



HAL
open science

Homogeneous superstrings with retract $CP^{1|4}$

Mikhail Bashkin

► **To cite this version:**

Mikhail Bashkin. Homogeneous superstrings with retract $CP^{1|4}$. Communications in Mathematics, In press, Volume 30 (2022), Issue 3 (Special issue: in memory of Arkady Onishchik), 10.46298/cm.9842 . hal-03726583v2

HAL Id: hal-03726583

<https://hal.science/hal-03726583v2>

Submitted on 4 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Homogeneous superstrings with retract $\mathcal{CP}^{1|4}$

Mikhail Bashkin

Abstract. Any complex-analytic supermanifold whose retract is diffeomorphic to the complex projective superline (superstring) $\mathcal{CP}^{1|4}$ is, up to a diffeomorphism, either a member of a 1-parameter family or one of 9 exceptional supermanifolds. I singled out the homogeneous of these supermanifolds and described Lie superalgebras of vector fields on them.

1 Preliminaries

Let (M, \mathcal{F}) be a complex-analytic manifold of dimension m . (More precisely, almost complex manifold, see [BGLS*], since the vanishing of the Nijenhuis tensor is never need; however, from the very beginning (see [Gr]) one requires the underlying manifold to be complex. *This comment and starred references are added by the editor of this Special Volume. D.Leites.*)

Let \mathbf{E} be a vector bundle of rank n over M , and \mathcal{E} the be locally free analytic sheaf of sections of \mathbf{E} . Set $\tilde{\mathcal{O}} := \Lambda_{\mathcal{F}}^{\bullet}(\mathcal{E})$.

The supermanifold isomorphic to the one of the form $\mathcal{M} := (M, \tilde{\mathcal{O}})$ is called *split*. The ringed space locally isomorphic to $(M, \Lambda_{\mathcal{F}}^{\bullet}(\mathcal{E}))$ is called a *supermanifold* of *superdimension* $m|n$. Physicists call supermanifolds of dimension $1|n$ *superstrings*, see [W*]. Let \mathcal{O} be a structure sheaf of any supermanifold. Let $\mathcal{I} \subset \mathcal{O}$ be the subsheaf of ideals generated by subsheaf $\mathcal{O}_{\bar{1}}$ and let $\mathcal{O}_{\text{rd}} := \mathcal{O}/\mathcal{I}$.

Consider the following filtration of \mathcal{O} by powers of \mathcal{I}

$$\mathcal{O} = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \dots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.$$

The graded sheaf $\text{gr } \mathcal{O} = \bigoplus_{p=0}^n \text{gr}_p \mathcal{O}$ with $\text{gr}_p \mathcal{O} := \mathcal{I}^p / \mathcal{I}^{p+1}$ defines the split supermanifold $(M, \text{gr } \mathcal{O})$ called the *retract* of (M, \mathcal{O}) .

Let $\pi : \mathcal{I}^p \rightarrow \text{gr}_p \mathcal{O}$ denote the natural projection. Then, we have the exact sequences of sheaves

$$0 \longrightarrow \mathcal{I}^{p+1} \longrightarrow \mathcal{I}^p \xrightarrow{\pi_p} \text{gr}_p \mathcal{O} \longrightarrow 0. \tag{1}$$

MSC 2020: Primary 32C11 Secondary 81T30

Keywords: Homogeneous supermanifold, split supermanifold, superstring.

Affiliation: P.A. Solovyov Rybinsk State Aviation Technical University, Rybinsk, Russia

E-mail: mbashkin@rsatu.ru

The supermanifold (M, \mathcal{O}) is split if and only if there exists an isomorphism of the superalgebra sheaves $h : \text{gr } \mathcal{O} \rightarrow \mathcal{O}$ such that its restriction $h_p : \text{gr}_p \mathcal{O} \rightarrow \mathcal{I}^p$ splits the sequence (1), i.e., satisfies $\pi_p \circ h_p = \text{id}$. Such an isomorphism exists in a neighborhood of any point of M .

Let (M, \mathcal{O}) be a supermanifold and \mathfrak{g} a complex finite-dimensional Lie superalgebra. An *action* of \mathfrak{g} on (M, \mathcal{O}) is an arbitrary Lie superalgebra homomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{v}(M, \mathcal{O})$. Then, a linear mapping $\varphi^x : \mathfrak{g} \rightarrow T_x(M, \mathcal{O})$ is associated with any $x \in M$. The action φ is called *transitive* if φ^x is surjective for any $x \in M$. By restricting the action $\varphi : \mathfrak{g} \rightarrow \mathfrak{v}(M, \mathcal{O})$ to the even component we get a homomorphism $\varphi_{\bar{0}} : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$. If M is compact, then it is possible to integrate $\varphi_{\bar{0}}$ getting a homomorphism $\Phi : G \rightarrow \text{Aut}(M, \mathcal{O})$, where G is the simply connected complex Lie group whose Lie algebra is $\mathfrak{g}_{\bar{0}}$. This homomorphism induces a homomorphism $\Phi_0 : G \rightarrow \text{Bih } M$ into the group of biholomorphic automorphisms of M , in other words — an action of G on M . The action φ is said to be $\bar{0}$ -*transitive* if Φ_0 is transitive.

If a Lie group G acts $\bar{0}$ -transitivity on M , then $\varphi^x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M)$ is surjective for any $x \in M$. Conversely, if M is compact and $\varphi^x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M)$ is surjective for any $x \in M$, we can integrate this action to a $\bar{0}$ -transitive action of a Lie group.

The supermanifold (M, \mathcal{O}) is called *homogeneous* (resp. $\bar{0}$ -*homogeneous*) if the natural action of the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ on (M, \mathcal{O}) is transitive (resp. $\bar{0}$ -transitive), see [BO2], [O3]. This means that the evaluation mapping $ev_x : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_x(M, \mathcal{O})$ (resp. the restriction of ev_x to $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$) is surjective for any $x \in M$.

Thanks to [O3] we know that φ is transitive if and only if it is $\bar{0}$ -transitive, M is compact, and the mapping $\varphi_{\bar{1}}^x : \mathfrak{g}_{\bar{1}} \rightarrow T_{x_0}(M, \mathcal{O})_{\bar{1}}$ is surjective at a certain point $x_0 \in M$. This implies that a $\bar{0}$ -homogeneous supermanifold is homogeneous if and only if the odd component of the mapping $ev_{x_0} : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_{x_0}(M, \mathcal{O})$ is surjective at a certain point $x_0 \in M$.

One easily proves (see [BO2]) that the retract of a homogeneous supermanifold (M, \mathcal{O}) with compact M is homogeneous, too.

In what follows, I consider the problem of classification (up to a diffeomorphism) of supermanifolds with retract $\mathcal{CP}^{1|4}$ and describe which of the supermanifolds considered are homogeneous or at least $\bar{0}$ -*homogeneous*.

2 Superstring $\mathcal{CP}^{1|4}$. The first cohomology of the tangent sheaf

Over \mathbb{CP}^1 , consider the holomorphic vector bundle

$$\mathbf{E} = \mathbf{L}_{-k_1} \oplus \mathbf{L}_{-k_2} \oplus \mathbf{L}_{-k_3} \oplus \mathbf{L}_{-k_4}, \text{ where } k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0.$$

Let $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4} := (\mathbb{CP}^1, \mathcal{O}_{\Lambda_{\mathcal{F}}(\mathcal{E})})$ designate the split supermanifold determined by \mathbf{E} . As shown in [BO2], if $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4}$ is homogeneous, then the k_i must be non-negative.

Let us cover \mathbb{CP}^1 by two charts U_0 and U_1 with local coordinates x and $y = \frac{1}{x}$, respectively. For $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4}$, the transition functions in $U_0 \cap U_1$ are $y = x^{-1}$ and $\eta_i = x^{-k_i} \xi_i$ for $i = 1, \dots, 4$, where ξ_i and η_i are basis sections of \mathbf{E} over U_0 and U_1 , respectively.

If \mathcal{O}_{gr} is the structure sheaf of $\mathcal{CP}^{1|4} := \mathcal{CP}_{1111}^{1|4}$, then $\mathcal{T}_{\text{gr}} := \mathcal{D}er \mathcal{O}_{\text{gr}}$ is the *tangent sheaf* (or the sheaf of vector fields). This is a sheaf of Lie superalgebras. The sections of the tangent sheaf are *holomorphic vector fields* on the supermanifold. Their sections are elements of the Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}}) := \Gamma(\mathbb{CP}^1, \mathcal{T}_{\text{gr}})$ of vector fields on $\mathcal{CP}^{1|4}$.

The sheaf \mathcal{T}_{gr} has a \mathbb{Z} -grading $\mathcal{T}_{\text{gr}} = \bigoplus_{-1 \leq p \leq 4} (\mathcal{T}_{\text{gr}})_p$, where

$$(\mathcal{T}_{\text{gr}})_p := \mathcal{D}er_p \mathcal{O}_{\text{gr}} = \{v \in \mathcal{T}_{\text{gr}} \mid v((\mathcal{O}_{\text{gr}})_q) \subset (\mathcal{O}_{\text{gr}})_{p+q} \text{ for any } q\}.$$

The Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})$ becomes also \mathbb{Z} -graded with the induced grading compatible with $\mathbb{Z}/2$ -grading by parity.

We can regard \mathcal{T}_{gr} as a locally free analytic sheaf on \mathbb{CP}^1 . From [O3] we have the following exact sequence of locally free analytic sheaves on \mathbb{CP}^1 :

$$0 \longrightarrow \mathcal{E}^* \otimes \Lambda^* \mathcal{E} \xrightarrow{i} \mathcal{T}_{\text{gr}} \xrightarrow{\alpha} \Theta \otimes \Lambda^* \mathcal{E} \longrightarrow 0,$$

where $\Theta = \mathcal{D}er \mathcal{F}$ is the tangent sheaf of the manifold \mathbb{CP}^1 , and \mathcal{F} is the sheaf of functions on \mathbb{CP}^1 . The mapping α is the restriction of a derivation of \mathcal{O} to \mathcal{F} , and i identifies any sheaf homomorphism $\mathcal{E} \rightarrow \Lambda^* \mathcal{E}$ with its prolongation to a derivation that vanishes on \mathcal{F} . Hence, the analytic sheaf \mathcal{T}_{gr} is locally free. Therefore, \mathcal{T}_{gr} is the sheaf of holomorphic sections of a holomorphic vector bundle over \mathbb{CP}^1 . We call it the *supertangent bundle* and denote **ST**.

Thanks to the Bott-Borel-Weil theorem the following theorem holds.

2.1 Theorem ([BO1]). *For $\dim H^p(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_q)$, see the following table*

q	-1	0	1	2	3	4
$p = 0$	8	19	8	0	0	0
$p = 1$	0	0	0	10	8	1

The group $\text{SL}_2(\mathbb{C})$ trivially acts on $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$.

