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# When are Poisson and Moyal Brackets equal?

Didier Robert\*

## Abstract

In this paper we prove that if a given smooth Hamiltonian  $H$  on the phase space  $\mathbb{R}^{2d}$ , with derivatives of moderate growth, satisfies  $\{A, H\} = \{A, H\}_\otimes$  for any observable  $A$  in the Schwartz space  $\mathcal{S}(\mathbb{R}^{2d})$ , then, as it is expected,  $H$  must be a polynomial of degree at most 2 in  $\mathbb{R}^{2d}$ .

Here  $\{A, B\}$  denotes the Poisson bracket of two smooth classical observables and  $\{A, B\}_\otimes$  their Moyal bracket, defined as the Weyl symbol of  $i[\hat{A}, \hat{B}]$ , where  $\hat{A}$  is the Weyl quantization of  $A$  and  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$  (commutator).

A related answer to this question is given in the Groenewold-van Hove Theorem [9] concerning quantization of polynomial observables.

We revisit the paper [1] where the authors considered an algebraic formal approach of this question for the time dependent evolution of observables.

## 1 Introduction

Let  $H, A, B$  be smooth classical observables on  $\mathbb{R}^{2d}$  in the variables  $X = (x, \xi)$ . The Poisson brackets is defined as  $\{A, B\} = \partial_\xi A \cdot \partial_x B - \partial_x A \cdot \partial_\xi B$ . So the classical time evolution of  $A$  determined by the Hamilton equation for  $H$  is solution of the equation:

$$\begin{aligned} \frac{d}{dt}A(t) &= \{A(t), H\} \\ A(0) &= A. \end{aligned} \tag{1.1}$$

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\*Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 3, France  
*Email:* didier.robert@univ-nantes.fr

The Weyl quantization  $\hat{A}$  of  $A$  is defined as the following operator, depending on the Planck constant  $\hbar > 0$ ,

$$\hat{A}f(x) := (\text{Op}_\hbar^w A)f(x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^{2d}} A\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)/\hbar} f(y) dy d\xi \quad (1.2)$$

for any  $f \in \mathcal{S}(\mathbb{R}^d)$ . Recall that  $f \in \mathcal{S}(\mathbb{R}^d)$  means that  $f \in C^\infty(\mathbb{R}^d)$  and for any multiindex  $\alpha, \beta$ ,  $x^\alpha \partial_x^\beta f(x)$  is bounded on  $\mathbb{R}^d$ .

The quantum time evolution of the quantum observable  $\hat{A}$  must satisfy the Heisenberg equation

$$\frac{d}{dt}\hat{A}(t) = \frac{i}{\hbar}[\hat{A}(t), \hat{H}] \quad (1.3)$$

$$\hat{A}(0) = \hat{A} \quad (1.4)$$

where  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ .

The Moyal bracket of the observables  $A, H$ , is defined such that

$$\frac{i}{\hbar}[\hat{A}, \hat{H}] = \text{Op}_\hbar^w(\{A, H\}_\circ). \quad (1.5)$$

Notice that it results from the Weyl quantization calculus with a small parameter  $\hbar$  that we have

$$\lim_{\hbar \searrow 0} \{A, H\}_\circ = \{A, H\}.$$

*A natural question is to ask when the classical dynamics generated by the Hamiltonian  $H$  (1.1) has an exact correspondence with the quantum dynamics generated by  $\hat{H}$  (1.3) (see below the quotation from Van Hove). In the correspondence principle stated by N. Bohr the Planck constant  $\hbar$  is supposed to be small. The question discussed here is for fixed  $\hbar > 0$ .*

A well known trick to check the correspondence Bohr principle is to compute the time evolution of Gaussian coherent states. Let us denote  $\varphi_Y = \hat{T}_Y \varphi_0$  the coherent state center in  $Y \in \mathbb{R}^{2d}$  and  $\varphi_0(x) = (\pi\hbar)^{-d/4} e^{-|x|^2/2\hbar}$  ( $\hat{T}_Y$  is defined in the next section). We have [3]

$$\lim_{\hbar \searrow 0} \langle \varphi_Y, \hat{A} \varphi_Y \rangle = A(Y).$$

Hence taking the average of (1.3) on  $\varphi_Y$  and passing to the limit  $\hbar \searrow 0$ , we recover (1.1).

