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Homogeneous superstrings with retract $\mathcal{CP}^{1|4}$

Mikhail Bashkin

Abstract. Any complex-analytic supermanifold whose retract is diffeomorphic to the complex projective superline (superstring) $\mathcal{CP}^{1|4}$ is, up to a diffeomorphism, either a member of a 1-parameter family or one of 9 exceptional supermanifolds. I singled out the homogeneous of these supermanifolds and described Lie superalgebras of vector fields on them.

1 Preliminaries

Let (M, \mathcal{F}) be a complex-analytic manifold of dimension m . (More precisely, almost complex manifold, see [BGLS*], since the vanishing of the Nijenhuis tensor is never need; however, from the very beginning (see [Gr]) one requires the underlying manifold to be complex. *This comment and starred references are added by the editor of this Special Volume. D.Leites.*)

Let \mathbf{E} be a vector bundle of rank n over M , and \mathcal{E} the be locally free analytic sheaf of sections of \mathbf{E} . Set $\tilde{\mathcal{O}} := \Lambda_{\mathcal{F}}^{\bullet}(\mathcal{E})$.

The supermanifold isomorphic to the one of the form $\mathcal{M} := (M, \tilde{\mathcal{O}})$ is called *split*. The ringed space locally isomorphic to $(M, \Lambda_{\mathcal{F}}^{\bullet}(\mathcal{E}))$ is called a *supermanifold of superdimension $m|n$* . Physicists call supermanifolds of dimension $1|n$ *superstrings*, see [W*]. Let \mathcal{O} be a structure sheaf of any supermanifold. Let $\mathcal{I} \subset \mathcal{O}$ be the subsheaf of ideals generated by subsheaf $\mathcal{O}_{\bar{1}}$ and let $\mathcal{O}_{\text{rd}} := \mathcal{O}/\mathcal{I}$.

Consider the following filtration of \mathcal{O} by powers of \mathcal{I}

$$\mathcal{O} = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \dots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.$$

The graded sheaf $\text{gr } \mathcal{O} = \bigoplus_{p=0}^n \text{gr}_p \mathcal{O}$ with $\text{gr}_p \mathcal{O} := \mathcal{I}^p / \mathcal{I}^{p+1}$ defines the split supermanifold $(M, \text{gr } \mathcal{O})$ called the *retract* of (M, \mathcal{O}) .

Let $\pi : \mathcal{I}^p \rightarrow \text{gr}_p \mathcal{O}$ denote the natural projection. Then, we have the exact sequences of sheaves

$$0 \longrightarrow \mathcal{I}^{p+1} \longrightarrow \mathcal{I}^p \xrightarrow{\pi_p} \text{gr}_p \mathcal{O} \longrightarrow 0. \tag{1}$$

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The supermanifold (M, \mathcal{O}) is split if and only if there exists an isomorphism of the superalgebra sheaves $h : \text{gr } \mathcal{O} \rightarrow \mathcal{O}$ such that its restriction $h_p : \text{gr}_p \mathcal{O} \rightarrow \mathcal{I}^p$ splits the sequence (1), i.e., satisfies $\pi_p \circ h_p = \text{id}$. Such an isomorphism exists in a neighborhood of any point of M .

Let (M, \mathcal{O}) be a supermanifold and \mathfrak{g} a complex finite-dimensional Lie superalgebra. An *action* of \mathfrak{g} on (M, \mathcal{O}) is an arbitrary Lie superalgebra homomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{v}(M, \mathcal{O})$. Then, a linear mapping $\varphi^x : \mathfrak{g} \rightarrow T_x(M, \mathcal{O})$ is associated with any $x \in M$. The action φ is called *transitive* if φ^x is surjective for any $x \in M$. By restricting the action $\varphi : \mathfrak{g} \rightarrow \mathfrak{v}(M, \mathcal{O})$ to the even component we get a homomorphism $\varphi_{\bar{0}} : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$. If M is compact, then it is possible to integrate $\varphi_{\bar{0}}$ getting a homomorphism $\Phi : G \rightarrow \text{Aut}(M, \mathcal{O})$, where G is the simply connected complex Lie group whose Lie algebra is $\mathfrak{g}_{\bar{0}}$. This homomorphism induces a homomorphism $\Phi_0 : G \rightarrow \text{Bih } M$ into the group of biholomorphic automorphisms of M , in other words — an action of G on M . The action φ is said to be $\bar{0}$ -*transitive* if Φ_0 is transitive.

If a Lie group G acts $\bar{0}$ -transitivity on M , then $\varphi^x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M)$ is surjective for any $x \in M$. Conversely, if M is compact and $\varphi^x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M)$ is surjective for any $x \in M$, we can integrate this action to a $\bar{0}$ -transitive action of a Lie group.

The supermanifold (M, \mathcal{O}) is called *homogeneous* (resp. $\bar{0}$ -*homogeneous*) if the natural action of the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ on (M, \mathcal{O}) is transitive (resp. $\bar{0}$ -transitive), see [BO2], [O3]. This means that the evaluation mapping $ev_x : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_x(M, \mathcal{O})$ (resp. the restriction of ev_x to $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$ is surjective for any $x \in M$.

Thanks to [O3] we know that φ is transitive if and only if it is $\bar{0}$ -transitive, M is compact, and the mapping $\varphi_{\bar{1}}^x : \mathfrak{g}_{\bar{1}} \rightarrow T_{x_0}(M, \mathcal{O})_{\bar{1}}$ is surjective at a certain point $x_0 \in M$. This implies that a $\bar{0}$ -homogeneous supermanifold is homogeneous if and only if the odd component of the mapping $ev_{x_0} : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_{x_0}(M, \mathcal{O})$ is surjective at a certain point $x_0 \in M$.

One easily proves (see [BO2]) that the retract of a homogeneous supermanifold (M, \mathcal{O}) with compact M is homogeneous, too.

In what follows, I consider the problem of classification (up to a diffeomorphism) of supermanifolds with retract $\mathcal{CP}^{1|4}$ and describe which of the supermanifolds considered are homogeneous or at least $\bar{0}$ -*homogeneous*.

2 Superstring $\mathcal{CP}^{1|4}$. The first cohomology of the tangent sheaf

Over \mathbb{CP}^1 , consider the holomorphic vector bundle

$$\mathbf{E} = \mathbf{L}_{-k_1} \oplus \mathbf{L}_{-k_2} \oplus \mathbf{L}_{-k_3} \oplus \mathbf{L}_{-k_4}, \text{ where } k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0.$$

Let $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4} := (\mathbb{CP}^1, \mathcal{O}_{\Lambda_{\mathcal{F}}(\mathcal{E})})$ designate the split supermanifold determined by \mathbf{E} . As shown in [BO2], if $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4}$ is homogeneous, then the k_i must be non-negative.

Let us cover \mathbb{CP}^1 by two charts U_0 and U_1 with local coordinates x and $y = \frac{1}{x}$, respectively. For $\mathcal{CP}_{k_1 k_2 k_3 k_4}^{1|4}$, the transition functions in $U_0 \cap U_1$ are $y = x^{-1}$ and $\eta_i = x^{-k_i} \xi_i$ for $i = 1, \dots, 4$, where ξ_i and η_i are basis sections of \mathbf{E} over U_0 and U_1 , respectively.

If \mathcal{O}_{gr} is the structure sheaf of $\mathcal{CP}^{1|4} := \mathcal{CP}_{1111}^{1|4}$, then $\mathcal{T}_{\text{gr}} := \mathcal{D}er \mathcal{O}_{\text{gr}}$ is the *tangent sheaf* (or the sheaf of vector fields). This is a sheaf of Lie superalgebras. The sections of the tangent sheaf are *holomorphic vector fields* on the supermanifold. Their sections are elements of the Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}}) := \Gamma(\mathbb{CP}^1, \mathcal{T}_{\text{gr}})$ of vector fields on $\mathcal{CP}^{1|4}$.

