2}U_V^*(\frac{t}{h})\\&+U_{V,\varepsilon}(\frac{t}{h})\varrho_h(0)^{1/2}[\varrho_h(0)^{1/2}U_V^*(\frac{t}{h})-\varrho_h(0)^{1/2}U_{V,\varepsilon}^*(\frac{t}{h})]\end{aligned}$$

we deduce

$$|\mu(t)-\mu_\varepsilon(t)|(T^*\mathbb{R}^d\cup\{\infty\})\leq 4\liminf_{h\in\mathscr{E},h\to 0}\|\varrho_h(t)-\varrho_{h,\varepsilon}(t)\|_{\mathscr{L}^1}\leq 8\liminf_{h\in\mathscr{E},h\to 0}\|\Psi^h(t)-\Psi^h_\varepsilon(t)\|_{L^2_{x,\omega,\hat{z}}}$$

with $\Psi^h_\varepsilon(t)=U_{V,\varepsilon}(\frac{t}{h})\varrho_h(0)^{1/2}\in\mathscr{L}^2(L^2(\mathbb{R}^d\times\Omega,dx\otimes\mathscr{G};\mathbb{C}))\sim L^2(\mathbb{R}^d\times\Omega\times\hat{Z},dx\otimes\mathscr{G}\otimes\mathbf{d}\hat{\mathbf{z}};\mathbb{C})$ with $\hat{Z}=\mathbb{R}^d\times\Omega$, $\mathbf{d}\hat{\mathbf{z}}=dx\otimes\mathscr{G}$.
But Proposition 7.1 implies

$$\forall t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[,\quad \|\Psi^h(t)-\Psi^h_\varepsilon(t)\|_{L^2_{x,\omega,\hat{z}}}\leq\hat{C}_{\alpha_1}\varepsilon.$$

□

7.2 Asymptotic conservation of energy

The result of this paragraph is a consequence of the approximation of the $U_{\mathcal{V}}$ dynamics by the one of $U_{\mathcal{V}_\varepsilon}$ in terms of wave functions in Proposition 7.1, states and semiclassical measures in Proposition 7.2

Proposition 7.3. *Assume like in Proposition 6.1*

$$V \in L^{r'_\sigma}(\mathbb{R}^d, dx; \mathbb{R}) \cap H^2(\mathbb{R}^d; \mathbb{R}) \quad , \quad r'_\sigma = \frac{2d}{d+2} \quad , \quad d \geq 3 ,$$

and assume that there exists $\alpha_1 > 0$ *such that* $\varrho_h(0) \in \mathscr{L}^1(L^2(\mathbb{R}^d \times \Omega, dx \otimes \mathscr{G}; \mathbb{C}))$, $\varrho_h(0) \geq 0$, $\mathrm{Tr}[\varrho_h(0)] = 1$ *satisfies*

$$\exists C_{\alpha_1} > 0 , \forall h \in]0, h_0[, \quad \mathrm{Tr}\left[e^{\alpha_1 N} \varrho_h(0) e^{\alpha_1 N} \right] \leq C_{\alpha_1} .$$

Call $\varrho_h(t) = U_{\mathcal{V}}(\frac{t}{h}) \varrho_h(0) U^*_{\mathcal{V}}(\frac{t}{h})$ *and let the subset* $\mathscr{E} \subset]0, h_0[$, $0 \in \overline{\mathscr{E}}$, *be such that*

$$\forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad \mathscr{M}(\varrho_h(t), h \in \mathscr{E}) = \{\mu_t\}$$

with the additional assumption at time $t = 0$,

$$\mathrm{supp}\, \mu_0 \subset \left\{ (x, \xi) \in T^* \mathbb{R}^d \, , \, |\xi|^2 \in F \right\} \tag{127}$$

where F *is a closed subset of* $\mathbb{R}$. *Then for all* $t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[$, *the support of* μ_t *restricted to* $T^* \mathbb{R}^d$ *satisfies*

$$\mathrm{supp}\, \mu_t \big|_{T^* \mathbb{R}^d} \subset \left\{ (x, \xi) \in T^* \mathbb{R}^d \, , \, |\xi|^2 \in F \right\} .$$

