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Structure Constants of Short Operators in Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We present a conjecture for the three-point functions of single-trace operators in planar $\mathcal{N} = 4$ super-Yang-Mills theory at finite coupling, in the case where two operators are protected. Our proposal is based on the hexagon representation for structure constants of long operators, which we complete to incorporate operators of any length using data from the TBA-QSC formalism. We perform various tests of our conjecture, at weak and strong coupling, finding agreement with the gauge theory through 5 loops for the shortest three-point function and with string theory in the classical limit.

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Introduction.—The discovery of integrability [1,2] in the planar limit of the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory has led to tremendous advances in the study of this interacting superconformal gauge theory and of its gravitational dual, the type IIB superstring theory in $\text{AdS}_5 \times S^5$ [3]. The best example is the solution to the full spectrum of anomalous dimensions of single-trace operators which was argued on both sides of the duality to follow from the diagonalization of a commuting family of transfer matrices $T_{a,s}(u)$, depending on a spectral parameter u and labeled by representations (a, s) of the superconformal group. The latter generate an infinite number of conserved charges, pinning down the conformal primary operators, and fulfill the celebrated Hirota equation [4]

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}, \quad (1)$$

with $T_{a,s}^\pm = T_{a,s}(u \pm i/2)$. Once supplied with appropriate boundary conditions and analyticity requirements [5–8], Eq. (1) can be cast into a set of integral thermodynamic Bethe ansatz (TBA) equations [4,9–11], or in the more compact quantum spectral curve (QSC) equations involving Q functions with simpler analytical structure [5,12]. One may expect optimistically that the remaining conformal data—the structure constants—can be addressed along the same lines and expressed in the same terms, for all values of the 't Hooft coupling constant, $\lambda = 16\pi^2 g^2$, as suggested by recent studies [13–16].

However, to date, the known non-perturbative method for structure constants of single-trace operators, which relies on the hexagon representation [17–21], only works for asymptotically long operators, with very high charges. For short operators, the description is known to disagree with perturbation theory, past a certain loop order, highlighting the need to include wrapping corrections in the formalism [22,23] as in the TBA's early days [24,25]. Even worse, the description is plagued with divergences that need to be subtracted carefully and it stays unknown so far how to perform this subtraction systematically, such as to make contact with the solution for the spectrum.

In this Letter, we present a conjectured solution to this problem for a simple class of structure constants, with two half-BPS operators and one spinning operator,

$$C^{\circ\circ\circ} \sim \langle \text{Tr}[Z_1^{L_1}] \text{Tr}[Z_2^{L_2}] \text{Tr}[D^S Z^L] \rangle, \quad (2)$$

with Z_1, Z_2, Z three complex scalar fields and D a light-cone derivative. They look pictorially like in Fig. 1 with two vacua of length L_1 and L_2 merging into a length- L Bethe state with S magnons. Our conjecture, which

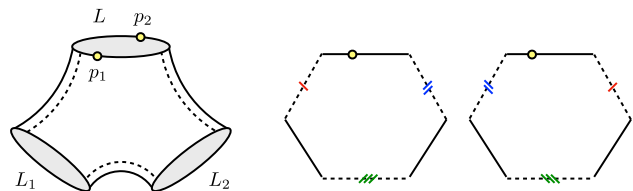


FIG. 1. Pair of pants with two empty strings at the bottom and an excited one at the top with magnons carrying momenta p_1 and p_2 . The string diagram may be obtained by gluing two hexagons together along the identified mirror edges (dashed lines) as shown in the right panel.

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extrapolates from existing results, may be viewed as a minimal way of combining the hexagon representation with the TBA-QSC spectral data, such as to obtain an exact description for operators of arbitrary length.