To define the Moyal bracket, there is a more explicit definition by introducing the Moyal product  $A \circledast B$  (see the next section) such that

$$(\text{Op}_\hbar^w A)(\text{Op}_\hbar^w B) = \text{Op}_\hbar^w(A \circledast B).$$

Then we have

$$\{A, B\}_\circ = \frac{i}{\hbar}(A \circ B - B \circ A).$$

These definitions make sense for  $A, B \in \mathcal{S}(\mathbb{R}^{2d})$  and can be extended to suitable classes of symbols with moderate growth. To be more explicit we introduce the classes  $\mathbb{S}_\delta^\mu$ , for  $\delta < 1$ ,  $\mu \in \mathbb{R}$ .  $A \in \mathbb{S}_\delta^\mu$  iff  $A \in C^\infty(\mathbb{R}^{2d})$  and for any multiindex  $\gamma \in \mathbb{N}^{2d}$  we have:

$$|\partial_X^\gamma A(X)| \leq C_\gamma \langle X \rangle^{\mu + \delta|\gamma|}$$

Using Theorem A.1 in [2], we can see that  $A \circ H$  is a smooth symbol if  $H \in \mathbb{S}_\delta^\mu$  and  $A \in \mathbb{S}_\delta^\nu$  where  $\mu, \nu \in \mathbb{R}$  and  $\delta < 1/2$ . Our aim here is to prove the following result.

**Theorem 1.1.** *Assume that  $\hbar > 0$  is fixed (for example  $\hbar = 1$ ). Let be  $H \in \mathbb{S}_\delta^\mu$  for some  $\mu \in \mathbb{R}$  and  $\delta < 1/2$ . Assume that for any  $A \in \mathcal{S}(\mathbb{R}^{2d})$  we have  $\{A, H\}_\circ = \{A, H\}$ . Then  $H(X)$  must be a polynomial in  $X = (x, \xi)$  of degree at most 2.*

**Remark 1.2.** *It is well known that if  $H$  is a polynomial of degree at most 2 then  $\{A, H\}_\circ = \{A, H\}$  for any  $A \in \mathbb{S}_0^\nu$ . This is an easy consequence of the Weyl calculus.*

**Remark 1.3.** *The usual proofs of the Groenewold-van Hove Theorem on the phase space  $\mathbb{R}^{2d}$  concern more general quantization procedures but are restricted to polynomial symbols  $A, H$ .*

*A quotation from [9] p.66-67:*

*"On établit ensuite qu'une correspondance biunivoque entre grandeurs classiques et quantiques, ayant le caractère d'un isomorphisme entre algèbres de Lie, existe entre les grandeurs représentées par des polynômes de degré 0, 1, 2 en les variables  $p_1, \dots, p_N, q_1, \dots, q_n$  mais ne peut être étendue sans perdre ses propriétés essentielles à l'ensemble de toutes les grandeurs classiques"*

*The Theorem of Groenewold-van Hove is detailed p.76 and the quadratic case p.87 of [9].*

*Notice that the quadratic case is related with the metaplectic representation [5].*

Let us consider now the time dependent consequence of theorem 1.1. Assume that  $H$  is real and generates a classical flow  $\Phi_H^t$  on  $\mathbb{R}^{2d}$  such that for any  $\gamma \in \mathbb{N}^{2d}$  we have

$$|\partial_X^\gamma \Phi_H^t(X)| \leq C_\gamma \langle X \rangle^m, \forall X \in \mathbb{R}^{2d}, |t| \leq \epsilon_0,$$

for some  $m \geq 0$  and  $\epsilon_0 > 0$ .

The solution of (1.1) is  $A_{\text{cl}}(t, X) := A(\Phi_H^t(X))$  and for  $|t| \leq \epsilon_0$  with  $A(t) \in \mathcal{S}(\mathbb{R}^{2d})$  if  $A(0) \in \mathcal{S}(\mathbb{R}^{2d})$ .

On the quantum side assume that  $\hat{H}$  is essentially self-adjoint in  $L^2(\mathbb{R}^d)$ . The solution of the Heisenberg equation (1.3) is  $\hat{A}_{\text{quant}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} \hat{A} e^{\frac{i}{\hbar}\hat{H}t}$ .

**Corollary 1.4.** *Under the conditions of Theorem 1.1, if for some  $\hbar > 0$  we have*

$$\hat{A}_{\text{quant}}(t) = \text{op}_\hbar^w(A_{\text{cl}}(t)), \quad \forall A \in \mathcal{S}(\mathbb{R}^{2d}), \quad |t| < \epsilon_0,$$

*if and only if  $H(X)$  is a polynomial in  $X$  of degree at most 2.*

**Remark 1.5.** *In [1] the authors consider the semi-classical expansion ( $\hbar \searrow 0$ ) in the Egorov theorem using a formal algebraic approach, and Bargman transform. In the Appendix C we give a direct proof of the main result of [1] which is stated below (proposition 1.6).*

Recall that according the semiclassical Egorov theorem the following asymptotic formula holds true as  $\hbar \searrow 0$ :

$$\hat{A}_{\text{quant}}(t) := e^{-\frac{i}{\hbar}\hat{H}t} \hat{A} e^{\frac{i}{\hbar}\hat{H}t} = \sum_{0 \leq k \leq N} \hbar^{2k} \hat{A}_{2k}(t) + O(\hbar^{2N+2}), \quad (1.6)$$

where the symbol of the coefficient  $A_{2j}$  can be computed by induction [2] and C.1.

