



**HAL**  
open science

# On the Joint Spectral Radius of Shuffled Switched Linear Systems

Georges Aazan, Antoine Girard, Luca Greco, Paolo Mason

► **To cite this version:**

Georges Aazan, Antoine Girard, Luca Greco, Paolo Mason. On the Joint Spectral Radius of Shuffled Switched Linear Systems. International Symposium on Mathematical Theory of Networks and Systems, 2022, Bayreuth, Germany. hal-03727152

**HAL Id: hal-03727152**

**<https://hal.science/hal-03727152v1>**

Submitted on 19 Jul 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the Joint Spectral Radius of Shuffled Switched Linear Systems

Georges Aazan\* Antoine Girard\*\* Luca Greco\*\*  
Paolo Mason\*\*

\* *Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France.*  
(e-mail: georges.aazan@lsv.fr).

\*\* *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France.*  
(e-mail: [firstname].[lastname]@l2s.centralesupelec.fr).

---

**Abstract:** We present and develop tools to analyze stability properties of discrete-time switched linear systems driven by shuffled switching signals. A switching signal is said to be shuffled if all modes of the system are activated infinitely often. We establish a notion of joint spectral radius related to these systems: the shuffled joint spectral radius (SJSR) which intuitively measures the impact of shuffling on the decay rate of the system's state. We show how this quantity relates to stability properties of such systems. Specifically, from the SJSR, we can build a lower bound on the minimal shuffling rate in order to stabilize an unstable system. Then, we present several methods to approximate the SJSR, mainly by computing lower and upper bounds using Lyapunov methods and some automata theoretic techniques.

*Keywords:* Switched Systems, Lyapunov Functions, Exponential Stability, Formal Languages and Automata.

---

## 1 INTRODUCTION

Switched systems are dynamical systems with several modes of operations where the active mode is determined by a switching signal. Early works focus on proving stability of switched systems driven by arbitrary switching signals or by switching signals with dwell-time conditions [Liberzon (2003); Sun (2006); Lin and Antsaklis (2009)]. More recent works have considered systems with constrained switching signals where the switching signals are generated by labeled graphs [Lee and Dullerud (2007); Dai (2012); Athanasopoulos and Lazar (2014); Philippe et al. (2016); Pepe (2018)]. Shuffled switching signals is also a class of constrained switching signals that has been considered in the literature [Gurvits (1995); Wang et al. (2017); Girard and Mason (2019)]. A switching signal is said to be shuffled if all the modes of the switched systems are activated infinitely often. In this paper, we focus on the discrete-time switched linear systems driven by shuffled switching signals.

It is well known that the question of stability of discrete-time switched systems with arbitrary switching can be answered efficiently with the notion of the joint spectral radius (JSR) (see Jungers (2009) and the references therein). Intuitively the JSR represents the maximal asymptotic growth rate of products of matrices taken in a set. Computing the JSR is not an easy task but several techniques exist to approximate it. With a similar approach, it is possible to define a joint spectral radius related to the

shuffled switched systems: the shuffled joint spectral radius (SJSR).

In this talk, we will present the notion of shuffled switched systems, we will also introduce the SJSR and investigate its properties. Then we will relate this quantity with the stability of such systems and finally we will describe numerical techniques for the SJSR approximation. Also, we mention that the proofs of theoretical results and numerical examples can be found in the preprint [Aazan et al. (2021)].

## 2 SHUFFLED JOINT SPECTRAL RADIUS

In this section, we present the concept of shuffled switched systems, then we introduce the notion of the  $\rho$ -SJSR, also we state a theorem that relates this quantity with the shuffled switched systems trajectories. Finally we present a resulting corollary that relates the system stabilizability with the  $\rho$ -SJSR.

### 2.1 Definition

We consider a discrete-time switched linear system described by the following equation

$$x(t+1) = A_{\theta(t)}x(t), \quad (1)$$

where  $t \in \mathbb{N}$ ,  $x(t) \in \mathbb{R}^n$  is the state and  $\theta : \mathbb{N} \rightarrow \mathcal{I}$  is the switching signal belonging to a particular class of arbitrary switching signals: the shuffled switching signals.  $\mathcal{I} = \{1, \dots, m\}$ , with  $m \geq 2$  is the finite set of modes and  $\mathcal{A} = \{A_i \in \mathbb{R}^{n \times n} | i \in \mathcal{I}\}$  is a finite set of matrices indexed by the modes. For a switching signal  $\theta$ , let  $\mathbb{A}_{\theta,0} = I_n$ , and