Main conjecture.—The key idea behind the hexagon construction is that the structure constant may be obtained by attaching two hexagons together (Fig. 1). The gluing is achieved by summing over all the states of the open strings stretching along the seams of the pair of pants. It results in a representation in terms of multiple sums of integrals, describing the multiple exchanges of particles, the mirror magnons, across the three channels of the pair of pants (Fig. 2), that is,

$$C^{\circ\circ} = \mathcal{N} \times \sum_L \times \sum_R \times \sum_B e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} |\mathcal{H}|^2, \quad (3)$$

with \mathcal{N} an overall normalization factor (see Sec. A of the Supplemental Material [26]). Each process in the sums occurs with a “probability” $|\mathcal{H}|^2$, determined by the hexagon form factors [17], and is suppressed by a factor depending on the energies of the mirror states, $\mathcal{E}_{L,R,B}$. The latter are conjugate to the so-called bridge lengths,

$$\ell_L = \frac{L_1 + L - L_2}{2}, \quad \ell_R = L - \ell_L, \quad \ell_B = \frac{L_1 + L_2 - L}{2}, \quad (4)$$

fixing the distances between the hexagons.

The details of this construction were worked out in Ref. [34] yielding definite predictions for the integrand $|\mathcal{H}|^2$ and the phase-space integral \sum . The latter reads

$$\sum = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \sum_{a_i=1}^{\infty} \int_C \frac{\mu_{a_i}^{\text{mir}}(u_i) du_i}{2\pi} \prod_{i<j} P_{a_i a_j}^{\text{mir}}(u_i, u_j), \quad (5)$$

with N the number of magnons in the given channel and where each magnon in the sum carries a rapidity u and “spin” a , labeling its representation under the asymptotic symmetry algebra, $\mathfrak{su}(2|2)_l \oplus \mathfrak{su}(2|2)_r$ [35], and its energy [24]

$$\mathcal{E}_a(u) = \log(x^{[+a]} x^{[-a]}), \quad (6)$$

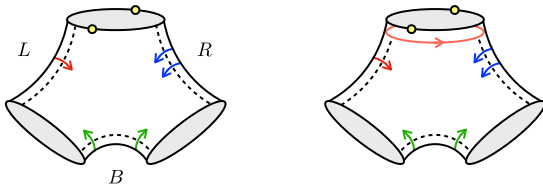


FIG. 2. Mirror magnons propagating across the left, right, and bottom seams of the pair of pants. Additional magnons moving closely around the excited operator as in the right panel are needed to account for wrapping corrections.

with $x^{[\pm a]} = x(u \pm ia/2)$ and $x(u) = (u + \sqrt{u^2 - 4g^2})/2g$. The integration is done with the help of the measure

$$p_{ab}^{\text{mir}}(u, v) = k_{ab}^{++} k_{ab}^{+-} k_{ab}^{-+} k_{ab}^{--}, \quad (7)$$

with $k_{ab}^{\pm\pm} = (x^{[\pm a]} - y^{[\pm b]}) / (x^{[\pm a]} y^{[\pm b]} - 1)$, $y^{[\pm b]} = x(v \pm ib/2)$, and with μ_a^{mir} coming from the double zero of p^{mir} at $v = u$,

$$p_{ab}^{\text{mir}}(u, v) \sim \mu_a^{\text{mir}}(u)^2 (u - v)^2 \delta_{ab}, \quad (8)$$

with δ_{ab} the Kronecker delta. The contour of integration C runs over the real axis, up to a small detour which is explained in the next section.

The integrand $|\mathcal{H}|^2$ is less explicit, yet remarkable in that it factorizes into a number of individual weights and pairwise interactions,

$$|\mathcal{H}|^2 = \prod_{i,j,k=1}^{N_{L,R,B}} \frac{\mathbb{W}_{a_i}^L(u_i) \mathbb{W}_{b_j}^R(v_j) \mathbb{W}_{c_k}^B(w_k)}{P_{a_i b_j}^{\text{mir}}(u_i, v_j)}, \quad (9)$$

with $\{u_i, v_j, w_k\}$ and $\{a_i, b_j, c_k\}$ denoting the rapidities and spins of the magnons in the three channels. The weights $\mathbb{W}_a(u)$ encode the interactions between the mirror magnons and the physical magnons in the excited state. As such, they depend implicitly on the Bethe roots, $\mathbf{z} = \{z_1, \dots, z_S\}$, parametrizing the excited-state wave function. In the original hexagon description, see formulas in Ref. [34], they are expressed in terms of the eigenvalues of the $\mathfrak{su}(2|2)$ transfer matrices, describing the action of the magnon S matrices on the Bethe state [35,36].

