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# EXISTENCE AND MULTIPLICITY FOR FRACTIONAL DIRICHLET PROBLEM WITH $p(x)$ -LAPLACIAN EQUATION AND NEHARI MANIFOLD

J. VANTERLER DA C. SOUSA \*, D. S. OLIVEIRA AND RAVI P. AGARWAL

ABSTRACT. This paper is divided in two parts. In the first part, we prove the coercivity results and minimization of the Euler energy functional, in addition to presenting a new extension of the Harnack inequality. In the second part, we focus on the existence and multiplicity of a positive solutions for the fractional Dirichlet problem involving the  $p(x)$ -Laplacian equation with nonnegative weight functions in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega, \mathbb{R})$  using the Nehari manifold approach and some variational techniques.

## 1. INTRODUCTION AND MOTIVATION

Over the last decade, the variable exponent Lebesgue spaces  $L^{p(x)}$  and the corresponding Sobolev space  $W^{1,p(x)}$  have been a subject of an active research area [1, 2, 3, 4, 5, 6, 7, 8]. The specific attention according to such problems is due to their applications in mathematical physics. What has been noticed is a growing interest in elliptic problems in Sobolev space  $W^{1,p(x)}$  using classical variational techniques. Researchers such as Radulescu [11], Alves [5], Fan [1], Rabinowitz [12], Ambrosetti [13], Winkert [14], Pucci [15], Motreanu [16], Papageorgiou [17], Bisci [18], Repovš [19], among other researchers, have dedicated themselves to investigating cutting-edge problems using operators  $p(x)$ -Laplacian and performing applications.

In 2006 Mihailescu [9] investigated the existence of solutions for a boundary value problem of the type

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a bounded domain with smooth boundary,  $1 < p(x)$  and  $p(x) \in C(\overline{\Omega})$ . For more details see. Another interesting work on the existence of solutions involving  $p(x)$ -Laplacian was investigated by Alves and Barreiro [5]. In 2015, Chabrowski and Fu [2], considered the existence of solutions in  $W_0^{1,p(x)}(\Omega)$  for the  $p(x)$ -Laplacian problems in the superlinear and sublinear cases using the mountain pass theorem technique.

In 2007 Wu [10] investigated the multiplicity of solutions using in Nehari manifold for the elliptic equation

$$\begin{cases} -\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where  $1 < q < p < r < p^*$ ,  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $\lambda \in \mathbb{R}/\{0\}$ , and the weight functions  $f, g \in C(\overline{\Omega})$  are satisfying  $f^\pm = \max\{\pm f, 0\} \neq 0$  and  $g^\pm = \max\{\pm g, 0\} \neq 0$ . For more details, see [10].

On the other hand, in recent years increasing attention has been paid to the study of fractional differential equations [20, 21, 22, 23]. Such equations are used to model phenomena in medicine, physics, engineering, biology, among other areas (see for instance [20, 21, 22, 23, 24, 25] and the references therein). Recently, fractional differential equation problems involving  $p$ -Laplacian have gained attention from some researchers, in particular, involving the  $\psi$ -Hilfer fractional operator [26, 27, 29, 30, 32, 33, 34].

In 2020, Sousa et al. [28] motivated by the  $\psi$ -Hilfer fractional operator introduce the space  $\psi$ -fractional and a variational structure. In this sense, we investigated the existence and nonexistence of weak solutions

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for the fractional  $p$ -Laplacian via combination of the Nehari manifold and the application of fibration, of the following problem

$$\begin{cases} \mathbf{H}\mathbf{D}_T^{\alpha,\beta;\psi} \left( \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^{p-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right) = \lambda |\xi(x)|^{p-2} \xi(x) + b(x) |\xi(x)|^{q-1} \xi(x) \\ I_{0+}^{\beta(\beta-1);\psi} u(0) = I_T^{\beta(\beta-1);\psi} u(T) = 0. \end{cases} \quad (1.2)$$

For more details on the fractional problem Eq.(1.2), see the paper [28].

Before presenting the main problem to be addressed in this paper, we highlight the concepts of fractional integral with respect to another function and the  $\psi$ -Hilfer partial fractional derivative.

Let  $\theta = (\theta_1, \theta_2, \dots, \theta_N)$ ,  $T = (T_1, T_2, \dots, T_N)$  and  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$  where  $0 < \alpha_1, \alpha_2, \dots, \alpha_N < 1$  with  $\theta_j < T_j$ , for all  $j \in \{1, 2, \dots, N\}$ ,  $N \in \mathbb{N}$ . Also put  $\Omega = I_1 \times I_2 \times \dots \times I_N = [\theta_1, T_1] \times [\theta_2, T_2] \times \dots \times [\theta_N, T_N]$  where  $T_1, T_2, \dots, T_N$  and  $\theta_1, \theta_2, \dots, \theta_N$  positive constants. Consider also  $\psi(\cdot)$  be an increasing and positive monotone function on  $(\theta_1, T_1), (\theta_2, T_2), \dots, (\theta_N, T_N)$ , having a continuous derivative  $\psi'(\cdot)$  on  $(\theta_1, T_1], (\theta_2, T_2], \dots, (\theta_N, T_N]$ . The  $\psi$ -Riemann-Liouville fractional partial integral of order  $\alpha$  of  $N$ -variables  $u = (u_1, u_2, \dots, u_N) \in L^1(\Omega)$  denoted by  $\mathbf{I}_{\theta, x_j}^{\alpha, \psi}(\cdot)$ , is defined by [35, 36, 37]

$$\mathbf{I}_{\theta, x_j}^{\alpha, \psi} u(x_j) = \frac{1}{\Gamma(\alpha_j)} \int \int \dots \int_{\Omega} \psi'(s_j) (\psi(x_j) - \psi(s_j))^{\alpha_j-1} u(s_j) ds_j$$

with  $\psi'(s_j) (\psi(x_j) - \psi(s_j))^{\alpha_j-1} = \psi'(s_1) (\psi(x_1) - \psi(s_1))^{\alpha_1-1} \psi'(s_2) (\psi(x_2) - \psi(s_2))^{\alpha_2-1} \dots \psi'(s_N) (\psi(x_N) - \psi(s_N))^{\alpha_N-1}$  where  $\Gamma(\alpha_j) = \Gamma(\alpha_1) \Gamma(\alpha_2) \dots \Gamma(\alpha_N)$ ,  $u(s_j) = u(s_1) u(s_2) \dots u(s_N)$ ,  $ds_j = ds_1 ds_2 \dots ds_N$ , for all  $j \in \{1, 2, \dots, N\}$ . Analogously, it is defined  $\mathbf{I}_{T, x_j}^{\alpha, \psi}(\cdot)$ .

On the other hand, let  $u, \psi \in C^n(\Omega)$  two functions such that  $\psi$  is increasing and  $\psi'(x_j) \neq 0$   $j \in \{1, 2, \dots, N\}$ ,  $x_j \in \Omega$ . The  $\psi$ -Hilfer fractional partial derivative of  $N$ -variables, denoted by  $\mathbf{H}\mathbf{D}_{\theta, x_j}^{\alpha, \beta; \psi}(\cdot)$ , of order  $\alpha$  and type  $\beta$  ( $0 \leq \beta \leq 1$ ), is defined by [35, 36, 37]

$$\mathbf{H}\mathbf{D}_{\theta, x_j}^{\alpha, \beta; \psi} u(x_j) = \mathbf{I}_{\theta, x_j}^{\beta(1-\alpha), \psi} \left( \frac{1}{\psi'(x_j)} \frac{\partial^N}{\partial x_j^N} \right) \mathbf{I}_{\theta, x_j}^{(1-\beta)(1-\alpha), \psi} u(x_j) \quad (1.3)$$

with  $\partial x_j = \partial x_1, \partial x_2 \dots \partial x_N$  and  $\psi'(x_j) = \psi'(x_1) \psi'(x_2) \dots \psi'(x_N)$ , for all  $j \in \{1, 2, \dots, N\}$ . Analogously it is defined  $\mathbf{H}\mathbf{D}_{T, x_j}^{\alpha, \beta; \psi}(\cdot)$ .