The sheaf \mathcal{T}_{gr} has a \mathbb{Z} -grading $\mathcal{T}_{\text{gr}} = \bigoplus_{-1 \leq p \leq 4} (\mathcal{T}_{\text{gr}})_p$, where

$$(\mathcal{T}_{\text{gr}})_p := \mathcal{D}er_p \mathcal{O}_{\text{gr}} = \{v \in \mathcal{T}_{\text{gr}} \mid v((\mathcal{O}_{\text{gr}})_q) \subset (\mathcal{O}_{\text{gr}})_{p+q} \text{ for any } q\}.$$

The Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})$ becomes also \mathbb{Z} -graded with the induced grading compatible with $\mathbb{Z}/2$ -grading by parity.

We can regard \mathcal{T}_{gr} as a locally free analytic sheaf on \mathbb{CP}^1 . From [O3] we have the following exact sequence of locally free analytic sheaves on \mathbb{CP}^1 :

$$0 \longrightarrow \mathcal{E}^* \otimes \Lambda^* \mathcal{E} \xrightarrow{i} \mathcal{T}_{\text{gr}} \xrightarrow{\alpha} \Theta \otimes \Lambda^* \mathcal{E} \longrightarrow 0,$$

where $\Theta = \mathcal{D}er \mathcal{F}$ is the tangent sheaf of the manifold \mathbb{CP}^1 , and \mathcal{F} is the sheaf of functions on \mathbb{CP}^1 . The mapping α is the restriction of a derivation of \mathcal{O} to \mathcal{F} , and i identifies any sheaf homomorphism $\mathcal{E} \rightarrow \Lambda^* \mathcal{E}$ with its prolongation to a derivation that vanishes on \mathcal{F} . Hence, the analytic sheaf \mathcal{T}_{gr} is locally free. Therefore, \mathcal{T}_{gr} is the sheaf of holomorphic sections of a holomorphic vector bundle over \mathbb{CP}^1 . We call it the *supertangent bundle* and denote **ST**.

Thanks to the Bott-Borel-Weil theorem the following theorem holds.

2.1 Theorem ([BO1]). *For $\dim H^p(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_q)$, see the following table*

q	-1	0	1	2	3	4
$p = 0$	8	19	8	0	0	0
$p = 1$	0	0	0	10	8	1

The group $\text{SL}_2(\mathbb{C})$ trivially acts on $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$.

Set

$$\begin{aligned} \delta &:= \xi_1 \xi_2 \xi_3 \xi_4, & \delta_l &:= \frac{\partial \delta}{\partial \xi_l}, & \nabla &:= \sum_{1 \leq i \leq 4} \xi_i \partial_{\xi_i}; \\ \delta' &:= \eta_1 \eta_2 \eta_3 \eta_4, & \delta'_l &:= \frac{\partial \delta'}{\partial \eta_l}, & \nabla' &:= \sum_{1 \leq i \leq 4} \eta_i \partial_{\eta_i}. \end{aligned} \tag{2}$$

The next Theorem expounds the result of Theorem 2.1 by giving the Čzech cocycles of the covering $\{U_0, U_1\}$; these cocycles determine the bases of non-zero spaces H^1 .

2.2. Theorem. *The basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_p)$, where $p = 2, 3, 4$, can be represented by the following cocycles z_{01} :*

p	z_{01}
2	$x^{-1} \delta_l \partial_{\xi_k} \sim x^{-1} \delta_k \partial_{\xi_l}$ for $l < k$ and $k, l = 1, \dots, 4$, $x^{-1} \delta_r \partial_{\xi_r}$ for $r = 1, \dots, 4$;
3	$x^{-1} \delta_l \partial_x \sim x^{-2} \delta \partial_{\xi_l}$ and $x^{-1} \delta \partial_{\xi_l}$ for $l = 1, \dots, 4$;
4	$x^{-1} \delta \partial_x$.

Proof. Let us find a basis in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$. For this, consider the part $\mathbf{ST}(\mathbf{E})_2$ of the tangent space $\mathbf{ST}(\mathbf{E}) := \bigoplus \mathbf{ST}(\mathbf{E})_i$. In [BO2], the following decomposition was considered:

$$\mathbf{ST}(\mathbf{E})_2 = \bigoplus_{i < j} \mathbf{ST}(\mathbf{E})_2^{(ij)} \oplus \bigoplus_{i < j < k, r \neq i, j, k} \mathbf{ST}(\mathbf{E})_2^{(ijk, r)}, \text{ where}$$

$$\begin{aligned} \mathbf{ST}(\mathbf{E})_2^{(ij)} &= \langle \xi_i \xi_j \partial_x \text{ and } \xi_i \xi_j \xi_k \partial_{\xi_k}, \text{ where } i < j, \text{ and } k \neq i, j \rangle, \\ \mathbf{ST}(\mathbf{E})_2^{(ijk, r)} &= \langle \xi_i \xi_j \xi_k \partial_{\xi_r}, \text{ where } i < j < k, \text{ and } r \neq i, j, k \rangle \end{aligned}$$

Moreover, it was shown in [BO2] that $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk, r)} \simeq \mathbf{L}_{-2}$. Consider the bundles $\mathbf{ST}(\mathbf{E})_2^{(ij)}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk, r)}$ separately. Let $(\mathcal{T}_{\text{gr}})_p^{i_1 \dots i_k}$ designate the sheaf of holomorphic sections of $\mathbf{ST}(\mathbf{E})_p^{(i_1 \dots i_k)}$. We see that (recall notation (2))

$$\begin{aligned} \delta_l \partial_{\xi_k} &= y^{-2} \delta'_l \partial_{\eta_k}, \text{ where } l < k, \quad k \neq i, j, \quad l \neq i, j, \\ \xi_i \xi_j \partial_x &= -\eta_i \eta_j \partial_y - y^{-1} \eta_i \eta_j \nabla' = -y^{-1} (\eta_i \eta_j \nabla' + y \eta_i \eta_j \partial_y), \\ \xi_i \xi_j \nabla + x \xi_i \xi_j \partial_x &= -y^{-1} \xi_i \xi_j \partial_x. \end{aligned}$$

Hence, for the basis sections of \mathbf{L}_{-2} (resp. \mathbf{L}_{-1}) we can take

$$\delta_l \partial_{\xi_k}, \quad \xi_i \xi_j \partial_x, \text{ (resp. } \xi_i \xi_j \nabla + x \xi_i \xi_j \partial_x \text{) for all } i, j, k, l.$$

Then, $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $(\mathcal{T}_{\text{gr}})_2^{ij} \simeq \mathcal{F}(-2) \oplus 2\mathcal{F}(-1)$, where \mathcal{F} is the sheaf of holomorphic functions on \mathbb{CP}^1 , is the corresponding isomorphism of sheaves.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2)) \quad \text{and} \quad \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ij}) = 1,$$

so the cocycle desired is of the form

$$x^{-1} \delta_l \partial_{\xi_k}.$$

Let us show that $x^{-1} \delta_l \partial_{\xi_k} \sim x^{-1} \delta_k \partial_{\xi_l}$.

