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When Poisson and Moyal Brackets are equal?

Didier Robert*

Abstract

In the phase space \mathbb{R}^{2d} , let us denote $\{A, B\}$ the Poisson bracket of two smooth classical observables and $\{A, B\}_\otimes$ their Moyal bracket, defined as the Weyl symbol of $i[\hat{A}, \hat{B}]$, where \hat{A} is the Weyl quantization of A and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ (commutator).

In this note we prove that if a given smooth Hamiltonian H on the phase space \mathbb{R}^{2d} , with derivatives of moderate growth, satisfies $\{A, H\} = \{A, H\}_\otimes$ for any observable A in the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$, then, as it is expected, H must be a polynomial of degree at most 2 in \mathbb{R}^{2d} .

A related answer to this question is given in the Groenewold-van Hove Theorem [4, 5, 8] concerning quantization of polynomial observables. We consider here more general classes of Hamiltonians.

1 Introduction

Let H, A, B be smooth classical observables on \mathbb{R}^{2d} in the variables $X = (x, \xi)$. The Poisson brackets is defined as $\{A, B\} = \partial_\xi A \cdot \partial_x B - \partial_x A \cdot \partial_\xi B$. So the classical time evolution of A determined by the Hamilton equation for H is solution of the equation:

$$\begin{aligned} \frac{d}{dt}A(t) &= \{A(t), H\} \\ A(0) &= A. \end{aligned} \tag{1.1}$$

The Weyl quantization \hat{A} of A is defined as the following operator:

$$\hat{A}f(x) := (\text{Op}_\hbar^w A)f(x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^{2d}} A\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)/\hbar} f(y) dy d\xi \tag{1.2}$$

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for any $f \in \mathcal{S}(\mathbb{R}^d)$. Recall that $f \in \mathcal{S}(\mathbb{R}^d)$ means that $f \in C^\infty(\mathbb{R}^d)$ and for any multiindex α, β , $x^\alpha \partial_x^\beta f(x)$ is bounded on \mathbb{R}^d .

The quantum time evolution of the quantum observable \widehat{A} must satisfy the Heisenberg equation

$$\frac{d}{dt} \widehat{A}(t) = \frac{i}{\hbar} [\widehat{A}(t), \widehat{H}] \quad (1.3)$$

$$\widehat{A}(0) = \widehat{A} \quad (1.4)$$

where $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$.

The Moyal bracket of the observables A, H , is defined such that

$$\frac{i}{\hbar} [\widehat{A}, \widehat{H}] = \text{Op}_\hbar^w(\{A, H\}_\otimes). \quad (1.5)$$

Notice that it results from the Weyl quantization calculus with a small parameter \hbar that we have

$$\lim_{\hbar \searrow 0} \{A, H\}_\otimes = \{A, H\}.$$

A natural question is to ask when the classical dynamics generated by the Hamiltonian H (1.1) has an exact correspondence with the quantum dynamics generated by \widehat{H} (1.3) (see below the quotation from Van Hove). In the correspondence principle stated by N. Bohr the Planck constant \hbar is supposed to be small. The question discussed here is for $\hbar > 0$ fixed.

A well known trick to check the correspondence Bohr principle is to compute the time evolution of Gaussian coherent states. Let us denote $\varphi_Y = \widehat{T}_Y \varphi_0$ the coherent state center in $Y \in \mathbb{R}^{2d}$ and $\varphi_0(x) = (\pi\hbar)^{-d/4} e^{-|x|^2/2}$ (\widehat{T}_Y is defined in the next section). We have [2]

$$\lim_{\hbar \searrow 0} \langle \varphi_Y, \widehat{A} \varphi_Y \rangle = A(Y).$$

Hence taking the average of (1.3) on φ_Y and passing to the limit $\hbar \searrow 0$, we recover (1.1).