Throughout this work, we will use the following notations  $\mathbf{I}_T^{\alpha, \psi}(\cdot) := \mathbf{I}_{T, x_j}^{\alpha, \psi}(\cdot)$ ,  $\mathbf{I}_{\theta}^{\alpha, \psi}(\cdot) := \mathbf{I}_{\theta, x_j}^{\alpha, \psi}(\cdot)$ ,  $\mathbf{H}\mathbf{D}_{\theta}^{\alpha, \beta; \psi}(\cdot) := \mathbf{H}\mathbf{D}_{\theta, x_j}^{\alpha, \beta; \psi}(\cdot)$  and  $\mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi}(\cdot) := \mathbf{H}\mathbf{D}_{T, x_j}^{\alpha, \beta; \psi}(\cdot)$ .

The natural question arises whether it is possible to extend the classical variable exponent case to include the fractional one. Motivated by the above works, in the present paper, we consider the fractional Dirichlet problem involving the  $p(x)$ -Laplacian equation given by

$$\begin{cases} \mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left( \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right|^{p(x)-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right) = \lambda a(x) |u|^{q(x)-2} u + b(x) |u|^{h(x)-2} u \quad \text{in } \Omega \\ u(x) = 0 \quad \text{on } \partial\Omega \end{cases} \quad (1.4)$$

where  $p, q, h \in C(\bar{\Omega})$  such that  $1 < q(x) < p(x) < h(x) < p_{\alpha}^*(x)$ ,  $p_{\alpha}^*(x) = \frac{2p(x)}{2-\alpha p(x)}$  if  $2 > \alpha p(x)$ ;  $p_{\alpha}^*(x) = \infty$  if  $2 \leq p(x)$ ,  $1 < p^- := \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \text{ess sup}_{x \in \Omega} p(x) < \infty$ ,  $1 < q^- \leq q^+ < p^- \leq p^+ < h^- \leq h^+$ ,  $\lambda > 0 \in \mathbb{R}$  and  $a, b \in C(\bar{\Omega})$  are nonnegative weight functions with compact support in  $\Omega := [0, T] \times [0, T]$  and  $\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi}(\cdot)$  and  $\mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi}(\cdot)$  are  $\psi$ -Hilfer fractional derivative of order  $1/p(x) < \alpha < 1$  and type  $0 \leq \beta \leq 1$  given by Eq.(1.3). The fractional operator

$\mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left( \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right|^{p(x)-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right)$  is a generalization of the operator

$\mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left( \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right|^{p-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(x) \right)$ , in which  $p(x) = p > 1$ .

The corresponding Euler functional of our problem (1.4) is defined by

$$\mathbf{E}_{\lambda}(u) = \int_{\Omega} \frac{1}{p(x)} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u \right|^{p(x)} dx - \lambda \int_{\Omega} \frac{1}{q(x)} a(x) |u|^{q(x)} dx - \int_{\Omega} \frac{1}{h(x)} b(x) |u|^{h(x)} dx. \quad (1.5)$$

We are now in a position to present the main contributions and consequences of our paper, which becomes clearer in detail as follows:

- (1) One of the main results of the paper is to continue the study of this new class of  $\psi$ -fractional spaces with variable exponents and nonlocal  $\psi$ -Hilfer fractional operators;
- (2) First, we present a broad class of fractional differential equations with  $p(x)$ -Laplacian of variable exponents as detailed by Eq.(1.4);
- (3) We prove some coercivity results and minimization of the Euler energy functional Eq.(1.5);
- (4) We prove a new version of Harnack inequality for the  $\psi$ -Hilfer fractional operator. The Harnack inequality is of paramount importance to guarantee the existence of positive solutions of Eq.(1.4);
- (5) We establish the multiplicity results of positive solutions for Eq.(1.4) with nonnegative weight functions;
- (6) We prove that the fractional problem Eq.(1.4) has at least two positive solutions.
- (7) A special case of fractional problems is that in the limit  $\alpha \rightarrow 1$ , we get the classic case.

To investigate the main results as highlighted above, we use of the Nehari manifold technique.

The rest of the article is divided as follows: In Section 2, we present some important concepts and results for use throughout the paper, in particular, we highlight the proof of an extension of the Harnack inequality for the  $\psi$ -Hilfer fractional operator. In Section 3, we investigate the main results of the paper, i.e, we discuss the existence and multiplicity of positive solutions to Eq.(1.4) using the Nehari manifold and the Harnack inequality.

## 2. MATHEMATICAL BACKGROUND - AUXILIARY RESULTS

In this section, we will give a brief presentation of the definitions and essential results for the development of this paper.

We discussed our problem Eq.(1.4) in the variable exponent  $\psi$ -fractional space  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ , so we need some theorems and basic properties in the spaces  $L^{p(x)}(\Omega)$  and  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ .

Write,

$$L^\infty = \{p \in L^\infty(\Omega), p^- > 1\}.$$

For any  $p \in L^{p(x)}(\Omega)$ , we denote the variable exponent Lebesgue (weight) space by [1, 6]

$$L^{p(x)}(\Omega) = \{u \in U(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < \infty\},$$

which is equipped with the norm, so-called Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \delta > 0 : \int_{\Omega} \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}$$

and  $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$  becomes a Banach space, we call it as variable exponent Lebesgue space.

Let  $c(x)$  be a measurable real valued function and  $u(x) > 0$  for  $x \in \Omega$ . Then the weight variable exponent Lebesgue space  $L_{u(x)}^{p(x)}(\Omega)$  is defined by which is equipped with the norm [1, 6]

$$\|u\|_{(p(x),c(x))} = \inf \left\{ \delta > 0 : \int_{\Omega} c(x) \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}.$$

**Definition 2.1.** Let  $0 < \alpha \leq 1, 0 \leq \beta \leq 1$  and  $p \in C^+(\bar{\Omega})$ . The left-sided  $\psi$ -fractional derivative space  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi} := \mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  with the following norm

$$\|u\|_{\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}} = \inf \left\{ k > 0 : \int_{\Omega} \left| \frac{u(x)}{k} \right|^{p(x)} + \left| \frac{\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(x)}{k} \right|^{p(x)} dx \leq 1 \right\},$$

where  $\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$  is the  $\psi$ -Hilfer fractional partial derivative with  $0 < \alpha \leq 1$  and type  $0 \leq \beta \leq 1$  (see Eq.(1.3)), which is given by

$$\mathbb{H}_{p(x)}^{\alpha,\beta;\psi} = \left\{ u \in L^{p(x)}(\Omega) : \mathbf{H}\mathbf{D}_{\theta}^{\alpha,\beta;\psi} u \in L^{p(x)}(\Omega), u(\Omega) = 0 \right\}.$$

The space  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$  is denoted by the closure of  $C_0^\infty(\Omega)$  in  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ . We will use  $\|u\|_{\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)} = \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|_{p(x)}$  for  $u \in \mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$  in the following discussion.

Next, we will present the integration by parts of the  $\psi$ -Riemann-Liouville fractional integral and  $\psi$ -Hilfer fractional derivative.