Indeed,

$$\begin{aligned} x^{-1} \delta_l \partial_{\xi_k} &= y^{-1} \delta'_l \partial_{\eta_k} \sim y^{-1} \delta'_l \partial_{\eta_k} + \eta_s \eta_t \partial_y \\ &= x^{-1} \delta_l \partial_{\xi_k} - \xi_s \xi_t \partial_x - x^{-1} \delta_l \partial_{\xi_k} + x^{-1} \delta_k \partial_{\xi_l} \sim x^{-1} \delta_k \partial_{\xi_l}, \end{aligned}$$

where $(l, k; s, t) \in \{(1, 2; 3, 4), (1, 4; 2, 3), (2, 3; 1, 4), (3, 4; 1, 2)\}$;

$$\begin{aligned} x^{-1} \delta_l \partial_{\xi_k} &= y^{-1} \delta'_l \partial_{\eta_k} \sim y^{-1} \delta'_l \partial_{\eta_k} - \eta_s \eta_t \partial_y \\ &= x^{-1} \delta_l \partial_{\xi_k} + \xi_s \xi_t \partial_x - x^{-1} \delta_l \partial_{\xi_k} + x^{-1} \delta_k \partial_{\xi_l} \sim x^{-1} \delta_k \partial_{\xi_l}, \end{aligned}$$

where $(l, k; s, t) \in \{(1, 3; 2, 4), (2, 4; 1, 3)\}$.

Since $\delta_r \partial_{\xi_r} = y^{-2} \delta'_r \partial_{\eta_r}$, then take $\delta_r \partial_{\xi_r}$ for a basis section of \mathbf{L}_{-2} . We have

$$\mathbf{ST}(\mathbf{E})_2^{(ijk, r)} \simeq \mathbf{L}_{-2}$$

and the corresponding isomorphism of sheaves $(\mathcal{T}_{\text{gr}})_2^{ijk,r} \simeq \mathcal{F}(-2)$.

The results of [BO2] imply that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ijk,r}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2))$,

$$\dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2^{ijk,r}) = 1 \text{ for any } i < j < k, \text{ and } r \neq i, j, k,$$

and the cocycle desired is of the form

$$x^{-1} \delta_r \partial_{\xi_r}.$$

Let us now find the basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3)$. Consider $\mathbf{ST}(\mathbf{E})_3$ and again apply the results of [BO2]; we get

$$\mathbf{ST}(\mathbf{E})_3 = \bigoplus_{i < j < k} \mathbf{ST}(\mathbf{E})_3^{(ijk)},$$

where $\mathbf{ST}(\mathbf{E})_3^{(ijk)}$ is spanned by

$$\begin{aligned} & \xi_i \xi_j \xi_k \partial_x, \quad \delta \partial_{\xi_l}, \text{ where } 1 \leq i < j < k \leq 4, \quad l \neq i, j, k, \quad l \in \{1, \dots, 4\}; \\ & \xi_i \xi_j \xi_k \partial_x = -y^{-2} (y \eta_i \eta_j \eta_k \partial_y + \eta_i \eta_j \eta_k \eta_l \partial_{\eta_l}), \\ & \delta \partial_{\xi_l} - x \delta_l \partial_x = y^{-3} \delta' \partial_{\eta_l} + y^{-2} \delta'_l \partial_y - y^{-3} \delta' \partial_{\eta_l} = y^{-2} \delta'_l \partial_y. \end{aligned}$$

Hence, take $\xi_i \xi_j \xi_k \partial_x$ and $\delta \partial_{\xi_l} - x \delta_l \partial_x$ for basis sections of \mathbf{L}_{-2} and \mathbf{L}_{-1} .

We see that $\mathbf{ST}(\mathbf{E})_3^{(ijk)} \simeq 2\mathbf{L}_{-2}$ and $(\mathcal{T}_{\text{gr}})_3^{ijk} \simeq 2\mathcal{F}(-2)$.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3^{ijk}) \simeq 2H^1(\mathbb{CP}^1, \mathcal{F}(-2)), \text{ and } \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_3^{ijk}) = 2,$$

where $1 \leq i < j < k \leq 4$, and the cocycles desired are of the form

$$x^{-1} \delta_l \partial_x \text{ and } x^{-1} \delta \partial_{\xi_l} - \delta_l \partial_x \sim x^{-1} \delta \partial_{\xi_l}, \quad \text{where } l \neq i, j, k.$$

Let us show that

$$x^{-1} \delta_l \partial_x \sim x^{-2} \delta \partial_{\xi_l}.$$

Indeed, $x^{-1} \delta_l \partial_x = -\delta'_l \partial_y + y^{-1} \delta' \partial_{\eta_l} \sim y^{-1} \delta' \partial_{\eta_l} = x^{-2} \delta \partial_{\xi_l}$.

In [BO1], it is proved that the basis element of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ can be represented by the cocycle $x^{-1} \delta \partial_x$. \square

3 Non-split supermanifolds with retract $\mathcal{CP}^{1|4}$

The structure sheaf of the split supermanifold $(M, \mathcal{O}_{\text{gr}})$ is endowed with a \mathbb{Z} -grading

$$\mathcal{O}_{\text{gr}} = \bigoplus_{0 \leq p \leq n} (\mathcal{O}_{\text{gr}})_p, \quad \text{where } (\mathcal{O}_{\text{gr}})_p = \Lambda_{\mathcal{F}}^p(\mathcal{E}).$$

Clearly, $(\mathcal{O}_{\text{gr}})_{\text{rd}}$ is naturally isomorphic to the subsheaf $\mathcal{F} \subset \mathcal{O}_{\text{gr}}$.

Observe that the natural filtration of the sheaf $\mathcal{T} = \mathcal{D}er \mathcal{O}$ yields the following filtration

$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) = \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(-1)} \supset \dots \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(4)} \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(5)} = 0,$$

where $\mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p)} = \Gamma(\mathbb{C}\mathbb{P}^1, \mathcal{T}_{(p)})$.

Thanks to results in [O1], we have the following exact sequence

$$0 \rightarrow \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p+1)} \rightarrow \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O})_{(p)} \xrightarrow{\sigma_p} \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O}_{\text{gr}})_p \quad \text{for any } p \geq -1.$$

We say that $u \in \mathfrak{v}(\mathbb{C}\mathbb{P}^1, \mathcal{O}_{\text{gr}})_p$ is *liftable* in $(\mathbb{C}\mathbb{P}^1, \mathcal{O})$, if $u \in \text{Im } \sigma_p$.

Consider

$$\text{Aut}_{(2)}\mathcal{O}_{\text{gr}} = \{a \in \text{Aut } \mathcal{O}_{\text{gr}} \mid a(f) - f \in \mathcal{J}^2 \text{ for any } f \in \mathcal{O}_{\text{gr}}\}.$$

Let $\text{Aut } \mathbf{E}$ be the group of fiber-preserving automorphisms of \mathbf{E} . Then, the following theorem holds.