To define the Moyal bracket, there is a more explicite definition by introducing the Moyal product $A \otimes B$ (see the next section) such that

$$(\text{Op}_\hbar^w A)(\text{Op}_\hbar^w B) = \text{Op}_\hbar^w(A \otimes B).$$

Then we have

$$\{A, B\}_\otimes = \frac{i}{\hbar} (A \otimes B - B \otimes A).$$

These definitions make sense for $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ and can be extended to suitable classes of symbols with moderate growth. To be more explicite we

introduce the classes \mathbb{S}_δ^μ , for $\delta < 1$, $\mu \in \mathbb{R}$. $A \in \mathbb{S}_\delta^\mu$ iff $A \in C^\infty(\mathbb{R}^{2d})$ and for any multiindex $\gamma \in \mathbb{N}^{2d}$ we have:

$$|\partial_X^\gamma A(X)| \leq C_\gamma \langle X \rangle^{\mu + \delta|\gamma|}$$

Using Theorem A.1 in [1], we can see that $A \circledast H$ is a smooth symbol if $H \in \mathbb{S}_\delta^\mu$ and $A \in \mathbb{S}_\delta^\nu$ where $\mu, \nu \in \mathbb{R}$ and $\delta < 1$. Our aim here is to prove the following result.

Theorem 1.1. *Assume that \hbar is fixed ($\hbar = 1$). Let be $H \in \mathbb{S}_\delta^\mu$ for some $\mu \in \mathbb{R}$ and $\delta < 1$. Assume that for any $A \in \mathcal{S}(\mathbb{R}^{2d})$ we have $\{A, H\}_\circledast = \{A, H\}$. Then $H(X)$ must be a polynomial in $X = (x, \xi)$ of degree at most 2.*

Remark 1.2. *It is well known that if H is a polynomial of degree at most 2 then $\{A, H\}_\circledast = \{A, H\}$ for any $A \in \mathbb{S}_\delta^\nu$. I do not know any reference for a proof of the converse statement. The proof given here is a consequence of basic properties of the Weyl quantization.*

Remark 1.3. *The usual proofs of the Groenewold-van Hove Theorem on the phase space \mathbb{R}^{2d} concern more general quantization procedures but are restricted to polynomial symbols A, H .*

A quotation from [8] p.66-67:

"On établit ensuite qu'une correspondance biunivoque entre grandeurs classiques et quantiques, ayant le caractère d'un isomorphisme entre algèbres de Lie, existe entre les grandeurs représentées par des polynômes de degré 0, 1, 2 en les variables $p_1, \dots, p_N, q_1, \dots, q_n$ mais ne peut être étendue sans perdre ses propriétés essentielles à l'ensemble de toutes les grandeurs classiques"

The Theorem of Groenewold-van Hove is detailed p.76 and the quadratic case p.87 of [8].

Notice that the quadratic case is related with the metaplectic representation [4].

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2 Weyl calculus

2.1 Introduction to the Weyl quantization

In this section, we recall some basic properties of the Weyl calculus (for more details see [6]).

Weyl quantization starts by quantization of exponent of linear forms $L_Y(X) = \sigma(Y, X) = \eta \cdot x - y \cdot \xi$ with $X = (x, \xi)$, $Y = (y, \eta)$. Apart the usual properties asked for an admissible quantization, Weyl quantization is uniquely determined by imposing that the Weyl symbol of $e^{i\widehat{L}_Y}$ is e^{iL_Y} . Recall that $\widehat{T}(Y) := e^{-i\widehat{L}_Y}$ is the Weyl-Heisenberg translation operator by Y in the phase space \mathbb{R}^{2d} . In other words the Weyl quantization $A \mapsto \widehat{A}$ has to satisfy $e^{i\widehat{L}_Y} = \widehat{(e^{iL_Y})}$. Then for any observable A , using a Fourier transform, the Weyl quantization A is defined for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, as

$$\widehat{A}\psi = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \widetilde{A}_\sigma(Y) \widehat{T}(Y) \psi dY \quad (2.1)$$

where $\widetilde{A}_\sigma(Y) = \int_{\mathbb{R}^{2d}} A(z) e^{-i\sigma(Y, z)} dz$ is the symplectic Fourier transform of A (in the sense of distributions). So that the family $\{\widehat{T}(Y)\}_{Y \in \mathbb{R}^{2d}}$ is an over-complete basis for operators between the Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. \widetilde{A}_σ is the covariant symbol of \widehat{A} and A the contravariant symbol of \widehat{A} .