Let  $\alpha \in (0, 1)$ ,  $p \in (1, \infty)$  and  $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ . If  $u \in L^q_\psi [0, T]$  and  $v \in L^p_\psi [0, T]$ , then the following integration by parts [28, 29, 30]

$$\int_a^b \left( \mathbf{I}_{0+}^{\alpha; \psi} v(x) \right) u(x) \psi'(x) dx = \int_a^b v(x) \mathbf{I}_T^{\alpha; \psi} u(x) \psi'(x) dx \quad (2.1)$$

holds.

Let  $\psi(\cdot)$  be an increasing and positive monotone function on  $[0, T]$ , having a continuous derivative  $\psi'(\cdot) \neq 0$  on  $(0, T)$ . If  $0 < \alpha \leq 1$  and  $0 \leq \beta \leq 1$ , then [28, 29, 30]

$$\int_a^b \left( \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u(t) \right) v(t) dt = \int_a^b u(t) \psi'(t) \mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left( \frac{v(t)}{\psi'(t)} \right) dt \quad (2.2)$$

for any  $u \in AC^1$  and  $\theta \in C^1$  satisfying the boundary conditions  $u(0) = u(T) = 0$ .

**Proposition 2.2.** [28, 29, 30] *Let  $0 < \alpha \leq 1$ ,  $0 \leq \beta \leq 1$  and  $1 < p(x) < \infty$ . Assume that  $\alpha > 1/p(x)$  and the sequence  $\{u_k\}$  converges weakly to  $u$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega; \mathbb{R})$  i.e.,  $u_k \rightharpoonup u$ . Then  $u_k \rightarrow u$  in  $C(\Omega, \mathbb{R})$ , i.e.,  $\|u - u_k\|_\infty \rightarrow 0$  as  $k \rightarrow \infty$ .*

**Proposition 2.3.** [29, 30, 31] *The conjugate space of  $L^{p(x)}(\Omega)$  is  $L^{p'(x)}(\Omega)$ , where  $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$ . For any  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{p'(x)}(\Omega)$ , we have*

$$\begin{aligned} \left| \int_\Omega u(x)v(x) dx \right| &\leq \left( \frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \\ &\leq 2 \|u\|_{p(x)} \|v\|_{p'(x)}. \end{aligned}$$

**Proposition 2.4.** [1, 6] *Denote  $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$ ,  $\forall u \in L^{p(x)}(\Omega)$ , then we have*

- (1)  $\|u\|_{p(x)} < 1$  ( $= 1, > 1$ )  $\iff \rho(u) < 1$ ;
- (2)  $\|u\|_{p(x)} > 1 \implies \|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$ ;
- (3)  $\|u\|_{p(x)} < 1 \implies \|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$ ;

**Proposition 2.5.** [1, 6] *If  $u, u_n \in L^{p(x)}(\Omega)$ ,  $n = 1, 2, \dots$ , then the follows statements are equivalent to each other:*

- (1)  $\lim_{n \rightarrow \infty} \|u_n - u\|_{p(x)} = 0$ ;
- (2)  $\lim_{n \rightarrow \infty} \rho(u_n - u) = 0$ ;
- (3)  $u_n \rightarrow u$  in measure on  $\Omega$  and  $\lim_{n \rightarrow \infty} \rho(u_n) = \rho(u)$ .

**Proposition 2.6.** [1, 6, 29, 31] *If  $p^- > 1$  and  $p^+ < \infty$ , then the spaces  $L^{p(x)}(\Omega)$ ,  $L_{c(x)}^{p(x)}(\Omega)$  and  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  are separable and reflexive Banach spaces.*

**Proposition 2.7.** [6] *Let  $p(x)$  and  $q(x)$  be measurable functions such that  $p(x) \in L^\infty(\Omega)$  and  $1 \leq p(x)q(x) \leq \infty$  for  $u \in x \in \Omega$ . Let  $u \in L^{q(x)}(\Omega)$ ,  $u \neq 0$ . Then*

$$\begin{aligned} |u|_{p(x)q(x)} \leq 1 &\implies |u|_{p(x)q(x)}^{p^+} \leq |u|_{q(x)}^{p(x)} \leq |u|_{p(x)q(x)}^{p^-} \\ |u|_{p(x)q(x)} \geq 1 &\implies |u|_{p(x)q(x)}^{p^-} \leq |u|_{q(x)}^{p(x)} \leq |u|_{p(x)q(x)}^{p^+}. \end{aligned}$$

**Theorem 2.8.** [4] *Assume that the boundary of  $\Omega$  possesses the cone property and  $p \in C(\bar{\Omega})$ . Suppose that  $b \in L^{p(x)}(\Omega)$ ,  $b(x) > 0$  for  $x \in \Omega$ ,  $\beta \in C(\bar{\Omega})$  and  $\beta^- > 1$ ,  $\beta_0^- \leq \beta_0(x) \leq \beta_0^+$  ( $\frac{1}{\beta(x)} + \frac{1}{\beta_0(x)} = 1$ ). If  $h \in C(\bar{\Omega})$  and*

$$1 < h(x) < \frac{\beta(x) - 1}{\beta(x)} p_\alpha^*, \quad \forall x \in \bar{\Omega} \quad (2.3)$$

or

$$1 < \beta(x) < \frac{Np(x)}{Np(x) - h(x)(N - p(x))}.$$

then the embedding from  $W^{1; p(x)}(\Omega)$  to  $L_{b(x)}^{h(x)}(\Omega)$  is compact. Moreover, there is a constant  $C_5 > 0$  such that the inequality

$$\int_\Omega b(x) |u|^{h(x)} dx \leq C_5 \left( \|u\|^{h^-} + \|u\|^{h^+} \right) \quad (2.4)$$

holds.

**Theorem 2.9.** [4] Assume that the boundary of  $\Omega$  possesses the cone property and  $p \in C(\bar{\Omega})$ . Suppose that  $a \in L^{\alpha(x)}(\Omega)$ ,  $a(x) > 0$  for  $x \in \Omega$ ,  $\alpha \in C(\bar{\Omega})$  and  $\alpha^- > 1$ ,  $\alpha_0^- \leq \alpha_0(x) \leq \alpha_0^+$   $\left(\frac{1}{\alpha(x)} + \frac{1}{\alpha_0(x)} = 1\right)$ .

If  $q \in C(\bar{\Omega})$ ,  $p(x) < \frac{\alpha(x)}{\alpha(x)-1}q(x)$  and

$$1 < q(x) < \frac{\alpha(x)-1}{\alpha(x)}p_\alpha^*(x), \quad \forall x \in \bar{\Omega}$$

or

$$\frac{Np(x)}{Np(x) - q(x)(N - p(x))} < \alpha(x) < \frac{p(x)}{p(x) - q(x)},$$

then the embedding from  $W^{1;p(x)}(\Omega)$  to  $L_a^{q(x)}(\Omega)$  is compact. Moreover, there is a constant  $C_7 > 0$  such that the inequality

$$\int_{\Omega} a(x)|u|^{q(x)} dx \leq C_7 \left( \|u\|^{q^-} + \|u\|^{q^+} \right).$$

**Proposition 2.10.** [4] Assume that the conditions of Theorem 2.8 and Theorem 2.9 hold, respectively. Let  $u \in W^{0;p(x)}(\Omega)$  then there are positive constants  $C_8, C_9, C_{10}, C_{11} > 0$  such that the following inequalities hold

$$\int_{\Omega} b(x)|u|^{h(x)} dx \leq \begin{cases} C_8 \|u\|^{h^+}, & \text{if } \|u\| > 1, \\ C_9 \|u\|^{h^-}, & \text{if } \|u\| < 1, \end{cases} \quad (2.5)$$

and

$$\int_{\Omega} a(x)|u|^{q(x)} dx \leq \begin{cases} C_{10} \|u\|^{q^+}, & \text{if } \|u\| > 1, \\ C_{11} \|u\|^{q^-}, & \text{if } \|u\| < 1. \end{cases} \quad (2.6)$$