**Proposition 1.6.** *[1] Let us consider the formal serie of symbols in  $\hbar$ :*

*$\sum_{j \geq 0} A_{2j}(t, X) \hbar^{2j}$  where the coefficient  $A_{2j}(t, X)$  are defined by (C.2).*

*If for some  $j_0 \geq 1$  we have  $A_{2j_0}(t, X) \equiv 0$  then  $A_2(t, X) \equiv 0$  and  $H$  must be a polynomial of degree at most 2. In particular if in a neighborhood of 0 in  $t$  we have  $\hat{A}_{\text{quant}}(t) = \hat{A}_{\text{cl}}(t) + O(\hbar^3)$  then  $H$  must be a polynomial of degree at most 2.*

**Remark 1.7.** *The above Proposition may be a little surprising: if one term of order  $\geq 2$  in the asymptotic expansion (1.6) vanishes then  $H$  must be a polynomial of degree  $\leq 2$  and all the terms vanish excepted the term of order 0.*

**Acknowledgement** *We dedicate this paper to Steve Zelditch who was at the origin of this question discussed with him twenty years ago.*

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## 2 Weyl calculus

### 2.1 Introduction to the Weyl quantization

In this section, we recall some basic properties of the Weyl calculus (for more details see [7]).

Weyl quantization starts by quantization of exponent of linear forms  $L_Y(X) = \sigma(Y, X) = \eta \cdot x - y \cdot \xi$  with  $X = (x, \xi)$ ,  $Y = (y, \eta)$ . Apart the usual properties asked for an admissible quantization, Weyl quantization is uniquely determined by imposing that the Weyl symbol of  $\widehat{e^{i\hat{L}_Y}}$  is  $e^{iL_Y}$ . Recall that  $\hat{T}(Y) := e^{-i\hat{L}_Y}$  is the Weyl-Heisenberg translation operator by  $Y$  in the phase space  $\mathbb{R}^{2d}$ . In other words the Weyl quantization  $A \mapsto \hat{A}$  has to satisfy  $\widehat{e^{i\hat{L}_Y}} = \widehat{(e^{iL_Y})}$ . Then for any observable  $A$ , using a Fourier transform, the Weyl quantization  $A$  is defined for any  $\psi \in \mathcal{S}(\mathbb{R}^d)$ , as

$$\hat{A}\psi = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \tilde{A}_\sigma(Y) \hat{T}(Y) \psi dY \quad (2.1)$$

where  $\tilde{A}_\sigma(Y) = \int_{\mathbb{R}^{2d}} A(z) e^{-i\sigma(Y, z)} dz$  is the symplectic Fourier transform of  $A$  (in the sense of distributions). So that the family  $\{\hat{T}(Y)\}_{Y \in \mathbb{R}^{2d}}$  is an over-complete basis for operators between the Schwartz spaces  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$ .  $\tilde{A}_\sigma$  is the covariant symbol of  $\hat{A}$  and  $A$  the contravariant symbol of  $\hat{A}$ .

**Remark 2.1.** Notice that from (2.1) for any symbol  $A$  and for any linear form  $L_Z$  we get

$$i[\hat{A}, \hat{L}_Z] = \widehat{\{A, L_Z\}}. \quad (2.2)$$

It is enough to prove (2.2) for  $\hat{A} = \hat{T}(Y)$ . This is done using the translation property of the Heisenberg unitary operators  $\hat{T}(Y)$  where  $Y = (y, \eta)$ ,  $D_x = i^{-1}\nabla_x$ , we have:

$$\hat{T}(sY)^* \begin{pmatrix} x \\ D_x \end{pmatrix} \hat{T}(sY) = \begin{pmatrix} x - sy \\ D_x - s\eta \end{pmatrix}, \quad s \in \mathbb{R}, \quad (2.3)$$

$s \mapsto (x - sy, \xi - s\eta)$  is the classical translation motion for the linear Hamiltonian  $L(Y)$ .

For quadratic Hamiltonians the classical flow is a time dependent linear symplectic map and the extension of (2.2) and (2.3) to quadratic Hamiltonians can be proved by the same method [3, Theorem 15, p.65].