Remark 2.1. Notice that from (2.1) for any symbol A and for any linear form L_Z we get

$$i[\widehat{A}, \widehat{L}_Z] = \widehat{\{A, L_Z\}}. \quad (2.2)$$

It is enough to prove (2.2) for $\widehat{A} = \widehat{T}(Y)$. This is done using the translation property of the Heisenberg unitary operators $\widehat{T}(Y)$ where $Y = (y, \eta)$, $D_x = i^{-1}\nabla_x$, we have:

$$\widehat{T}(sY)^* \begin{pmatrix} x \\ D_x \end{pmatrix} \widehat{T}(sY) = \begin{pmatrix} x - sy \\ D_x - s\eta \end{pmatrix}, \quad s \in \mathbb{R}, \quad (2.3)$$

$s \mapsto (x - sy, \xi - s\eta)$ is the classical translation motion for the linear Hamiltonian $L(Y)$.

For quadratic Hamiltonians the classical flow is a time dependent linear symplectic map and the extension of (2.2) and (2.3) to quadratic Hamiltonians can be proved by the same method [2, Theorem 15, p.65].

2.2 The Moyal Product

We first recall the formal product rule for quantum observables with Weyl quantization. Let $A, B \in \mathcal{S}(\mathbb{R}^{2d})$. The Moyal product $C := A \circledast B$ is the observable C such that $\widehat{A} \cdot \widehat{B} = \widehat{C}$. Some computations with the Fourier transform give the following well known formulas [6] (see also [7])

$$(A \circledast B)(X) = (\pi\hbar)^{-2d} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} e^{\frac{2i}{\hbar}\sigma(u, v)} A(X + u) B(X + v) dudv. \quad (2.4)$$

Some computations with the Fourier transform give the following formula :

$$(A \otimes B)(x, \xi) = \exp\left(\frac{i\hbar}{2}\sigma(D_q, D_p; D_{q'}, D_{p'})\right) A(q, p)B(q', p')|_{(q,p)=(q',p')=(x,\xi)}, \quad (2.5)$$

where σ is the symplectic bilinear form $\sigma((q, p), (q', p')) = p \cdot q' - p' \cdot q$ and $D = i^{-1}\hbar\nabla$. By expanding the exponential term in a formal power series in \hbar we get

$$C(x, \xi) = \sum_{j \geq 0} \frac{\hbar^j}{j!} \left(\frac{i}{2}\sigma(D_q, D_p; D_{q'}, D_{p'})\right)^j A(q, p)B(q', p')|_{(q,p)=(q',p')=(x,\xi)}. \quad (2.6)$$

So that $C(x, \xi)$ is a formal power series in \hbar with coefficients given by

$$C_j(A, B; x, \xi) = \frac{1}{2^j} \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (D_x^\beta \partial_\xi^\alpha A) \cdot (D_x^\alpha \partial_\xi^\beta B)(x, \xi). \quad (2.7)$$

Furthermore we need a remainder estimates for the expansion of the Moyal product.

For every $N \geq 1$. we denote

$$R_N(A, B; X) := A \otimes B(X) - \sum_{0 \leq j \leq N} \hbar^j C_j(X). \quad (2.8)$$

The following estimate is a particular case of Theorem A.1 in [1] see also Remark A.3.

Lemma 2.2. *Let be $A \in \mathbb{S}_\delta^{\mu_A}$ and $B \in \mathbb{S}_\delta^{\mu_B}$, $\delta < 1$, then for any $N \geq 1$, $\gamma \in \mathbb{N}^{2d}$, $M \geq M_0$ there exists $C_{N,\gamma,M} > 0$ (independent of (A, B)) such that*

$$|\partial_X^\gamma R_N(A, B; X)| \leq C_{N,\gamma,M} \hbar^{N+1} \sum_{\substack{|\alpha+\beta|=N+1 \\ |\mu+\nu| \leq M+|\gamma|}} \sup_{u,v \in \mathbb{R}^{2d}} (1 + |u|^2 + |v|^2)^{(M_0-M)/2} |\partial_u^{(\alpha,\beta)+\mu} A(X+u)| |\partial_v^{(\beta,\alpha)+\nu} B(X+v)| \quad (2.9)$$

In particular $R_N(A, B, X) \in \mathbb{S}_\delta^{\mu_{AB}}$ for some $\mu_{AB} \geq \mu_A + \mu_B$.