**Definition 2.11.** [38] Let  $f$  and  $h$  be of  $\psi$ -exponential order, piecewise continuous functions over each finite interval  $[0, T]$ . Then, the  $\psi$ -convolution of  $f$  and  $h$ , denoted by  $f *_{\psi} h$  is given by

$$f *_{\psi} h = \int_0^t f(\psi^{-1}(\psi(t) - \psi(\tau))) \psi'(\tau) h(\tau) d\tau.$$

Consider  $\psi(0) = 0$ . Then we have

$$\begin{aligned} g_\alpha^\psi *_{\psi} g_\beta^\psi &= \int_0^t g_\alpha^\psi(\psi^{-1}(\psi(t) - \psi(\tau))) g_\beta^\psi(\tau) \psi'(\tau) d\tau \\ &= \int_0^t \frac{(\psi(t) - \psi(\tau))^{\alpha-1}}{\Gamma(\alpha)} \frac{(\psi(\tau))^{\beta-1}}{\Gamma(\beta)} \psi'(\tau) d\tau. \end{aligned} \quad (2.7)$$

Making the following variable change  $u = \psi(\tau)$  in Eq.(2.7), we have

$$g_\alpha^\psi *_{\psi} g_\beta^\psi = \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\psi(t)} \left(1 - \frac{u}{\psi(t)}\right)^{\alpha-1} u^{\beta-1} du. \quad (2.8)$$

Again, making another variable change  $v = \frac{u}{\psi(t)}$  in Eq.(2.9), we have

$$\begin{aligned} g_\alpha^\psi *_{\psi} g_\beta^\psi &= \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-v)^{\alpha-1} (v\psi(t))^{\beta-1} dv \\ &= \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} dv \\ &= \frac{(\psi(t))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &= g_{\alpha+\beta}^\psi. \end{aligned} \quad (2.9)$$

For the discussion of the results, we will use the following function notations.

$$g_\gamma^\psi(x-s) = \begin{cases} \frac{(\psi(x) - \psi(s))^{\gamma-1}}{\Gamma(\gamma)} & , (x, s) > 0, \psi(x) - \psi(s) > 0 \\ 0 & , (x, s) \leq 0, \psi(x) - \psi(s) \leq 0 \end{cases} , \gamma > 0.$$

and

$$g_\gamma^\psi(x) = \begin{cases} \frac{(\psi(x))^{\gamma-1}}{\Gamma(\gamma)} & , x > 0, \psi(x) > 0 \\ 0 & , x \leq 0, \psi(x) \leq 0 \end{cases} , \gamma > 0.$$

Next we will present a version of the Harnack inequality with respect to the function  $\psi$ .

**Theorem 2.12.** *Let  $t_* \geq 0, 0 < \sigma_1 < \sigma_2 < \sigma_3$  and  $\rho > 0$ . Let further  $\alpha \in (0, 1), 0 \leq \beta \leq 1, \psi(0) = 0$  and  $u_0 \geq 0$ . Then for any function  $u \in Z(t_*, t_* + \sigma_3\rho)$  and that satisfies*

$$\partial_t^{\alpha, \beta; \psi}(u - u_0)(t) = 0, \text{ a.a. } t \in (t_*, t_* + \sigma_3\rho) \quad (2.10)$$

there holds the inequality

$$\sup_{W^-} \leq \sigma_3 \sigma_1 \inf_{W^+} u \quad (2.11)$$

where  $W^- = (t_* + \sigma_1\rho, t_* + \sigma_2\rho)$  e  $W^+ = (t_* + \sigma_2\rho, t_* + \sigma_3\rho)$ .

*Proof.* Suppose  $u \in Z = (t_*, t_* + \sigma_3\rho)$  is nonnegative on  $(0, t_* + \sigma_3\rho)$  and satisfies Eq.(2.10). We introduce the shifted time  $s = t - t_*$ , and define the function  $\tilde{u}$  by means of  $\tilde{u}(s) = u(s + t_*)$ ,  $s \in (0, \sigma_3\rho)$ .

Then Eq.(2.10) implies that ( $t = s + t_*$ )

$$\partial_t^{\gamma; \psi} u(t) - \partial_t^{\gamma; \psi} u_0(t) = 0 \quad (2.12)$$

implies  $\partial_t^{\gamma; \psi} u(s + t_*) = \partial_t^{\gamma; \psi} u_0(s + t_*)$ ,  $s \in (0, \sigma_3\rho)$  where

$$\partial_t^{\gamma; \psi} u(t) := {}^{RL}D_{0+}^{\gamma; \psi} u(t) = \frac{\partial_t}{\psi'(t)} \int_0^t g_{1-\gamma}^\psi(t-\tau) u(\tau) d\tau. \quad (2.13)$$

Note that Eq.(2.13) can be written as

$$\partial_t^{\gamma; \psi} u(t) = \partial_s^{\gamma; \psi} \tilde{u}(s) + \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\tau) g_{1-\gamma}^\psi(t-\tau) u(\tau) d\tau. \quad (2.14)$$

Thus, from Eq.(2.12) and Eq.(2.14), we have

$$\partial_s^{\gamma; \psi} \tilde{u}(s) = \partial_t^{\gamma; \psi} u_0(t) - \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\tau) g_{1-\gamma}^\psi(t-\tau) u(\tau) d\tau. \quad (2.15)$$

Remembering that  $\partial_t^{\alpha, \beta; \psi} u(t) = \mathbf{H}\mathbb{D}_{0+}^{\alpha, \beta; \psi} u(t) = I_{0+}^{\gamma-\alpha; \psi} {}^{RL}D_{0+}^{\gamma; \psi} u(t)$  with  $\gamma = \alpha + \beta(1 - \alpha)$ . Applying  $I_{0+}^{\gamma-\alpha; \psi}(\cdot)$  o both sides of Eq.(2.15), we have

$$\begin{aligned} \partial_s^{\alpha, \beta; \psi} \tilde{u}(s) &= u_0 I_{0+}^{\gamma-\alpha; \psi} g_{1-\gamma}^\psi(t) - I_{0+}^{\gamma-\alpha; \psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\tau) g_{1-\gamma}^\psi(t-\tau) u(\tau) d\tau \right) \\ &= u_0 g_{1-\alpha}^\psi(t) - I_{0+}^{\gamma-\alpha; \psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\tau) g_{1-\gamma}^\psi(t-\tau) u(\tau) d\tau \right). \end{aligned} \quad (2.16)$$

Remembering that  $t = s + t_*$ , so Eq.(2.16) can be written as follows

$$\partial_s^{\alpha, \beta; \psi} \tilde{u}(s) = u_0 g_{1-\alpha}^\psi(s + t_*) + h^\psi(s) \quad (2.17)$$

where  $h^\psi(s) = -I_{0+}^{\gamma-\alpha; \psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\tau) g_{1-\gamma}^\psi(\psi(s + t_* - \tau)) u(\tau) d\tau \right)$ .