As said earlier, the problem with this description is that it is only valid in the *asymptotic* limit, that is, when the length L of the excited operator is large. At finite length L , one also expects wrapping corrections $\sim e^{-nL\mathcal{E}_a}$, $n = 1, 2, \dots$, associated with particles winding around the excited operator (Fig. 2). Their appearance relates to the double poles in Eq. (9), see Eq. (8), for magnons with same quantum numbers, $u_i = v_j$ and $a_i = b_j$, in the left and right channels, which must be regularized somehow. An all-order derivation of these corrections, akin to the TBA for the spectrum, is still lacking. Nonetheless, one can gain several insights into the general formula from the leading exponentials, which were worked out in a number of situations [22,37,38]. They indicate that the hexagon formula stays intact (and as factorized as above) up to modifications of the weights \mathbb{W} in Eq. (9) and normalization factor \mathcal{N} in Eq. (3). Furthermore, the expressions hint at a simple all-order extrapolation in terms of TBA-QSC quantities.

To be precise, the evidence suggests that the problem is solved by (i) shifting the poles away from the real axis,

$$p_{ab}^{\text{mir}}(u, v) \rightarrow p_{ab}^{\text{mir}}(u + i0, v - i0), \quad (10)$$

in Eq. (9), and (ii) setting

$$\mathbb{W}_a^{R/L}(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)} \frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^{+/-}(u)}, \quad (11)$$

with $\mathbf{T}_{a,0}^\pm(u) = \mathbf{T}_{a,0}(u \pm i/2)$, and

$$\mathbb{W}_a^B(u) = e^{-\frac{1}{2}L\mathcal{E}_a(u)} \mathbf{t}_{a,1}(u). \quad (12)$$

Here, $\mathbf{t}_{a,s}$ and $\mathbf{T}_{a,s}$ denote the eigenvalues (for the Bethe state of interest) of two families of transfer matrices, associated with compact and noncompact representations of the superconformal algebra, respectively. The latter generate the familiar T-system underlying the TBA equations [4,9–11], with (a, s) labeling the nodes on the T hook, whereas the former define a companion system solving the Hirota equation (1) on the L hook (Fig. 3).

Transfer matrices in general are only defined up to gauge transformations [4]

$$T_{a,s} \rightarrow g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-a]} g_4^{[-a-s]} T_{a,s}, \quad (13)$$

which leave Eq. (1) invariant $\forall g_j$, with $g_j^{[n]} = g_j(u + in/2)$. In our case, the gauge is fixed, with \mathbf{T} given in the distinguished \mathbf{T} -gauge of Refs. [5,8] and with \mathbf{t} normalized as in Ref. [39], up to a sign, $\mathbf{t}_{a,s|\text{here}} = (-1)^{as} \mathbf{t}_{a,s|\text{there}}$. One may also find in these references explicit representations for the transfer matrices in terms of the QSC's Q functions, which are valid for operators of any length L and prove extremely useful in practical applications.

Alternatively, one may state the conjecture in a gauge-invariant way, using the Y functions,

$$Y_{a,s} = \mathbf{T}_{a,s+1} \mathbf{T}_{a,s-1} / \mathbf{T}_{a+1,s} \mathbf{T}_{a-1,s}. \quad (14)$$

This is so at least for \mathbb{W}_a^R and \mathbb{W}_a^L , which readily obey

$$e^{-L\mathcal{E}_a} \mathbb{W}_a^R \mathbb{W}_a^L = \frac{\mathbf{T}_{a,1} \mathbf{T}_{a,1}}{\mathbf{T}_{a,0}^+ \mathbf{T}_{a,0}^-} = \frac{Y_{a,0}}{1 + Y_{a,0}}, \quad (15)$$

