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# Lambek-Grishin Calculus: focusing, display and full polarization

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*Focused sequent calculi* [1, 2, 11] make use of syntactic restrictions on the applicability of inference rules achieving three main goals: (i) the proof search space is considerably reduced without losing completeness, (ii) every cut-free proof comes in a special normal form, (iii) a criterion for defining identity of sequent calculi proofs. Being able to identify or tell apart two proofs has far-reaching consequences.

We introduce a novel focused display calculus **fd.LG** and a *fully polarized* algebraic semantics (see the last paragraph of this abstract for more details on this)  $\text{FP.LG}$  for Lambek-Grishin logic [12] by generalising the theory of *multi-type calculi* [5] and their algebraic semantics, admitting not only heterogeneous operators [4], but also *heterogeneous consequence relations* (see [9]) now interpreted as *weakening relations* [10] (i.e. a natural generalisation of partial orders). The calculus **fd.LG** has *strong focalization* and it is *sound and complete* w.r.t.  $\text{FP.LG}$ . This completeness result is in a sense stronger than completeness with respect to standard polarized algebraic semantics, insofar we do not need to quotient over proofs with consecutive applications of *shifts operator* over the same formula (see the last paragraph of this abstract for more details on this). We also show a number of additional results. **fd.LG** is sound and complete w.r.t. LG-algebras: this amounts to a semantic proof of the *completeness of focusing*, given that the standard (display) sequent calculus for Lambek-Grishin logic is complete w.r.t. LG-algebras. **fd.LG** and the focused calculus **flG** of Moortgat and Moot are equivalent with respect to proofs, indeed there is an effective translation from **flG**-derivations to **fd.LG**-derivations and vice versa: this provides the link with operational semantics, given that every **flG**-derivation is in a Curry-Howard correspondence with a directional  $\bar{\lambda}\mu\tilde{\mu}$ -term.

We conjecture that this approach, here tailored for the signature of the Lambek-Grishin logic, can be extended to a large class of logics, namely all lattice expansions logics extended with *analytic inductive axioms* (see [6]). We conjecture that if a calculus belongs to this class, then it enjoys cut-elimination, aiming at generalizing the cut-elimination meta-theorem in the tradition of display calculi (see [13]). Moreover, we conjecture that any *displayable logic* [6] can be equivalently presented as an instance of this class.

In what follows we summarise the main features of this analysis in general terms, without special reference to Lambek-Grishin logic. In the case of focused sequent calculi, the distinction between *positive* versus *negative* formulas is the key ingredient for organising proofs in *phases*. The distinction is proof-theoretically relevant in that it reflects a fundamental distinction between logical introduction rules. We observe that this distinction is also semantically

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grounded, indeed the main connective of a positive formula (in the original language of the logic) is a left adjoint/residual and the main connective of a negative formula (in the original language of the logic) is a right adjoint/residual. Proofs in *focalized normal form* (see [12]) are cut-free proofs organised in three phases: two focused phases (either positive or negative) and one non-focused phase (also called neutral phase). A focused positive (resp. negative) phase in a derivation is a proof-section where a formula is decomposed as much as possible only by means of non-invertible logical rules for positive (resp. negative) connectives. This formula and all its immediate subformulas in this proof-section are said ‘in focus’. All the other rules are applied only in non-focused phases. So, each derivable sequent has at most one formula in focus. Moreover, the interaction between two focused phases is always mediated by a non-focused phase.

*Shift operators* –usually denoted as  $\uparrow$  and  $\downarrow$  ([7, 8, 3])– are often considered to polarize a focused sequent calculus, i.e. as a tool to control the interplay between positive and negative formulas and the interaction between phases. Shifts are adjoint unary operators that change the polarity of a formula, where  $\uparrow$  goes from positive to negative,  $\downarrow$  goes from negative to positive, and  $\uparrow \dashv \downarrow$ . In this paper, we consider positive and negative formulas as formulas of different sorts. We also distinguish between positive (resp. negative) *pure* formulas and positive (resp. negative) *shifted* formulas, i.e. formulas under the scope of a shift operator. So, we end up considering four different sorts, each of which is interpreted in a different sub-algebra. Therefore, in this setting shifts are heterogeneous operators, where  $\uparrow$  gets split into  $\uparrow$  (from positive pure formulas into negative shifted formulas) and  $\uparrow$  (from positive shifted formulas into negative pure formulas),  $\downarrow$  gets split into  $\downarrow$  (from negative pure formulas into positive shifted formulas) and  $\downarrow$  (from negative shifted formulas into positive pure formulas),  $\uparrow \dashv \downarrow$  and  $\downarrow \dashv \uparrow$ . Moreover, the composition of two shifts is still either a closure or an interior operator (by adjunction), but we do not assume that it is an identity. We call a presentation of a logic with the features described above *full* polarization.

## References

- [1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
- [2] J.-M. Andreoli. Focussing and proof construction. *Annals of Pure and Applied Logic*, 107(1):131–163, 2001.
- [3] A. Bastenhof. Polarized Montagovian semantics for the Lambek-Grishin calculus. In P. de Groote and M. Nederhof, editors, *Formal Grammar*, volume 7395 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, 2012.
- [4] G. Birkhoff and J. D. Lipson. Heterogeneous algebras. *Journal of Combinatorial Theory*, 8(1):115–133, 1970.
- [5] S. Frittella, G. Greco, A. Kurz, A. Palmigiano, and V. Sikimić. Multi-type sequent calculi. In A. In- drzejczak, J. Kaczmarek, and M. Zawidzki, editors, *Proceedings Trends in Logic XIII*, volume 13, pages 81–93, 2014.
- [6] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, 28(7):1367–1442, 2016.
- [7] M. Hamano and P. Scott. A categorical semantics for polarized MALL. *Annals of Pure and Applied Logic*, 145(3):276–313, 2007.
- [8] M. Hamano and R. Takemura. A phase semantics for polarized linear logic and second order conservativity. *Journal of Symbolic Logic*, 75:77–102, 03 2010.

- [9] A. Jung, M. Kegelman, and A. M. Moshier. Multi lingual sequent calculus and coherent spaces. *Fundamenta Informaticae*, 37(4):369–412, 1999.
- [10] A. Kurz, A. M. Moshier, and A. Jung. Stone duality for relations, 2019.
- [11] D. Miller. *An Overview of Linear Logic Programming*, pages 119–150. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.
- [12] M. Moortgat and R. Moot. Proof nets and the categorial flow of information. In A. Baltag, D. Grossi, A. Marcoci, B. Rodenhäuser, and S. Smets, editors, *Logic and Interactive RAtionality. Yearbook 2011*, pages 270–302. ILLC, University of Amsterdam, 2012.
- [13] H. Wansing. *Sequent Systems for Modal Logics*, volume 8, pages 61–145. Springer, Dordrecht, 2002.