For proving this Lemma one assume first that $A, B \in \mathcal{S}(\mathbb{R}^{2d})$. For the general case we put $A_\varepsilon(X) = e^{-\varepsilon|X|^2} A(X)$, $B_\varepsilon(X) = e^{-\varepsilon|X|^2} B(X)$ and pass to the limit for $\varepsilon \searrow 0$. In the appendix we give more details. We also need to use the following Lemma.

Lemma 2.3. $A \in \mathbb{S}_\delta^{\mu_A}$ and $B \in \mathbb{S}_\delta^{\mu_B}$, $\delta < 1$. Then uniformly in every compact of \mathbb{R}_X^{2d} , we have

$$\lim_{\epsilon \searrow 0} (A_\epsilon \circledast B)(X) = \lim_{\epsilon \searrow 0} (A \circledast B_\epsilon)(X) = (A \circledast B)(X).$$

In particular we have

$$\lim_{\epsilon \searrow 0} \{A_\epsilon, B\}_\circledast(X) = \{A, B\}_\circledast(X).$$

For completeness a proof is given in the appendix B.

3 Proof of Theorem(1.1)

Here $\hbar = 1$. Notice first that from Lemma 2.3 we also have for any $A \in \mathbb{S}_0^0$, $\{A, H\}_\circledast = \{A, H\}$. So it is enough to consider the test observables $A = T_Y := e^{-iL_Y}$ ($Y \in \mathbb{R}^{2d}$).

We have

$$\widehat{T}_Y \widehat{H} \widehat{T}_Y^* = [\widehat{T}_Y, \widehat{H}] \widehat{T}_Y^* + \widehat{H}$$

Using the assumption of Theorem(1.1) and Lemma 2.3 we get

$$\frac{1}{i} (\{T_Y^*, H\} \circledast T_Y)(X) = H(X+Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.1)$$

Computing the Poisson bracket in (3.1) gives

$$(((y \cdot \partial_x H + \eta \cdot \partial_\xi H) T_Y^*) \circledast T_Y)(X) = H(X+Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.2)$$

Our aim is to prove that (3.2) implies that $H(X)$ is a polynomial of degree at most 2. For that purpose we shall compute the asymptotic expansion as $Y \rightarrow 0$ of the left hand side of (3.2) and compare it with the Taylor expansion for $H(X+Y)$ modulo $O(|Y|^4)$. From that we shall conclude that all the third order derivatives of H vanish for X in any bounded subset of \mathbb{R}^{2d} hence the conclusion will follow.

We have

$$\partial_x^\alpha \partial_\xi^\beta T_Y = i^{-|\alpha+\beta|} \eta^\alpha y^\beta T_Y$$

Let us denote by $C(X, Y)$ the left hand side in (3.2). So using Lemma(2.2) uniformly in every compact in $X \in \mathbb{R}^{2d}$, we have

$$C(X, Y) = \sum_{0 \leq j \leq 2} (C_j(X, Y) + O(|Y|^4)),$$

where

$$C_0(X, Y) = Y \cdot \nabla_X H(X) \quad (3.3)$$

$$C_1(X, Y) = \frac{1}{2} Y \cdot \nabla_X^2 H(X) Y, \quad (3.4)$$

where $\nabla_X^2 H(X)$ is the Hessian matrix of H .

Let us compute now $C_2(X, Y)$, which is an homogeneous polynomial of degree 3 in Y .

For simplicity let us consider the 1-D case. The same computation can clearly be done for $d > 1$.