3.1 Theorem ([Gr]). *There is a bijective correspondence between the isomorphism classes of supermanifolds (M, \mathcal{O}) such that $\text{gr } \mathcal{O} \simeq \mathcal{O}_{\text{gr}}$ and the orbits of the action of $\text{Aut } \mathbf{E}$ on $H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$, and $(M, \mathcal{O}_{\text{gr}})$ corresponds to the unit class e in $H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.*

4 On $\bar{0}$ -homogeneity of supermanifolds with retract $\mathcal{C}\mathcal{P}^{1|4}$

Recall a fine resolution of the sheaf $\mathcal{T}_{\text{gr}} = \text{Der } \mathcal{O}_{\text{gr}}$ endowed with a bracket operation that extends the bracket given in \mathcal{T}_{gr} . Let us denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued forms of type (p, q) on M . We form the standard Dolbeault-Serre resolution $\widehat{\Phi}$ of \mathcal{O}_{gr} by setting for any $\varphi \in \Phi^{0,q}$ and $u \in (\mathcal{O}_{\text{gr}})_p$

$$\widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{\text{gr}})_p, \quad \widehat{\Phi}^{\bullet,\bullet} := \bigoplus_{p,q \geq 0} \widehat{\Phi}^{p,q}, \quad \bar{\partial}(\varphi \otimes u) = \bar{\partial}(\varphi) \otimes u.$$

Then, regarding \mathcal{S} as a sheaf of graded algebras with respect to the total degree, consider the sheaf of bigraded Lie superalgebras $\widehat{\mathcal{T}} = \text{Der } \widehat{\Phi}$. Clearly, $\bar{D} = \text{ad}_{\bar{\partial}}$ is a derivation of bidegree $(0, 1)$ of $\widehat{\mathcal{T}}$ satisfying $\bar{D}^2 = 0$. Set

$$\mathcal{S} := \{u \in \widehat{\mathcal{T}} \mid u(\bar{f}) = 0 \text{ and } u(df) = 0 \text{ for any } f \in \mathcal{F}\}.$$

It is easy to see that \mathcal{S} is a \bar{D} -invariant subsheaf of bigraded subalgebras of $\widehat{\mathcal{T}}$. Moreover, $\widehat{\mathcal{T}}_{\text{gr}}$ is identified with the kernel of the mapping $\bar{D} : \mathcal{S}^{\bullet,0} \rightarrow \mathcal{S}^{\bullet,1}$. Thus, we get the sequence

$$0 \longrightarrow \widehat{\mathcal{T}} \xrightarrow{\tau} \mathcal{S}^{\bullet,0} \xrightarrow{\bar{D}} \mathcal{S}^{\bullet,1} \xrightarrow{\bar{D}} \dots$$

Let us specify an explicit form of τ . Let \mathcal{F}^∞ be a sheaf of differentiable complex-valued functions on M , then $\mathcal{O}_{\text{gr}}^\infty = \mathcal{O}_{\text{gr}} \otimes \mathcal{F}^\infty$ and

$$\begin{aligned} \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty = \\ \{a \in \text{Aut } \mathcal{O}_{\text{gr}}^\infty \mid a(\bar{f}) = \bar{f} \text{ for any } f \in \mathcal{F}; a(u) - u \in \bigoplus_{k \geq 2} (\mathcal{O}_{\text{gr}}^\infty)_k \text{ for any } u \in \mathcal{O}_{\text{gr}}^\infty\}. \end{aligned}$$

If $z = (z_{ij}) \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ is a cocycle in the covering \mathfrak{U} , then $z_{ij} = h_i^{-1}h_j$, where $h_i : \mathcal{O}_{\text{gr}}|_{U_i} \rightarrow \mathcal{O}|_{U_i}$. On the other hand, $z_{ij} = a_i^{-1}a_j$, where $a_i \in \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty(U_i)$. Then,

over $U_i \cap U_j$, we have $h_i^{-1}h_j = a_i^{-1}a_j$, and hence $\varrho := a_i h_i^{-1} = a_j h_j^{-1}$ is an injective sheaf homomorphism $\mathcal{O} \rightarrow \mathcal{O}_{\text{gr}}^\infty$. Then, $\tau : \widehat{\mathcal{T}} \rightarrow \mathcal{S}^{*,0}$ is defined by the formula $\tau(v) := \varrho v \varrho^{-1}$.

Now, let the bundle \mathbf{E} correspond to the supermanifold $\mathcal{CP}^{1|4}$; let the tangent bundle $\mathbf{ST}(\mathbf{E})$ be endowed with a smooth SU_2 -invariant hermitian metric (see [O1]). Since \mathbb{CP}^1 is compact, we can apply the Hodge theory. In [O1], [O3], [O2] a complex (\mathcal{S}, \bar{D}) is constructed which can be considered as a complex of $(0, *)$ -forms with values in the bundle $\mathbf{ST}(\mathbf{E})$.

Let $\mathbf{H} \subset \mathcal{S}$ denote the bigraded space of harmonic elements, H the orthogonal projection to \mathbf{H} . As is known,

$$\mathbf{H}_{p,q} \simeq H^{p,q}(\mathcal{S}, \bar{D}) \simeq H^q(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_p) \text{ for any } p, q \geq 0. \quad (3)$$

Set

$$\mathbf{H}_{(1)} = \bigoplus_{p \geq 1} \mathbf{H}_{2p,1} = \mathbf{H}_{2,1} \oplus \mathbf{H}_{4,1} \simeq H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_2) \oplus H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_4).$$

The SU_2 -invariance of the metric implies SU_2 -equivariance of \mathbf{H} , and isomorphisms (3).

4.1. Theorem. *Let $(\mathbb{CP}^1, \mathcal{O})$ be any supermanifold with retract $\mathcal{CP}^{1|4}$. Then, the SU_2 -action on \mathbb{CP}^1 can be lifted to $(\mathbb{CP}^1, \mathcal{O})$. In particular, $(\mathbb{CP}^1, \mathcal{O})$ is $\bar{0}$ -homogeneous.*

Proof. Consider the non-linear complex $K = (K^0, K^1, K^2)$ (see [BO2]), where

$$\begin{aligned} K^0 &= \Gamma(\mathbb{CP}^1, \mathcal{P}\text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty), \\ K^q &= \bigoplus_{k \geq 1} S^{2k,q} \text{ for } q = 1, 2, \end{aligned}$$

with the coboundary operators $\delta_q : K^q \rightarrow K^{q+1}$ for $q = 0, 1$, and the action ρ of the group K^0 on K^1 , defined by the formulas

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1}, \\ \delta_1(u) &= \bar{\partial}u - \frac{1}{2}[u, u], \\ \rho(a)(u) &= a(u - \bar{\partial})a^{-1} + \bar{\partial}. \end{aligned}$$

By definition, the corresponding set of 1-cohomology has the form $Z^1(K)/\rho(K^0)$, where $Z^1(K) = \{u \in K^1 \mid \delta_1(u) = 0\}$.

Since $\dim \mathbb{CP}^1 = 1$, it follows that $\mathbf{H}_{(1)} \subset Z^1(K)$. Moreover, as is shown in [O2], the natural map $\mathbf{H}_{(1)} \rightarrow H^1(K)$ is surjective.

Further, $H^0(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2) = 0$ by Theorem 2.1. Hence, Theorem 3.13 from [O2] is applicable implying that this map is bijective. Thus, the bijection $\mathbf{H}_{(1)} \rightarrow H^1(K)$ is SU_2 -invariant.

On the other hand, Theorem 2.1 implies that SU_2 acts on $\mathbf{H}_{(1)}$ trivially. Hence, it acts on $H^1(K)$ also trivially, and every cohomology class contains an invariant cocycle.

Applying the obtained in [O3] criterion for lifting the action of the compact groups on the non-split supermanifold we see that the SU_2 -action on $\mathcal{CP}^{1|4}$ can be lifted to any supermanifold $(\mathbb{CP}^1, \mathcal{O})$ with $\mathcal{CP}^{1|4}$ as its retract. Since SU_2 transitively acts on \mathbb{CP}^1 , all these supermanifolds are $\bar{0}$ -homogeneous. \square

5 Description of supermanifolds via cocycles

Thanks to Theorem 3.1, the classes of isomorphic supermanifolds (M, \mathcal{O}) are in bijective correspondence with the $\text{Aut } \mathbf{E}$ -orbits on the set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. Let $M = \mathbb{C}\mathbb{P}^1$ and the odd dimension of supermanifolds $(\mathbb{C}\mathbb{P}^1, \mathcal{O})$ be ≤ 5 .