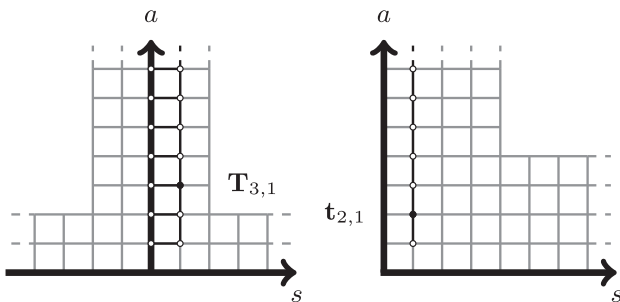


FIG. 3. Hooks for rectangular representations of $\mathfrak{su}(2, 2|4)$ and $\mathfrak{su}(4|4)$, respectively [39], with highlighted the transfer matrices used in the conjecture. The \mathbf{T} - \mathbf{t} systems are defined as solutions to the Hirota equation with null boundary conditions outside their respective domains.

using Eq. (1) and the left-right symmetry, $\mathbf{T}_{a,s} = \mathbf{T}_{a,-s}$, observed for our states. Integral representations [5,39] yield the extra relation

$$\log \left[\frac{\mathbb{W}_a^L(u)}{\mathbb{W}_a^R(u)} \right] = i \sum_{b=1}^{\infty} \int_C \frac{dv}{c} L_b(v) \partial_v \log p_{ba}^{\text{mir}}(v, u), \quad (16)$$

with $L_a(u) = \log [1 + Y_{a,0}(u)]$ and with a principal-value prescription for the pole at $v = u$ when $b = a$. The analysis is much harder for the \mathbf{t} 's, however, which are less obviously embedded in the TBA formalism.

Formulas (11) and (12) are our main results for the resummation of the wrapping corrections. We stress that they go along with the prescription (10), with the $\pm i0$ shifts responding to the \mp shifts of $\mathbf{T}_{a,0}$ in Eq. (11).

At last, there is a formula for the normalization constant, \mathcal{N} , which depends on the Bethe state alone. It is conjectured to be given, up to a simple factor, by Fredholm determinants canonically associated with the TBA equations, in line with recent findings for structure constants of determinant operators [15,40]. Its detailed presentation is deferred to Sec. A of the Supplemental Material [26].

Contour and partitions.—In addition to the mirror sums, the hexagon rules also predict a sum over all possible ways of distributing the physical magnons on the two sides of the pair of pants. This sum was skipped in the formula above, which may be viewed as describing a configuration with all roots standing on the same hexagon. The reason for this omission is that this sum is not independent and may be restored for a suitable choice of the contour \mathcal{C} in Eq. (5), as was first noticed in Refs. [41,42]. The key observation is that the fundamental weights (with $a = 1$) develop simple poles on the physical sheet, at the locations of the Bethe roots, $\log \mathbb{W}_1^{\text{phys}}(u) \sim \log(u - z_i)$ for $u \sim z_i$. In analogy with the analytic continuation trick used in the thermodynamic Bethe ansatz [43–45], see also Refs. [15,40,46] for recent discussions, the contour must be chosen to go around these Bethe poles.

There are two alternatives: \mathcal{C} may go around the poles of \mathbb{W}_1^R sitting on the second sheet of $x(u + i/2)$, as depicted in Fig. 4, or around those of \mathbb{W}_1^L on the second sheet of $x(u - i/2)$. They may be shown to be equivalent, for roots solving the exact Bethe equations, $Y_{1,0}^{\text{phys}} = -1$. Either way, extracting the residues has the desired effect of moving magnons to the next hexagon, over the right or left bridge.