Since  $(g_{1-\gamma}^\psi *_\psi \tilde{u})(0) = 0$  and  $g_{1-\gamma}^\psi *_\psi g_\gamma^\psi = 1$ , we have

$$\begin{aligned} g_\gamma^\psi *_\psi \partial_s^{\alpha, \beta; \psi} \tilde{u} &= g_\gamma^\psi *_\psi I_{0+}^{\gamma-\alpha; \psi} \partial_s^{\gamma; \psi} \tilde{u} \\ &= g_\gamma^\psi *_\psi I_{0+}^{\gamma-\alpha; \psi} \partial_s^\psi (g_{1-\gamma}^\psi *_\psi \tilde{u}) \\ &= I_{0+}^{\gamma-\alpha; \psi} \partial_s^\psi (g_\gamma^\psi *_\psi g_{1-\gamma}^\psi *_\psi \tilde{u}) \\ &= I_{0+}^{\gamma-\alpha; \psi} \partial_s^\psi (1 *_\psi \tilde{u}) \\ &= I_{0+}^{\gamma-\alpha; \psi} \tilde{u}. \end{aligned}$$

So from the convolution of Eq.(2.17) with  $g_\alpha^\psi$ , we have

$$\begin{aligned} \partial_s^{\alpha, \beta; \psi} \tilde{u}(s) *_\psi g_\alpha^\psi &= (u_0 g_{1-\alpha}^\psi(s + t_*) + h^\psi(s)) *_\psi g_\alpha^\psi \\ &= u_0 (g_{1-\alpha}^\psi(\cdot + t_*) *_\psi g_\alpha^\psi)(s) + (h^\psi *_\psi g_\alpha^\psi)(s) \end{aligned} \quad (2.18)$$

with  $s \in (0, \sigma_3\rho)$ .

The first term of Eq.(2.18) can be written as

$$\begin{aligned} \left( g_{1-\alpha}^\psi(\cdot + t_*) *_\psi g_\alpha^\psi \right) (s) &= \int_0^s g_\alpha^\psi \left( \psi^{-1}(\psi(s) - \psi(\tau)) \right) \psi'(\tau) g_{1-\alpha}^\psi(\tau + t_*) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^s \psi'(\tau) (\psi(s) - \psi(\tau))^{\alpha-1} (\psi(\tau + t_*))^{-\alpha} d\tau \end{aligned} \quad (2.19)$$

Making the following variable change  $\tau = rs$  in Eq.(2.19), we have

$$\begin{aligned} \left( g_{1-\alpha}^\psi(\cdot + t_*) *_\psi g_\alpha^\psi \right) (s) &= s \int_0^1 \psi'(rs) \frac{(\psi(s) - \psi(rs))^{\alpha-1}}{\Gamma(\alpha)} \frac{(\psi(rs + t_*))^{-\alpha}}{\Gamma(1-\alpha)} dr \\ &= s \int_0^1 \psi'(rs) g_\alpha^\psi(s - rs) g_{1-\alpha}^\psi(rs + t_*) dr \\ &=: \tilde{\Phi}(s) \end{aligned} \quad (2.20)$$

with  $s \in (0, \sigma_3\rho)$ .

Similarly, we have for the second term,

$$\begin{aligned} \left( h^\psi *_\psi g_\alpha^\psi \right) (s) &= \int_0^s g_\alpha^\psi \left( \psi^{-1}(\psi(s) - \psi(\tau)) \right) \psi'(\tau) h^\psi(\tau) d\tau \\ &= \int_0^s \psi'(\tau) \frac{(\psi(s) - \psi(\tau))^{\alpha-1}}{\Gamma(\alpha)} h^\psi(\tau) d\tau \\ &= \int_0^s \psi'(\tau) \frac{(\psi(s) - \psi(\tau))^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \left( -I_{0+}^{\gamma-\alpha;\psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\sigma) g_{1-\gamma}^\psi(\psi(\tau + t_* - \sigma)u(\sigma) d\sigma \right) \right) d\tau. \end{aligned} \quad (2.21)$$

Making the following variable change  $\tau = rs$  in Eq.(2.21), we have

$$\begin{aligned} \left( h^\psi *_\psi g_\alpha^\psi \right) (s) &= s \int_0^1 \psi'(rs) \frac{(\psi(s) - \psi(rs))^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \left( -I_{0+}^{\gamma-\alpha;\psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\sigma) g_{1-\gamma}^\psi(\psi(rs + t_* - \sigma)u(\sigma) d\sigma \right) \right) dr \\ &=: \tilde{\Psi}(s) \end{aligned} \quad (2.22)$$

with  $s \in (0, \sigma_3\rho)$ .

Consequently, the Eq.(2.18) is equivalent to

$$\tilde{u}(s) = u_0 \tilde{\Phi}(s) + \tilde{\Psi}(s), \quad s \in (0, \sigma_3\rho). \quad (2.23)$$

Let now  $s \in (\sigma_1\rho, \sigma_2\rho)$  and  $\bar{s} \in (\sigma_2\rho, \sigma_3\rho)$ . Then, from Eq.(2.20), we have  $\tilde{\Phi}(s) \leq \tilde{\Phi}(\bar{s})$ . As to  $\tilde{\Psi}$ , we use the positivity of  $u$  on  $(0, t_*)$ , and the monotonicity of  $\frac{\partial_t}{\psi'(t)} g_{1-\gamma}^\psi$  to estimate as follows

$$\begin{aligned} \tilde{\Psi}(s) &\leq s \int_0^1 \psi'(rs) \frac{(\psi(s) - \psi(rs))^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \left( -I_{0+}^{\gamma-\alpha;\psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\sigma) g_{1-\gamma}^\psi(\psi(rs + t_* - \sigma)u(\sigma) d\sigma \right) \right) dr \\ &\leq \sigma_1\rho \int_0^1 \psi'(r\sigma_1\rho) \frac{(\psi(\sigma_2\rho) - \psi(r\sigma_2\rho))^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \left( -I_{0+}^{\gamma-\alpha;\psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\sigma) g_{1-\gamma}^\psi(\psi(r\sigma_2\rho + t_* - \sigma)u(\sigma) d\sigma \right) \right) dr \\ &\leq \sigma_1\sigma_3\bar{s} \int_0^1 \psi'(r\bar{s}) \frac{(\psi(\bar{s}) - \psi(r\bar{s}))^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \left( -I_{0+}^{\gamma-\alpha;\psi} \left( \int_0^{t_*} \frac{\partial_t}{\psi'(t)} \psi'(\sigma) g_{1-\gamma}^\psi(r\bar{s} + t_* - \sigma)u(\sigma) d\sigma \right) \right) dr \\ &= \sigma_1\sigma_3\tilde{\Psi}(\bar{s}). \end{aligned}$$

By positivity of  $u_0$ , we thus obtain

$$\tilde{u}(s) \leq \sigma_1 \sigma_3 \tilde{u}(\bar{s}),$$

which immediately implies inequality Eq.(2.11). This complete the proof of Theorem.  $\square$

### 3. EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS

Consider the Euler functional defined by Eq.(1.5). Then, by Theorem 2.8 and Theorem 2.9 and Proposition 2.4, yields

$$\begin{aligned} \mathbf{E}_\lambda(u) &\geq \frac{1}{p^+} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \frac{\lambda}{q^-} \int_{\Omega} a(x)|u|^{q(x)} dx - \frac{1}{h^-} \int_{\Omega} b(x)|u|^{h(x)} dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{q^-} C_7 \left( \|u\|^{q^-} + \|u\|^{q^+} \right). \end{aligned}$$

Since  $q^+ < p^- \leq p^+ < h^- \leq h^+$ , this shows  $\mathbf{E}_\lambda(\cdot)$  is not bounded below on whole  $\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ . However, we shall it is bounded on the Nehari manifold  $\mathbf{N}_\lambda(\Omega)$  which is given by

$$\mathbf{N}_\lambda(\Omega) = \left\{ u \in \mathbb{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega) \setminus \{0\} : \langle \mathbf{E}'_\lambda(u), u \rangle = 0 \right\}.$$