---

\* This work was supported in part by the Agence Nationale de la Recherche (ANR) under Grant HANDY ANR-18-CE40-0010.

$$\mathbb{A}_{\theta, T} = \prod_{t=0}^{T-1} A_{\theta(t)} = A_{\theta(T-1)} \cdots A_{\theta(0)}, \quad \forall T \geq 1.$$

Given an initial state  $x_0 \in \mathbb{R}^n$ , the trajectory defined by (1) with  $x(0) = x_0$  is denoted  $\mathbf{x}(\cdot, x_0, \theta)$ , it satisfies for all  $t \in \mathbb{N}$ ,  $\mathbf{x}(t, x_0, \theta) = \mathbb{A}_{\theta, t} x_0$ .

Formally, the shuffled switching signal is defined as in [Girard and Mason (2019)]:

*Definition 1.* A switching signal  $\theta : \mathbb{N} \rightarrow \mathcal{I}$  is *shuffled* if

$$\forall i \in \mathcal{I}, \forall T \in \mathbb{N}, \exists t \geq T : \theta(t) = i.$$

Following the previous definition, it is natural to define the following quantities related to a shuffled switching signal  $\theta$ :

- The sequence of *shuffling instants*  $(\tau_k^\theta)_{k \in \mathbb{N}}$  is defined by  $\tau_0^\theta = 0$  and for all  $k \in \mathbb{N}$ ,

$$\tau_{k+1}^\theta = \min \left\{ t > \tau_k^\theta \mid \begin{array}{l} \forall i \in \mathcal{I}, \exists s \in \mathbb{N} : \\ \tau_k^\theta \leq s < t \text{ and } \theta(s) = i \end{array} \right\}.$$

- The *shuffling index*  $\kappa^\theta : \mathbb{N} \rightarrow \mathbb{N}$  is given by
- The *shuffling rate*  $\gamma^\theta$  is defined as

$$\gamma^\theta = \liminf_{t \rightarrow +\infty} \frac{\kappa^\theta(t)}{t}.$$

Let  $\mathcal{S}_s(\mathcal{I})$  be the set of all shuffled switching signals taking values in  $\mathcal{I}$ .

Now we are in a good position to define the  $\rho$ -SJSR. Given a finite set of matrices  $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ , let  $\rho(\mathcal{A})$  be its joint spectral radius (JSR), we recall that the JSR of a set of matrices  $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$  is defined as following:

$$\rho(\mathcal{A}) = \lim_{k \rightarrow +\infty} \left( \sup \left\{ \left\| \prod_{j=1}^k A_j \right\|^{1/k} \mid A_j \in \mathcal{A}, j = 1, \dots, k \right\} \right).$$

We define the  $\rho$ -SJSR as following:

*Definition 2.* For all  $\rho > \rho(\mathcal{A})$ , the Shuffled Joint Spectral Radius relative to  $(\mathcal{A}, \rho)$  ( $\rho$ -SJSR for short) is defined as

$$\lambda(\mathcal{A}, \rho) = \limsup_{k \rightarrow +\infty} \left( \sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left( \frac{\|\mathbb{A}_{\theta, \tau_k^\theta}\|}{\rho^{\tau_k^\theta}} \right)^{1/k} \right). \quad (2)$$

It is useful to say that the *limsup* in the above definition can be replaced by a simple limit. Due to the norm equivalence, one can replace the norm in (2) by any submultiplicative matrix norm.

## 2.2 Shuffled switched systems and $\rho$ -SJSR

Now, when  $\lambda(\mathcal{A}, \rho) > 0$ , we bring out the relation between the system's trajectories and the  $\rho$ -SJSR. Then, we will derive a sufficient condition for stabilization based on the minimal shuffling rate and the  $\rho$ -SJSR. The following theorem clarifies the relationship between the  $\rho$ -SJSR and the behavior of the trajectories of (1).

*Theorem 1.* For all  $\rho > \rho(\mathcal{A})$ , for all  $\lambda \in (\lambda(\mathcal{A}, \rho), 1]$ , there exists  $C \geq 1$  such that

$$\|\mathbf{x}(t, x_0, \theta)\| \leq C \rho^t \lambda^{\kappa^\theta(t)} \|x_0\|, \quad \forall \theta \in \mathcal{S}_s(\mathcal{I}), \forall x_0 \in \mathbb{R}^n, \forall t \in \mathbb{N}. \quad (3)$$

Conversely, if there exists  $C \geq 1$ ,  $\rho \geq 0$  and  $\lambda \in [0, 1]$  such that (3) holds, then either  $\rho > \rho(\mathcal{A})$  and  $\lambda \geq \lambda(\mathcal{A}, \rho)$ , or  $\rho = \rho(\mathcal{A})$  and  $\lambda \geq \sup_{\rho' > \rho(\mathcal{A})} \lambda(\mathcal{A}, \rho')$ .