Proof. For $\varepsilon > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}$ the resolvent estimate

$$\| [z + \Delta_x]^{-1} - [z - (-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon)]^{-1} \|_{\mathscr{L}(L^2_{x,\omega})} \leq \frac{C_\varepsilon \sqrt{h}}{|\mathrm{Im}\, z|^2}$$

with $\mathcal{V}_\varepsilon = \chi_\varepsilon(N) \mathcal{V} \chi_\varepsilon(N) \in \mathscr{L}(L^2_{x,\omega})$ as in (126) combined with Helffer-Sjöstrand formula [HeSj] gives

$$\forall \varepsilon > 0 , \forall \chi \in \mathscr{C}^\infty_0(\mathbb{R}; \mathbb{C}) , \exists C_{\chi,\varepsilon} > 0 , \quad \| \chi(-\Delta_x) - \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) \|_{\mathscr{L}(L^2_{x,\omega})} \leq C_{\chi,\varepsilon} \sqrt{h} .$$

The semiclassical calculus then implies

$$\left\| \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) a^{\mathrm{Weyl}}(hx, D_x) \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) - [\chi^2(|\xi|^2) a]^{\mathrm{Weyl}}(hx, D_x) \right\|_{\mathscr{L}(L^2_{x,\omega})} = \mathscr{O}_{a,\chi,\varepsilon}(\sqrt{h})$$

for all $a \in \mathscr{C}^\infty_0(T^* \mathbb{R}^d; \mathbb{C})$ and all $\chi \in \mathscr{C}^\infty_0(\mathbb{R}; \mathbb{C})$.
Hence, the assumption (127) implies

$$\forall \chi \in \mathscr{C}^\infty_0(\mathbb{R} \setminus F; [0,1]) , \quad \lim_{h \in \mathscr{E}, h \to 0} \| \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) \varrho_h(0) \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) \|_{\mathscr{L}^1(L^2_{x,\omega})} = 0 ,$$

and therefore

$$\forall \chi \in \mathscr{C}^\infty_0(\mathbb{R} \setminus F; [0,1]) , \forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad \lim_{h \in \mathscr{E}, h \to 0} \| \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) \varrho_{h,\varepsilon}(t) \chi(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon) \|_{\mathscr{L}^1(L^2_{x,\omega})} = 0 ,$$

with $\varrho_{h,\varepsilon}(t) = U_{\mathcal{V}_\varepsilon}(\frac{t}{h}) \varrho_h(0) U^*_{\mathcal{V}_\varepsilon}(\frac{t}{h})$ and $U_{\mathcal{V}_\varepsilon}(t) = e^{-it(-\Delta_x + \sqrt{h} \mathcal{V}_\varepsilon)}$.
When $\mathscr{E}' \subset \mathscr{E}$, $0 \in \overline{\mathscr{E}'}$, is such that

$$\mathscr{M}(\varrho_{h,\varepsilon}(t), h \in \mathscr{E}') = \{\mu_{t,\varepsilon}\} ,$$

Proposition 7.2 tells us

$$|\mu_t - \mu_{t,\varepsilon}|(T^*\mathbb{R}^d) \le C'_{\alpha_1}\varepsilon\,.$$

while

$$\int_{T^*\mathbb{R}^d} a(x,\xi)\,|\chi|^2(|\xi|^2)\,d\mu_{t,\varepsilon}(x,\xi)$$
$$= \lim_{h\in\mathscr{E}',h\to 0} \mathrm{Tr}\left[\chi(-\Delta_x+\sqrt{h}\mathscr{V}_\varepsilon)a^{\mathrm{Weyl}}(hx,D_x)\,\chi(-\Delta_x+\sqrt{h}\mathscr{V}_\varepsilon)\varrho_{h,\varepsilon}(t)\right]=0\,,$$

for $a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C})$ and $\chi\in\mathscr{C}_0^\infty(\mathbb{R}\setminus F;[0,1])$. We deduce