As in [O2], define the *exponential map*

$$\exp : \mathcal{T}_{\text{gr}} = (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4 \longrightarrow \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}},$$

its inverse $\log := \exp^{-1}$, and the map

$$\lambda_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}} \longrightarrow (\mathcal{T}_{\text{gr}})_{2p}$$

sending any $a \in \mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}}$ to the $2p$ -component $(\log a)_{2p}$ of $\log a$.

The map \exp is an isomorphism of sheaves of sets, λ_{2p} is a surjective homomorphism of sheaves of groups. In what follows, we will represent the cocycle $g = \exp u$ by the cocycle $u = u_2 + u_4$, where

$$u_2 \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_2) \text{ and } u_4 \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_4). \quad (4)$$

Since

$$(\mathcal{T}_{\text{gr}})_2 \cdot (\mathcal{T}_{\text{gr}})_4 = (\mathcal{T}_{\text{gr}})_4 \cdot (\mathcal{T}_{\text{gr}})_4 = 0,$$

it follows that $(\mathcal{T}_{\text{gr}})_4$ is a central ideal in \mathcal{T}_{gr} .

The exact sequence (see [O2])

$$e \longrightarrow \mathcal{A}ut_{(2p+2)}\mathcal{O} \longrightarrow \mathcal{A}ut_{(2p)}\mathcal{O} \xrightarrow{\lambda_{2p}} (\mathcal{T}_{\text{gr}})_{2p} \longrightarrow 0$$

yields — for $p = 2$ — an isomorphism of the sheaves of groups

$$\exp : (\mathcal{T}_{\text{gr}})_4 \longrightarrow \mathcal{A}ut_{(4)}\mathcal{O}_{\text{gr}} \subset \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}.$$

Hence, $\mathcal{A}ut_{(4)}\mathcal{O}_{\text{gr}}$ belongs to the center of $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$.

Define the action of the sheaf of groups $(\mathcal{T}_{\text{gr}})_4$ on $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$ by mean of right shifts

$$\Psi: v \mapsto t_v : z \mapsto z(\exp v), \text{ where } v \in (\mathcal{T}_{\text{gr}})_4, \text{ and } z \in \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}.$$

Let us translate this action to Čzech cocycles of the covering U_0, U_1 . Let us check if it is also well defined on cohomology. Let $v' \sim v$ and $z' \sim z$, where

$$v', v \in Z^1(\mathbb{C}\mathbb{P}^1, (\mathcal{T}_{\text{gr}})_4), \quad z', z \in Z^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}).$$

Then, $v' = b^{(0)} + v - b^{(1)}$ and $z' = c^{(0)}z(c^{(1)})^{-1}$, where $b^{(i)}, c^{(i)}$ are holomorphic sections over U_i for $i = 0, 1$ of the sheaves $(\mathcal{T}_{\text{gr}})_4$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$, respectively. We see that

$$\begin{aligned} z' \exp v' &= (c^{(0)}z(c^{(1)})^{-1})(\exp v)(\exp b^{(0)})(\exp b^{(1)})^{-1} \\ &= (c^{(0)}\exp b^{(0)})(z \exp v)(c^{(1)}\exp b^{(1)})^{-1} \sim z \exp v. \end{aligned}$$

Therefore, the action Ψ on cohomology is well defined.

5.1. Theorem. *Let $\dim(\mathbb{CP}^1, \mathcal{O}) = 1|n$ with $n \leq 5$. Then, the action Ψ defines a free action of the group $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$; the orbits of this action are the fibers of the map*

$$\lambda_2^* : H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}) \longrightarrow H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2).$$

Proof. To show that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ freely acts on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$, i.e., the stabilizer of any element $z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ is trivial, let $z(\exp v) \sim z$. By the above, there exists an element

$$u \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)$$

such that $z = \exp u$.

Let $u = u_2 + u_4$, see (4). Since

$$v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4),$$

then

$$z(\exp v) = \exp(u_2 + u_4)\exp v = \exp(u_2 + u_4 + v_4).$$

Hence,

$$\exp(u_2 + u_4 + v_4) \sim \exp(u_2 + u_4).$$

Therefore,

$$\exp(u_2 + u_4 + v_4) = c^{(0)}\exp(u_2 + u_4)(c^{(1)})^{-1},$$

where $c^{(i)} \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are holomorphic sections for $i = 0, 1$.

Let us represent $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$, where $a_2^{(i)}$ and $a_4^{(i)}$ are holomorphic sections of the sheaves $(\mathcal{T}_{\text{gr}})_2$ and $(\mathcal{T}_{\text{gr}})_4$, respectively, over U_i for $i = 0, 1$. Then, applying the Campbell-Hausdorff decomposition twice, we see that

$$\begin{aligned} & \exp(a_2^{(0)} + a_4^{(0)}) \exp(u_2 + u_4) \exp(-a_2^{(1)} - a_4^{(1)}) \\ &= \exp(a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \\ & \quad - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]). \end{aligned}$$

Hence,

$$\begin{aligned} u_2 + u_4 + v_4 &= a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] \\ & \quad - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]. \end{aligned}$$

Therefore,

$$v_4 = a_2^{(0)} + a_4^{(0)} - a_2^{(1)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}].$$

With respect to the degrees this equality breaks into two:

$$\begin{aligned} 0 &= a_2^{(0)} - a_2^{(1)}, \\ v_4 &= a_4^{(0)} - a_4^{(1)} + \frac{1}{2}[a_2^{(0)}, u_2] - \frac{1}{2}[u_2, a_2^{(1)}] - \frac{1}{2}[a_2^{(0)}, a_2^{(1)}]. \end{aligned}$$

Hence, we see that $v_4 = a_4^{(0)} - a_4^{(1)}$, i.e., $v \sim 0$.

Let us show now that the orbits of the $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$ -action on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are fibers of the map λ_2^* . Indeed, let $z(\exp v) \sim y$, where

$$y, z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}), \quad v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4).$$

Then, there exist $u, w \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)$ such that $z = \exp u$ and $y = \exp w$.

Let

$$u = u_2 + u_4 \text{ and } w = w_2 + w_4,$$

where $u_2, w_2 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and $u_4, w_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$. We have

$$\exp(u_2 + u_4 + v_4) \sim \exp(w_2 + w_4)$$

or, as in the first part of the proof,

$$\exp(u_2 + u_4 + v_4) = c^{(0)} \exp(w_2 + w_4) (c^{(1)})^{-1},$$

where $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$. We similarly obtain

$$\exp(u_2 + u_4 + v_4) = \exp(a_2^{(0)} + a_4^{(0)} + w_2 + w_4 - a_2^{(1)} - a_4^{(1)}).$$

Having applied λ_2 to both sides of this equality we get

$$u_2 = a_2^{(0)} + w_2 - a_2^{(1)}.$$

Hence, $u_2 \sim w_2$, i.e., $z(\exp v)$ and y determine the same class in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ with respect to λ_2^* . \square

5.2. Corollary. *The classes in $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ are determined by pairs (u_2, u_4) , see (4).*

Proof. Theorem 2.1 implies that $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ can be viewed as the bundle with base $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2)$ and fiber $H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4)$. Thus, any class in $H^1(\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ is determined by (u_2, u_4) , see (4). \square

5.3. Theorem. *Any supermanifold with retract $\mathcal{CP}^{1|4}$ coincides, up to an isomorphism, with one of the following supermanifolds \mathcal{M} determined by the cocycle $u^{(01)} = u_2 + u_4$, see (4).*

*For homogeneous of these supermanifolds — the cases marked by *, the Lie superalgebra $\mathfrak{v}(\mathcal{M})$ is described in § 7.*

1*) $u_2 = 0$, and $u_4 = 0$;

2*) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = 0$;

3*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = 0$;

4*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = 0$;

5*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}$, and $u_4 = 0$;

- 6) $u_2 = 0$, and $u_4 = x^{-1}\delta\partial_x$;
 7) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = x^{-1}\delta\partial_x$;
 8) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = x^{-1}\delta\partial_x$;
 9) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = x^{-1}\delta\partial_x$;
 10*) $u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})$ with $t \in \mathbb{C}^\times$, and $u_4 = x^{-1}\delta\partial_x$.