### 2.2 The Moyal Product

We first recall the formal product rule for quantum observables with Weyl quantization. Let  $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ . The Moyal product  $C := A \circledast B$  is the

observable  $C$  such that  $\widehat{A} \cdot \widehat{B} = \widehat{C}$ . Some computations with the Fourier transform give the following well known formulas [7] (see also [8])

$$(A \circledast B)(X) = (\pi \hbar)^{-2d} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} e^{\frac{2i}{\hbar} \sigma(u,v)} A(X+u) B(X+v) du dv. \quad (2.4)$$

Some more computations with the Fourier transform give the following formula :

$$(A \circledast B)(x, \xi) = \exp \left( \frac{i\hbar}{2} \sigma(D_q, D_p; D_{q'}, D_{p'}) \right) A(q, p) B(q', p')|_{(q,p)=(q',p')=(x,\xi)}, \quad (2.5)$$

where  $\sigma$  is the symplectic bilinear form  $\sigma((q, p), (q', p')) = p \cdot q' - p' \cdot q$  and  $D = i^{-1} \hbar \nabla$ . By expanding the exponential term in a formal power series in  $\hbar$  we get

$$C(x, \xi) = \sum_{j \geq 0} \frac{\hbar^j}{j!} \left( \frac{i}{2} \sigma(D_q, D_p; D_{q'}, D_{p'}) \right)^j A(q, p) B(q', p')|_{(q,p)=(q',p')=(x,\xi)}. \quad (2.6)$$

Hence  $C(x, \xi)$  is a formal power series in  $\hbar$  with coefficients given by

$$C_j(A, B; x, \xi) = (2i)^{-j} \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\alpha|}}{\alpha! \beta!} (\nabla_x^\beta \nabla_\xi^\alpha A) \cdot (\nabla_x^\alpha \nabla_\xi^\beta B)(x, \xi). \quad (2.7)$$

Furthermore we need remainder estimates for the expansion of the Moyal product.

For every  $N \geq 1$  we introduce

$$R_N(A, B; X) := A \circledast B(X) - \sum_{0 \leq j \leq N} \hbar^j C_j(X). \quad (2.8)$$

The following estimate is a particular case of Theorem A.1 in [2] see also Remark A.3.

**Lemma 2.2.** *Let be  $A \in \mathbb{S}_\delta^{\mu_A}$  and  $B \in \mathbb{S}_\delta^{\mu_B}$ ,  $\delta < 1/2$ , then for any  $N \geq 1$ ,  $\gamma \in \mathbb{N}^{2d}$ ,  $M \geq M_0$  there exists  $C_{N,\gamma,M} > 0$  (independent of  $(A, B)$ ) such that*

$$|\partial_X^\gamma R_N(A, B; X)| \leq C_{N,\gamma,M} \hbar^{N+1} \sum_{\substack{|\alpha+\beta|=N+1 \\ |\mu+\nu| \leq M+|\gamma|}} |\partial_u^{(\alpha,\beta)+\mu} A(X+u)| |\partial_v^{(\beta,\alpha)+\nu} B(X+v)| \quad (2.9)$$

*In particular  $R_N(A, B, X) \in \mathbb{S}_\delta^{\mu_{AB}}$  for some  $\mu_{AB} \geq \mu_A + \mu_B$ .*

For proving this lemma one assumes first that  $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ . For the general case we put  $A_\epsilon(X) = e^{-\epsilon|X|^2}A(X)$ ,  $B_\epsilon(X) = e^{-\epsilon|X|^2}B(X)$  and pass to the limit for  $\epsilon \searrow 0$ . In the appendix we give more details. We also need to use the following lemma.

**Lemma 2.3.** *Let  $A \in \mathbb{S}_\delta^{\mu_A}$  and  $B \in \mathbb{S}_\delta^{\mu_B}$ ,  $\delta < 1/2$ . Then uniformly in every compact of  $\mathbb{R}_X^{2d}$ , we have*

$$\lim_{\epsilon \searrow 0} (A_\epsilon \circledast B)(X) = \lim_{\epsilon \searrow 0} (A \circledast B_\epsilon)(X) = (A \circledast B)(X).$$

In particular we have

$$\lim_{\epsilon \searrow 0} \{A_\epsilon, B\}_\circledast(X) = \{A, B\}_\circledast(X).$$

For completeness a proof is given in the appendix B.

### 3 Proof of Theorem 1.1

Here  $\hbar = 1$ . Notice first that from Lemma 2.3 we also have for any  $A \in \mathbb{S}_0^0$ ,  $\{A, H\}_\circledast = \{A, H\}$ . So it is enough to consider the test observables  $A = T_Y := e^{-iL_Y}$  ( $Y \in \mathbb{R}^{2d}$ ).