It is clear that all critical points of  $\mathbf{E}_\lambda$  must be on  $\mathbf{N}_\lambda(\Omega)$  and local minimizes on  $\mathbf{E}_\lambda(\Omega)$  are usually critical points of  $\mathbf{E}_\lambda$ . Thus,  $u \in \mathbf{N}_\lambda(\Omega)$  if, and only if,

$$\mathbf{I}_\lambda(u) := \langle \mathbf{E}'_\lambda(u), u \rangle = \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \lambda \int_{\Omega} a(x)|u|^{q(x)} dx - \int_{\Omega} b(x)|u|^{h(x)} dx = 0. \quad (3.1)$$

Then, for  $u \in \mathbf{N}_\lambda(\Omega)$ , yields

$$\begin{aligned} \langle \mathbf{I}'_\lambda(u), u \rangle &= \int_{\Omega} p(x) |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \lambda \int_{\Omega} q(x)a(x)|u|^{q(x)} dx - \int_{\Omega} h(x)b(x)|u|^{h(x)} dx \\ &\leq (p^+ - q^-) \lambda \int_{\Omega} a(x)|u|^{q(x)} dx - (p^+ - h^-) \int_{\Omega} b(x)|u|^{h(x)} dx. \end{aligned}$$

Now let's decompose the Nehari manifold  $\mathbf{N}_\lambda(\Omega)$  into three parts

$$\begin{aligned} \mathbf{N}_\lambda^+(\Omega) &= \{u \in \mathbf{N}_\lambda(\Omega) : \langle \mathbf{I}'_\lambda(u), u \rangle = 0\} \\ \mathbf{N}_\lambda^-(\Omega) &= \{u \in \mathbf{N}_\lambda(\Omega) : \langle \mathbf{I}'_\lambda(u), u \rangle = 0\} \\ \mathbf{N}_\lambda^0(\Omega) &= \{u \in \mathbf{N}_\lambda(\Omega) : \langle \mathbf{I}'_\lambda(u), u \rangle = 0\}. \end{aligned}$$

**Theorem 3.1.** *Suppose that  $u_0$  is a local maximum or minimum for  $\mathbf{E}_\lambda$  on  $\mathbf{N}_\lambda(\Omega)$ . If  $u_0 \notin \mathbf{N}_\lambda^0(\Omega)$ , then  $u_0$  is a critical point of  $\mathbf{E}_\lambda$ .*

**Lemma 3.2.** *The energy functional  $\mathbf{E}_\lambda$  is coercive and bounded below on  $\mathbf{N}_\lambda(\Omega)$ .*

*Proof.* Let  $u \in \mathbf{N}_\lambda(\Omega)$  and  $\|u\| > 1$ . Then, using (3.1) and Propositions 2.4 and 2.10, yields

$$\begin{aligned} \mathbf{E}_\lambda(u) &= \int_{\Omega} \frac{1}{p(x)} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \lambda \int_{\Omega} \frac{1}{q(x)} a(x)|u|^{q(x)} dx - \int_{\Omega} \frac{1}{h(x)} b(x)|u|^{h(x)} dx \\ &\geq \frac{1}{p^+} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \frac{\lambda}{q^-} \int_{\Omega} a(x)|u|^{q(x)} dx - \frac{1}{h^-} \left( \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx - \lambda \int_{\Omega} a(x)|u|^{q(x)} dx \right) \\ &\geq \left( \frac{1}{p^+} - \frac{1}{h^-} \right) \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u|^{p(x)} dx + \lambda \left( \frac{1}{h^-} - \frac{1}{q^-} \right) \int_{\Omega} a(x)|u|^{q(x)} dx \\ &\geq \left( \frac{h^- - p^+}{h^- p^+} \right) \|u\|^{p^-} - C_{10} \lambda \left( \frac{h^- - q^-}{h^- q^-} \right) \|u\|^{q^+}. \end{aligned}$$

Since  $p^- > q^+$ , so  $\mathbf{E}_\lambda(u) \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ . This implies  $\mathbf{E}_\lambda$  is coercive and bounded below on  $\mathbf{E}_\lambda(\Omega)$ .  $\square$

**Lemma 3.3.** *There exists  $\lambda_1 > 0$  such that for  $0 < \lambda < \lambda_1$  we have  $\mathbf{N}_\lambda^0 = \emptyset$ .*

*Proof.* Suppose otherwise, this is,  $\mathbf{N}_\lambda^0 = \emptyset$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$ . Let  $u \in \mathbf{N}_\lambda^0(\Omega)$  such that  $\|u\| > 1$ . Then, using Eq.(3.1), Eq.(2.5) and definition of  $\mathbf{N}_\lambda^0(\Omega)$ , yields

$$0 = \langle \mathbf{I}'_\lambda(u), u \rangle$$

$$\begin{aligned}
&= \int_{\Omega} p(x) \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \lambda \int_{\Omega} q(x) a(x) |u|^{q(x)} dx - \int_{\Omega} h(x) b(x) |u|^{h(x)} dx \\
&\geq p^- \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - q^+ \left( \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \int_{\Omega} b(x) |u|^{h(x)} dx \right) - h^+ \int_{\Omega} b(x) |u|^{h(x)} dx \\
&\geq (p^- - q^+) \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx + (q^+ - h^+) \int_{\Omega} b(x) |u|^{h(x)} dx.
\end{aligned}$$

By Proposition 2.10, yields

$$\begin{aligned}
0 &\geq (p^- - q^+) \|u\|^{p^-} + C_8 (q^+ - h^+) \|u\|^{h^+} \\
\|u\| &\geq C_{12} \left( \frac{p^- - q^+}{h^+ - q^+} \right)^{\frac{1}{h^+ - p^-}}.
\end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}
0 &= \langle \mathbf{I}'_{\lambda}(u), u \rangle \\
&= p^+ \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \lambda q^- \int_{\Omega} a(x) |u|^{q(x)} dx - h^- \int_{\Omega} b(x) |u|^{h(x)} dx \\
&\leq p^+ \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \lambda q^- \int_{\Omega} a(x) |u|^{q(x)} dx - h^- \left( \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \lambda \int_{\Omega} a(x) |u|^{q(x)} dx \right).
\end{aligned}$$

Using Proposition 2.10, yields

$$\begin{aligned}
0 &\leq (p^+ - h^-) \|u\|^{p^+} + \lambda C_{10} (h^- - q^-) \|u\|^{q^-} \\
\|u\| &\geq C_{13} \left( \lambda \frac{h^- - q^-}{h^- - p^+} \right)^{\frac{1}{p^+ - q^-}}.
\end{aligned} \tag{3.3}$$

If  $\lambda$  is sufficiently small  $\lambda = \left( \frac{h^- - p^+}{h^- - q^-} \right) \left( \frac{p^- - q^+}{h^+ - q^+} \right)^{\frac{p^- - q^+}{h^+ - p^-}}$ , then from inequalities (3.2) and (3.3) we get  $\|u\| < 1$  which contradicts which our assumption. Hence, we conclude  $\mathbf{N}_{\lambda}^0 = \emptyset$ .  $\square$