Set

$$\begin{aligned} \delta &:= \xi_1 \xi_2 \xi_3 \xi_4, & \delta_l &:= \frac{\partial \delta}{\partial \xi_l}, & \nabla &:= \sum_{1 \leq i \leq 4} \xi_i \partial_{\xi_i}; \\ \delta' &:= \eta_1 \eta_2 \eta_3 \eta_4, & \delta'_l &:= \frac{\partial \delta'}{\partial \eta_l}, & \nabla' &:= \sum_{1 \leq i \leq 4} \eta_i \partial_{\eta_i}. \end{aligned} \tag{2}$$

The next Theorem expounds the result of Theorem 2.1 by giving the Čzech cocycles of the covering $\{U_0, U_1\}$; these cocycles determine the bases of non-zero spaces H^1 .

2.2. Theorem. *The basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_p)$, where $p = 2, 3, 4$, can be represented by the following cocycles z_{01} :*

p	z_{01}
2	$x^{-1} \delta_l \partial_{\xi_k} \sim x^{-1} \delta_k \partial_{\xi_l}$ for $l < k$ and $k, l = 1, \dots, 4$, $x^{-1} \delta_r \partial_{\xi_r}$ for $r = 1, \dots, 4$;
3	$x^{-1} \delta_l \partial_x \sim x^{-2} \delta \partial_{\xi_l}$ and $x^{-1} \delta \partial_{\xi_l}$ for $l = 1, \dots, 4$;
4	$x^{-1} \delta \partial_x$.

Proof. Let us find a basis in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$. For this, consider the part $\mathbf{ST}(\mathbf{E})_2$ of the tangent space $\mathbf{ST}(\mathbf{E}) := \bigoplus \mathbf{ST}(\mathbf{E})_i$. In [BO2], the following decomposition was considered:

$$\mathbf{ST}(\mathbf{E})_2 = \bigoplus_{i < j} \mathbf{ST}(\mathbf{E})_2^{(ij)} \oplus \bigoplus_{i < j < k, r \neq i, j, k} \mathbf{ST}(\mathbf{E})_2^{(ijk, r)}, \text{ where}$$

$$\begin{aligned} \mathbf{ST}(\mathbf{E})_2^{(ij)} &= \langle \xi_i \xi_j \partial_x \text{ and } \xi_i \xi_j \xi_k \partial_{\xi_k}, \text{ where } i < j, \text{ and } k \neq i, j \rangle, \\ \mathbf{ST}(\mathbf{E})_2^{(ijk, r)} &= \langle \xi_i \xi_j \xi_k \partial_{\xi_r}, \text{ where } i < j < k, \text{ and } r \neq i, j, k \rangle \end{aligned}$$

Moreover, it was shown in [BO2] that $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk, r)} \simeq \mathbf{L}_{-2}$. Consider the bundles $\mathbf{ST}(\mathbf{E})_2^{(ij)}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk, r)}$ separately. Let $(\mathcal{T}_{\text{gr}})_p^{i_1 \dots i_k}$ designate the sheaf of holomorphic sections of $\mathbf{ST}(\mathbf{E})_p^{(i_1 \dots i_k)}$. We see that (recall notation (2))

$$\begin{aligned} \delta_l \partial_{\xi_k} &= y^{-2} \delta'_l \partial_{\eta_k}, \text{ where } l < k, \quad k \neq i, j, \quad l \neq i, j, \\ \xi_i \xi_j \partial_x &= -\eta_i \eta_j \partial_y - y^{-1} \eta_i \eta_j \nabla' = -y^{-1} (\eta_i \eta_j \nabla' + y \eta_i \eta_j \partial_y), \\ \xi_i \xi_j \nabla + x \xi_i \xi_j \partial_x &= -y^{-1} \xi_i \xi_j \partial_x. \end{aligned}$$

Hence, for the basis sections of \mathbf{L}_{-2} (resp. \mathbf{L}_{-1}) we can take

$$\delta_l \partial_{\xi_k}, \quad \xi_i \xi_j \partial_x, \text{ (resp. } \xi_i \xi_j \nabla + x \xi_i \xi_j \partial_x \text{) for all } i, j, k, l.$$

Then, $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $(\mathcal{T}_{\text{gr}})_2^{ij} \simeq \mathcal{F}(-2) \oplus 2\mathcal{F}(-1)$, where \mathcal{F} is the sheaf of holomorphic functions on \mathbb{CP}^1 , is the corresponding isomorphism of sheaves.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2)) \quad \text{and} \quad \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ij}) = 1,$$

so the cocycle desired is of the form

$$x^{-1} \delta_l \partial_{\xi_k}.$$

Let us show that $x^{-1} \delta_l \partial_{\xi_k} \sim x^{-1} \delta_k \partial_{\xi_l}$.

Indeed,

$$\begin{aligned} x^{-1} \delta_l \partial_{\xi_k} &= y^{-1} \delta'_l \partial_{\eta_k} \sim y^{-1} \delta'_l \partial_{\eta_k} + \eta_s \eta_t \partial_y \\ &= x^{-1} \delta_l \partial_{\xi_k} - \xi_s \xi_t \partial_x - x^{-1} \delta_l \partial_{\xi_k} + x^{-1} \delta_k \partial_{\xi_l} \sim x^{-1} \delta_k \partial_{\xi_l}, \end{aligned}$$

where $(l, k; s, t) \in \{(1, 2; 3, 4), (1, 4; 2, 3), (2, 3; 1, 4), (3, 4; 1, 2)\}$;

$$\begin{aligned} x^{-1} \delta_l \partial_{\xi_k} &= y^{-1} \delta'_l \partial_{\eta_k} \sim y^{-1} \delta'_l \partial_{\eta_k} - \eta_s \eta_t \partial_y \\ &= x^{-1} \delta_l \partial_{\xi_k} + \xi_s \xi_t \partial_x - x^{-1} \delta_l \partial_{\xi_k} + x^{-1} \delta_k \partial_{\xi_l} \sim x^{-1} \delta_k \partial_{\xi_l}, \end{aligned}$$

where $(l, k; s, t) \in \{(1, 3; 2, 4), (2, 4; 1, 3)\}$.

Since $\delta_r \partial_{\xi_r} = y^{-2} \delta'_r \partial_{\eta_r}$, then take $\delta_r \partial_{\xi_r}$ for a basis section of \mathbf{L}_{-2} . We have

$$\mathbf{ST}(\mathbf{E})_2^{(ijk, r)} \simeq \mathbf{L}_{-2}$$

and the corresponding isomorphism of sheaves $(\mathcal{T}_{\text{gr}})_2^{ijk,r} \simeq \mathcal{F}(-2)$.

The results of [BO2] imply that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ijk,r}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2))$,

$$\dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ijk,r}) = 1 \text{ for any } i < j < k, \text{ and } r \neq i, j, k,$$

and the cocycle desired is of the form

$$x^{-1} \delta_r \partial_{\xi_r}.$$

Let us now find the basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3)$. Consider $\mathbf{ST}(\mathbf{E})_3$ and again apply the results of [BO2]; we get

$$\mathbf{ST}(\mathbf{E})_3 = \bigoplus_{i < j < k} \mathbf{ST}(\mathbf{E})_3^{(ijk)},$$

where $\mathbf{ST}(\mathbf{E})_3^{(ijk)}$ is spanned by

$$\begin{aligned} & \xi_i \xi_j \xi_k \partial_x, \quad \delta \partial_{\xi_l}, \text{ where } 1 \leq i < j < k \leq 4, \quad l \neq i, j, k, \quad l \in \{1, \dots, 4\}; \\ & \xi_i \xi_j \xi_k \partial_x = -y^{-2} (y \eta_i \eta_j \eta_k \partial_y + \eta_i \eta_j \eta_k \eta_l \partial_{\eta_l}), \\ & \delta \partial_{\xi_l} - x \delta_l \partial_x = y^{-3} \delta'_l \partial_{\eta_l} + y^{-2} \delta'_l \partial_y - y^{-3} \delta'_l \partial_{\eta_l} = y^{-2} \delta'_l \partial_y. \end{aligned}$$

Hence, take $\xi_i \xi_j \xi_k \partial_x$ and $\delta \partial_{\xi_l} - x \delta_l \partial_x$ for basis sections of \mathbf{L}_{-2} and \mathbf{L}_{-1} .

We see that $\mathbf{ST}(\mathbf{E})_3^{(ijk)} \simeq 2\mathbf{L}_{-2}$ and $(\mathcal{T}_{\text{gr}})_3^{ijk} \simeq 2\mathcal{F}(-2)$.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3^{ijk}) \simeq 2H^1(\mathbb{CP}^1, \mathcal{F}(-2)), \text{ and } \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3^{ijk}) = 2,$$

where $1 \leq i < j < k \leq 4$, and the cocycles desired are of the form

$$x^{-1} \delta_l \partial_x \text{ and } x^{-1} \delta \partial_{\xi_l} - \delta_l \partial_x \sim x^{-1} \delta \partial_{\xi_l}, \quad \text{where } l \neq i, j, k.$$

Let us show that

$$x^{-1} \delta_l \partial_x \sim x^{-2} \delta \partial_{\xi_l}.$$

Indeed, $x^{-1} \delta_l \partial_x = -\delta'_l \partial_y + y^{-1} \delta'_l \partial_{\eta_l} \sim y^{-1} \delta'_l \partial_{\eta_l} = x^{-2} \delta \partial_{\xi_l}$.

In [BO1], it is proved that the basis element of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ can be represented by the cocycle $x^{-1} \delta \partial_x$. \square

3 Non-split supermanifolds with retract $\mathcal{CP}^{1|4}$

The structure sheaf of the split supermanifold $(M, \mathcal{O}_{\text{gr}})$ is endowed with a \mathbb{Z} -grading

$$\mathcal{O}_{\text{gr}} = \bigoplus_{0 \leq p \leq n} (\mathcal{O}_{\text{gr}})_p, \quad \text{where } (\mathcal{O}_{\text{gr}})_p = \Lambda_{\mathcal{F}}^p(\mathcal{E}).$$

Clearly, $(\mathcal{O}_{\text{gr}})_{\text{rd}}$ is naturally isomorphic to the subsheaf $\mathcal{F} \subset \mathcal{O}_{\text{gr}}$.

Observe that the natural filtration of the sheaf $\mathcal{T} = \mathcal{D}er \mathcal{O}$ yields the following filtration

$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) = \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(-1)} \supset \dots \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(4)} \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(5)} = 0,$$

where $\mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p)} = \Gamma(\mathbb{C}\mathbb{P}^1, \mathcal{T}_{(p)})$.