We have

$$\widehat{T}_Y \widehat{H} \widehat{T}_Y^* = [\widehat{T}_Y, \widehat{H}] \widehat{T}_Y^* + \widehat{H}$$

Using the assumption of Theorem 1.1 and Lemma 2.3 we get

$$\frac{1}{i}(\{T_Y^*, H\} \circledast T_Y)(X) = H(X + Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.1)$$

Computing the Poisson bracket in (3.1) gives

$$(((y \cdot \partial_x H + \eta \cdot \partial_\xi H) T_Y^*) \circledast T_Y)(X) = H(X + Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.2)$$

Our aim is to prove that (3.2) implies that  $H(X)$  is a polynomial of degree at most 2. For that purpose we shall compute the asymptotic expansion as  $Y \rightarrow 0$  of the left hand side of (3.2) and compare it with the Taylor expansion for  $H(X + Y)$  modulo  $O(|Y|^4)$ . From that we shall conclude that all the third order derivatives of  $H$  vanish for  $X$  in any bounded subset of  $\mathbb{R}^{2d}$ , hence the conclusion will follow.

We have

$$\partial_x^\alpha \partial_\xi^\beta T_Y = i^{-|\alpha+\beta|} \eta^\alpha y^\beta T_Y$$



Let us denote by  $C(X, Y)$  the left hand side in (3.2). So using Lemma 2.2 uniformly in every compact in  $X \in \mathbb{R}^{2d}$ , we have

$$C(X, Y) = \sum_{0 \leq j \leq 2} C_j(X, Y) + O(|Y|^4),$$

where

$$C_0(X, Y) = Y \cdot \nabla_X H(X) \quad (3.3)$$

$$C_1(X, Y) = \frac{1}{2} Y \cdot \nabla_X^2 H(X) Y, \quad (3.4)$$

where  $\nabla_X^2 H(X)$  is the Hessian matrix of  $H$ .

Let us compute now  $C_2(X, Y)$ , which is an homogeneous polynomial of degree 3 in  $Y$ .

For simplicity let us consider the 1-D case. The same computation can clearly be done for  $d > 1$ .

Using (2.6) we get with  $Y = (y, \eta)$ ,

$$C_2(X, Y) = \frac{1}{8} (y^3 \partial_x^3 H + \eta^3 \partial_\eta^3 H - y^2 \eta \partial_\xi \partial_x^2 H - y \eta^2 \partial_\xi^2 \partial_x H). \quad (3.5)$$

According (3.2),  $C_2(X, Y)$  must coincide with the term of order 3 in  $Y$  of the Taylor expansion in  $X$  for  $H(X + Y) - H(X)$ . But this is possible only if  $\partial_x^3 H = \partial_\eta^3 H = \partial_\xi \partial_x^2 H = \partial_\xi^2 \partial_x H = 0$  for any  $(x, \xi) \in \mathbb{R}^2$ . So  $H$  must be a polynomial of degree  $\leq 2$ .  $\square$

## 4 Extension to polynomials of arbitrary degree

The asymptotic expansion in  $\hbar$  in the Moyal product suggests to introduce the following semi-classical approximations of the Moyal bracket:

$$\{A, B\}_{\otimes, m} = \{A, B\} + \hbar^2 \{A, B\}_3 + \cdots + \hbar^{2m} \{A, B\}_{2m+1},$$

where  $\{A, B\}_j = \frac{i}{\hbar} (C_j(A, B) - C_j(B, A))$  (notation of (2.7)). More explicitly we have

$$\{A, B\}_j = (2i)^{-j} \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\alpha|} - (-1)^{|\beta|}}{\alpha! \beta!} (\nabla_x^\beta \nabla_\xi^\alpha A) \cdot (\nabla_x^\alpha \nabla_\xi^\beta B)(x, \xi). \quad (4.1)$$

Notice that  $\{A, B\}_j = 0$  for  $j$  even. In particular if  $A = T_Y$  we have if  $j = 2k + 1$ , with  $Y = (y, \eta)$ ,

$$T_Y^* \{T_Y, B\}_{2k+1}(x, \xi) = (2)^{-2k} \sum_{|\alpha+\beta|=2k+1} \frac{y^\alpha \eta^\beta}{\alpha! \beta!} \cdot (\nabla_x^\alpha \nabla_\xi^\beta B)(x, \xi). \quad (4.2)$$

It is clear that if  $H$  is a polynomial of degree at most  $2m + 2$  then we have  $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$  for any  $A$ . Conversely we have