Proof. Theorem 2.1 implies that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2) \simeq \mathbb{C}^{10}, \text{ and } H^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_4) \simeq \mathbb{C}.$$

Thus, by Corollary 5.2, $H^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ is the bundle with base \mathbb{C}^{10} and fiber \mathbb{C} .

Theorem 2.2 provides us with the system of generating cocycles for the basis and the fiber of this bundle. Thus, any supermanifold with retract $\mathcal{CP}^{1|4}$ is determined, up to an isomorphism, by the cocycle which in U_0 is of the form

$$u = \sum_{i,j=1,\dots,4} c_{ij}x^{-1}\delta_i\partial_{\xi_j} + cx^{-1}\delta\partial_x, \text{ where } c_{ij} = c_{ji}, \text{ and } c \in \mathbb{C}. \quad (5)$$

Let $\alpha \in \text{Aut } \mathbf{E}$. Since the automorphism α is a linear function in sections ξ_1, \dots, ξ_4 , then in U_0 we see that

$$\alpha(\xi_i) = \sum_{1 \leq j \leq 4} a_{ji}(x)\xi_j, \text{ where } a_{ji}(x) \text{ are holomorphic functions in } x.$$

Therefore, in U_1 this equality takes the form

$$y^{-1}\alpha(\eta_i) = \sum_{1 \leq j \leq 4} a_{ji}(y^{-1})y^{-1}\eta_j, \quad \text{or} \quad \alpha(\eta_i) = \sum_{1 \leq j \leq 4} a_{ji}(y^{-1})\eta_j.$$

This is how the action of the group $\text{Aut } \mathbf{E}$ on sections η_1, \dots, η_4 on U_1 is defined. Hence, $a_{ji}(y^{-1})$ should be holomorphic functions in y on U_1 . Hence, $a_{ji} = \text{const}$ for any i, j .

Therefore, any automorphism $\alpha \in \text{Aut } \mathbf{E}$ is given by a complex matrix $A = (a_{ij})$, i.e., $\text{Aut } \mathbf{E} \simeq \text{GL}_4(\mathbb{C})$.

Let $B = (b_{ij})$ be the inverse of $A = (a_{ij})$. Then,

$$\alpha(\partial_{\xi_i})\xi_j = (\alpha\partial_{\xi_i}\alpha^{-1})\xi_j = \alpha\partial_{\xi_i}\left(\sum_{1 \leq k \leq 4} b_{kj}\xi_k\right) = \alpha(b_{ij}) = b_{ij},$$

i.e.,

$$\alpha(\partial_{\xi_i}) = \sum_{1 \leq j \leq 4} b_{ij}\partial_{\xi_j}.$$

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix} \in S_4$. Further,

$$\begin{aligned} \alpha(\delta) &= \alpha(\xi_1\xi_2\xi_3\xi_4) = \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3)\alpha(\xi_4) \\ &= \left(\sum_{1 \leq i \leq 4} a_{i1}\xi_i\right)\left(\sum_{1 \leq j \leq 4} a_{j2}\xi_j\right)\left(\sum_{1 \leq k \leq 4} a_{k3}\xi_k\right)\left(\sum_{1 \leq l \leq 4} a_{l4}\xi_l\right) \\ &= \left(\sum_{\sigma \in S_4} \text{sign } \sigma \cdot a_{i1}a_{j2}a_{k3}a_{l4}\right)\delta = \det A \cdot \delta; \\ \alpha(\delta_i) &= \alpha(\partial_{\xi_i}\delta) = \alpha(\partial_{\xi_i})\alpha(\delta) = \det A \cdot \sum_{1 \leq k \leq 4} b_{ik}\partial_{\xi_k}\delta = \det A \cdot \sum_{1 \leq k \leq 4} b_{ik}\delta_k. \end{aligned}$$

We see that

$$\begin{aligned}
\alpha(x^{-1}\delta\partial_x) &= x^{-1}\alpha(\delta)\partial_x \\
&= x^{-1}\left(\sum_{1\leq i\leq 4} a_{i1}\xi_i\right)\left(\sum_{1\leq j\leq 4} a_{j2}\xi_j\right)\left(\sum_{1\leq k\leq 4} a_{k3}\xi_k\right)\left(\sum_{1\leq l\leq 4} a_{l4}\xi_l\right)\partial_x \\
&= x^{-1}\left(\sum_{\sigma\in S_4} \text{sign } \sigma \cdot a_{i_1}a_{j_2}a_{k_3}a_{l_4}\right)\delta\partial_x = \det A \cdot x^{-1}\delta\partial_x; \\
\alpha(x^{-1}\delta_i\partial_{\xi_j}) &= x^{-1}\alpha(\delta_i)\alpha(\partial_{\xi_j}) \\
&= \det A \cdot \sum_{1\leq k\leq 4} b_{ik}\delta_k \sum_{1\leq l\leq 4} b_{jl}\partial_{\xi_l} = \det A \cdot \sum_{1\leq k,l\leq 4} b_{ik}b_{jl}\delta_k\partial_{\xi_l}.
\end{aligned}$$

Then, having applied α to the cocycle (5), we get

$$\alpha(u) = \det A \cdot \sum_{1\leq i,j,k,l\leq 4} x^{-1}c_{ij}b_{ik}b_{jl}\delta_k\partial_{\xi_l} + \det A \cdot cx^{-1}\delta\partial_x.$$

This implies, in particular, that the matrix $C = (c_{ij})$ transforms into $(\det A)B^tCB$.

Since every cocycle of the form (5) is uniquely determined by the matrix C and the number c , and the $(\text{Aut } \mathbf{E})$ -action is known, it suffices to consider the following cases.

1. In this case, $C = 0$ and $c = 0$.

2. Let $\text{rk } C = 1$ and $c = 0$. Then, as follows from Algebra course, the group $\text{GL}_4(\mathbb{C})$ does not change the rank of C and there is an invertible operator reducing C to the 4×4 matrix E_{11} . To this matrix the cocycle $x^{-1}\delta_1\partial_{\xi_1}$ corresponds; we will consider this cocycle as a representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

3, 4 and 5. Let $\text{rk } C = 2, 3$ and 4 , respectively and $c = 0$. As in case 2, we get, respectively, the following representatives:

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.
\end{aligned}$$

6. Let $C = 0$ and $c \neq 0$. Then, the transformation determined by the matrix A with $\det A = \frac{1}{c}$ leads to the cocycle $x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

7. Let $\text{rk } C = 1$ and $c \neq 0$. Since C is not invertible, it is possible (elementary linear algebra) to find a matrix A with $\det A = \frac{1}{c}$ so that C becomes the matrix unit E_{11} . The cocycle corresponding to this matrix and the number 1 is $x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\text{Aut } \mathbf{E})$ -orbit on $Z^1(\mathbb{CP}^1, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$.

8 and 9. Let $\text{rk } C = 2$ and 3 , respectively, and $c \neq 0$. In analogy with case 7, the representatives are, respectively,

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta\partial_x, \\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta\partial_x.
\end{aligned}$$

10. Let $\text{rk } C = 4$ and $c \neq 0$. In this case, by an invertible transformation with matrix A such that $\det A = \frac{1}{c}$ the matrix C can be reduced to the diagonal form $t1_4$, where $t \in \mathbb{C}^\times$.