Thanks to results in [O1], we have the following exact sequence

$$0 \rightarrow \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p+1)} \rightarrow \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p)} \xrightarrow{\sigma_p} \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O}_{\text{gr}})_p \quad \text{for any } p \geq -1.$$

We say that $u \in \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O}_{\text{gr}})_p$ is *liftable* in $(\mathbb{C}\mathbb{P}^1, \mathcal{O})$, if $u \in \text{Im } \sigma_p$.

Consider

$$\text{Aut}_{(2)}\mathcal{O}_{\text{gr}} = \{a \in \text{Aut } \mathcal{O}_{\text{gr}} \mid a(f) - f \in \mathcal{J}^2 \text{ for any } f \in \mathcal{O}_{\text{gr}}\}.$$

Let $\text{Aut } \mathbf{E}$ be the group of fiber-preserving automorphisms of \mathbf{E} . Then, the following theorem holds.

3.1 Theorem ([Gr]). *There is a bijective correspondence between the isomorphism classes of supermanifolds (M, \mathcal{O}) such that $\text{gr } \mathcal{O} \simeq \mathcal{O}_{\text{gr}}$ and the orbits of the action of $\text{Aut } \mathbf{E}$ on $H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$, and $(M, \mathcal{O}_{\text{gr}})$ corresponds to the unit class e in $H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.*

4 On $\bar{0}$ -homogeneity of supermanifolds with retract $\mathcal{C}\mathcal{P}^{1|4}$

Recall a fine resolution of the sheaf $\mathcal{T}_{\text{gr}} = \text{Der } \mathcal{O}_{\text{gr}}$ endowed with a bracket operation that extends the bracket given in \mathcal{T}_{gr} . Let us denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued forms of type (p, q) on M . We form the standard Dolbeault-Serre resolution $\widehat{\Phi}$ of \mathcal{O}_{gr} by setting for any $\varphi \in \Phi^{0,q}$ and $u \in (\mathcal{O}_{\text{gr}})_p$

$$\widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{\text{gr}})_p, \quad \widehat{\Phi}^{\bullet,\bullet} := \bigoplus_{p,q \geq 0} \widehat{\Phi}^{p,q}, \quad \bar{\partial}(\varphi \otimes u) = \bar{\partial}(\varphi) \otimes u.$$

Then, regarding \mathcal{S} as a sheaf of graded algebras with respect to the total degree, consider the sheaf of bigraded Lie superalgebras $\widehat{\mathcal{T}} = \text{Der } \widehat{\Phi}$. Clearly, $\bar{D} = \text{ad}_{\bar{\partial}}$ is a derivation of bidegree $(0, 1)$ of $\widehat{\mathcal{T}}$ satisfying $\bar{D}^2 = 0$. Set

$$\mathcal{S} := \{u \in \widehat{\mathcal{T}} \mid u(\bar{f}) = 0 \text{ and } u(df) = 0 \text{ for any } f \in \mathcal{F}\}.$$

It is easy to see that \mathcal{S} is a \bar{D} -invariant subsheaf of bigraded subalgebras of $\widehat{\mathcal{T}}$. Moreover, $\widehat{\mathcal{T}}_{\text{gr}}$ is identified with the kernel of the mapping $\bar{D} : \mathcal{S}^{\bullet,0} \rightarrow \mathcal{S}^{\bullet,1}$. Thus, we get the sequence

$$0 \longrightarrow \widehat{\mathcal{T}} \xrightarrow{\tau} \mathcal{S}^{\bullet,0} \xrightarrow{\bar{D}} \mathcal{S}^{\bullet,1} \xrightarrow{\bar{D}} \dots$$

Let us specify an explicit form of τ . Let \mathcal{F}^∞ be a sheaf of differentiable complex-valued functions on M , then $\mathcal{O}_{\text{gr}}^\infty = \mathcal{O}_{\text{gr}} \otimes \mathcal{F}^\infty$ and

$$\begin{aligned} \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty = \\ \{a \in \text{Aut } \mathcal{O}_{\text{gr}}^\infty \mid a(\bar{f}) = \bar{f} \text{ for any } f \in \mathcal{F}; a(u) - u \in \bigoplus_{k \geq 2} (\mathcal{O}_{\text{gr}}^\infty)_k \text{ for any } u \in \mathcal{O}_{\text{gr}}^\infty\}. \end{aligned}$$

If $z = (z_{ij}) \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ is a cocycle in the covering \mathfrak{U} , then $z_{ij} = h_i^{-1}h_j$, where $h_i : \mathcal{O}_{\text{gr}}|_{U_i} \rightarrow \mathcal{O}|_{U_i}$. On the other hand, $z_{ij} = a_i^{-1}a_j$, where $a_i \in \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty(U_i)$. Then,

over $U_i \cap U_j$, we have $h_i^{-1}h_j = a_i^{-1}a_j$, and hence $\varrho := a_i h_i^{-1} = a_j h_j^{-1}$ is an injective sheaf homomorphism $\mathcal{O} \rightarrow \mathcal{O}_{\text{gr}}^\infty$. Then, $\tau : \widehat{\mathcal{T}} \rightarrow \mathcal{S}^{*,0}$ is defined by the formula $\tau(v) := \varrho v \varrho^{-1}$.

Now, let the bundle \mathbf{E} correspond to the supermanifold $\mathcal{CP}^{1|4}$; let the tangent bundle $\mathbf{ST}(\mathbf{E})$ be endowed with a smooth SU_2 -invariant hermitian metric (see [O1]). Since \mathbb{CP}^1 is compact, we can apply the Hodge theory. In [O1], [O3], [O2] a complex (\mathcal{S}, \bar{D}) is constructed which can be considered as a complex of $(0, *)$ -forms with values in the bundle $\mathbf{ST}(\mathbf{E})$.

Let $\mathbf{H} \subset \mathcal{S}$ denote the bigraded space of harmonic elements, H the orthogonal projection to \mathbf{H} . As is known,

$$\mathbf{H}_{p,q} \simeq H^{p,q}(\mathcal{S}, \bar{D}) \simeq H^q(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_p) \text{ for any } p, q \geq 0. \quad (3)$$

Set

$$\mathbf{H}_{(1)} = \bigoplus_{p \geq 1} \mathbf{H}_{2p,1} = \mathbf{H}_{2,1} \oplus \mathbf{H}_{4,1} \simeq H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_2) \oplus H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_4).$$

The SU_2 -invariance of the metric implies SU_2 -equivariance of \mathbf{H} , and isomorphisms (3).

4.1. Theorem. *Let $(\mathbb{CP}^1, \mathcal{O})$ be any supermanifold with retract $\mathcal{CP}^{1|4}$. Then, the SU_2 -action on \mathbb{CP}^1 can be lifted to $(\mathbb{CP}^1, \mathcal{O})$. In particular, $(\mathbb{CP}^1, \mathcal{O})$ is $\bar{0}$ -homogeneous.*

Proof. Consider the non-linear complex $K = (K^0, K^1, K^2)$ (see [BO2]), where

$$\begin{aligned} K^0 &= \Gamma(\mathbb{CP}^1, \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty), \\ K^q &= \bigoplus_{k \geq 1} S^{2k,q} \text{ for } q = 1, 2, \end{aligned}$$

with the coboundary operators $\delta_q : K^q \rightarrow K^{q+1}$ for $q = 0, 1$, and the action ρ of the group K^0 on K^1 , defined by the formulas

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1}, \\ \delta_1(u) &= \bar{\partial}u - \frac{1}{2}[u, u], \\ \rho(a)(u) &= a(u - \bar{\partial})a^{-1} + \bar{\partial}. \end{aligned}$$

By definition, the corresponding set of 1-cohomology has the form $Z^1(K)/\rho(K^0)$, where $Z^1(K) = \{u \in K^1 \mid \delta_1(u) = 0\}$.

Since $\dim \mathbb{CP}^1 = 1$, it follows that $\mathbf{H}_{(1)} \subset Z^1(K)$. Moreover, as is shown in [O2], the natural map $\mathbf{H}_{(1)} \rightarrow H^1(K)$ is surjective.

Further, $H^0(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2) = 0$ by Theorem 2.1. Hence, Theorem 3.13 from [O2] is applicable implying that this map is bijective. Thus, the bijection $\mathbf{H}_{(1)} \rightarrow H^1(K)$ is SU_2 -invariant.

On the other hand, Theorem 2.1 implies that SU_2 acts on $\mathbf{H}_{(1)}$ trivially. Hence, it acts on $H^1(K)$ also trivially, and every cohomology class contains an invariant cocycle.

Applying the obtained in [O3] criterion for lifting the action of the compact groups on the non-split supermanifold we see that the SU_2 -action on $\mathcal{CP}^{1|4}$ can be lifted to any supermanifold $(\mathbb{CP}^1, \mathcal{O})$ with $\mathcal{CP}^{1|4}$ as its retract. Since SU_2 transitively acts on \mathbb{CP}^1 , all these supermanifolds are $\bar{0}$ -homogeneous. \square

5 Description of supermanifolds via cocycles

Thanks to Theorem 3.1, the classes of isomorphic supermanifolds (M, \mathcal{O}) are in bijective correspondence with the $\text{Aut } \mathbf{E}$ -orbits on the set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. Let $M = \mathbb{C}\mathbb{P}^1$ and the odd dimension of supermanifolds $(\mathbb{C}\mathbb{P}^1, \mathcal{O})$ be ≤ 5 .

As in [O2], define the *exponential map*

$$\exp : \mathcal{T}_{\text{gr}} = (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4 \longrightarrow \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}},$$

its inverse $\log := \exp^{-1}$, and the map

$$\lambda_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}} \longrightarrow (\mathcal{T}_{\text{gr}})_{2p}$$

sending any $a \in \mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}}$ to the $2p$ -component $(\log a)_{2p}$ of $\log a$.

The map \exp is an isomorphism of sheaves of sets, λ_{2p} is a surjective homomorphism of sheaves of groups. In what follows, we will represent the cocycle $g = \exp u$ by the cocycle $u = u_2 + u_4$, where

$$u_2 \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_2) \text{ and } u_4 \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_4). \quad (4)$$

Since

$$(\mathcal{T}_{\text{gr}})_2 \cdot (\mathcal{T}_{\text{gr}})_4 = (\mathcal{T}_{\text{gr}})_4 \cdot (\mathcal{T}_{\text{gr}})_4 = 0,$$

it follows that $(\mathcal{T}_{\text{gr}})_4$ is a central ideal in \mathcal{T}_{gr} .

The exact sequence (see [O2])

$$e \longrightarrow \mathcal{A}ut_{(2p+2)}\mathcal{O} \longrightarrow \mathcal{A}ut_{(2p)}\mathcal{O} \xrightarrow{\lambda_{2p}} (\mathcal{T}_{\text{gr}})_{2p} \longrightarrow 0$$

yields — for $p = 2$ — an isomorphism of the sheaves of groups

$$\exp : (\mathcal{T}_{\text{gr}})_4 \longrightarrow \mathcal{A}ut_{(4)}\mathcal{O}_{\text{gr}} \subset \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}.$$

Hence, $\mathcal{A}ut_{(4)}\mathcal{O}_{\text{gr}}$ belongs to the center of $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$.