**Theorem 4.1.** *Assume  $\hbar = 1$  and  $H \in \mathbb{S}_\delta^\mu$ ,  $\mu \in \mathbb{R}$ ,  $\delta < 1/2$ . If for any  $A \in \mathcal{S}(\mathbb{R}^{2d})$  we have  $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$  then  $H$  must be a polynomial of degree at most  $2m + 2$ .*

*Proof.* Here we give a proof different from the case  $m = 0$ , without connection with the Taylor formula, for simpler computations. Using Lemma 2.2 we have, uniformly in every compact in  $X \in \mathbb{R}^{2d}$ ,

$$T_Y^* (\{T_Y, H\}_{\otimes}(X) - \{T_Y, H\}_{\otimes, m}(X)) = \mathcal{O}(|Y|^{2m+3}), \quad Y \rightarrow 0. \quad (4.3)$$

Moreover from (2.7) we get:

$$T_Y^* \{T_Y, H\}_{2j+1}(X) = \frac{1}{2^{j+1}} \sum_{|\alpha+\beta=2j+1} \frac{y^\alpha \eta^\beta}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta H(X). \quad (4.4)$$

Using the assumption of Theorem 4.1. and (4.3) we get that

$$T_Y^* \{T_Y, H\}_{2m+3}(X) = \mathcal{O}_X(|Y|^{2m+5}).$$

But  $T_Y^* \{T_Y, H\}_{2m+3}$  is an homogeneous polynomial of degree  $2m + 3$  in  $Y$  so we get that this polynomial is 0 and from (4.4) we get that  $\partial_x^\alpha \partial_\xi^\beta H(X) = 0$  for  $|\alpha + \beta| = 2m + 3$ . Then we can conclude that  $H(X)$  is a polynomial of degree at most  $2m + 2$  in  $X \in \mathbb{R}^{2d}$ .  $\square$ .

## A Proofs for formulas (2.4) and (2.5)

It is enough to assume that  $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ .

Recall first the relationship between Weyl symbols and integral kernel of  $\hat{A}$ . We have

$$K_{\hat{A}}(x, y) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}(x-y) \cdot \eta} A\left(\frac{x+y}{2}, \eta\right) d\eta$$

and

$$A(x, \xi) = \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar}\xi \cdot t} K_{\hat{A}}(x + t/2, x - t/2) dt.$$

Using these formulas and the relation  $K_{\hat{A}\hat{B}}(x, z) = \int_{\mathbb{R}^d} K_{\hat{A}}(x, y) K_{\hat{B}}(y, z) dy$  we get

$$(A \otimes B)(X) = (2\pi\hbar)^{-2d} \int_{\mathbb{R}^{4d}} \exp\left(\frac{i}{\hbar}(-t \cdot \xi + (x - y - t/2) \cdot \eta + (y - x + t/2)) \cdot \zeta\right) \cdot A((x+y)/2 + t/4, \eta) B((x+y)/2 - t/4, \zeta) d\zeta d\eta dy dt \quad (\text{A.1})$$

Then after the change of variables in the integral  $v_\xi = \zeta - \xi, u_\xi = \eta - \xi$ ,  $u_x = (y - x)/2 + t/4, v_x = (y - x)/2 - t/4$ , we get formula (2.4), with  $u = (u_x, u_\xi), v = (v_x, v_\xi)$ ,

$$(A \circledast B)(X) = (\pi\hbar)^{-2d} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} e^{\frac{2i}{\hbar} \sigma(u,v)} A(X+u) B(X+v) du dv.$$

To get formula (2.5) we notice that  $(u, v) \mapsto 2\sigma(u, v)$  is non degenerate and its matrix is  $G := \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$ , so  $G^{-1} = G$ . Hence using Fourier transform in  $(u, v)$  and the Fourier multiplier formula we get (2.5).  $\square$

## B Proofs for Lemmas 2.2 and 2.3

### B.1 Proof of lemma 2.3

Using (2.4) for  $A_\epsilon \circledast B$  we split the integral in two pieces :

$1 = \chi_0(|u|^2 + |v|^2) + \chi_1(|u|^2 + |v|^2)$ , where  $\chi_0 \in C_0^\infty(\mathbb{R})$ ,  $\chi_0(t) = 1$  for  $|t| \leq 1/2$ . On the support of  $\chi_0$  we can obviously pass to the limit in  $\epsilon$ . On the support of  $\chi_1$  we first perform integrations by parts with the differential operator  $L$  to get a uniformly and absolutely convergent integral,

$$L = \frac{Ju \cdot \partial_v - Jv \cdot \partial_u}{|u|^2 + |v|^2},$$

using that  $Le^{\frac{2i}{\hbar} \sigma(u,v)} = Le^{\frac{2i}{\hbar} \sigma(u,v)} = \frac{2i}{\hbar} e^{\frac{2i}{\hbar} \sigma(u,v)}$ . On the support of  $\chi_1$ , performing  $4d+1$  integrations by parts for gaining enough decay to ensure integrability in  $(u, v) \in \mathbb{R}^{4d}$ . Then passing to the limit in  $\epsilon$  we get  $\lim_{\epsilon \searrow 0} (A_\epsilon \circledast B)(X) = (A \circledast B)(X)$  and the same for  $\lim_{\epsilon \searrow 0} (A \circledast B_\epsilon)(X) = \{A, B\}_\circledast(X)$ .