$$\forall a\in\mathscr{C}_0^\infty(T^*\mathbb{R}^d;\mathbb{C}),\forall\chi\in\mathscr{C}_0^\infty(\mathbb{R}\setminus F;[0,1]),\forall t\in]-\hat{T}_{\alpha_1},\hat{T}_{\alpha_1}[,\quad \int_{T^*\mathbb{R}^d} a(x,\xi)\,\chi^2(|\xi|^2)\,d\mu_t(x,\xi)=0\,,$$

which yields the result. □

7.3 Changing V

The formulation of Theorem 5.1 $u^h_G(t)=U_{\mathscr{V}}(t)u^h_{G,0}=U(\frac{t}{h})u^h_{G,0}+u^h_\infty(t)$ where $\left(u^h_q\right)_{q\in\{\infty,2,1\}}$ is a solution of a fixed point problem, solved in Proposition 4.5, where only $\|V\|_{L^{r'_\sigma}}$, $r'_\sigma=\frac{2d}{d+2}$, is used, allows to consider perturbations of V, which can be done separately in the the terms $a_G(V)$ and $a^*_G(V)$ and with complex valued perturbations.
Remember that our state $\varrho_h(t)=U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)U^*_{\mathscr{V}}(\frac{t}{h})$ is written

$$\varrho_h(t)=[U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}][\varrho_h(0)^{1/2}U^*_{\mathscr{V}}(\frac{t}{h})],$$

and the link with the fixed point problem is done after setting

$$U(t)u^h_{G,0}+u^h_\infty(t)=u^h_G(t)=U_{\mathscr{V}}(\frac{t}{h})\varrho_h(0)^{1/2}\quad\text{in }\mathscr{L}^2(L^2_{x,\omega})\sim L^2_{z,y_G}\,,$$

where the last identification is done via the unitary transform U_G of Section 3, omitted here and explained in the proof of Proposition 6.1.
A generalization is done by writing for a pair $\tilde{\mathscr{V}}=(V_1,V_2)\in L^{r'_\sigma}(\mathbb{R}^d,dy;\mathbb{C})^2$,

$$\varrho_{h,\tilde{\mathscr{V}}}(t)=u^h_{G,\tilde{\mathscr{V}}}(\frac{t}{h})[u^h_{G,\tilde{\mathscr{V}}}(\frac{t}{h})]^*\in\mathscr{L}^1(L^2_{x,\omega}),\tag{128}$$

where $u^h_{G,\tilde{\mathscr{V}}}(t)=U(t)\varrho_h(0)^{1/2}+u^h_{\infty,\tilde{\mathscr{V}}}(t)$ and $\left(u^h_{q,\tilde{\mathscr{V}}}\right)_{q\in\{\infty,2,1\}}$ solves the fixed point problem (59)(60)(61) with $f^h_1(t)=0$ and f^h_∞ and f^h_2 given by

$$f^h_\infty(t)=f^h_{\infty,\tilde{\mathscr{V}}}(t)=-i\int_0^t U(t)U(s)^*a^*_G(V_1)\sqrt{h}U(s)u^h_{G,0}\,ds\,,\tag{129}$$

$$f^h_2(t)=f_{2,\tilde{\mathscr{V}}}(t)=-i\,a_G(V_2)\int_0^t U(t)U(s)^*a^*_G(V_1)\sqrt{h}U(s)u^h_{G,0}\,ds+a_G(V_2)U(t)u^h_{G,0}\,.\tag{130}$$

This fixed point problem will be written

$$\begin{pmatrix}u^h_{\infty,\tilde{\mathscr{V}}}\\ u^h_{2,\tilde{\mathscr{V}}}\\ u^h_{1,\tilde{\mathscr{V}}}\end{pmatrix}=L_{\tilde{\mathscr{V}}}\begin{pmatrix}u^h_{\infty,\tilde{\mathscr{V}}}\\ u^h_{2,\tilde{\mathscr{V}}}\\ u^h_{1,\tilde{\mathscr{V}}}\end{pmatrix}+\begin{pmatrix}f^h_{\infty,\tilde{\mathscr{V}}}\\ f^h_{2,\tilde{\mathscr{V}}}\\ 0\end{pmatrix}.\tag{131}$$