Define the action of the sheaf of groups $(\mathcal{T}_{\text{gr}})_4$ on $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$ by mean of right shifts

$$\Psi: v \mapsto t_v : z \mapsto z(\exp v), \text{ where } v \in (\mathcal{T}_{\text{gr}})_4, \text{ and } z \in \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}.$$

Let us translate this action to Čzech cocycles of the covering U_0, U_1 . Let us check if it is also well defined on cohomology. Let $v' \sim v$ and $z' \sim z$, where

$$v', v \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_4), \quad z', z \in Z^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}).$$

Then, $v' = b^{(0)} + v - b^{(1)}$ and $z' = c^{(0)}z(c^{(1)})^{-1}$, where $b^{(i)}, c^{(i)}$ are holomorphic sections over U_i for $i = 0, 1$ of the sheaves $(\mathcal{T}_{\text{gr}})_4$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$, respectively. We see that

$$\begin{aligned} z' \exp v' &= (c^{(0)}z(c^{(1)})^{-1})(\exp v)(\exp b^{(0)})(\exp b^{(1)})^{-1} \\ &= (c^{(0)} \exp b^{(0)})(z \exp v)(c^{(1)} \exp b^{(1)})^{-1} \sim z \exp v. \end{aligned}$$

Therefore, the action Ψ on cohomology is well defined.

5.1. Theorem. *Let $\dim(\mathbb{CP}^1, \mathcal{O}) = 1|n$ with $n \leq 5$. Then, the action Ψ defines a free action of the group $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$; the orbits of this action are the fibers of the map*

$$\lambda_2^* : H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}) \longrightarrow H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2).$$

Proof. To show that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ freely acts on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$, i.e., the stabilizer of any element $z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ is trivial, let $z(\exp v) \sim z$. By the above, there exists an element

$$u \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)$$

such that $z = \exp u$.

Let $u = u_2 + u_4$, see (4). Since

$$v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4),$$

then

$$z(\exp v) = \exp(u_2 + u_4)\exp v = \exp(u_2 + u_4 + v_4).$$

Hence,

$$\exp(u_2 + u_4 + v_4) \sim \exp(u_2 + u_4).$$

Therefore,

$$\exp(u_2 + u_4 + v_4) = c^{(0)}\exp(u_2 + u_4)(c^{(1)})^{-1},$$

where $c^{(i)} \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are holomorphic sections for $i = 0, 1$.

Let us represent $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$, where $a_2^{(i)}$ and $a_4^{(i)}$ are holomorphic sections of the sheaves $(\mathcal{T}_{\text{gr}})_2$ and $(\mathcal{T}_{\text{gr}})_4$, respectively, over U_i for $i = 0, 1$. Then, applying the Campbell-Hausdorff decomposition twice, we see that

$$\begin{aligned} & \exp(a_2^{(0)} + a_4^{(0)}) \exp(u_2 + u_4) \exp(-a_2^{(1)} - a_4^{(1)}) \\ &= \exp(a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \\ & \quad - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]). \end{aligned}$$

Hence,

$$\begin{aligned} u_2 + u_4 + v_4 &= a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] \\ & \quad - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]. \end{aligned}$$

Therefore,

$$v_4 = a_2^{(0)} + a_4^{(0)} - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}].$$

With respect to the degrees this equality breaks into two:

$$\begin{aligned} 0 &= a_2^{(0)} - a_2^{(1)}, \\ v_4 &= a_4^{(0)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]. \end{aligned}$$

Hence, we see that $v_4 = a_4^{(0)} - a_4^{(1)}$, i.e., $v \sim 0$.

Let us show now that the orbits of the $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ -action on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are fibers of the map λ_2^* . Indeed, let $z(\exp v) \sim y$, where

$$y, z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}), \quad v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4).$$

Then, there exist $u, w \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)$ such that $z = \exp u$ and $y = \exp w$.

Let

$$u = u_2 + u_4 \text{ and } w = w_2 + w_4,$$

where $u_2, w_2 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and $u_4, w_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$. We have

$$\exp(u_2 + u_4 + v_4) \sim \exp(w_2 + w_4)$$

or, as in the first part of the proof,

$$\exp(u_2 + u_4 + v_4) = c^{(0)} \exp(w_2 + w_4) (c^{(1)})^{-1},$$

where $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$. We similarly obtain

$$\exp(u_2 + u_4 + v_4) = \exp(a_2^{(0)} + a_4^{(0)} + w_2 + w_4 - a_2^{(1)} - a_4^{(1)}).$$

Having applied λ_2 to both sides of this equality we get

$$u_2 = a_2^{(0)} + w_2 - a_2^{(1)}.$$

Hence, $u_2 \sim w_2$, i.e., $z(\exp v)$ and y determine the same class in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ with respect to λ_2^* . \square

5.2. Corollary. *The classes in $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are determined by pairs (u_2, u_4) , see (4).*

Proof. Theorem 2.1 implies that $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ can be viewed as the bundle with base $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and fiber $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$. Thus, any class in $H^1(\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ is determined by (u_2, u_4) , see (4). \square

5.3. Theorem. *Any supermanifold with retract $\mathcal{CP}^{1|4}$ coincides, up to an isomorphism, with one of the following supermanifolds \mathcal{M} determined by the cocycle $u^{(01)} = u_2 + u_4$, see (4).*

*For homogeneous of these supermanifolds — the cases marked by *, the Lie superalgebra $\mathfrak{v}(\mathcal{M})$ is described in § 7.*

1*) $u_2 = 0$, and $u_4 = 0$;

2*) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = 0$;

3*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = 0$;

4*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = 0$;

5*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}$, and $u_4 = 0$;

- 6) $u_2 = 0$, and $u_4 = x^{-1}\delta\partial_x$;
 7) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = x^{-1}\delta\partial_x$;
 8) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = x^{-1}\delta\partial_x$;
 9) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = x^{-1}\delta\partial_x$;
 10*) $u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})$ with $t \in \mathbb{C}^\times$, and $u_4 = x^{-1}\delta\partial_x$.

Proof. Theorem 2.1 implies that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2) \simeq \mathbb{C}^{10}, \text{ and } H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4) \simeq \mathbb{C}.$$

Thus, by Corollary 5.2, $H^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ is the bundle with base \mathbb{C}^{10} and fiber \mathbb{C} .

Theorem 2.2 provides us with the system of generating cocycles for the basis and the fiber of this bundle. Thus, any supermanifold with retract $\mathcal{CP}^{1|4}$ is determined, up to an isomorphism, by the cocycle which in U_0 is of the form

$$u = \sum_{i,j=1,\dots,4} c_{ij}x^{-1}\delta_i\partial_{\xi_j} + cx^{-1}\delta\partial_x, \text{ where } c_{ij} = c_{ji}, \text{ and } c \in \mathbb{C}. \quad (5)$$

Let $\alpha \in \text{Aut } \mathbf{E}$. Since the automorphism α is a linear function in sections ξ_1, \dots, ξ_4 , then in U_0 we see that

$$\alpha(\xi_i) = \sum_{1 \leq j \leq 4} a_{ji}(x)\xi_j, \text{ where } a_{ji}(x) \text{ are holomorphic functions in } x.$$

Therefore, in U_1 this equality takes the form

$$y^{-1}\alpha(\eta_i) = \sum_{1 \leq j \leq 4} a_{ji}(y^{-1})y^{-1}\eta_j, \quad \text{or} \quad \alpha(\eta_i) = \sum_{1 \leq j \leq 4} a_{ji}(y^{-1})\eta_j.$$

This is how the action of the group $\text{Aut } \mathbf{E}$ on sections η_1, \dots, η_4 on U_1 is defined. Hence, $a_{ji}(y^{-1})$ should be holomorphic functions in y on U_1 . Hence, $a_{ji} = \text{const}$ for any i, j .

Therefore, any automorphism $\alpha \in \text{Aut } \mathbf{E}$ is given by a complex matrix $A = (a_{ij})$, i.e., $\text{Aut } \mathbf{E} \simeq \text{GL}_4(\mathbb{C})$.

Let $B = (b_{ij})$ be the inverse of $A = (a_{ij})$. Then,

$$\alpha(\partial_{\xi_i})\xi_j = (\alpha\partial_{\xi_i}\alpha^{-1})\xi_j = \alpha\partial_{\xi_i}\left(\sum_{1 \leq k \leq 4} b_{kj}\xi_k\right) = \alpha(b_{ij}) = b_{ij},$$

i.e.,

$$\alpha(\partial_{\xi_i}) = \sum_{1 \leq j \leq 4} b_{ij}\partial_{\xi_j}.$$

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix} \in S_4$. Further,

$$\begin{aligned} \alpha(\delta) &= \alpha(\xi_1\xi_2\xi_3\xi_4) = \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3)\alpha(\xi_4) \\ &= \left(\sum_{1 \leq i \leq 4} a_{i1}\xi_i\right)\left(\sum_{1 \leq j \leq 4} a_{j2}\xi_j\right)\left(\sum_{1 \leq k \leq 4} a_{k3}\xi_k\right)\left(\sum_{1 \leq l \leq 4} a_{l4}\xi_l\right) \\ &= \left(\sum_{\sigma \in S_4} \text{sign } \sigma \cdot a_{i1}a_{j2}a_{k3}a_{l4}\right)\delta = \det A \cdot \delta; \\ \alpha(\delta_i) &= \alpha(\partial_{\xi_i}\delta) = \alpha(\partial_{\xi_i})\alpha(\delta) = \det A \cdot \sum_{1 \leq k \leq 4} b_{ik}\partial_{\xi_k}\delta = \det A \cdot \sum_{1 \leq k \leq 4} b_{ik}\delta_k. \end{aligned}$$

We see that

$$\begin{aligned}
\alpha(x^{-1}\delta\partial_x) &= x^{-1}\alpha(\delta)\partial_x \\
&= x^{-1}\left(\sum_{1\leq i\leq 4} a_{i1}\xi_i\right)\left(\sum_{1\leq j\leq 4} a_{j2}\xi_j\right)\left(\sum_{1\leq k\leq 4} a_{k3}\xi_k\right)\left(\sum_{1\leq l\leq 4} a_{l4}\xi_l\right)\partial_x \\
&= x^{-1}\left(\sum_{\sigma\in S_4} \text{sign } \sigma \cdot a_{i_1}a_{j_2}a_{k_3}a_{l_4}\right)\delta\partial_x = \det A \cdot x^{-1}\delta\partial_x; \\
\alpha(x^{-1}\delta_i\partial_{\xi_j}) &= x^{-1}\alpha(\delta_i)\alpha(\partial_{\xi_j}) \\
&= \det A \cdot \sum_{1\leq k\leq 4} b_{ik}\delta_k \sum_{1\leq l\leq 4} b_{jl}\partial_{\xi_l} = \det A \cdot \sum_{1\leq k,l\leq 4} b_{ik}b_{jl}\delta_k\partial_{\xi_l}.
\end{aligned}$$

Then, having applied α to the cocycle (5), we get

$$\alpha(u) = \det A \cdot \sum_{1\leq i,j,k,l\leq 4} x^{-1}c_{ij}b_{ik}b_{jl}\delta_k\partial_{\xi_l} + \det A \cdot cx^{-1}\delta\partial_x.$$

This implies, in particular, that the matrix $C = (c_{ij})$ transforms into $(\det A)B^tCB$.