The other properties follow.  $\square$

### B.2 Proof of Lemma 2.2

From (2.4), by Fourier transform computations and application of the Taylor formula, we get the following formula for the remainder,

$$R_N(A, B, X) = \frac{1}{N!} \left( \frac{i\hbar}{2} \right)^{N+1} \int_0^1 (1-t)^N R_{N,t}(X; \hbar) dt, \quad (\text{B.1})$$

where

$$R_{N,t}(X; \epsilon) = (2\pi t)^{-2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp \left( -\frac{i}{2t\hbar} \sigma(u, v) \right) \sigma^{N+1}(D_u, D_v) A(u+X) B(v+X) du dv.$$

Notice that the integral is an oscillating integral as we shall see below. So we shall use the following lemma :

**Lemma B.1.** *There exists a constant  $C_d > 0$  such that for any  $F \in \mathcal{S}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$  the integral*

$$I(\lambda) = \lambda^{2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] F(u, v) du dv \quad (\text{B.2})$$

*satisfies the following estimate:*

*for any  $M > 0$  there exists  $C_M > 0$ , independent of  $F$  such that*

$$|I(\lambda)| \leq C_M \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha| + |\beta| \leq M + 4d + 1}} (1 + |u|^2 + |v|^2)^{(4d+1-M)/2} |\partial_u^\alpha \partial_v^\beta F(u, v)|. \quad (\text{B.3})$$

A proof will be given later in the next section.

Using this lemma for  $A, B \in \mathcal{S}(\mathbb{R}^{2d})$  with the integrand

$$F_{N,\gamma}(X; u, v) = \pi^{-2d} \partial_X^\gamma (\sigma^{N+1}(D_u, D_v) A(u + X) B(v + X))$$

and the parameter  $\lambda = 1/(2t\hbar)$ , we then have that

$$|\partial_X^\gamma R_{N,t}(X; \varepsilon)| \leq C_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha| + |\beta| \leq 4d + 1}} |\partial_u^\alpha \partial_v^\beta F_{N,\gamma}(X; u, v)|.$$

Moreover, we have the elementary estimate

$$|\sigma^{N+1}(D_u, D_v) A(u) B(v)| \leq (2d)^{N+1} \sup_{|\alpha| + |\beta| = N+1} |\partial_x^\alpha \partial_\xi^\beta A(x, \xi) \partial_y^\beta \partial_\eta^\alpha B(y, \eta)|. \quad (\text{B.4})$$

Together with the Leibniz formula, we then get the claimed result with universal constants. For symbols  $A \in \mathbb{S}_0^\mu$  and  $B \in \mathbb{S}_0^\nu$  we argue by localisation. We use  $A_\epsilon(u) = e^{-\epsilon u^2} A(u)$  and  $B_\epsilon(v) = e^{-\epsilon v^2} B(v)$  for  $\epsilon > 0$  and pass to the limit as  $\epsilon \rightarrow 0$ .

### B.3 Proof of the Lemma B.1

We consider the same cut-off  $\chi_0$  as above. We split  $I(\lambda)$  into two pieces and write  $I(\lambda) = I_0(\lambda) + I_1(\lambda)$  with

$$\begin{aligned} I_0(\lambda) &= \lambda^{2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] \chi_0((u^2 + v^2)) F(u, v) du dv, \\ I_1(\lambda) &= \lambda^{2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] (1 - \chi_0)(u^2 + v^2) F(u, v) du dv. \end{aligned}$$

We notice that  $(u, v) \mapsto \sigma(u, v)$  is a quadratic non-degenerate real form on  $\mathbb{R}^{4d}$ .