In particular, keeping *only* the integrals around the roots one reproduces the sum over the partitions [17,47]

$$\mathcal{C}^{\circ\circ} \rightarrow \mathcal{N} \times \sum_{\alpha} \frac{(-1)^{|\alpha|} \prod_{z_j \in \alpha} e^{ip(z_j) \ell_R} \mathcal{T}(z_j)}{\prod_{z_i, z_j \in \bar{\alpha}} h(z_i, z_j)}, \quad (17)$$

which runs over the subsets of \mathbf{z} , with $\alpha \cup \bar{\alpha} = \mathbf{z}$, $p(u) = i\mathcal{E}_1^{\text{phys}}$ the spin-chain momentum and $h(z_i, z_j)$ the physical

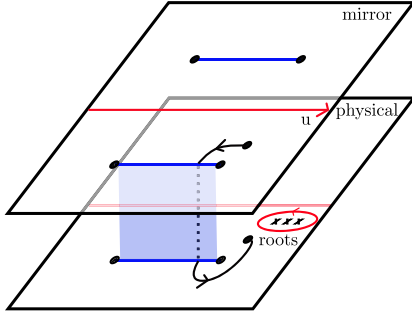


FIG. 4. Rapidity sheets of $\mathbb{W}_1^R(u)$ with branch points at $u = \pm 2g \pm i/2$ and with poles on the second sheet at the positions of the Bethe roots. The contour \mathcal{C} goes along the real line in the mirror sheet and around the roots in the physical sheet.

hexagon form factor [17]. $\mathcal{T}(z_j)$ is a dressing factor collecting the wrapping corrections to the residue of $\mathbb{W}_1^R(u)^{\text{phys}}$ at $u = z_j$. Using Eqs. (14) and (15), it may be written as

$$\mathcal{T} = \exp \left[\frac{i}{2} \Phi(z_j) - \frac{i}{2} \sum_{a=1}^{\infty} \int \frac{du}{2\pi} L_a(u) \partial_u \log p_{a1}(u', z_j) \right], \quad (18)$$

where Φ is the wrapping-induced phase shift in the exact Bethe equations [25] and $p_{a1}(u', z_j)$ is the measure (7) in the mixed, mirror-physical kinematics, see Ref. [17] for notations. It agrees perfectly with the Lüscher formula proposed in Ref. [22], in the IR limit $L_a \approx Y_{a,0} \sim e^{-L\mathcal{E}_a}$, up to an overall factor which follows from \mathcal{N} .

At last, let us comment on \mathbb{W}_1^B . In the asymptotic description, this weight also appears to have poles on the physical sheets. It suggests the strange possibility that the physical magnons should also be transported over the bottom bridge, in tension with the locality of the hexagon rules. However, the tension goes away when wrapping corrections are included. The exact expression (12) is indeed free from singularities, since transfer matrices are regular everywhere on their Riemann surfaces [5,39]. The puzzle arises because the analytic continuation to the physical sheets does not commute with the large L limit; the mirror asymptotic formula being valid only on the mirror sheet. The effect is nicely illustrated with the fundamental transfer matrix at weak coupling. On the physical sheet, it is given as a sum of two terms,

$$\mathbf{t}_{1,1}^{\text{phys}} \propto e^{\frac{1}{2}pL} \left(\frac{Q(u+i)}{Q(u)} - 1 \right) + e^{-\frac{1}{2}pL} \left(\frac{Q(u-i)}{Q(u)} - 1 \right), \quad (19)$$

with $Q = \prod_{j=1}^S (u - z_j)$ and $e^{ip} = (u + i/2)/(u - i/2)$. It is regular at $u = z_j$, despite the poles in each term, in virtue of the Baxter equation. Nonetheless, the cancellation cannot be seen asymptotically from the mirror kinematics, because

the two terms map then to different orders $e^{\pm ipL/2} \rightarrow e^{\mp \mathcal{E}L/2}$ at large L .

Advanced checks.—We turn to direct tests of our conjecture with available data from field and string theory. To simplify the discussion, we shall focus on a ratio of structure constants

$$R(\ell_B) = C^{\circ\circ\circ} / \lim_{\ell_B \rightarrow \infty} C^{\circ\circ\circ}, \quad (20)$$

with the limit taken at fixed L . Owing to the factorization in Eqs. (3) and (9), this ratio receives no contributions from the left-right channels or normalization factor. As such, it is given by a single sum of mirror integrals,

$$R = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi} \mu_a^{\text{mir}}(u) e^{-\ell_B \mathcal{E}_a(u)} \mathbb{W}_a^B(u) + \dots, \quad (21)$$

with dots standing for terms with $N = 2, 3, \dots$ magnons.