Proposition 7.4. *For two pairs* $\tilde{\mathcal{V}}_k = (V_{1,k}, V_{2,k}) \in L^{r'_\sigma}(\mathbb{R}^d, dy; \mathbb{C})^2$, *for* $\|e^{\alpha_1 N} u^h_{G,0}\| \leq C_{\alpha_1}$ *and by choosing* $\hat{T}_{\alpha_1} > 0$ *small enough, the two solutions to* (131) *with the right-hand sides given by* (129)(130) *satisfy*

$$\forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad \|u^h_{\infty,\tilde{\mathcal{V}}_2}(\frac{t}{h}) - u^h_{\infty,\tilde{\mathcal{V}}_1}(\frac{t}{h})\|_{L^2_{z,y_G}} \leq C\left[\|V_{1,2} - V_{1,1}\|_{L^{r'_\sigma}} + \|V_{2,2} - V_{2,1}\|_{L^{r'_\sigma}}\right]$$

for some constant $C > 0$ *given by* $\alpha_1 > 0$, C_{α_1}, *the dimension* d, *and* $\max_{i,j} \|V_{i,j}\|_{L^{r'_\sigma}}$.

Proof. It suffices to notice that the difference $v^h = u^h_{\tilde{\mathcal{V}}_2} - u^h_{\tilde{\mathcal{V}}_1}$ with $u^h_{\tilde{\mathcal{V}}_k} = \left(u^h_{q,\tilde{\mathcal{V}}_k}\right)_{q\in\{\infty,2,1\}}$, $k = 1, 2$, solves

$$v^h - L_{\tilde{\mathcal{V}}_1}(v^h) = (L_{\tilde{\mathcal{V}}_2} - L_{\tilde{\mathcal{V}}_1})(u^h_{\tilde{\mathcal{V}}_2}) + \begin{pmatrix} f^h_{\infty,\tilde{\mathcal{V}}_2} - f^h_{\infty,\tilde{\mathcal{V}}_1} \\ f^h_{2,\tilde{\mathcal{V}}_2} - f^h_{2,\tilde{\mathcal{V}}_1} \\ 0 \end{pmatrix}.$$

Estimates for all the terms of the right-hand side have essentially been proved for Proposition 4.5 and for Theorem 5.1. Although they are written for $V_1 = V_2$ real-valued in Theorem 5.1 the generalization is straightforward (like in Proposition 4.5) and upper bounds are proportional the $L^{r'_\sigma}$ of the potential which is either $(V_{1,2} - V_{1,1})$ or $(V_{2,2} - V_{2,1})$.
The time interval $]-T_{\alpha_1}, T_{\alpha_1}[=]-2\hat{T}_{\alpha_1}, 2\hat{T}_{\alpha_1}[$ is actually chosen like in Proposition 4.5 such that $\|L_{\tilde{\mathcal{V}}_1}\|_{\mathscr{L}(\mathscr{E}_{\alpha_1,-\alpha_1},\gamma)} \leq \frac{1}{2}$ and this ends the proof. □

For a general pair $\tilde{\mathcal{V}} = (V_1, V_2) \in L^{r'_\sigma}(\mathbb{R}^d, dy; \mathbb{C})^2$, the trace-class operator $\varrho^h_{\tilde{\mathcal{V}}}(t)$ is no more a state and neither self-adjoint. However it remains uniformly bounded in $\mathscr{L}^1(L^2_{x,\omega})$ and complex-valued semiclassical measures $\mu_{\tilde{\mathcal{V}}}(t)$ make sense for $t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[$. Moreover the results of Proposition 5.2 and Proposition 6.1 can be adapted mutatis mutandis for such a general pair, so that semiclassical measures (extraction process) can be defined simultaneously for all $t \in]\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[$.
The above comparison result can be translated in terms of trace-class operators and asymptotically for semiclassical measures.