Let us estimate  $I_1(\lambda)$ . We can integrate by parts with the differential operator

$$L = \frac{i}{|u|^2 + |v|^2} \left( Ju \cdot \frac{\partial}{\partial v} - Jv \cdot \frac{\partial}{\partial u} \right),$$

using that  $Le^{-i\lambda\sigma(u,v)} = Le^{-i\lambda Ju \cdot v} = \lambda e^{-i\lambda\sigma(u,v)}$ . For  $I_1(\lambda)$ , the integrand is supported outside the ball of radius  $1/\sqrt{2}$  in  $\mathbb{R}^{4d}$ . Performing  $4d+1$  integrations by parts for gaining enough decay to ensure integrability in  $(u, v) \in \mathbb{R}^{4d}$ , we get a constant  $c_d$  such that

$$|I_1(\lambda)| \leq c_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\mu| + |\nu| \leq 4d+1}} |\partial_u^\mu \partial_v^\nu F(u, v)|. \quad (\text{B.5})$$

But we need to control the behaviour for  $u^2 + v^2$  large, so with  $M$  more integrations by parts we get

$$|I_1(\lambda)| \leq C_M \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha| + |\beta| \leq M}} (1 + |u|^2 + |v|^2)^{(4d+1-M)/2} |\partial_u^\alpha \partial_v^\beta F(u, v)| \quad (\text{B.6})$$

To estimate  $I_0(\lambda)$  we apply the stationary phase theorem. The symmetric matrix of the quadratic form  $\sigma(u, v)$  is

$$A_\sigma = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

So by the stationary phase theorem ([7], Vol.I, section 7.7), noticing that the leading term in the stationary phase theorem is of order  $\lambda^{-2d}$ , we get

$$|I_0(\lambda)| \leq c'_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\mu| + |\nu| \leq 2d+3}} |\partial_u^\mu \partial_v^\nu F(u, v)|. \quad (\text{B.7})$$

## C Proof of proposition 1.6

Let us recall the statement of the semiclassical Egorov theorem (see [2, 8] for a proof)

**Theorem C.1.** *Let us assume that  $H$  is a subquadratic Hamiltonian, that means here that for  $|\alpha| + |\beta| \geq 2$  there exists  $C_{\alpha, \beta}$  such that*

$$|\partial_x^\alpha \partial_\xi^\beta H(x, \xi)| \leq C_{\alpha, \beta}, \quad \forall (x, \xi) \in \mathbb{R}^{2d}.$$

For any  $A \in \mathbb{S}_0^0$  we have, for the operator norm in  $L^2(\mathbb{R}^d)$  and for every  $N \geq 0$ ,

$$\hat{A}_{\text{quant}}(t) = e^{-\frac{i}{\hbar}\hat{H}} \hat{A} e^{\frac{i}{\hbar}\hat{H}} = \sum_{0 \leq k \leq N} \hbar^{2k} \hat{A}_{2k}(t) + O(\hbar^{2N+2}). \quad (\text{C.1})$$

In particular we have  $A_0(t, X) = A(\Phi_H^t(X)) := A_{\text{cl}}(t, X)$  and for  $j \geq 1$ ,  $A_{2j}(t, X)$  is defined by induction on  $j$  such that  $A_{2j}(0) = 0$  and

$$\partial_t A_{2j+2}(t) - \{A_{2j+2}, H\}_1 = \sum_{k+\ell=j+1, k \leq j} \{A_{2k}(t), H\}_{2\ell+1}. \quad (\text{C.2})$$

Let us remark that here the expansion is even in  $\hbar$  because  $H$  is reduced to its principal part (no perturbative term in  $\hbar$ ).

$A_2 \equiv 0$  hence  $H$  is a polynomial of degree at most 2. We can recover easily here these results.

*Proof of proposition 1.6.*

From (C.2) we get

$$\partial_t A_{2j}(t) - \{A_{2j}(t), H\}_1 = \sum_{k+\ell=j, k \leq j-1} \{A_{2k}(t), H\}_{2\ell+1} \quad (\text{C.3})$$

If for  $j = j_0$  we have  $\partial_t A_{2j}(t) - \{A_{2j}(t), H\}_1 \equiv 0$ , so taking  $t = 0$  in the r.h.s of (C.3) we get  $\{A, H\}_{2j_0+1} = 0$ . Hence using the observables  $A = T_Y$ , like in (4.4), we get that  $H$  is a polynomial of degree at most  $2j_0$ .

Moreover using that  $\partial_t A_{2j}(0) = 0$  and  $A_{2k}(0) = 0$  for  $k \geq 1$ ,  $A_0(0) = A$ , we get the formula (4.5) of the paper [1]:

$$\sum_{k+\ell=j, k \leq j-1} \{\{A, H\}_{2\ell+1}, H\}_{2k+1} = 0. \quad (\text{C.4})$$

Then as in [1] from (C.4) one can conclude that  $H$  is a polynomial of degree at most 2.

For the last part of the proposition we notice that from the assumption we get that  $A_2(t, X) \equiv 0$ .  $\square$

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