Since every cocycle of the form (5) is uniquely determined by the matrix C and the number c , and the $(\text{Aut } \mathbf{E})$ -action is known, it suffices to consider the following cases.

1. In this case, $C = 0$ and $c = 0$.

2. Let $\text{rk } C = 1$ and $c = 0$. Then, as follows from Algebra course, the group $\text{GL}_4(\mathbb{C})$ does not change the rank of C and there is an invertible operator reducing C to the 4×4 matrix E_{11} . To this matrix the cocycle $x^{-1}\delta_1\partial_{\xi_1}$ corresponds; we will consider this cocycle as a representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

3, 4 and 5. Let $\text{rk } C = 2, 3$ and 4 , respectively and $c = 0$. As in case 2, we get, respectively, the following representatives:

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.
\end{aligned}$$

6. Let $C = 0$ and $c \neq 0$. Then, the transformation determined by the matrix A with $\det A = \frac{1}{c}$ leads to the cocycle $x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

7. Let $\text{rk } C = 1$ and $c \neq 0$. Since C is not invertible, it is possible (elementary linear algebra) to find a matrix A with $\det A = \frac{1}{c}$ so that C becomes the matrix unit E_{11} . The cocycle corresponding to this matrix and the number 1 is $x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

8 and 9. Let $\text{rk } C = 2$ and 3 , respectively, and $c \neq 0$. In analogy with case 7, the representatives are, respectively,

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta\partial_x, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta\partial_x.
\end{aligned}$$

10. Let $\text{rk } C = 4$ and $c \neq 0$. In this case, by an invertible transformation with matrix A such that $\det A = \frac{1}{c}$ the matrix C can be reduced to the diagonal form $t1_4$, where $t \in \mathbb{C}^\times$.

As a result, we get a 1-parameter family of cocycles

$$c_t := t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x,$$

Every cocycle c_t defines an orbit in $Z^1(\mathbb{CP}^1, \mathcal{Aut}_{(2)}\mathcal{O}_{\text{gr}})$. \square

6 Which of the supermanifolds with retract $\mathcal{CP}^{1|4}$ are homogeneous

Consider an open covering $\mathcal{U} = (U_i)_{i \in I}$ of a topological space M . Let the supermanifold (M, \mathcal{O}) be determined by a cocycle $g \in Z^1(M, \mathcal{Aut}_{(2)}\mathcal{O}_{\text{gr}})$. Recall the definition of a liftable field, see § 2.

6.1 Theorem ([BO1]). *The vector field $v \in \mathfrak{v}(M, \mathcal{O}_{\text{gr}})_p$ can be lifted to (M, \mathcal{O}) if and only if there exists a $v^{(i)} \in C^0(M, (\mathcal{T}_{\text{gr}})_{(p)})$ such that*

$$\begin{aligned} v^{(i)} &\equiv v \pmod{(\mathcal{T}_{\text{gr}})_{(p+1)}(U_i)}, \\ g^{(ij)}v^{(j)} &= v^{(i)}g^{(ij)} \quad \text{in } U_i \cap U_j \neq \emptyset. \end{aligned}$$

Consider the representation

$$g^{(01)} = \exp u^{(01)}, \text{ where } u^{(01)} \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4).$$

Let $u^{(01)} = u_2 + u_4$, see (4).

6.2. Corollary. *The vector field $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_p$ can be lifted if and only if the following conditions hold*

$$\begin{aligned} p = -1 \quad [v, u_2] &= v_1^{(1)} - v_1^{(0)}, & (2) \\ [v, u_4] &= v_3^{(1)} - v_3^{(0)} + [u_2, v_1^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], & (3) \\ p = 0 \quad [v, u_2] &= v_2^{(1)} - v_2^{(0)}, \\ [v, u_4] &= v_4^{(1)} - v_4^{(0)} + [u_2, v_2^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], \\ p = 1 \quad [v, u_2] &= v_3^{(1)} - v_3^{(0)}. \end{aligned}$$

Proof. Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_p$, where $p = -1, 0$. Let us seek $v^{(i)}$ in the form $v + v_{p+2}^{(i)} + v_{p+4}^{(i)}$ for $v_{p+2}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_{p+2}$, and $v_{p+4}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_{p+4}$.

By Theorem 6.1 we have a condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$\begin{aligned} v^{(0)} &= g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \text{ad}_{u^{(01)}})v^{(1)} \\ &= v^{(1)} + [u^{(01)}, v^{(1)}] + \frac{1}{2}[u^{(01)}, [u^{(01)}, v^{(1)}]]. \end{aligned}$$

Therefore,

$$v + v_{p+2}^{(0)} + v_{p+4}^{(0)} = v + v_{p+2}^{(1)} + v_{p+4}^{(1)} + [u_2 + u_4, v + v_{p+2}^{(1)}] + \frac{1}{2}[u_2, [u_2, v]].$$

Hence, we get the conditions

$$\begin{aligned} [v, u_2] &= v_{p+2}^{(1)} - v_{p+2}^{(0)}, \\ [v, u_4] &= v_{p+4}^{(1)} - v_{p+4}^{(0)} + [u_2, v_{p+2}^{(1)}] + \frac{1}{2}[u_2, [u_2, v]]. \end{aligned}$$

Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$. We seek $v^{(i)}$ in the form $v + v_3^{(i)}$, where $v_3^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_3$. By Theorem 6.1 we get the condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$v^{(0)} = g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \text{ad}_{u^{(01)}})v^{(1)} = v^{(1)} + [u^{(01)}, v^{(1)}].$$

Therefore,

$$v + v_3^{(0)} = v + v_3^{(1)} + [u_2 + u_4, v + v_3^{(1)}].$$

Hence, we get the condition

$$[v, u_2] = v_3^{(1)} - v_3^{(0)}. \quad \square$$

The definition of homogeneous supermanifold implies that the supermanifold is homogeneous if and only if the following map is surjective

$$\text{ev}_x : \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) \longrightarrow T_x(\mathbb{CP}^1, \mathcal{O}) \quad \text{for any } x \in \mathbb{CP}^1.$$

Since $\bar{0}$ -homogeneity takes place by the proved, it is necessary and sufficient to prove that all fields $\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$ can be lifted to $(\mathbb{CP}^1, \mathcal{O})$.

Let us use the conditions obtained to verify the homogeneity of the supermanifolds in the 10 cases of Theorem 6.1. For every homogeneous $(\mathbb{CP}^1, \mathcal{O})$, I compute the Lie superalgebra of vector fields on it.

6.2 Case 1. It corresponds to the supermanifold $\mathcal{CP}^{1|4}$. It is well-known that this supermanifold is homogeneous. The Lie superalgebra of vector fields on $\mathcal{CP}^{1|4}$ is known (see [O3]):

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1} &= \langle v_{5i} = -\partial_{\xi_i}, v_{6i} = -x\partial_{\xi_i}, \text{ where } i = 1, \dots, 4 \rangle, \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_0 &= \langle v_{ij} = -\xi_i\partial_{\xi_j}, \text{ where } i, j = 1, \dots, 4, v_{55} = x\partial_x + \nabla, \\ v_{66} &= -x\partial_x, v_{56} = -\partial_x, v_{65} = x^2\partial_x + x\nabla, \text{ where } \sum_{1 \leq i \leq 6} v_{ii} = 0 \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1 &= \langle v_{i5} = -\xi_i(x\partial_x + \nabla), v_{i6} = -\xi_i\partial_x \text{ for } i = 1, \dots, 4 \rangle. \end{aligned}$$

6.3 Case 2. Then, $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

We will repeatedly use the following ‘‘formula’’

$$\text{Since } [v_i, u_2] \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_1), \text{ then } [v_i, u_2] \sim 0 \text{ by Theorem 2.1, and hence condition (2) of Corollary 6.2 holds.} \quad (6)$$

Since

$$[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}$$

is holomorphic in U_1 , then for each v_i we set

$$v_i^{(0)} = 0, \quad v_i^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}.$$

We have

$$[x^{-1}\delta_1\partial_{\xi_1}, -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}] = -x^{-2}\delta_1\partial_{\xi_1}\left(\frac{\partial\delta_1}{\partial\xi_i}\right)\partial_{\xi_1} + x^{-2}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}(\delta_1)\partial_{\xi_1} = 0.$$

The first summand vanishes since δ_1 does not contain ξ_1 by definition and $\partial_{\xi_1}\left(\frac{\partial\delta_1}{\partial\xi_i}\right) = 0$.

The second summand also vanishes since $\partial_{\xi_1}(\delta_1) = 0$.

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$, where $i = 1, \dots, 4$, can be lifted; moreover, in U_0 they have the same form. Hence, the supermanifolds corresponding to the cocycle of the 2nd case is homogeneous.

6.3. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1}$. Then, (recall notation (2))*

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, \quad x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1}, \quad 2x\partial_x + \nabla, \\ & -x\partial_x - \xi_2\partial_{\xi_2}, \quad -x\partial_x - \xi_3\partial_{\xi_3}, \quad -x\partial_x - \xi_4\partial_{\xi_4}, \quad -\xi_2\partial_{\xi_1}, \quad -\xi_2\partial_{\xi_3}, \\ & -\xi_2\partial_{\xi_4}, \quad -\xi_3\partial_{\xi_1}, \quad -\xi_3\partial_{\xi_2}, \quad -\xi_3\partial_{\xi_4}, \quad -\xi_4\partial_{\xi_1}, \quad -\xi_4\partial_{\xi_2}, \quad -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i} \text{ for } i = 1, \dots, 4, \quad -x\partial_{\xi_1}, \quad -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \quad -\xi_j\partial_x, \\ & -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1}, \quad -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1}, \quad -\xi_j(x\partial_x + \nabla), \text{ for } j = 2, 3, 4 \rangle. \end{aligned}$$

6.4 Case 3. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}.$$

In the same way as in Case 2 we see that

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}] = 0.$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of Case 3 is homogeneous.