Proposition 7.5. *Assume*

$$V \in L^{r'_\sigma}(\mathbb{R}^d, dx; \mathbb{R}) \cap H^2(\mathbb{R}^d; \mathbb{R}) \quad , \quad V_1, V_2 \in L^{r'_\sigma}(\mathbb{R}^d, dx; \mathbb{C}) \quad , \quad r'_\sigma = \frac{2d}{d+2} \quad , \quad d \geq 3,$$

and assume that there exists $\alpha_1 > 0$ *such that* $\varrho_h(0) \in \mathscr{L}^1(L^2(\mathbb{R}^d \times \Omega, dx \otimes \mathscr{G}; \mathbb{C}))$, $\varrho_h(0) \geq 0$, $\mathrm{Tr}[\varrho_h(0)] = 1$ *satisfies*

$$\exists C_{\alpha_1} > 0, \forall h \in]0, h_0[, \quad \mathrm{Tr}\left[e^{\alpha_1 N} \varrho_h(0) e^{\alpha_1 N}\right] \leq C_{\alpha_1}.$$

Let $\varrho_h(t) = U_{\mathcal{V}}(\frac{t}{h})\varrho(0)U^*_{\mathcal{V}}(\frac{t}{h})$ *and let* $\varrho_{h,\tilde{\mathcal{V}}}(t)$ *be defined by* (128). *Then*

$$\exists C > 0, \forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad \|\varrho_h(t) - \varrho_{h,\tilde{\mathcal{V}}}(t)\|_{\mathscr{L}^1(L^2_{x,\omega})} \leq C\left[\|V_1 - V\|_{L^{r'_\sigma}} + \|V_2 - V\|_{L^{r'_\sigma}}\right].$$

When the subset $\mathscr{E} \subset]0, h_0[$, $0 \in \overline{\mathscr{E}}$, *is chosen such that*

$$\forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad \mathscr{M}(\varrho_h(t), h \in \mathscr{E}) = \{\mu_t\} \quad \textit{and} \quad \mathscr{M}(\varrho_{h,\tilde{\mathcal{V}}}(t), h \in \mathscr{E}) = \{\mu_{t,\tilde{\mathcal{V}}}\}$$

Then the total variation of $\mu_t - \mu_{t,\tilde{\mathcal{V}}}$ *is estimated by*

$$\exists C' > 0, \forall t \in]-\hat{T}_{\alpha_1}, \hat{T}_{\alpha_1}[, \quad |\mu_t - \mu_{t,\tilde{\mathcal{V}}}|(\underbrace{T^*\mathbb{R}^d}_{\text{or } T^*\mathbb{R}^d \sqcup \{\infty\}}) \leq C'\left[\|V_1 - V\|_{L^{r'_\sigma}} + \|V_2 - V\|_{L^{r'_\sigma}}\right].$$

Proof. It suffices to write

$$\varrho_{h,\tilde{\mathcal{V}}}(t)-\varrho_h(t)=\left[u^h_{G,\tilde{\mathcal{V}}}(\frac{t}{h})-u^h_G(\frac{t}{h})\right][u^h_{G,\tilde{\mathcal{V}}}(\frac{t}{h})]^*+[u^h_G(\frac{t}{h})]\left[u^h_{G,\tilde{\mathcal{V}}}(\frac{t}{h})-u^h_G(\frac{t}{h})\right]^*$$

and to remember that Hilbert-Schmidt norms correspond to L^2_{z,y_G}-norms estimated in Proposition 7.4. □

7.4 Quantum dynamics with low regularity

We conclude with an easy application of Proposition 7.4 which says that the dynamics $(U_{\mathcal{V}}(t))_{t\in\mathbb{R}}$ is actually well defined under the sole assumption

$$V\in L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})\quad,\quad r'_\sigma=\frac{2d}{d+2}\quad d\geq 3\,, \tag{132}$$

with good approximations when $V_n\in L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})\cap H^2(\mathbb{R}^d;\mathbb{R})$ satisfies $\lim_{n\to\infty}\|V_n-V\|_{L^{r'_\sigma}}=0$.