As a result, we get a 1-parameter family of cocycles

$$c_t := t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x,$$

Every cocycle c_t defines an orbit in $Z^1(\mathbb{CP}^1, \mathcal{Aut}_{(2)}\mathcal{O}_{\text{gr}})$. \square

6 Which of the supermanifolds with retract $\mathcal{CP}^{1|4}$ are homogeneous

Consider an open covering $\mathcal{U} = (U_i)_{i \in I}$ of a topological space M . Let the supermanifold (M, \mathcal{O}) be determined by a cocycle $g \in Z^1(M, \mathcal{Aut}_{(2)}\mathcal{O}_{\text{gr}})$. Recall the definition of a liftable field, see § 2.

6.1 Theorem ([BO1]). *The vector field $v \in \mathfrak{v}(M, \mathcal{O}_{\text{gr}})_p$ can be lifted to (M, \mathcal{O}) if and only if there exists a $v^{(i)} \in C^0(M, (\mathcal{T}_{\text{gr}})_{(p)})$ such that*

$$\begin{aligned} v^{(i)} &\equiv v \pmod{(\mathcal{T}_{\text{gr}})_{(p+1)}(U_i)}, \\ g^{(ij)}v^{(j)} &= v^{(i)}g^{(ij)} \quad \text{in } U_i \cap U_j \neq \emptyset. \end{aligned}$$

Consider the representation

$$g^{(01)} = \exp u^{(01)}, \text{ where } u^{(01)} \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4).$$

Let $u^{(01)} = u_2 + u_4$, see (4).

6.2. Corollary. *The vector field $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_p$ can be lifted if and only if the following conditions hold*

$$\begin{aligned} p = -1 \quad [v, u_2] &= v_1^{(1)} - v_1^{(0)}, & (2) \\ [v, u_4] &= v_3^{(1)} - v_3^{(0)} + [u_2, v_1^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], & (3) \\ p = 0 \quad [v, u_2] &= v_2^{(1)} - v_2^{(0)}, \\ [v, u_4] &= v_4^{(1)} - v_4^{(0)} + [u_2, v_2^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], \\ p = 1 \quad [v, u_2] &= v_3^{(1)} - v_3^{(0)}. \end{aligned}$$

Proof. Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_p$, where $p = -1, 0$. Let us seek $v^{(i)}$ in the form $v + v_{p+2}^{(i)} + v_{p+4}^{(i)}$ for $v_{p+2}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_{p+2}$, and $v_{p+4}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_{p+4}$.

By Theorem 6.1 we have a condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$\begin{aligned} v^{(0)} &= g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \text{ad}_{u^{(01)}})v^{(1)} \\ &= v^{(1)} + [u^{(01)}, v^{(1)}] + \frac{1}{2}[u^{(01)}, [u^{(01)}, v^{(1)}]]. \end{aligned}$$

Therefore,

$$v + v_{p+2}^{(0)} + v_{p+4}^{(0)} = v + v_{p+2}^{(1)} + v_{p+4}^{(1)} + [u_2 + u_4, v + v_{p+2}^{(1)}] + \frac{1}{2}[u_2, [u_2, v]].$$

Hence, we get the conditions

$$\begin{aligned} [v, u_2] &= v_{p+2}^{(1)} - v_{p+2}^{(0)}, \\ [v, u_4] &= v_{p+4}^{(1)} - v_{p+4}^{(0)} + [u_2, v_{p+2}^{(1)}] + \frac{1}{2}[u_2, [u_2, v]]. \end{aligned}$$

Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$. We seek $v^{(i)}$ in the form $v + v_3^{(i)}$, where $v_3^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\text{gr}})_3$. By Theorem 6.1 we get the condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$v^{(0)} = g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \text{ad}_{u^{(01)}})v^{(1)} = v^{(1)} + [u^{(01)}, v^{(1)}].$$

Therefore,

$$v + v_3^{(0)} = v + v_3^{(1)} + [u_2 + u_4, v + v_3^{(1)}].$$

Hence, we get the condition

$$[v, u_2] = v_3^{(1)} - v_3^{(0)}. \quad \square$$

The definition of homogeneous supermanifold implies that the supermanifold is homogeneous if and only if the following map is surjective

$$\text{ev}_x : \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) \longrightarrow T_x(\mathbb{CP}^1, \mathcal{O}) \quad \text{for any } x \in \mathbb{CP}^1.$$

Since $\bar{0}$ -homogeneity takes place by the proved, it is necessary and sufficient to prove that all fields $\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$ can be lifted to $(\mathbb{CP}^1, \mathcal{O})$.

Let us use the conditions obtained to verify the homogeneity of the supermanifolds in the 10 cases of Theorem 6.1. For every homogeneous $(\mathbb{CP}^1, \mathcal{O})$, I compute the Lie superalgebra of vector fields on it.

6.2 Case 1. It corresponds to the supermanifold $\mathcal{CP}^{1|4}$. It is well-known that this supermanifold is homogeneous. The Lie superalgebra of vector fields on $\mathcal{CP}^{1|4}$ is known (see [O3]):

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1} &= \langle v_{5i} = -\partial_{\xi_i}, v_{6i} = -x\partial_{\xi_i}, \text{ where } i = 1, \dots, 4 \rangle, \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_0 &= \langle v_{ij} = -\xi_i\partial_{\xi_j}, \text{ where } i, j = 1, \dots, 4, v_{55} = x\partial_x + \nabla, \\ v_{66} = -x\partial_x, v_{56} = -\partial_x, v_{65} = x^2\partial_x + x\nabla, \text{ where } \sum_{1 \leq i \leq 6} v_{ii} = 0 \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1 &= \langle v_{i5} = -\xi_i(x\partial_x + \nabla), v_{i6} = -\xi_i\partial_x \text{ for } i = 1, \dots, 4 \rangle. \end{aligned}$$

6.3 Case 2. Then, $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

We will repeatedly use the following ‘‘formula’’

$$\text{Since } [v_i, u_2] \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_1), \text{ then } [v_i, u_2] \sim 0 \text{ by Theorem 2.1, and hence condition (2) of Corollary 6.2 holds.} \quad (6)$$

Since

$$[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}$$

is holomorphic in U_1 , then for each v_i we set

$$v_i^{(0)} = 0, \quad v_i^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}.$$

We have

$$[x^{-1}\delta_1\partial_{\xi_1}, -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}] = -x^{-2}\delta_1\partial_{\xi_1}\left(\frac{\partial\delta_1}{\partial\xi_i}\right)\partial_{\xi_1} + x^{-2}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}(\delta_1)\partial_{\xi_1} = 0.$$

The first summand vanishes since δ_1 does not contain ξ_1 by definition and $\partial_{\xi_1}\left(\frac{\partial\delta_1}{\partial\xi_i}\right) = 0$.

The second summand also vanishes since $\partial_{\xi_1}(\delta_1) = 0$.

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$, where $i = 1, \dots, 4$, can be lifted; moreover, in U_0 they have the same form. Hence, the supermanifolds corresponding to the cocycle of the 2nd case is homogeneous.

6.3. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1}$. Then, (recall notation (2))*

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, \quad x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1}, \quad 2x\partial_x + \nabla, \\ & -x\partial_x - \xi_2\partial_{\xi_2}, \quad -x\partial_x - \xi_3\partial_{\xi_3}, \quad -x\partial_x - \xi_4\partial_{\xi_4}, \quad -\xi_2\partial_{\xi_1}, \quad -\xi_2\partial_{\xi_3}, \\ & -\xi_2\partial_{\xi_4}, \quad -\xi_3\partial_{\xi_1}, \quad -\xi_3\partial_{\xi_2}, \quad -\xi_3\partial_{\xi_4}, \quad -\xi_4\partial_{\xi_1}, \quad -\xi_4\partial_{\xi_2}, \quad -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i} \text{ for } i = 1, \dots, 4, \quad -x\partial_{\xi_1}, \quad -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \quad -\xi_j\partial_x, \\ & -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1}, \quad -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1}, \quad -\xi_j(x\partial_x + \nabla), \text{ for } j = 2, 3, 4 \rangle. \end{aligned}$$

6.4 Case 3. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}.$$

In the same way as in Case 2 we see that

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}] = 0.$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of Case 3 is homogeneous.

