



Lambek-Grishin Calculus: focusing, display and full polarization

Giuseppe Greco, Michael Moortgat, Valentin D. Richard, Apostolos Tzimoulis

► To cite this version:

Giuseppe Greco, Michael Moortgat, Valentin D. Richard, Apostolos Tzimoulis. Lambek-Grishin Calculus: focusing, display and full polarization. *Topology, Algebra, and Categories in Logic (TACL 2022)*, Maria Manuel Clementino; Peter Jipsen; Jorge Picado, Jun 2022, Coimbra, Portugal. pp.79-81. hal-03727761

HAL Id: hal-03727761

<https://hal.science/hal-03727761>

Submitted on 19 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lambek-Grishin Calculus: focusing, display and full polarization

GIUSEPPE GRECO^{1,*}, MICHAEL MOORTGAT², VALENTIN D. RICHARD³,
AND APOSTOLOS TZIMOULIS¹

¹ Vrije Universiteit Amsterdam
{g.greco,a.tzimoulis}@vu.nl

² Utrecht University
mjmoortgat@gmail.com

³ LORIA, Nancy
valentin.richard@live.fr

Focused sequent calculi [1, 2, 11] make use of syntactic restrictions on the applicability of inference rules achieving three main goals: (i) the proof search space is considerably reduced without losing completeness, (ii) every cut-free proof comes in a special normal form, (iii) a criterion for defining identity of sequent calculi proofs. Being able to identify or tell apart two proofs has far-reaching consequences.

We introduce a novel focused display calculus **fd.LG** and a *fully polarized* algebraic semantics (see the last paragraph of this abstract for more details on this) **FP.LG** for Lambek-Grishin logic [12] by generalising the theory of *multi-type calculi* [5] and their algebraic semantics, admitting not only heterogeneous operators [4], but also *heterogeneous consequence relations* (see [9]) now interpreted as *weakening relations* [10] (i.e. a natural generalisation of partial orders). The calculus **fd.LG** has *strong focalization* and it is *sound and complete* w.r.t. **FP.LG**. This completeness result is in a sense stronger than completeness with respect to standard polarized algebraic semantics, insofar we do not need to quotient over proofs with consecutive applications of *shifts operator* over the same formula (see the last paragraph of this abstract for more details on this). We also show a number of additional results. **fd.LG** is sound and complete w.r.t. LG-algebras: this amounts to a semantic proof of the *completeness of focusing*, given that the standard (display) sequent calculus for Lambek-Grishin logic is complete w.r.t. LG-algebras. **fd.LG** and the focused calculus **flg** of Moortgat and Moot are equivalent with respect to proofs, indeed there is an effective translation from **flg**-derivations to **fd.LG**-derivations and vice versa: this provides the link with operational semantics, given that every **flg**-derivation is in a Curry-Howard correspondence with a directional $\bar{\lambda}\mu\tilde{\mu}$ -term.

We conjecture that this approach, here tailored for the signature of the Lambek-Grishin logic, can be extended to a large class of logics, namely all lattice expansions logics extended with *analytic inductive axioms* (see [6]). We conjecture that if a calculus belongs to this class, then it enjoys cut-elimination, aiming at generalizing the cut-elimination meta-theorem in the tradition of display calculi (see [13]). Moreover, we conjecture that any *displayable logic* [6] can be equivalently presented as an instance of this class.

In what follows we summarise the main features of this analysis in general terms, without special reference to Lambek-Grishin logic. In the case of focused sequent calculi, the distinction between *positive* versus *negative* formulas is the key ingredient for organising proofs in *phases*. The distinction is proof-theoretically relevant in that it reflects a fundamental distinction between logical introduction rules. We observe that this distinction is also semantically

*Speaker.

grounded, indeed the main connective of a positive formula (in the original language of the logic) is a left adjoint/residual and the main connective of a negative formula (in the original language of the logic) is a right adjoint/residual. Proofs in *focalized normal form* (see [12]) are cut-free proofs organised in three phases: two focused phases (either positive or negative) and one non-focused phase (also called neutral phase). A focused positive (resp. negative) phase in a derivation is a proof-section where a formula is decomposed as much as possible only by means of non-invertible logical rules for positive (resp. negative) connectives. This formula and all its immediate subformulas in this proof-section are said ‘in focus’. All the other rules are applied only in non-focused phases. So, each derivable sequent has at most one formula in focus. Moreover, the interaction between two focused phases is always mediated by a non-focused phase.