Weak coupling.—Taking the weak coupling limit, $g^2 \rightarrow 0$, greatly simplifies the analysis, by suppressing the N -particle terms in Eq. (21) which kick in at $N(\ell_B + N)$ loops at the earliest [34]. The first wrapping correction, on the other hand, enters at $(L + \ell_B + 2)$ loops and thus dominates over the higher-particle terms for $L < \ell_B + 2$. This hierarchy is observed for the “shortest” three-point function of two stress tensors and one Konishi operator, with $\ell_B = 1$ and $L = S = 2$. The associated ratio (20) was studied in Refs. [23,48–50] through higher loops, using diagrammatic techniques, and a mismatch was found with the original hexagon formula at five loops, that is, precisely when the first wrapping occurs. The discrepancy reads [23]

$$\frac{\delta R}{g^{10}} = 972\zeta_3 - 2700\zeta_5 + 5355\zeta_7 - 2376\zeta_3\zeta_5 - 1512\zeta_9, \quad (22)$$

with $\zeta_n = \sum_{a=1}^{\infty} a^{-n}$ the Riemann zeta function. Nicely, formula (12) restores the agreement with the field theory. To see that, one needs the expression for $\mathbf{t}_{a,1}$. It is easily obtained using the QSC [39]

$$\mathbf{t}_{a,1}(u) = - \sum_{j=1}^4 \mathbf{Q}_j(u + ia/2) \tilde{\mathbf{Q}}^j(u - ia/2), \quad (23)$$

with $\mathbf{Q}_j, \tilde{\mathbf{Q}}^j$ the fundamental (fermionic) Q functions. The weak coupling expansion may then be taken efficiently using powerful QSC solvers [51–54]. The outcome is a complicated meromorphic function of the rapidity, which is too lengthy to be shown here; we describe it in detail in Sec. B of the Supplemental Material [26]. Its integration is immediate and it reproduces perfectly the gauge-theory prediction, including the wrapping contribution (22).

Strong coupling.—The ratio (20) may also be calculated in the string theory at strong coupling, $g \gg 1$, in the classical limit $L, S, \ell_B \sim g$. In this regime, one may rely on the integrability of the classical worldsheet theory to get a closed-form expression for the structure constants [55,56]. It is given in terms of the eigenvalues of the string monodromy matrix $\Omega(x) \in \text{SU}(4|4)$, generating the conserved charges of the string [57–59],

$$\Omega \cong \text{diag}(e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4} | e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4}), \quad (24)$$

with x the spectral parameter and $\tilde{p}_i(x), \hat{p}_j(x)$ the so-called quasimomenta. General expressions for the states of interest give [60,61]

$$\hat{p}_1(x) = -\hat{p}_4(x) = -\hat{p}_2\left(\frac{1}{x}\right) = \hat{p}_3\left(\frac{1}{x}\right) = \frac{L}{2i}\mathcal{E}(x) + G(x), \quad (25)$$

and similarly for \tilde{p}_i with $G \rightarrow 0$, with $\mathcal{E}(x) = ix/[g(x^2 - 1)]$ and with $G(x)$ the resolvent of the classical curve. The string theory then predicts that [55,56]

$$\log R_{\text{string}} = \int_{U^+} \frac{du(x)}{2\pi} \sum_{j=1}^4 [\text{Li}_2(\xi e^{i\hat{p}_j}) - \text{Li}_2(\xi e^{i\tilde{p}_j})], \quad (26)$$

at large g , with $\text{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$ the dilogarithm, U^+ the upper half of the circle $|x| = 1$, $u(x) = g(x + 1/x)$, and $\xi(x) = e^{-\frac{1}{2}(L_1+L_2)\mathcal{E}(x)}$.