A remarkable result from the previous theorem is a sufficient condition for stabilization based on the minimal shuffling rate.

*Corollary 1.* Assume  $\lambda(\mathcal{A}, \rho) > 0$  for every  $\rho > \rho(\mathcal{A})$ . Let  $\theta \in \mathcal{S}_s(\mathcal{I})$ , if there exists  $\rho > \rho(\mathcal{A})$  such that  $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\lambda(\mathcal{A}, \rho))}$ , then

$$\lim_{t \rightarrow +\infty} \|\mathbf{x}(t, x_0, \theta)\| = 0, \quad \forall x_0 \in \mathbb{R}^n. \quad (4)$$

The proof of this corollary follows from the previous theorem and from the shuffling rate definition.

## 3 APPROXIMATION OF THE $\rho$ -SJSR

We have seen in the previous section that the  $\rho$ -SJSR plays an important role in the stability analysis of shuffled switched systems. However, like the JSR, this quantity is difficult to calculate. In this section, we will consider the problem of approximating the  $\rho$ -SJSR. In the following, we will give an explicit expression for a lower bound based on the JSR of a set constructed from  $\mathcal{A}$ , we will show that this lower bound is asymptotically tight, moreover an exact expression for the  $\rho$ -SJSR will be given under certain conditions. Next, an approach to find upper bounds will be given based on multiple Lyapunov functions and automata theoretic techniques.

### 3.1 Lower bounds computation

Let  $\mathcal{N}_{\mathcal{I}}$  be the set of products of matrices where all modes in  $\mathcal{I}$  appear exactly once, formally:

$$\mathcal{N}_{\mathcal{I}} = \left\{ \prod_{k=1}^m A_{j_k} \mid \forall i \in \mathcal{I}, \exists k \in \{1, \dots, m\}, j_k = i \right\}.$$

The following theorem gives an explicit expression of a lower bound for the  $\rho$ -SJSR based on the JSR of  $\mathcal{N}_{\mathcal{I}}$ . Also, it shows that this lower bound is asymptotically tight, moreover, under certain conditions, it reveals an explicit expression of the  $\rho$ -SJSR.

*Theorem 2.* The following results hold true.

- (i) For all  $\rho > \rho(\mathcal{A})$ ,

$$\lambda(\mathcal{A}, \rho) \geq \frac{\rho(\mathcal{N}_{\mathcal{I}})}{\rho^m}. \quad (5)$$

- (ii) We have the asymptotic estimate

$$\lim_{\rho \rightarrow +\infty} \rho^m \lambda(\mathcal{A}, \rho) = \rho(\mathcal{N}_{\mathcal{I}}). \quad (6)$$

- (iii) If there exists a norm  $\|\cdot\|_*$  that is extremal<sup>1</sup> for  $\mathcal{N}_{\mathcal{I}}$ , then there exists  $R \geq \rho(\mathcal{A})$  such that for all  $\rho \geq R$ ,

$$\lambda(\mathcal{A}, \rho) = \frac{\rho(\mathcal{N}_{\mathcal{I}})}{\rho^m}. \quad (7)$$

**Remark:** If  $\mathcal{A}$  consists of invertible matrices only, an explicit expression of  $R$  can be given.

In the next section, we give a method for computing upper bounds using Lyapunov theory.

<sup>1</sup> An induced norm  $\|\cdot\|_*$  is said to be extremal for a set of matrices  $\mathcal{A}$ , if it satisfies  $\rho(\mathcal{A}) = \max_{A \in \mathcal{A}} \|A\|_*$ .

### 3.2 Upper bounds computation

This section details a method for computing upper bounds on the JSR and the  $\rho$ -SJSR.