6.4. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$. Then, (recall notation (2))*

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2}, 2x\partial_x + \nabla, \\ & \xi_2\partial_{\xi_2}, -x\partial_x - \xi_3\partial_{\xi_3}, -x\partial_x - \xi_4\partial_{\xi_4}, \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, -\xi_3\partial_{\xi_1}, \\ & -\xi_3\partial_{\xi_2}, -\xi_3\partial_{\xi_4}, -\xi_4\partial_{\xi_1}, -\xi_4\partial_{\xi_2}, -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i}, i = 1, \dots, 4, -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2}, -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \\ & -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2}, -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2}, \\ & -\xi_j(x\partial_x + \nabla), -\xi_j\partial_x, j = 3, 4 \rangle. \end{aligned}$$

6.5 Case 4. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} \end{aligned}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}.$$

In the same way as in Case 2 we see that $[X, Y] = 0$, where

$$\begin{aligned} X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\ Y &:= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}. \end{aligned}$$

Hence, condition (3) of Corollary 6.2 is of the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.5. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}.$$

Then, (recall notation (2))

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2} - \delta_3\partial_{\xi_3}, 2x\partial_x + \nabla, \\ & \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, -x\partial_x - \xi_4\partial_{\xi_4}, \\ & -\xi_4\partial_{\xi_1}, -\xi_4\partial_{\xi_2}, -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i} \text{ for } i = 1, \dots, 4, -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3}, \\ & -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3}, -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2}, \\ & -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}, -\xi_4(x\partial_x + \nabla), -\xi_4\partial_x \rangle. \end{aligned}$$

6.6 Case 5. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}, \text{ and } u_4 = 0.$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] = \\ &= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4} \end{aligned}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}.$$

In the same way as in Case 2 we see that $[A, B] = 0$, where

$$\begin{aligned} A &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}, \\ B &:= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}. \end{aligned}$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 5th case is homogeneous.

6.6. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.$$

Then, (recall notation (2))

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla + \nabla, 2x\partial_x + \nabla, \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \\ & \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} \rangle, \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}, \\ & -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}, -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}, \\ & -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}, -\partial_{\xi_i} \text{ for } i = 1, \dots, 4 \rangle. \end{aligned}$$

6.7 Case 6. Let $u_2 = 0$, and $u_4 = x^{-1}\delta\partial_x$.

Consider the fields $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$. Since $[v_i, u_2] = 0$, then condition (2) of Corollary 6.2 is satisfied; it takes the form

$$[v_i, u_4] = v_3^{(1)} - v_3^{(0)},$$

where $v_3^{(j)}$ is holomorphic in U_j for $j = 0, 1$. Then, $[v_i, u_4]$ should be cohomologous to 0. Substituting the values of v_i and u_4 , we see that $[\partial_{\xi_i}, x^{-1}\delta\partial_x] = x^{-1}\delta_i\partial_x$ which is a basis cocycle (see Theorem 2.2). Hence, condition (3) of Corollary 6.2 is not satisfied.

Therefore, none of the fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Therefore, the supermanifold corresponding to the cocycle of the 6th case is not homogeneous.

6.8 Case 7. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1}$$

is holomorphic in U_1 , set $v_1^{(0)} = v'$, and $v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$.

Since $[x^{-1}\delta_1\partial_{\xi_1}, x^{-1}\xi_2\xi_3\partial_{\xi_1}] = 0$, then condition (3) of Corollary 6.2 is of the form

$$[-\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} [v', x^{-1}\delta_1\partial_{\xi_1}] &= -A_1 x^{-1}\delta\partial_{\xi_1} + 3A_1\delta\partial_{\xi_1} - B_1 x^{-2}\delta\partial_{\xi_1} - \\ &\quad - (A_1\delta_1\partial_x - 2A_1 x^{-1}\delta\partial_{\xi_1} + B_1 x^{-1}\delta_1\partial_x) \\ &= A_1 x^{-1}\delta\partial_{\xi_1} - B_1 x^{-2}\delta\partial_{\xi_1} - B_1 x^{-1}\delta_1\partial_x + 3A_1\delta\partial_{\xi_1} - A_1\delta_1\partial_x \\ &\sim A_1 x^{-1}\delta\partial_{\xi_1} - 2B_1 x^{-1}\delta_1\partial_x. \end{aligned}$$

But then, for any $A_1, B_1 \in \mathbb{C}$, we have

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} - 2B_1 x^{-1}\delta_1\partial_x \not\sim 0.$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 7th case is not homogeneous.

6.9 Case 8. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}$$

is holomorphic in U_1 , we set

$$v_1^{(0)} = v', \quad v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} + v', \quad \text{where } v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1.$$

Since

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}] = 0,$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \quad \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} & [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \\ & \sim A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x. \end{aligned}$$

But then, for any $A_1, A_2, B_1, B_2 \in \mathbb{C}$, we have

$$\begin{aligned} & x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \\ & \sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x \not\sim 0. \end{aligned}$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 8th case is not homogeneous.

6.10 Case 9. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \quad \text{and } u_4 = x^{-1}\delta\partial_x.$$

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$\begin{aligned} [v, u_2] &= [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3} \end{aligned}$$

is holomorphic in U_1 , let us set $v_1^{(0)} = v'$, and

$$v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3} + v',$$

where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$.

Since $[X, Y] = 0$, where

$$\begin{aligned} X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\ Y &:= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3}, \end{aligned}$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, then we can consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \quad \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} &[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} + A_3 x^{-1}\delta\partial_{\xi_3} \\ &\quad - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x - 2B_3 x^{-1}\delta_3\partial_x. \end{aligned}$$

But then, for any $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathbb{C}$, we have

$$\begin{aligned} &x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} + A_3 x^{-1}\delta\partial_{\xi_3} - 2B_1 x^{-1}\delta_1\partial_x \\ &\quad - 2B_2 x^{-1}\delta_2\partial_x - 2B_3 x^{-1}\delta_3\partial_x \not\sim 0. \end{aligned}$$

Hence, the field $-\partial_{\xi_4}$ can not be lifted. This shows that the supermanifold of the 9th case is not homogeneous.

6.11 Case 10. Let $t \in \mathbb{C}^\times$. Let

$$u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), \quad \text{and } u_4 = x^{-1}\delta\partial_x.$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$. Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})] \\ &= -tx^{-1} \left(\frac{\partial\delta_1}{\partial\xi_i} \partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i} \partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i} \partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i} \partial_{\xi_4} \right) \end{aligned}$$

is holomorphic in U_1 , then set $v_1^{(0)} = v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, and

$$v_1^{(1)} = v' - tx^{-1} \left(\frac{\partial\delta_1}{\partial\xi_i} \partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i} \partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i} \partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i} \partial_{\xi_4} \right).$$

We have $[A, B] = 0$, where

$$\begin{aligned} A &:= t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), \\ B &:= -tx^{-1}\left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\right). \end{aligned}$$

Hence, condition (3) of Corollary 6.2 is of the form

$$\begin{aligned} &[-\partial_{\xi_i}, x^{-1}\delta\partial_x] \\ &= v_3^{(1)} - v_3^{(0)} + [t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), v']. \end{aligned}$$

Therefore,

$$-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, take

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} &[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \\ &= \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - B_k x^{-2}\delta\partial_{\xi_k} - B_k x^{-1}\delta_k\partial_x - A_k \delta_k\partial_x \right) \\ &\sim \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - 2B_k x^{-1}\delta_k\partial_x \right). \end{aligned}$$

But then, for $B_i = -\frac{1}{2t}$, and $B_j = 0$ for $j \in \{1, \dots, 4 \mid j \neq i\}$, and $A_k = 0$ for $k = 1, \dots, 4$, we have

$$\begin{aligned} &-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \\ &\sim -x^{-1}\delta_i\partial_x + t \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - 2B_k x^{-1}\delta_k\partial_x \right) \sim 0. \end{aligned}$$

Thus, all fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Hence, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.7. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x.$$

Then (recall notation (2)),

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} &= \langle -\partial_x, x^2\partial_x + x\nabla - t\omega - \delta\partial_x, 2x\partial_x + \nabla, \\ &\xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \\ &\xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} \rangle, \text{ where } \omega = \sum_{1 \leq i \leq 4} \delta_i\partial_{\xi_i}; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} &= \langle -\partial_{\xi_i} - \frac{1}{2t}\xi_i\partial_x, i = 1, \dots, 4, \\ &-x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_4} - \xi_2\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x, \\ &-x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x, \\ &-x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \frac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x, \\ &-x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \frac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x \rangle. \end{aligned}$$

6.8 Theorem (Summary). *The supermanifold isomorphic to one of the supermanifolds of cases 1 – 5 or 10 is homogeneous; it is not homogeneous in cases 6 – 9.*

7 Description of $\mathfrak{v}(\mathcal{M})$ for homogeneous superstrings \mathcal{M} with retract $\mathcal{CP}^{1|4}$

The cases are numbered as in Theorem 5.3. We reproduce here the bases found in §2.3, and multiplication tables. In all cases (recall notation (2)),

$$h := [e, f] = 2x\partial_x + \nabla \text{ and } [h, e] = 2e, [h, f] = -2f.$$

1. Fact (well-known): $\mathfrak{v}(\mathcal{CP}^{1|4}) \simeq \mathfrak{pgl}_{\mathbb{C}}(4|2)$.
2. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 12 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1}, f = -\partial_x, a_1 = -x\partial_x - \xi_2\partial_{\xi_2}, \\ a_2 &= -x\partial_x - \xi_3\partial_{\xi_3}, a_3 = -x\partial_x - \xi_4\partial_{\xi_4}, a_4 = -\xi_2\partial_{\xi_1}, a_5 = -\xi_2\partial_{\xi_3}, \\ a_6 &= -\xi_2\partial_{\xi_4}, a_7 = -\xi_3\partial_{\xi_2}, a_8 = -\xi_3\partial_{\xi_1}, a_9 = -\xi_3\partial_{\xi_4}, a_{10} = -\xi_4\partial_{\xi_2}, \\ a_{11} &= -\xi_4\partial_{\xi_3}, a_{12} = -\xi_4\partial_{\xi_1}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i, i = 1, \dots, 14 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, z_2 = -\partial_{\xi_2}, z_3 = -\partial_{\xi_3}, z_4 = -\partial_{\xi_4}, z_5 = -\xi_2\partial_x, \\ z_6 &= -\xi_3\partial_x, z_7 = -\xi_4\partial_x, z_8 = -x\partial_{\xi_1}, z_9 = -\xi_2(x\partial_x + \nabla), \\ z_{10} &= -\xi_3(x\partial_x + \nabla), z_{11} = -\xi_4(x\partial_x + \nabla), z_{12} = -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \\ z_{13} &= -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1}, z_{14} = -x\partial_{\xi_3} + \xi_2\xi_3\partial_{\xi_1}. \end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0 \text{ for all } i = 1, \dots, 12.$$