By Lemma 3.3, for  $0 < \lambda < \lambda_1$ , we can write  $\mathbf{N}_{\lambda}(\Omega) = \mathbf{N}_{\lambda}^+(\Omega) \cup \mathbf{N}_{\lambda}^-(\Omega)$ . Therefore, we can let

$$\alpha_{\lambda}^+ = \inf_{u \in \mathbf{N}_{\lambda}^+(\Omega)} \mathbf{E}_{\lambda}(u) \quad \text{and} \quad \alpha_{\lambda}^- = \inf_{u \in \mathbf{N}_{\lambda}^-(\Omega)} \mathbf{E}_{\lambda}(u).$$

**Lemma 3.4.** *If  $0 < \lambda < \lambda_1$ , then for all  $u \in \mathbf{N}_{\lambda}^+(\Omega)$ ,  $\mathbf{E}_{\lambda}(u) < 0$ .*

*Proof.* Let  $u \in \mathbf{N}_{\lambda}^+(\Omega)$ . By definition of  $\mathbf{E}_{\lambda}(u)$ , we can write

$$\mathbf{E}_{\lambda}(u) \leq \frac{1}{p^-} \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \frac{\lambda}{q^+} \int_{\Omega} a(x) |u|^{q(x)} dx - \frac{1}{h^+} \int_{\Omega} b(x) |u|^{h(x)} dx. \tag{3.4}$$

Since  $u \in \mathbf{N}_{\lambda}^+(\Omega)$ , yields

$$p^+ \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \lambda q^- \int_{\Omega} a(x) |u|^{q(x)} dx - h^- \int_{\Omega} b(x) |u|^{h(x)} dx > 0. \tag{3.5}$$

Now, if we multiply (3.1) by  $(-q^-)$  and add with (3.5), yields

$$\int_{\Omega} b(x) |u|^{h(x)} dx < \left( \frac{p^+ - q^-}{h^- - q^-} \right) \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx. \tag{3.6}$$

Moreover, using (3.1) together with (3.4),

$$\mathbf{E}_{\lambda}(u) \leq \left( \frac{1}{p^-} + \frac{1}{q^+} \right) \int_{\Omega} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx + \left( \frac{1}{q^+} - \frac{1}{h^+} \right) \int_{\Omega} b(x) |u|^{h(x)} dx \tag{3.7}$$

and applying (3.6) in (3.7), it follows

$$\mathbf{E}_{\lambda}(u) < -\frac{(p^- - q^+)(h^+ - p^-)}{h^+ p^- q^+} \|u\|_{\mathbb{H}_{p(x)}^{\alpha,\beta;\psi}}^p < 0.$$

Hence, we have  $\alpha_{\lambda}^+ = \inf_{u \in \mathbf{N}_{\lambda}^+(\Omega)} \mathbf{E}_{\lambda}(u) < 0$ .  $\square$

**Theorem 3.5.** *If  $0 < \lambda < \lambda_1$ , there exists a minimizer of  $\mathbf{E}_{\lambda}$  on  $\mathbf{N}_{\lambda}^+(\Omega)$ .*

*Proof.* Since  $\mathbf{E}_\lambda(\cdot)$  is bounded below on  $\mathbf{N}_\lambda(\Omega)$  and so on  $\mathbf{N}_\lambda^+(\Omega)$ . Then, there exists a minimizing sequence  $\{u_n^+\} \subseteq \mathbf{N}_\lambda^+(\Omega)$  such that

$$\lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^+) = \inf_{u \in \mathbf{N}_\lambda^+(\Omega)} \mathbf{E}_\lambda(u) = \alpha_\lambda^+ < 0.$$

Since  $\mathbf{E}_\lambda$  is coercive,  $u_n^+$  is bounded in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$ . Thus, we may assume that, without loss of generality,  $u_n^+ \rightharpoonup u_0^+$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  and by the compact embeddings we have

$$u_n^+ \rightarrow u_0^+ \text{ in } L_{a(x)}^{q(x)}(\Omega)$$

and

$$u_n^+ \rightarrow u_0^+ \text{ in } L_{b(x)}^{h(x)}(\Omega).$$

Now, we shall prove  $u_n^+ \rightarrow u_0^+$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$ . Otherwise, suppose  $u_n^+ \rightharpoonup u_0^+$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$ . Then,

$$\int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_0^+|^{p(x)} dx < \liminf_{n \rightarrow \infty} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_n^+|^{p(x)} dx.$$

Moreover, by the compact embeddings, yields

$$\begin{aligned} \int_{\Omega} a(x)|u_0^+|^{q(x)} dx &= \liminf_{n \rightarrow \infty} \int_{\Omega} a(x)|u_n^+|^{q(x)} dx \\ \int_{\Omega} b(x)|u_0^+|^{h(x)} dx &= \liminf_{n \rightarrow \infty} \int_{\Omega} b(x)|u_n^+|^{h(x)} dx. \end{aligned}$$

Using the fact that  $\langle \mathbf{E}'_\lambda(u_n^+), u_n^+ \rangle = 0$  and Theorem 2.9, we can write the followings

$$\begin{aligned} \mathbf{E}_\lambda(u_n^+) &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_n^+|^{p(x)} dx + \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \int_{\Omega} a(x)|u_n^+|^{q(x)} dx \\ \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^+) &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \lim_{n \rightarrow \infty} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_n^+|^{p(x)} dx + \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \lim_{n \rightarrow \infty} \int_{\Omega} a(x)|u_n^+|^{q(x)} dx \\ \alpha_\lambda^+ &= \inf_{u \in \mathbf{N}_\lambda^+} \mathbf{E}_\lambda(u) > \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \|u_0^+\|^{p^-} + C_7 \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) (\|u_0^+\|^{q^-} + \|u_0^+\|^{q^+}), \end{aligned}$$

since  $p^- > q^+$ , for  $\|u_0^+\| > 1$ , yields

$$\alpha_\lambda^+ = \inf_{u \in \mathbf{N}_\lambda^+} \mathbf{E}_\lambda(u) > 0.$$

However, in Lemma 3.4, it was showed that for any  $u \in \mathbf{N}_\lambda^+(\Omega)$ ,  $\mathbf{E}_\lambda(u) < 0$ . So this is a contradiction. Hence,  $u \in \mathbf{N}_\lambda^+(\Omega)$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  and

$$\mathbf{E}_\lambda(u_0^+) = \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^+) = \inf_{u \in \mathbf{N}_\lambda^+} \mathbf{E}_\lambda(u).$$

Thus,  $u_0^+$  is a minimizer for  $\mathbf{E}_\lambda$  on  $\mathbf{N}_\lambda^+(\Omega)$ . □

**Lemma 3.6.** *If  $0 < \lambda < \lambda_1$ , then for all  $u \in \mathbf{N}_\lambda^-(\Omega)$ ,  $\mathbf{E}_\lambda(u) > 0$ .*

*Proof.* Let  $u \in \mathbf{N}_\lambda(\Omega)$ . By definition of  $\mathbf{E}_\lambda(\Omega)$  and (3.1), yields

$$\mathbf{E}_\lambda(u) \geq \frac{1}{p^+} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u|^{p(x)} dx - \frac{\lambda}{q^-} \int_{\Omega} a(x)|u|^{q(x)} dx - \frac{1}{h^-} \int_{\Omega} b(x)|u|^{h(x)} dx \quad (3.8)$$

and

$$\int_{\Omega} b(x)|u|^{h(x)} dx = \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u|^{p(x)} dx - \lambda \int_{\Omega} a(x)|u|^{q(x)} dx. \quad (3.9)$$