Proposition 7.6. *Let V belong to $L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})$ and let $(V_n)_{n\in\mathbb{N}}$ be a sequence in $L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})\cap H^2(\mathbb{R}^d;\mathbb{R})$ such that $\lim_{n\to\infty}\|V-V_n\|_{L^{r'_\sigma}}=0$. Then for any $t\in\mathbb{R}$ the unitary operator $U_{\mathcal{V}_n}(t)$ converges strongly to a unitary operator $U_{\mathcal{V}}(t)$.*
Therefore $(U_{\mathcal{V}}(t))_{t\in\mathbb{R}}$ is a strongly continous unitary group in $L^2(\mathbb{R}^d\times Z'',dx\otimes\mathbf{dz}'';\Gamma(L^2(\mathbb{R}^d,dy;\mathbb{C})))=L^2_{z,\mathrm{sym}}L^2_{y_G}$ with a self-adjoint generator denoted $(-\Delta_x+\sqrt{h}\mathcal{V},D(-\Delta_x+\sqrt{h}\mathcal{V}))$.
The convergence $(-\Delta_x+\sqrt{h}\mathcal{V}_n,D(-\Delta_x+\sqrt{h}\mathcal{V}_n))$ to $(-\Delta_x+\sqrt{h}\mathcal{V},D(-\Delta_x+\sqrt{h}\mathcal{V}))$ holds in the strong resolvent sense.

Remark 7.7. *Although the dynamics $(U_{\mathcal{V}}(t))_{t\in\mathbb{R}}$ and its self-adjoint generator $(-\Delta_x+\sqrt{h}\mathcal{V},D(-\Delta_x+\sqrt{h}\mathcal{V}))$ is well defined for $V\in L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})$, we have no information on the domain $D(-\Delta_x+\sqrt{h}\mathcal{V})$. The approximation process by $V_n\in L^{r'_\sigma}(\mathbb{R}^d;\mathbb{R})\cap H^2(\mathbb{R}^d;\mathbb{R})$ for which a core of $\Delta_x+\sqrt{h}\mathcal{V}_n$ is given by Proposition 4.4 in [Bre] recalled in Lemma 2.2, provides a substitute for the analysis.*

It could be interesting to see if this Schrödinger type approach relying on endpoint Strichartz estimates could be applied to other quantum field theoretic problem and whether it would bring additional information of tools as compared with the euclidean approach (see [Sim] and refences therein).

Proof. Actually we can work here with $h=1$. The convergence of

$$U_{\mathcal{V}_n}(t)u_{G,0}=U(t)u_{G,0}+u_{\infty,\mathcal{V}_n}(t)$$

is deduced from the convergence (see Proposition 7.4) of $u_{\infty,\mathcal{V}_n}(t)$ to $u_{\infty,\mathcal{V}}(t)$ when $e^{\alpha_1 N}u_{G,0}\in L^2_{z,\mathrm{sym}}L^2_{y_G}$ for some $\alpha_1>0$.
From $\|U_{\mathcal{V}_n}(t)u_{G,0}\|_{L^2_{z,\mathrm{sym}}L^2_{y_G}}=\|u_{G,0}\|_{L^2_{z,\mathrm{sym}}L^2_{y_G}}$ we deduce $\|U_{\mathcal{V}}(t)u_{G,0}\|_{L^2_{z,\mathrm{sym}}L^2_{y_G}}=\|u_{G,0}\|_{L^2_{z,\mathrm{sym}}L^2_{y_G}}$.
This finally provides the extension of $U_{\mathcal{V}}(t)u_{G,0}$ for any $u_{G,0}\in L^2_{z,\mathrm{sym}}L^2_{y_G}$ with the convergence of $U_{\mathcal{V}_n}(t)u_{G,0}$ to $U_{\mathcal{V}}(t)u_{G,0}$, because $e^{-\alpha_1 N}L^2_{z,\mathrm{sym}}L^2_{y_G}$ is dense in $L^2_{z,\mathrm{sym}}L^2_{y_G}$. Passing from the strong convergence of unitary groups to the strong resolvent convergence of generators is standard. □

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