Using (2.6) we get with $Y = (y, \eta)$,

$$C_2(X, Y) = \frac{1}{8} (y^3 \partial_x^3 H + \eta^3 \partial_\eta^3 H - y^2 \eta \partial_\xi \partial_x^2 H - y \eta^2 \partial_\xi^2 \partial_x H). \quad (3.5)$$

According (3.2), $C_2(X, Y)$ must coincide with the term of order 3 in Y of the Taylor expansion in X for $H(X + Y) - H(X)$. But this is possible only if $\partial_x^3 H = \partial_\eta^3 H = \partial_\xi \partial_x^2 H = \partial_\xi^2 \partial_x H = 0$ for any $(x, \xi) \in \mathbb{R}^2$. So H must be a polynomial of degree ≤ 2 . \square

4 Extension to polynomials of arbitrary degree

The asymptotic expansion in \hbar in the Moyal product suggests to introduce the following semi-classical approximations of the Moyal bracket:

$$\{A, B\}_{\otimes, m} = \{A, B\} + \hbar^2 \{A, B\}_3 + \cdots + \hbar^{2m} \{A, B\}_{2m+1},$$

where $\{A, B\}_j = \frac{i}{\hbar} (C_j(A, B) - C_j(B, A))$ (notation of (2.7)). Notice that $\{A, B\}_j = 0$ for j even.

It is clear that if H is a polynomial of degree at most $2m + 2$ then we have $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$ for any A . Conversely we have

Theorem 4.1. *Assume $\hbar = 1$ and $H \in \mathbb{S}_\delta^\mu$, $\mu \in \mathbb{R}$, $\delta < 1$. If for any $A \in \mathcal{S}(\mathbb{R}^{2d})$ we have $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$ then H must be a polynomial of degree at most $2m + 2$.*

Proof. Here we give a proof different from the case $m = 0$, without connection with the Taylor formula, for simpler computations.

Using Lemma 2.2 we have, uniformly in every compact in $X \in \mathbb{R}^{2d}$,

$$T_Y^* (\{T_Y, H\}_{\otimes}(X) - \{T_Y, H\}_{\otimes, m}(X)) = \mathcal{O}(|Y|^{2m+3}), \quad Y \rightarrow 0. \quad (4.1)$$

Moreover from (2.7) we get:

$$T_Y^*\{T_Y, H\}_{2j+1}(X) = \frac{1}{2^{j+1}} \sum_{|\alpha+\beta=2j+1} \frac{y^\alpha \eta^\beta}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta H(X). \quad (4.2)$$

Using the assumption of Theorem 4.1. and (4.1) we get that

$$T_Y^*\{T_Y, H\}_{2m+3}(X) = \mathcal{O}_X(|Y|^{2m+5}).$$

But $T_Y^*\{T_Y, H\}_{2m+3}$ is an homogeneous polynomial of degree $2m+3$ in Y so we get that this polynomial is 0 and from (4.2) we get that $\partial_x^\alpha \partial_\xi^\beta H(X) = 0$ for $|\alpha + \beta| = 2m + 3$. Then we can conclude that $H(X)$ is a polynomial of degree at most $2m + 2$ in $X \in \mathbb{R}^{2d}$. \square .

A Proofs for formula (2.4) and (2.5)

It is enough to assume that $A, B \in \mathcal{S}(\mathbb{R}^{2d})$.

Recall first the relationship between Weyl symbols and integral kernel of \hat{A} . We have

$$K_{\hat{A}}(x, y) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}(x-y)\cdot\eta} A\left(\frac{x+y}{2}, \eta\right) d\eta$$

and

$$A(x, \xi) = \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar}\xi\cdot t} K_{\hat{A}}(x+t/2, x-t/2) dt.$$