$[,]$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	a_7	0	0	a_{10}	0	0
a_2	0	0	0	0	a_5	0	$-a_7$	$-a_8$	$-a_9$	0	a_{11}	0
a_3	0	0	0	0	0	a_6	0	0	a_9	$-a_{10}$	$-a_{11}$	$-a_{12}$
a_4	a_4	0	0	0	0	0	a_8	0	0	a_{12}	0	0
a_5	a_5	$-a_5$	0	0	0	0	$a_2 - a_1$	$-a_4$	$-a_6$	a_{11}	0	0
a_6	a_6	0	$-a_6$	0	0	0	a_9	0	0	$a_3 - a_1$	$-a_5$	$-a_4$
a_7	$-a_7$	a_7	0	$-a_8$	$a_1 - a_2$	$-a_9$	0	0	0	0	a_{12}	0
a_8	0	a_8	0	0	a_4	0	0	0	0	0	a_{10}	0
a_9	0	a_9	$-a_9$	0	a_6	0	0	0	0	$-a_7$	$a_3 - a_2$	$-a_8$
a_{10}	$-a_{10}$	0	$-a_{10}$	$-a_{12}$	$-a_{11}$	$a_1 - a_3$	0	0	a_7	0	0	0
a_{11}	0	$-a_{11}$	a_{11}	0	0	a_5	$-a_{12}$	$-a_{10}$	$a_2 - a_3$	0	0	0
a_{12}	0	0	a_{12}	0	0	a_4	0	0	a_8	0	0	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
e	$-z_8$	$-z_{12}$	$-z_{13}$	$-z_{14}$	$-z_9$	$-z_{10}$	$-z_{11}$	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	$-z_1$	$-z_5$	$-z_6$	$-z_7$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	$-z_7$	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
a_1	0	z_2	0	0	0	z_6	z_7	$-z_8$	$-z_9$	0	0	0	$-z_{13}$	$-z_{14}$
a_2	0	0	z_3	0	z_5	0	z_7	$-z_8$	0	$-z_{10}$	0	$-z_{12}$	0	$-z_{14}$
a_3	0	0	0	z_4	z_5	z_6	0	$-z_8$	0	0	$-z_{11}$	$-z_{12}$	$-z_{13}$	0
a_4	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0	0
a_5	0	z_3	0	0	0	$-z_5$	0	0	0	$-z_9$	0	$-z_{12}$	0	0
a_6	0	z_4	0	0	0	0	$-z_5$	0	0	0	$-z_9$	z_{14}	0	0
a_7	0	0	z_2	0	z_6	0	0	0	$-z_{10}$	0	0	0	z_{12}	0
a_8	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0
a_9	0	0	z_4	0	0	0	$-z_6$	0	0	0	$-z_{10}$	0	z_4	0
a_{10}	0	0	0	z_2	$-z_7$	0	0	0	$-z_{11}$	0	0	0	0	z_{12}
a_{11}	0	0	0	z_3	0	$-z_7$	0	0	0	$-z_{11}$	0	0	0	z_{13}
a_{12}	0	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
z_1	0	0	0	0	0	0	0	0	a_4	a_8	a_{12}	0	0	0
z_2	0	0	0	0	$-f$	0	0	0	a_1+h	a_7	a_{10}	0	$-a_{12}$	a_8
z_3	0	0	0	0	0	$-f$	0	0	a_5	a_2+h	a_{11}	a_{12}	0	$-a_4$
z_4	0	0	0	0	0	0	$-f$	0	a_8	a_9	a_3+h	$-a_8$	a_4	0
z_5	0	$-f$	0	0	0	0	0	$-a_4$	0	0	0	$-a_1$	$-a_5$	$-a_6$
z_6	0	0	$-f$	0	0	0	0	$-a_8$	0	0	0	$-a_7$	$-a_2$	$-a_9$
z_7	0	0	0	$-f$	0	0	0	$-a_{12}$	0	0	0	$-a_{10}$	$-a_{11}$	$-a_3$
z_8	0	0	0	0	$-a_4$	$-a_8$	$-a_{12}$	0	0	0	0	0	0	0
z_9	a_4	a_1+h	a_5	a_6	0	0	0	0	0	0	0	e	0	0
z_{10}	a_8	a_7	a_2+h	a_9	0	0	0	0	0	0	0	0	e	0
z_{11}	a_{12}	a_{10}	a_{11}	a_3+h	0	0	0	0	0	0	0	0	0	e
z_{12}	0	0	a_{12}	$-a_8$	$-a_1$	$-a_7$	$-a_{10}$	0	e	0	0	0	0	0
z_{13}	0	$-a_{12}$	0	a_4	$-a_5$	$-a_2$	$-a_{11}$	0	0	e	0	0	0	0
z_{14}	0	a_8	$-a_4$	0	$-a_6$	$-a_9$	$-a_3$	0	0	0	e	0	0	0

3. We have $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 9 \rangle$, where

$$\begin{aligned} e &= x^2 \partial_x + x \nabla - \delta_1 \partial_{\xi_1} - \delta_2 \partial_{\xi_2}, \quad f = -\partial_x, \quad a_1 = -x \partial_x - \xi_3 \partial_{\xi_3}, \\ a_2 &= -x \partial_x - \xi_4 \partial_{\xi_4}, \quad a_3 = \xi_1 \partial_{\xi_2} - \xi_2 \partial_{\xi_1}, \quad a_4 = -\xi_3 \partial_{\xi_1}, \quad a_5 = -\xi_3 \partial_{\xi_2}, \\ a_6 &= -\xi_3 \partial_{\xi_4}, \quad a_7 = -\xi_4 \partial_{\xi_1}, \quad a_8 = -\xi_4 \partial_{\xi_2}, \quad a_9 = -\xi_4 \partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 12 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, \quad z_2 = -\partial_{\xi_2}, \quad z_3 = -\partial_{\xi_3}, \quad z_4 = -\partial_{\xi_4}, \quad z_5 = -\xi_3 \partial_x, \\ z_6 &= -\xi_4 \partial_x, \quad z_7 = -x \partial_{\xi_1} - \xi_3 \xi_4 \partial_{\xi_2}, \quad z_8 = -x \partial_{\xi_2} + \xi_3 \xi_4 \partial_{\xi_1}, \\ z_9 &= -x \partial_{\xi_3} - \xi_2 \xi_4 \partial_{\xi_1} + \xi_1 \xi_4 \partial_{\xi_2}, \quad z_{10} = -x \partial_{\xi_4} + \xi_2 \xi_3 \partial_{\xi_1} - \xi_1 \xi_3 \partial_{\xi_2}, \\ z_{11} &= -\xi_3 (x \partial_x + \nabla), \quad z_{12} = -\xi_4 (x \partial_x + \nabla). \end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0, \quad i = 1, \dots, 9.$$

$[,]$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	0	0	a_9
a_2	0	0	0	0	0	a_6	$-a_7$	$-a_8$	$-a_9$
a_3	0	0	0	$-a_5$	a_4	0	$-a_8$	a_7	0
a_4	a_4	0	a_5	0	0	0	0	0	a_7
a_5	a_5	0	$-a_4$	0	0	0	0	0	a_8
a_6	a_6	$-a_6$	0	0	0	0	$-a_4$	$-a_5$	$a_2 - a_1$
a_7	0	a_7	a_8	0	0	a_4	0	0	0
a_8	0	a_8	$-a_7$	0	0	a_5	0	0	0
a_9	$-a_9$	a_9	0	$-a_7$	$-a_8$	$a_1 - a_2$	0	0	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
e	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	$-z_{11}$	$-z_{12}$	0	0	0	0	0	0
f	0	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
a_1	0	0	z_3	0	0	z_6	$-z_7$	$-z_8$	0	$-z_{10}$	$-z_{11}$	0
a_2	0	0	0	z_4	z_5	0	$-z_7$	$-z_8$	$-z_9$	0	0	z_{12}
a_3	$-z_2$	z_1	0	0	0	0	$-z_8$	z_7	0	0	0	0
a_4	0	0	z_1	0	0	0	0	0	z_7	0	0	0
a_5	0	0	z_2	0	0	0	0	0	z_8	0	0	0
a_6	0	0	z_4	0	0	$-z_5$	0	0	z_{10}	0	0	$-z_{11}$
a_7	0	0	0	z_1	0	0	0	0	0	z_7	0	0
a_8	0	0	0	z_2	0	0	0	0	0	z_8	0	0
a_9	0	0	0	z_3	$-z_6$	0	0	0	0	z_9	$-z_{12}$	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
z_1	0	0	0	0	0	0	0	0	a_8	$-a_5$	a_4	a_7
z_2	0	0	0	0	0	0	0	0	$-a_7$	a_4	a_5	a_8
z_3	0	0	0	0	$-f$	0	$-a_8$	a_7	0	$-a_3$	a_1+h	a_9
z_4	0	0	0	0	0	$-f$	a_5	$-a_4$	a_3	0	a_6	a_2+h
z_5	0	0	$-f$	0	0	0	$-a_4$	$-a_5$	$-a_1$	$-a_6$	0	0
z_6	0	0	0	$-f$	0	0	$-a_7$	$-a_8$	$-a_9$	$-a_2$	0	0
z_7	0	0	$-a_8$	a_7	$-a_4$	$-a_7$	0	0	0	0	0	0
z_8	0	0	a_7	$-a_4$	$-a_5$	$-a_8$	0	0	0	0	0	0
z_9	a_8	$-a_7$	0	a_3	$-a_1$	$-a_9$	0	0	0	0	e	0
z_{10}	$-a_5$	a_4	$-a_3$	0	$-a_6$	$-a_2$	0	0	0	0	0	e
z_{11}	a_4	a_5	a_1+h	a_6	0	0	0	0	e	0	0	0
z_{12}	a_7	a_8	a_9	a_2+h	0	0	0	0	0	e	0	0

4. $\mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2} - \delta_3\partial_{\xi_3}, \quad f = -\partial_x, \quad a = -x\partial_x - \xi_4\partial_{\xi_4}, \\ a_1 &= \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \quad a_2 = \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \quad a_3 = \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \\ a_4 &= -\xi_4\partial_{\xi_1}, \quad a_5 = -\xi_4\partial_{\xi_2}, \quad a_6 = -\xi_4\partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 10 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, z_2 = -\partial_{\xi_2}, z_3 = -\partial_{\xi_3}, z_4 = -\xi_4 \partial_x, z_5 = -\partial_{\xi_4}, \\ z_6 &= -x \partial_{\xi_1} - \xi_3 \xi_4 \partial_{\xi_2} + \xi_2 \xi_4 \partial_{\xi_3}, z_7 = -x \partial_{\xi_2} + \xi_3 \xi_4 \partial_{\xi_1} - \xi_1 \xi_4 \partial_{\xi_3}, \\ z_8 &= -x \partial_{\xi_3} - \xi_2 \xi_4 \partial_{\xi_1} + \xi_1 \xi_4 \partial_{\xi_2}, z_9 = -\xi_4 (x \partial_x + \nabla), \\ z_{10} &= -x \partial_{\xi_4} + \xi_2 \xi_3 \partial_{\xi_1} - \xi_1 \xi_3 \partial_{\xi_2} + \xi_1 \xi_2 \partial_{\xi_3}. \end{aligned}$$

$$[h, v] = [e, v] = [f, v] = 0, \text{ where } v \in \{a, a_i \mid i = 1, \dots, 6\}.$$

