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When Poisson and Moyal Brackets are equal?

Didier Robert*

Abstract

In the phase space \mathbb{R}^{2d} , let us denote $\{A, B\}$ the Poisson bracket of two smooth classical observables and $\{A, B\}_\otimes$ their Moyal bracket, defined as the Weyl symbol of $i[A, B]$, where \widehat{A} is the Weyl quantization of A and $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$ (commutator). In this note we prove that if a smooth Hamiltonian H on the phase space \mathbb{R}^{2d} , with derivatives of moderate growth, satisfies $\{A, H\} = \{A, H\}_\otimes$ for any smooth and bounded observable A then H must be a polynomial of degree at most 2.

This is related with the Groenewold-van Hove Theorem [3, 4, 6] concerning quantization of polynomial observables.

1 Introduction

Let H, A, B be smooth classical observables on \mathbb{R}^{2d} in the variables $X = (x, \xi)$. The Poisson brackets is defined as $\{A, B\} = \partial_\xi A \cdot \partial_x B - \partial_x A \cdot \partial_\xi B$. So the classical time evolution of A determined by the Hamilton equation for H is solution of the equation:

$$\begin{aligned} \frac{d}{dt}A(t) &= \{A(t), H\} \\ A(0) &= A. \end{aligned} \tag{1.1}$$

The Weyl quantization \widehat{A} of A is defined as as the following operator:

$$\widehat{A}f(x) := \text{Op}_\hbar^w f(x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^{2d}} A\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)/\hbar} f(y) dy d\xi \tag{1.2}$$

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for any $f \in \mathcal{S}(\mathbb{R}^d)$.

The quantum time evolution of the quantum observable \widehat{A} must satisfy the Heisenberg equation

$$\frac{d}{dt}\widehat{A}(t) = \frac{i}{\hbar}[\widehat{A}(t), \widehat{H}] \quad (1.3)$$

$$\widehat{A}(0) = \widehat{A} \quad (1.4)$$

where $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$.

The Moyal bracket of the observables A, H is defined such that

$$\frac{i}{\hbar}[\widehat{A}, \widehat{H}] = \text{Op}_\hbar^w(\{A, H\}_\otimes). \quad (1.5)$$

Notice that it results from the Weyl quantization calculus with a small parameter \hbar that we have

$$\lim_{\hbar \searrow 0} \{A, H\}_\otimes = \{A, H\}.$$

There is an equivalent definition by introducing the Moyal product $A \circledast B$ (see the next section) such that

$$(\text{Op}_\hbar^w A)(\text{Op}_\hbar^w B) = \text{Op}_\hbar^w(A \circledast B).$$

Then we have

$$\{A, B\}_\otimes = \frac{i}{\hbar}(A \circledast B - B \circledast A).$$

These definitions make sense for $A, B \in \mathcal{S}(\mathbb{R}^{2d})$ and can be extended to suitable classes of symbols with moderate growth. To be more explicit we introduce the classes \mathbb{S}_δ^μ , for $\delta < 1$, $\mu \in \mathbb{R}$. $A \in \mathbb{S}_\delta^\mu$ iff $A \in C^\infty(\mathbb{R}^{2d})$ and for any multiindex $\gamma \in \mathbb{N}^{2d}$ we have:

$$|\partial_X^\gamma A(X)| \leq C_\gamma \langle X \rangle^{\mu + \delta|\gamma|}$$

Using Theorem A.1 in [1], we can see that $A \circledast H$ is a smooth symbol if $H \in \mathbb{S}_\delta^\mu$ and $A \in \mathbb{S}_0^0$. Our aim here is to prove the following result.

Theorem 1.1. *Assume that \hbar is fixed ($\hbar = 1$). Let be $H \in \mathbb{S}_\delta^\mu$ for some $\mu \in \mathbb{R}$ and $\delta < 1$. Assume that for any $A \in \mathbb{S}_0^0$ we have $\{A, H\}_\otimes = \{A, H\}$. Then $H(X)$ is a polynomial in $X = (x, \xi)$ of degree at most 2.*

Remark 1.2. *It is well known that if H is a polynomial of degree at most 2 then $\{A, H\}_\otimes = \{A, H\}$ for any $A \in \mathbb{S}_0^0$. I do not know any reference for a proof of the converse statement. The proof given here is a direct consequence of basic properties of the Weyl quantization.*

Remark 1.3. *The usual proofs of the Groenewold-van Hove Theorem on the phase space \mathbb{R}^{2d} concern more general quantization procedures but are restricted to polynomial symbols A, H .*

A quotation from [6] p.66-67:

"On établit ensuite qu'une correspondance biunivoque entre grandeurs classiques et quantiques, ayant le caractère d'un isomorphisme entre algèbres de Lie, existe entre les grandeurs représentées par des polynômes de degré 0, 1, 2 en les variables $p_1, \dots, p_N, q_1, \dots, q_n$ mais ne peut être étendue sans perdre ses propriétés essentielles à l'ensemble de toutes les grandeurs classiques"

The Theorem of Groenewold-van Hove is detailed p.76 and the quadratic case p.87 of [6].