Now, the hexagon sum in Eq. (21) also greatly simplifies in this regime. It exponentiates and its exponent can be calculated exactly using the clustering method [42]

$$\log R_{\text{strong}} = \int_{U^+} \frac{du(x)}{2\pi} \int_0^\xi \frac{dq}{q} \log \left[\sum_{a=0}^{\infty} q^a \mathbf{t}_{a,1}(x) \right], \quad (27)$$

with $\mathbf{t}_{0,1} = 1$. In this form, the comparison with the string formula is immediate. One first observes [62,63] that in the classical limit the \mathbf{t} 's become characters of the group element Ω in finite-dimensional representations of $\text{su}(4|4)$, with the generating function

$$\sum_{a=0}^{\infty} q^a \mathbf{t}_{a,1} = \text{sdet}(1 - q\Omega(x)) = \prod_{j=1}^4 \frac{1 - qe^{i\tilde{p}_j(x)}}{1 - qe^{i\hat{p}_j(x)}}, \quad (28)$$

and “ sdet ” the Berezinian (superdeterminant). Plugging then this expression inside Eq. (27) and integrating over q , one gets

$$\log R_{\text{strong}} = - \int_{U^+} \frac{du(x)}{2\pi} \text{str}[\text{Li}_2(\xi\Omega)], \quad (29)$$

with “ str ” the graded trace on the $4|4$ module, in perfect agreement with the string result (26).

It is also instructive to look at these expressions in the asymptotic limit, which corresponds here to $L \gg g$. In this regime, half of the eigenvalues are exponentially large, see Eq. (25),

$$e^{i\hat{p}_{1,2}} \sim e^{i\tilde{p}_{1,2}} \sim e^{L\mathcal{E}/2}. \quad (30)$$

It implies that the product in Eq. (28) and the sum in Eq. (26) can be restricted to $j = 1, 2$, as for the $\text{su}(2|2)$ problem studied in Ref. [42]. The four remaining eigenvalues, with indices $j = 3, 4$, are exponentially small $\sim e^{-L\mathcal{E}/2}$ and stand for wrapping corrections. This decomposition makes it clear that maintaining the agreement with the classical string theory away from the large L limit amounts to restoring the full $\text{su}(4|4)$ symmetry, as done in Eq. (12) through the replacement of the “old” $\text{su}(2|2)$ hexagon weights by “new” $\text{su}(4|4)$ ones.

Finally, let us mention that similar tests may be performed for the left-right sums and normalization factor. The analysis is significantly harder and not as transparent as the one carried out here, but the comparison may be done order by order in the wrapping parameter $\sim e^{-L\mathcal{E}}$, using the character solution for the \mathbf{T} system [60,62,63].

Conclusion and outlook.—We reported a conjecture for structure constants of single-trace operators in planar $\mathcal{N} = 4$ SYM theory at finite coupling and found evidence for it at weak and strong coupling from matching with gauge and string theory.

For simplicity we focused in this Letter on operators in the smallest closed subsector. It seems possible, however, to extend the conjecture to the higher-rank sectors, such as to describe operators with derivatives but also scalars and fermions. Structure constants of this sort were analyzed in Ref. [64] in the asymptotic limit and found to be subject to a flavor selection rule, imposing a stringent left-right symmetry on the operators, that is, $\mathbf{T}_{a,s} = \mathbf{T}_{a,-s}$ [5]. We believe our formula applies as well to this more general case. It is less clear how to extend it to asymmetric operators though, should the left-right symmetry be broken by wrapping corrections. It would also be fascinating to explore applications of our findings to structure constants with two or more unprotected operators, like the ones studied recently in Refs. [65–69].

One may also expect applications in lower dimensional AdS/CFT setups, where the hexagon formalism was developed [70], or in integrable deformations of the SYM theory, such as the fishnet theory [71,72]. The latter is of particular interest to make contact with approaches based on the separation of variables [16,73] or to shed light on the group theory meaning of our formula.

Admittedly, the evaluation of the hexagon sums at high orders in perturbation theory or at finite coupling necessitates a fair amount of work. One will also run into serious difficulties at special points in the parameter space, like at the extremal points where the mixing with double-trace

operators is important. At these points the hexagon integrals no longer make sense and must be analytically continued. The Pfaffian structure identified in Ref. [64] may help simplifying the algebra. It may also help obtaining a more compact representation for the structure constants, as found recently in the study of large-charge four-point functions [74–76].

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