$[,]$	a	a_1	a_2	a_3	a_4	a_5	a_6			
a	0	0	0	0	$-a_4$	$-a_5$	$-a_6$			
a_1	0	0	$-a_3$	a_2	$-a_5$	a_4	0			
a_2	0	a_3	0	$-a_1$	$-a_6$	0	a_4			
a_3	0	$-a_2$	a_1	0	0	$-a_6$	a_5			
a_4	a_4	a_5	a_6	0	0	0	0			
a_5	a_5	$-a_4$	0	a_6	0	0	0			
a_6	a_6	0	$-a_4$	$-a_5$	0	0	0			
$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
e	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$
f	$-z_6$	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	0	0	0	0	0
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	z_6	z_7	z_8	z_9	z_{10}
a	0	0	0	0	z_5	$-z_6$	$-z_7$	$-z_8$	$-z_9$	0
a_1	$-z_2$	z_1	0	0	0	$-z_7$	z_6	0	0	0
a_2	$-z_3$	0	z_1	0	0	$-z_8$	0	z_6	0	0
a_3	0	$-z_3$	z_2	0	0	0	$-z_8$	z_7	0	0
a_4	0	0	0	0	z_1	0	0	0	0	z_6
a_5	0	0	0	0	z_2	0	0	0	0	z_7
a_6	0	0	0	0	z_3	0	0	0	0	z_8
$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
z_1	0	0	0	0	0	0	$-a_6$	a_5	a_4	$-a_3$
z_2	0	0	0	0	0	a_6	0	$-a_4$	a_5	a_2
z_3	0	0	0	0	0	$-a_5$	a_4	0	a_6	$-a_1$
z_4	0	0	0	0	$-f$	$-a_4$	$-a_5$	$-a_6$	0	$-a$
z_5	0	0	0	$-f$	0	a_3	$-a_2$	a_1	$a + h$	0
z_6	0	a_6	$-a_5$	$-a_4$	a_3	0	0	0	0	0
z_7	$-a_6$	0	a_4	$-a_5$	$-a_2$	0	0	0	0	0
z_8	a_5	$-a_4$	0	$-a_6$	a_1	0	0	0	0	0
z_9	a_4	a_5	a_6	0	$a + h$	0	0	0	0	e
z_{10}	$-a_3$	a_2	$-a_1$	$-a$	0	0	0	0	e	0

5. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2 \partial_x + x \nabla - \omega, f = -\partial_x, a_1 = \xi_1 \partial_{\xi_2} - \xi_2 \partial_{\xi_1}, a_2 = \xi_1 \partial_{\xi_3} - \xi_3 \partial_{\xi_1}, \\ a_3 &= \xi_1 \partial_{\xi_4} - \xi_4 \partial_{\xi_1}, a_4 = \xi_2 \partial_{\xi_3} - \xi_3 \partial_{\xi_2}, a_5 = \xi_2 \partial_{\xi_4} - \xi_4 \partial_{\xi_2}, \\ a_6 &= \xi_3 \partial_{\xi_4} - \xi_4 \partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 8 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, \quad z_2 = -\partial_{\xi_2}, \quad z_3 = -\partial_{\xi_3}, \quad z_4 = -\partial_{\xi_4}, \\ z_5 &= -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}, \\ z_6 &= -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}, \\ z_7 &= -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}, \\ z_8 &= -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}. \end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0, \quad i = 1, \dots, 6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
e	$-z_5$	$-z_6$	$-z_7$	$-z_8$	0	0	0	0
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	z_5	z_6	z_7	z_8
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	0	0	0	0	0	$-a_6$	a_5	$-a_3$
z_2	0	0	0	0	a_6	0	$-a_4$	a_2
z_3	0	0	0	0	$-a_5$	a_4	0	$-a_1$
z_4	0	0	0	0	a_3	$-a_2$	a_1	0
z_5	0	a_6	$-a_5$	a_3	0	0	0	0
z_6	$-a_6$	0	a_4	$-a_2$	0	0	0	0
z_7	a_5	$-a_4$	0	a_1	0	0	0	0
z_8	$-a_3$	a_2	$-a_1$	0	0	0	0	0

Let $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0)$ and $\sigma_1 + \sigma_2 + \sigma_3 = 0$. In [BO1], the family of Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, discovered by Kaplansky [Kapp*], [Kap*], is described. Observe that the Lie superalgebra $\mathfrak{osp}(4|2; \alpha)$, where $\alpha = \frac{\sigma_i}{\sigma_j}$ for $\sigma_j \neq 0$, is simple except for $\alpha = 0$ or -1 .

Since $\Gamma(\sigma_1, \sigma_2, \sigma_3) \simeq \Gamma(\sigma'_1, \sigma'_2, \sigma'_3)$ if and only if $(\sigma'_1, \sigma'_2, \sigma'_3) = a(\sigma_1, \sigma_2, \sigma_3)$, where $a \in \mathbb{C}^\times$ and the triple σ differs from σ' by a permutation of its components, see [BGL*], it follows that $\mathfrak{v}(\mathcal{M})$ is isomorphic to $\Gamma(1, -1, 0) = \mathfrak{osp}(4|2; -1)$.

10. We have $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - t\omega - \delta\partial_x, \quad f = -\partial_x, \\ a_1 &= \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \quad a_2 = \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \quad a_3 = \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \\ a_4 &= \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \quad a_5 = \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \quad a_6 = \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 8 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1} - \frac{1}{2t}\xi_1\partial_x, & z_2 &= -\partial_{\xi_2} - \frac{1}{2t}\xi_2\partial_x, \\ z_3 &= -\partial_{\xi_3} - \frac{1}{2t}\xi_3\partial_x, & z_4 &= -\partial_{\xi_4} - \frac{1}{2t}\xi_4\partial_x, \\ z_5 &= -x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x, \\ z_6 &= -x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x, \\ z_7 &= -x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \frac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x, \\ z_8 &= -x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \frac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x. \end{aligned}$$

$$[e, a_i] = [f, a_i] = [h, a_i] = 0, \quad i = 1, \dots, 6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
e	z_5	z_6	z_7	z_8	0	0	0	0
h	z_1	z_2	z_3	z_4	$-z_5$	$-z_6$	$-z_7$	$-z_8$
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	$-\frac{1}{t}f$	0	0	0	$\frac{1}{2t}h$	$-ta_6 + \frac{1}{2t}a_1$	$ta_5 + \frac{1}{2t}a_2$	$-ta_4 + \frac{1}{2t}a_3$
z_2	0	$-\frac{1}{t}f$	0	0	$ta_6 - \frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3 + \frac{1}{2t}a_4$	$ta_2 + \frac{1}{2t}a_5$
z_3	0	0	$-\frac{1}{t}f$	0	$-ta_5 - \frac{1}{2t}a_2$	$ta_3 - \frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1 + \frac{1}{2t}a_6$
z_4	0	0	0	$-\frac{1}{t}f$	$ta_4 - \frac{1}{2t}a_3$	$-ta_2 - \frac{1}{2t}a_5$	$ta_1 - \frac{1}{2t}a_6$	$\frac{1}{2t}h$
z_5	$\frac{1}{2t}h$	$-ta_6 + \frac{1}{2t}a_1$	$ta_5 + \frac{1}{2t}a_2$	$-ta_4 + \frac{1}{2t}a_3$	$\frac{1}{t}e$	0	0	0
z_6	$ta_6 - \frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3 + \frac{1}{2t}a_4$	$ta_2 + \frac{1}{2t}a_5$	0	$\frac{1}{t}e$	0	0
z_7	$-ta_5 - \frac{1}{2t}a_2$	$ta_3 - \frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1 + \frac{1}{2t}a_6$	0	0	$\frac{1}{t}e$	0
z_8	$ta_4 - \frac{1}{2t}a_3$	$-ta_2 - \frac{1}{2t}a_5$	$ta_1 - \frac{1}{2t}a_6$	$\frac{1}{2t}h$	0	0	0	$\frac{1}{t}e$

Comparing the above table with the tables in [BO1] we deduce that this Lie superalgebra is

$$\Gamma\left(\frac{1}{2t}, -\frac{1}{2}\left(\frac{1}{2t} + t\right), -\frac{1}{2}\left(\frac{1}{2t} - t\right)\right) \simeq \Gamma(2, -(2t^2 + 1), 2t^2 - 1) \simeq \mathfrak{osp}(4|2; \frac{2t^2 + 1}{1 - 2t^2}).$$

References

- [BGL*] Bouarroudj S., Grozman P., Leites D., Classification of finite dimensional modular Lie superalgebras with indecomposable Cartan matrix. Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), **5**(60) (2009), 63 pages.

- [BGLS*] Bouarroudj S., Grozman P., Leites D., Shchepochkina I., Minkowski superspaces and superstrings as almost real-complex supermanifolds. *Theor. and Mathem. Physics*, **173**(3), (2012) 1687–1708.
- [BO1] Bunegina V.A., Onishchik A.L., Two families of flag supermanifolds. *Different. Geom. and its Appl.* (1994) **4**, 329–360.
- [BO2] Bunegina V.A., Onishchik A.L., Homogeneous supermanifolds related with complex projective line. In: *Algebraic geometry. I. J. Math. Sci.* (1996) **82**(4), 3503–3527.
- [Gr] Green P., On holomorphic graded manifolds. *Proc. of the AMS.* (1982) **85**(4), 587–590.
- [Kapp*] Kaplansky I., Graded Lie algebras I, II, preprints, Univ. Chicago, Chicago, Ill., 1975, see <https://web.osu.cz/~Zusmanovich/links/files/kaplansky/>.
- [Kap*] Kaplansky I., Superalgebras. *Pacific J. Math.*, (1980) **86**(1), 93–98
- [O1] Onishchik A.L., A Construction of Non-Split Supermanifolds. *Annals of Global Analysis and Geometry*, (1998) **16** 309–333.
- [O2] Onishchik A.L., Non-Abelian Cohomology and Supermanifolds. SFB 288. Preprint N 360. Berlin, (1998) 1–38.
- [O3] Onishchik A.L., Lifting of holomorphic actions on complex supermanifolds. *Lie Groups, Geometric Structures and Differential Geometry. Adv. Studies in Pure Math.* **37**. Kyoto, (2002) 317–335.
- [W*] Witten E. On Super Riemann Surfaces and String Theory, *Pure and Applied Mathematics Quarterly*, (2019) **15**(1), 1-607.