Notice that the quadratic case is related with the metaplectic representation [3].

2 Weyl calculus

2.1 Introduction to the Weyl quantization

In this section, we recall some basic properties of the Weyl calculus (for more details see [5]).

Weyl quantization start by quantization of exponent of linear forms $L_Y(X) = \sigma(Y, X) = \eta \cdot x - y \cdot \xi$ with $X = (x, \xi)$, $Y = (y, \eta)$. Apart the usual properties asked for an admissible quantization, Weyl quantization is uniquely determined by imposing that the Weyl symbol of $e^{i\widehat{L}_Y}$ is e^{iL_Y} . Recall that $\widehat{T}(Y) := e^{-\frac{i}{\hbar}\widehat{L}_Y}$ is the Weyl-Heisenberg translation operator by Y in the phase space \mathbb{R}^{2d} . Then for any observable A , using a Fourier transform, the Weyl quantization \widehat{A} is defined for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, as

$$\widehat{A}\psi = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \widetilde{A}_\sigma(Y) T(Y)\psi dY$$

where $\widetilde{A}_\sigma(Y) = \int_{\mathbb{R}^{2d}} A(z) e^{-i\sigma(Y, z)} dz$ is the symplectic Fourier transform of A (in the sense of distributions). So that the family $\{T(Y)\}_{Y \in \mathbb{R}^{2d}}$ is an over-complete basis for operators between the Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$

2.2 The Moyal Product

We first recall the formal product rule for quantum observables with Weyl quantization. Let $A, B \in \mathcal{S}(\mathbb{R}^{2d})$. The Moyal product $C := A \circledast B$ is the observable C such that $\widehat{A} \cdot \widehat{B} = \widehat{C}$. Some computations with the Fourier

transform give the following well known formula [5]

$$C(x, \xi) = \exp\left(\frac{i\hbar}{2}\sigma(D_q, D_p; D_{q'}, D_{p'})\right) A(q, p)B(q', p')|_{(q,p)=(q',p')=(x,\xi)}, \quad (2.1)$$

where σ is the symplectic bilinear form $\sigma((q, p), (q', p')) = p \cdot q' - p' \cdot q$ and $D = i^{-1}\hbar\nabla$. By expanding the exponential term in a formal power series in \hbar we get

$$C(x, \xi) = \sum_{j \geq 0} \frac{\hbar^j}{j!} \left(\frac{i}{2}\sigma(D_q, D_p; D_{q'}, D_{p'})\right)^j A(q, p)B(q', p')|_{(q,p)=(q',p')=(x,\xi)}. \quad (2.2)$$

So that $C(x, \xi)$ is a formal power series in \hbar with coefficients given by

$$C_j(A, B; x, \xi) = \frac{1}{2^j} \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (D_x^\beta \partial_\xi^\alpha A) \cdot (D_x^\alpha \partial_\xi^\beta B)(x, \xi). \quad (2.3)$$

Furthermore we need a remainder estimates for the expansion of the Moyal product.

For every $N \geq 1$. we denote

$$R_N(A, B; z) := A \circledast B(z) - \sum_{0 \leq j \leq N} \hbar^j C_j(z). \quad (2.4)$$

The following estimate is a particular case of Theorem A.1 in [1] see also Remark A.3.

Lemma 2.1. *Let be $A \in \mathbb{S}_\delta^{\mu_A}$ and $B \in \mathbb{S}_\delta^{\mu_B}$ then for any $N \geq 1$, $\gamma \in \mathbb{N}^{2d}$, $M \geq M_0$ there exists $C_{N,\gamma,M} > 0$ (independent of (A, B)) such that*

$$|\partial_z^\gamma R_N(A, B; z)| \leq C_{N,\gamma,M} \hbar^{N+1} \sum_{\substack{|\alpha+\beta|=N+1 \\ |\mu+\nu| \leq M+|\gamma|}} \quad (2.5)$$

$$\sup_{u,v \in \mathbb{R}^{2d}} (1 + |u|^2 + |v|^2)^{(M_0-M)/2} |\partial_u^{(\alpha,\beta)+\mu} A(z+u)| |\partial_v^{(\beta,\alpha)+\nu} B(z+v)| \quad (2.6)$$

In particular $R_N(A, B, z) \in \mathbb{S}_\delta^{\mu_{AB}}$ for some $\mu_{AB} \geq \mu_A + \mu_B$.

For proving this Lemma one assume first that $A, B \in \mathcal{S}(\mathbb{R}^{2d})$. For the general case we put $A_\varepsilon(X) = e^{-\varepsilon|X|^2} A(X)$, $B_\varepsilon(X) = e^{-\varepsilon|X|^2} B(X)$ and pass to the limit for $\varepsilon \searrow 0$.

3 Proof of Theorem(1.1)

Here $\hbar = 1$. It is enough to consider the test observables $T_Y := e^{-iLY}$ ($Y \in \mathbb{R}^{2d}$).