*Theorem 3.* If there exist  $V : (2^{\mathcal{I}} \setminus \{\mathcal{I}\}) \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ ,  $\alpha_1, \alpha_2, \rho > 0$  and  $\lambda \in [0, 1]$  such that the following inequalities hold true for every  $x \in \mathbb{R}^n$

$$\alpha_1 \|x\|^2 \leq V(J, x) \leq \alpha_2 \|x\|^2, \quad \forall J \subsetneq \mathcal{I} \quad (8)$$

$$V(J \cup \{i\}, A_i x) \leq \rho^2 V(J, x), \quad \text{if } J \cup \{i\} \neq \mathcal{I} \quad (9)$$

$$V(\emptyset, A_i x) \leq \rho^2 \lambda^2 V(J, x), \quad \text{if } J \cup \{i\} = \mathcal{I} \quad (10)$$

then the bound (3) holds. Conversely, if the matrices  $A_i$  are invertible, for all  $i \in \mathcal{I}$  and the bound (3) holds for some  $\rho > 0$ ,  $\lambda \in [0, 1]$  and  $C \geq 1$ , then there exists a function  $V : (2^{\mathcal{I}} \setminus \{\mathcal{I}\}) \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$  such that the inequalities (8), (9) and (10) are satisfied.

The proof of the direct result relies on a finite state automaton whose states corresponds to subsets of  $\mathcal{I}$ , where each time the switching signal shuffles, there will be a transition to the automaton's initial state which corresponds to the empty set, therefore, based on the automaton, it is not hard to construct the conditions of the theorem. The proof of the converse result relies on a multiple Lyapunov function (each corresponds to a state) which can be seen as the supremum of trajectories norm that lead from the corresponding state to a specific state.

By combining the result of the previous theorem with Theorem 1, one can easily compute upper bounds on the  $\rho$ -SJSR and the JSR using some LMIs.

## 4 CONCLUSION

In this work, we have defined the  $\rho$ -SJSR, a special kind of JSR related to shuffled switched systems. We successfully related this notion to the stabilization of such systems. Also, we provided some theoretical tools for approximation using automata theoretic techniques and Lyapunov functions. Some interesting numerical examples can be found in [Aazan et al. (2021)] and will be presented in the talk.

The current work opens several research directions for the future. First, the development of numerical and theoretic techniques to compute tighter bounds on the  $\rho$ -SJSR. Secondly since our approach is based on automata theoretic techniques, it is natural to think that one can derive stability conditions by working directly on the Büchi, Rabin or Muller automaton specifying the  $\omega$ -regular language.

## References

Aazan, G., Girard, A., Greco, L., and Mason, P. (2021). Stability of shuffled switched linear systems: A joint spectral radius approach. URL <https://hal.archives-ouvertes.fr/hal-03257026>.  
 Athanasopoulos, N. and Lazar, M. (2014). Stability analysis of switched linear systems defined by graphs. In *IEEE Conference on Decision and Control*, 5451–5456.  
 Berger, M.A. and Wang, Y. (1992). Bounded semigroups of matrices. *Linear Algebra and its Applications*, 166, 21–27.

Dai, X. (2012). A Gel'fand-type spectral radius formula and stability of linear constrained switching systems. *Linear Algebra and its Applications*, 436(5), 1099–1113.  
 Girard, A. and Mason, P. (2019). Lyapunov functions for shuffle asymptotic stability of discrete-time switched systems. *IEEE Control Systems Letters*, 3(3), 499–504.  
 Gurvits, L. (1995). Stability of discrete linear inclusion. *Linear algebra and its applications*, 231, 47–85.  
 Jungers, R. (2009). *The joint spectral radius: theory and applications*, volume 385. Springer Science & Business Media.  
 Lee, J.W. and Dullerud, G.E. (2007). Uniformly stabilizing sets of switching sequences for switched linear systems. *IEEE Transactions on Automatic Control*, 52(5), 868–874.  
 Liberzon, D. (2003). *Switching in systems and control*. Springer Science & Business Media.  
 Lin, H. and Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic control*, 54(2), 308–322.  
 Pepe, P. (2018). Converse Lyapunov theorems for discrete-time switching systems with given switches digraphs. *IEEE Transactions on Automatic Control*.  
 Philippe, M., Essick, R., Dullerud, G.E., and Jungers, R.M. (2016). Stability of discrete-time switching systems with constrained switching sequences. *Automatica*, 72, 242–250.  
 Sun, Z. (2006). *Switched linear systems: control and design*. Springer Science & Business Media.  
 Wang, Y., Roohi, N., Dullerud, G.E., and Viswanathan, M. (2017). Stability analysis of switched linear systems defined by regular languages. *IEEE Transactions on Automatic Control*, 62(5), 2568–2575.