Using these formulas and the relation $K_{\hat{A}\hat{B}}(x, z) = \int_{\mathbb{R}^d} K_{\hat{A}}(x, y) K_{\hat{B}}(y, z) dy$ we get

$$(A \circledast B)(X) = (2\pi\hbar)^{-2d} \int_{\mathbb{R}^{4d}} \exp\left(\frac{i}{\hbar}(-t \cdot \xi + (x - y - t/2) \cdot \eta + (y - x + t/2)) \cdot \zeta\right) \cdot A\left(\frac{(x+y)}{2} + t/4, \eta\right) B\left(\frac{(x+y)}{2} - t/4, \zeta\right) d\zeta d\eta dy dt \quad (\text{A.1})$$

Then after a change of variables in the integral $v_\xi = \zeta - \xi, u_\xi = \eta - \xi, u_x = (y - x)/2 + t/4, v_x = (y - x)/2 - t/4$, we get formula (2.4), with $u = (u_x, u_\xi), v = (v_x, v_\xi)$,

$$(A \circledast B)(X) = (\pi\hbar)^{-2d} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} e^{\frac{2i}{\hbar}\sigma(u,v)} A(X+u) B(X+v) dudv.$$

To get formula (2.5) we notice that $(u, v) \mapsto 2\sigma(u, v)$ is non degenerate and its matrix is $G := \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$, so $G^{-1} = G$. Hence using Fourier transform in (u, v) and the Fourier multiplier formula we get (2.5). \square

B Proofs for Lemmas 2.2 and 2.3

B.1 Proof of lemma 2.3

Using (2.4) for $A_\epsilon \otimes B$ we split the integral in two pieces :
 $1 = \chi_0(|u|^2 + |v|^2) + \chi_1(|u|^2 + |v|^2)$, where $\chi_0 \in C_0^\infty(\mathbb{R})$, $\chi_0(t) = 1$ for $|t| \leq 1/2$.
 On the support of χ_0 we can obviously pass to the limit in ϵ . On the support
 of χ_1 we first perform integrations by parts with the differential operator L
 several times to get a uniformly and absolutely convergent integral,

$$L = \frac{Ju \cdot \partial_v - Jv \cdot \partial_u}{|u|^2 + |v|^2},$$

using that $Le^{\frac{2i}{\hbar}\sigma(u,v)} = Le^{\frac{2i}{\hbar}\sigma(u,v)} = \frac{2i}{\hbar}e^{\frac{2i}{\hbar}\sigma(u,v)}$. On the support of χ_1 , perform-
 ing $4d+1$ integrations by parts for gaining enough decay to ensure integrabil-
 ity in $(u, v) \in \mathbb{R}^{4d}$. Then passing to the limit in ϵ we get $\lim_{\epsilon \searrow 0}(A_\epsilon \otimes B)(X) =$
 $(A \otimes B)(X)$ and the same for $\lim_{\epsilon \searrow 0}(A \otimes B_\epsilon)(X) = \{A, B\}_\otimes(X)$.

The other properties follows. \square

B.2 Proof of Lemma 2.2

From (2.4), by Fourier transform computations and application of the Taylor
 formula, we get the following formula for the remainder,

$$R_N(A, B, X) = \frac{1}{N!} \left(\frac{i\hbar}{2}\right)^{N+1} \int_0^1 (1-t)^N R_{N,t}(X; \hbar) dt, \quad (\text{B.1})$$

where

$$R_{N,t}(X; \epsilon) = (2\pi\bar{t})^{-2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp\left(-\frac{i}{2t\hbar}\sigma(u, v)\right) \sigma^{N+1}(D_u, D_v) A(u+X) B(v+X) dudv.$$

Notice that the integral is an oscillating integral as we shall see below. So
 we shall use the following lemma :

Lemma B.1. *There exists a constant $C_d > 0$ such that for any $F \in \mathcal{S}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ the integral*

$$I(\lambda) = \lambda^{2d} \int \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] F(u, v) dudv. \quad (\text{B.2})$$

satisfies the following estimate:

for any $M > 0$ there exists $C_M > 0$, independent of F , such that

$$|I(\lambda)| \leq C_M \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha + \beta| \leq M + 4d + 1}} (1 + |u|^2 + |v|^2)^{(4d+1-M)/2} |\partial_u^\alpha \partial_v^\beta F(u, v)| \quad (\text{B.3})$$

A proof will be given later.