We have

$$\widehat{T}_Y \widehat{H} \widehat{T}_Y^* = [\widehat{T}_Y, \widehat{H}] \widehat{T}_Y^* + \widehat{H}$$

Using the assumption of Theorem(1.1) we get

$$\frac{1}{i}(\{T_Y^*, H\} \otimes T_Y)(X) = H(X+Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.1)$$

Computing the Poisson bracket in (3.1) gives

$$(((y \cdot \partial_x H + \eta \cdot \partial_\xi H) T_Y^*) \otimes T_Y)(X) = H(X+Y) - H(X), \forall X, Y \in \mathbb{R}^{2d}. \quad (3.2)$$

Our aim is to prove that (3.2) implies that $H(X)$ is a polynomial of degree at most 2. For that purpose we shall compute the asymptotic expansion as $Y \rightarrow 0$ of the left hand side of (3.2) and compare it with the Taylor expansion for $H(X+Y)$ modulo $O(|Y|^4)$. From that we shall conclude that all the third order derivatives of H vanish in X hence the conclusion will follow.

We have

$$\partial_x^\alpha \partial_\xi^\beta T_Y = i^{-|\alpha+\beta|} \eta^\alpha y^\beta T_Y$$

Let us denote by $C(X, Y)$ the left hand side in (3.2). So using Lemma(2.1) we have

$$C(X, Y) = \sum_{0 \leq j \leq 2} (C_j(X, Y) + O(|Y|^4)),$$

where

$$C_0(X, Y) = Y \cdot \nabla_X H(X) \quad (3.3)$$

$$C_1(X, Y) = \frac{1}{2} Y \cdot \nabla_X^2 H(X) Y, \quad (3.4)$$

where $\nabla_X^2 H(X)$ is the Hessian matrix of H .

Let us compute now $C_2(X, Y)$, which is an homogeneous polynomials of degree 3 in Y .

For simplicity let us consider the 1-D case. The same computation can clearly be done for $d > 1$.

Using (2.2) we get with $Y = (y, \eta)$,

$$C_2(X, Y) = \frac{1}{8} (y^3 \partial_x^3 H + \eta^3 \partial_\eta^3 H - y^2 \eta \partial_\xi \partial_x^2 H - y \eta^2 \partial_\xi^2 \partial_x H). \quad (3.5)$$

According (3.2), $C_2(X, Y)$ must coincide with the term of order 3 in Y of the Taylor expansion in X for $H(X+Y) - H(X)$. But this is possible only if $\partial_x^3 H = \partial_\eta^3 H = \partial_\xi \partial_x^2 H = \partial_\xi^2 \partial_x H = 0$ for any $(x, \xi) \in \mathbb{R}^2$. So H must be a polynomial of degree ≤ 2 . \square

4 Extension to polynomials of arbitrary degree

The asymptotic expansion in \hbar in the Moyal product suggest to introduce the following semi-classical approximations of the Moyal bracket:

$$\{A, B\}_{\otimes, m} = \{A, B\} + \hbar^2 \{A, B\}_3 + \cdots + \hbar^{2m} \{A, B\}_{2m+1},$$

where $\{A, B\}_j = \frac{i}{\hbar}(C_j(A, B) - C_j(B, A))$ (notation of (2.3)). Notice that $\{A, B\}_j = 0$ for j even.

It is clear that if H is a polynomial of degree at most $2m + 2$ then we have $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$ for any A . Conversely we have

Theorem 4.1. *Assume $\hbar = 1$. If for any $A \in \mathbb{S}_0^0$ we have $\{A, H\}_{\otimes, m} = \{A, H\}_{\otimes}$ then H must be a polynomial of degree at most $2m + 2$.*

Proof. Here we give a proof different from the case $m = 0$, without connection with the Taylor formula, for simpler computations.

Using Lemma 2.1 we have

$$T_Y^* (\{T_Y, H\}_{\otimes} - \{T_Y, H\}_{\otimes, m}) = \mathcal{O}(|Y|^{2m+3}), \quad Y \rightarrow 0. \quad (4.1)$$

Moreover from (2.3) we get:

$$T_Y^* \{T_Y, H\}_{2j+1} = \frac{1}{2^{j+1}} \sum_{|\alpha+\beta=2j+1} \frac{y^\alpha \eta^\beta}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta H \quad (4.2)$$

Using the assumption of Theorem 4.1. and (4.1) we get that

$$T_Y^* \{T_Y, H\}_{2m+3} = \mathcal{O}(|Y|^{2m+5})$$

But $T_Y^* \{T_Y, H\}_{2m+3}$ is an homogeneous polynomial of degree $2m + 3$ in Y so we get that this polynomial is 0 and from (4.2) we get that $\partial_x^\alpha \partial_\xi^\beta H = 0$ for $|\alpha + \beta| = 2m + 3$. Then we can conclude that $H(X)$ is a polynomial of degree at most $2m + 2$ in $X \in \mathbb{R}^{2d}$. \square .

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