Shift operators –usually denoted as $\uparrow$ and $\downarrow$ ([7, 8, 3])– are often considered to polarize a focused sequent calculus, i.e. as a tool to control the interplay between positive and negative formulas and the interaction between phases. Shifts are adjoint unary operators that change the polarity of a formula, where $\uparrow$ goes from positive to negative, $\downarrow$ goes from negative to positive, and $\uparrow \dashv \downarrow$. In this paper, we consider positive and negative formulas as formulas of different sorts. We also distinguish between positive (resp. negative) *pure* formulas and positive (resp. negative) *shifted* formulas, i.e. formulas under the scope of a shift operator. So, we end up considering four different sorts, each of which is interpreted in a different sub-algebra. Therefore, in this setting shifts are heterogeneous operators, where $\uparrow$ gets split into $\uparrow$ (from positive pure formulas into negative shifted formulas) and $\uparrow$ (from positive shifted formulas into negative pure formulas), $\downarrow$ gets split into $\downarrow$ (from negative pure formulas into positive shifted formulas) and $\downarrow$ (from negative shifted formulas into positive pure formulas), $\uparrow \dashv \downarrow$ and $\uparrow \dashv \downarrow$. Moreover, the composition of two shifts is still either a closure or an interior operator (by adjunction), but we do not assume that it is an identity. We call a presentation of a logic with the features described above *full* polarization.

References

- [1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
- [2] J.-M. Andreoli. Focussing and proof construction. *Annals of Pure and Applied Logic*, 107(1):131–163, 2001.
- [3] A. Bastenhof. Polarized Montagovian semantics for the Lambek-Grishin calculus. In P. de Groote and M. Nederhof, editors, *Formal Grammar*, volume 7395 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, 2012.
- [4] G. Birkhoff and J. D. Lipson. Heterogeneous algebras. *Journal of Combinatorial Theory*, 8(1):115–133, 1970.
- [5] S. Frittella, G. Greco, A. Kurz, A. Palmigiano, and V. Sikimić. Multi-type sequent calculi. In A. In- drzejczak, J. Kaczmarek, and M. Zawidzki, editors, *Proceedings Trends in Logic XIII*, volume 13, pages 81–93, 2014.
- [6] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, 28(7):1367–1442, 2016.
- [7] M. Hamano and P. Scott. A categorical semantics for polarized MALL. *Annals of Pure and Applied Logic*, 145(3):276–313, 2007.
- [8] M. Hamano and R. Takemura. A phase semantics for polarized linear logic and second order conservativity. *Journal of Symbolic Logic*, 75:77–102, 03 2010.

- [9] A. Jung, M. Kegelman, and A. M. Moshier. Multi lingual sequent calculus and coherent spaces. *Fundamenta Informaticae*, 37(4):369–412, 1999.
- [10] A. Kurz, A. M. Moshier, and A. Jung. Stone duality for relations, 2019.
- [11] D. Miller. *An Overview of Linear Logic Programming*, pages 119–150. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.
- [12] M. Moortgat and R. Moot. Proof nets and the categorial flow of information. In A. Baltag, D. Grossi, A. Marcoci, B. Rodenhäuser, and S. Smets, editors, *Logic and Interactive Rationality. Yearbook 2011*, pages 270–302. ILLC, University of Amsterdam, 2012.
- [13] H. Wansing. *Sequent Systems for Modal Logics*, volume 8, pages 61–145. Springer, Dordrecht, 2002.