Using this Lemma for $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ with the integrand

$$F_{N,\gamma}(X; u, v) = \pi^{-2d} \partial_X^\gamma (\sigma^{N+1}(D_u, D_v)A(u + X)B(v + X))$$

and the parameter $\lambda = 1/(2t\hbar)$. We then have that

$$|\partial_X^\gamma R_{N,t}(X; \varepsilon)| \leq C_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha + |\beta|| \leq 4d + 1}} |\partial_u^\alpha \partial_v^\beta F_{N,\gamma}(X; u, v)|.$$

Moreover, we have the elementary estimate

$$|\sigma^{N+1}(D_u, D_v)A(u)B(v)| \leq (2d)^{N+1} \sup_{|\alpha + |\beta|| = N+1} |\partial_x^\alpha \partial_\xi^\beta A(x, \xi) \partial_y^\beta \partial_\eta^\alpha B(y, \eta)|. \quad (\text{B.4})$$

Together with the Leibniz formula, we then get the claimed result with universal constants. For symbols $A \in \mathbb{S}_0^\mu$ and $B \in \mathbb{S}_0^\nu$ we argue by localisation. We use $A_\varepsilon(u) = e^{-\varepsilon u^2} A(u)$ and $B_\varepsilon(v) = e^{-\varepsilon v^2} B(v)$ for $\varepsilon > 0$ and pass to the limit as $\varepsilon \rightarrow 0$.

B.3 Proof of the Lemma B.1

We consider the same cut-off χ_0 as above. We split $I(\lambda)$ into two pieces and write $I(\lambda) = I_0(\lambda) + I_1(\lambda)$ with

$$I_0(\lambda) = \lambda^{2d} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] \chi_0((u^2 + v^2)) F(u, v) du dv,$$

$$I_1(\lambda) = \lambda^{2d} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \exp[-i\lambda\sigma(u, v)] (1 - \chi_0)(u^2 + v^2) F(u, v) du dv.$$

We notice that $(u, v) \mapsto \sigma(u, v)$ is a quadratic non-degenerate real form on \mathbb{R}^{4d} .

Let us estimate $I_1(\lambda)$. We can integrate by parts with the differential operator

$$L = \frac{i}{|u|^2 + |v|^2} \left(Ju \cdot \frac{\partial}{\partial v} - Jv \cdot \frac{\partial}{\partial u} \right),$$

using that $Le^{-i\lambda\sigma(u,v)} = Le^{-i\lambda Ju \cdot v} = \lambda e^{-i\lambda\sigma(u,v)}$. For $I_1(\lambda)$, the integrand is supported outside the ball of radius $1/\sqrt{2}$ in \mathbb{R}^{4d} . Performing $4d + 1$ integrations by parts for gaining enough decay to ensure integrability in $(u, v) \in \mathbb{R}^{4d}$, we get a constant c_d such that

$$|I_1(\lambda)| \leq c_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\mu| + |\nu| \leq 4d+1}} |\partial_u^\mu \partial_v^\nu F(u, v)|. \quad (\text{B.5})$$

But we need to control the behaviour for $u^2 + v^2$ large, so with M more integrations by parts we get

$$|I_1(\lambda)| \leq C_M \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\alpha| + |\beta| \leq M}} (1 + |u|^2 + |v|^2)^{(4d+1-M)/2} |\partial_u^\alpha \partial_v^\beta F(u, v)| \quad (\text{B.6})$$

To estimate $I_0(\lambda)$ we apply the stationary phase. The symmetric matrix of the quadratic form $\sigma(u, v)$ is

$$A_\sigma = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

So the stationary phase theorem ([6], Vol.I, section 7.7), noticing that the leading term in the stationary phase theorem is of order λ^{-2d} , we get

$$|I_0(\lambda)| \leq c'_d \sup_{\substack{u, v \in \mathbb{R}^{2d} \\ |\mu| + |\nu| \leq 2d+3}} |\partial_u^\mu \partial_v^\nu F(u, v)|. \quad (\text{B.7})$$

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