Using Eq.(3.8)-Eq.(3.9), Propositions 2.4 and 2.10 and the condition  $p^- > q^+$ , yields

$$\begin{aligned} \mathbf{E}_\lambda(u) &\geq \frac{1}{p^+} \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u|^{p(x)} dx - \frac{\lambda}{q^-} \int_{\Omega} a(x)|u|^{q(x)} dx - \frac{1}{h^-} \left( \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u|^{p(x)} dx - \lambda \int_{\Omega} a(x)|u|^{q(x)} dx \right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \int_{\Omega} |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u|^{p(x)} dx + \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \int_{\Omega} a(x)|u|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \|u\|^{p^-} + C_{10} \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \|u\|^{q^+} \end{aligned}$$

$$\geq \left( \frac{h^- - p^+}{p^+ h^-} + C_{10} \frac{q^- - h^-}{h^- q^-} \right) \|u\|^{p^-}.$$

So, if we choose  $\lambda < \frac{q^-(h^- - p^+)}{C_{10} p^+(h^- - q^-)}$ , we get  $\mathbf{E}_\lambda(u) > 0$ . Moreover, if we consider the facts  $\mathbf{N}_\lambda(\Omega) = \mathbf{N}_\lambda^+(\Omega) \cup \mathbf{N}_\lambda^-(\Omega)$  (see Lemma 3.3),  $\mathbf{N}_\lambda^+(\Omega) \cap \mathbf{N}_\lambda^-(\Omega) = \emptyset$ , and Lemma 3.4, we must have  $u \in \mathbf{N}_\lambda^-(\Omega)$ .  $\square$

**Theorem 3.7.** *If  $0 < \lambda < \lambda_1$ , there exists a minimizer of  $\mathbf{E}_\lambda(\cdot)$  on  $\mathbf{N}_\lambda^-(\Omega)$ .*

*Proof.* Since  $\mathbf{E}_\lambda$  is bounded below on  $\mathbf{N}_\lambda(\Omega)$  and so on  $\mathbf{N}_\lambda^-(\Omega)$ , then there exists a minimizing sequence  $\{u_n^-\} \subseteq \mathbf{N}_\lambda^-(\Omega)$  such that

$$\lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^-) = \inf_{u \in \mathbf{N}_\lambda^-(\Omega)} \mathbf{E}_\lambda(u) = \alpha_\lambda^- > 0.$$

Since  $\mathbf{E}_\lambda$  is coercive,  $u_n^-$  is bounded in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$ . Thus, we may assume that without loss of generality,  $u_n^- \rightharpoonup u_0^-$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  and by the compact embeddings, we have

$$u_n^- \rightarrow u_0^- \text{ in } L_{a(x), \psi}^{q(x)}(\Omega)$$

and

$$u_n^- \rightarrow u_0^- \text{ in } L_{b(x), \psi}^{h(x)}(\Omega).$$

Moreover, if  $u_0^- \in \mathbf{N}_\lambda^-(\Omega)$ , then there is a constant  $t > 0$  such that  $tu_0^- \in \mathbf{N}_\lambda^-(\Omega)$  and  $\mathbf{E}_\lambda(u_0^-) \geq \mathbf{E}_\lambda(tu_0^-)$ . Indeed, since

$$\mathbf{I}'_\lambda(u) = \int_\Omega p(x) \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u \right|^{p(x)} dx - \lambda \int_\Omega q(x) a(x) |u|^{q(x)} dx - \int_\Omega h(x) b(x) |u|^{h(x)} dx,$$

then

$$\begin{aligned} \mathbf{I}'_\lambda(tu_0^-) &= \int_\Omega p(x) \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} (tu_0^-) \right|^{p(x)} dx - \lambda \int_\Omega q(x) a(x) |tu_0^-|^{q(x)} dx - \int_\Omega h(x) b(x) |tu_0^-|^{h(x)} dx \\ &\geq t^{p^+} p^+ \int_\Omega \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_0^- \right|^{p(x)} dx - \lambda t^{q^-} q^- \int_\Omega a(x) |u_0^-|^{q(x)} dx - t^{h^-} h^- \int_\Omega b(x) |u_0^-|^{h(x)} dx. \end{aligned}$$

Since  $q^- < p^+ < h^-$ , and by the assumptions on  $a$  and  $b$ , it follows  $\mathbf{I}'_\lambda(tu_0^-) < 0$ . Hence, by the definition of  $\mathbf{N}_\lambda^-(\Omega)$ ,  $tu_0^- \in \mathbf{N}_\lambda^-(\Omega)$ .

Now, we shall show  $u_n^- \rightarrow u_0^-$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$ . Then using the fact that

$$\int_\Omega \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_0^- \right|^{p(x)} dx < \liminf_{n \rightarrow \infty} \int_\Omega \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_n^- \right|^{p(x)} dx,$$

yields

$$\begin{aligned} \mathbf{E}_\lambda(tu_0^-) &\leq \frac{t^{p^+}}{p^-} \int_\Omega \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_0^- \right|^{p(x)} dx - \lambda \frac{t^{q^-}}{q^+} \int_\Omega a(x) |u_0^-|^{q(x)} dx - \frac{t^{h^-}}{h^-} \int_\Omega b(x) |u_0^-|^{h(x)} dx \\ &< \lim_{n \rightarrow \infty} \left[ \frac{t^{p^+}}{p^-} \int_\Omega \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} u_n^- \right|^{p(x)} dx - \lambda \frac{t^{q^-}}{q^+} \int_\Omega a(x) |u_n^-|^{q(x)} dx - \frac{t^{h^-}}{h^-} \int_\Omega b(x) |u_n^-|^{h(x)} dx \right] \\ &\leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(tu_n^-) \leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^-) = \inf_{u \in \mathbf{N}_\lambda^-(\Omega)} \mathbf{E}_\lambda(u) = \alpha_\lambda^-. \end{aligned}$$

This implies that  $\mathbf{E}_\lambda(tu_0^-) < \inf_{u \in \mathbf{N}_\lambda^-(\Omega)} \mathbf{E}_\lambda(u) = \alpha_\lambda^-$ , which is a contradiction. Hence,  $u_n^- \rightarrow u_0^-$  in  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  and so

$$\mathbf{E}_\lambda(u_0^-) = \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(u_n^-) = \inf_{u \in \mathbf{N}_\lambda^-(\Omega)} \mathbf{E}_\lambda(u).$$

Thus,  $u_0^-$  is a minimizer for  $\mathbf{E}_\lambda$  on  $\mathbf{N}_\lambda^-(\Omega)$ .  $\square$

**Corollary 3.8.** *By Theorem 3.5 and Theorem 3.7, we conclude that there exists  $u_0^+ \in \mathbf{N}_\lambda^+(\Omega)$  and  $u_0^- \in \mathbf{N}_\lambda^-(\Omega)$  such that  $\mathbf{E}_\lambda(u_0^+) = \inf_{u \in \mathbf{N}_\lambda^+(\Omega)} \mathbf{E}_\lambda(u)$  and  $\mathbf{E}_\lambda(u_0^-) = \inf_{u \in \mathbf{N}_\lambda^-(\Omega)} \mathbf{E}_\lambda(u)$ . Moreover, since  $\mathbf{E}_\lambda(u_0^\pm) = \mathbf{E}_\lambda(|u_0^\pm|)$  and  $|u_0^\pm| \in \mathbf{N}_\lambda^\pm(\Omega)$ , we may assume  $u_0^\pm \geq 0$ . By Theorem 3.1,  $u_0^\pm$  are critical points of  $\mathbf{E}_\lambda$  on  $\mathbb{H}_{p(x)}^{\alpha, \beta; \psi}(\Omega)$  and hence are weak solutions of (1.4). Finally, by the Harnack inequality (Theorem 2.12), we obtain that  $u_0^\pm$  are positive solutions of (1.4).*

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