6.4. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$. Then, (recall notation (2))*

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2}, 2x\partial_x + \nabla, \\ & \xi_2\partial_{\xi_2}, -x\partial_x - \xi_3\partial_{\xi_3}, -x\partial_x - \xi_4\partial_{\xi_4}, \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, -\xi_3\partial_{\xi_1}, \\ & -\xi_3\partial_{\xi_2}, -\xi_3\partial_{\xi_4}, -\xi_4\partial_{\xi_1}, -\xi_4\partial_{\xi_2}, -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i}, i = 1, \dots, 4, -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2}, -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \\ & -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2}, -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2}, \\ & -\xi_j(x\partial_x + \nabla), -\xi_j\partial_x, j = 3, 4 \rangle. \end{aligned}$$

6.5 Case 4. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} \end{aligned}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}.$$

In the same way as in Case 2 we see that $[X, Y] = 0$, where

$$\begin{aligned} X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\ Y &:= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}. \end{aligned}$$

Hence, condition (3) of Corollary 6.2 is of the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.5. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}.$$

Then, (recall notation (2))

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2} - \delta_3\partial_{\xi_3}, 2x\partial_x + \nabla, \\ & \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, -x\partial_x - \xi_4\partial_{\xi_4}, \\ & -\xi_4\partial_{\xi_1}, -\xi_4\partial_{\xi_2}, -\xi_4\partial_{\xi_3} \rangle; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -\partial_{\xi_i} \text{ for } i = 1, \dots, 4, -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3}, \\ & -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3}, -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2}, \\ & -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}, -\xi_4(x\partial_x + \nabla), -\xi_4\partial_x \rangle. \end{aligned}$$

6.6 Case 5. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}, \text{ and } u_4 = 0.$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] = \\ &= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4} \end{aligned}$$

is holomorphic in U_1 , then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}.$$

In the same way as in Case 2 we see that $[A, B] = 0$, where

$$\begin{aligned} A &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}, \\ B &:= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}. \end{aligned}$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 5th case is homogeneous.

6.6. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.$$

Then, (recall notation (2))

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = & \langle -\partial_x, x^2\partial_x + x\nabla + \nabla, 2x\partial_x + \nabla, \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \\ & \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} \rangle, \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = & \langle -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}, \\ & -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}, -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}, \\ & -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}, -\partial_{\xi_i} \text{ for } i = 1, \dots, 4 \rangle. \end{aligned}$$

6.7 Case 6. Let $u_2 = 0$, and $u_4 = x^{-1}\delta\partial_x$.

Consider the fields $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$. Since $[v_i, u_2] = 0$, then condition (2) of Corollary 6.2 is satisfied; it takes the form

$$[v_i, u_4] = v_3^{(1)} - v_3^{(0)},$$

where $v_3^{(j)}$ is holomorphic in U_j for $j = 0, 1$. Then, $[v_i, u_4]$ should be cohomologous to 0. Substituting the values of v_i and u_4 , we see that $[\partial_{\xi_i}, x^{-1}\delta\partial_x] = x^{-1}\delta_i\partial_x$ which is a basis cocycle (see Theorem 2.2). Hence, condition (3) of Corollary 6.2 is not satisfied.

Therefore, none of the fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Therefore, the supermanifold corresponding to the cocycle of the 6th case is not homogeneous.

6.8 Case 7. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1}$$

is holomorphic in U_1 , set $v_1^{(0)} = v'$, and $v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$.

Since $[x^{-1}\delta_1\partial_{\xi_1}, x^{-1}\xi_2\xi_3\partial_{\xi_1}] = 0$, then condition (3) of Corollary 6.2 is of the form

$$[-\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} [v', x^{-1}\delta_1\partial_{\xi_1}] &= -A_1 x^{-1}\delta\partial_{\xi_1} + 3A_1\delta\partial_{\xi_1} - B_1 x^{-2}\delta\partial_{\xi_1} - \\ &\quad - (A_1\delta_1\partial_x - 2A_1 x^{-1}\delta\partial_{\xi_1} + B_1 x^{-1}\delta_1\partial_x) \\ &= A_1 x^{-1}\delta\partial_{\xi_1} - B_1 x^{-2}\delta\partial_{\xi_1} - B_1 x^{-1}\delta_1\partial_x + 3A_1\delta\partial_{\xi_1} - A_1\delta_1\partial_x \\ &\sim A_1 x^{-1}\delta\partial_{\xi_1} - 2B_1 x^{-1}\delta_1\partial_x. \end{aligned}$$

But then, for any $A_1, B_1 \in \mathbb{C}$, we have

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} - 2B_1 x^{-1}\delta_1\partial_x \not\sim 0.$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 7th case is not homogeneous.

6.9 Case 8. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}$$

is holomorphic in U_1 , we set

$$v_1^{(0)} = v', \quad v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} + v', \quad \text{where } v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1.$$

Since

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}] = 0,$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \quad \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} & [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \\ & \sim A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x. \end{aligned}$$

But then, for any $A_1, A_2, B_1, B_2 \in \mathbb{C}$, we have

$$\begin{aligned} & x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \\ & \sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x \not\sim 0. \end{aligned}$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 8th case is not homogeneous.

6.10 Case 9. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \quad \text{and } u_4 = x^{-1}\delta\partial_x.$$

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$. Thanks to (6) and since

$$\begin{aligned} [v, u_2] &= [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3} \end{aligned}$$

is holomorphic in U_1 , let us set $v_1^{(0)} = v'$, and

$$v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3} + v',$$

where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$.

Since $[X, Y] = 0$, where

$$\begin{aligned} X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\ Y &:= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3}, \end{aligned}$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, then we can consider (recall notation (2))

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \quad \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} &[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} + A_3 x^{-1}\delta\partial_{\xi_3} \\ &\quad - 2B_1 x^{-1}\delta_1\partial_x - 2B_2 x^{-1}\delta_2\partial_x - 2B_3 x^{-1}\delta_3\partial_x. \end{aligned}$$

But then, for any $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathbb{C}$, we have

$$\begin{aligned} &x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim x^{-1}\delta_4\partial_x + A_1 x^{-1}\delta\partial_{\xi_1} + A_2 x^{-1}\delta\partial_{\xi_2} + A_3 x^{-1}\delta\partial_{\xi_3} - 2B_1 x^{-1}\delta_1\partial_x \\ &\quad - 2B_2 x^{-1}\delta_2\partial_x - 2B_3 x^{-1}\delta_3\partial_x \not\sim 0. \end{aligned}$$

Hence, the field $-\partial_{\xi_4}$ can not be lifted. This shows that the supermanifold of the 9th case is not homogeneous.

6.11 Case 10. Let $t \in \mathbb{C}^\times$. Let

$$u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), \quad \text{and } u_4 = x^{-1}\delta\partial_x.$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$. Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})] \\ &= -tx^{-1} \left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4} \right) \end{aligned}$$

is holomorphic in U_1 , then set $v_1^{(0)} = v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, and

$$v_1^{(1)} = v' - tx^{-1} \left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4} \right).$$

We have $[A, B] = 0$, where

$$\begin{aligned} A &:= t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), \\ B &:= -tx^{-1}\left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\right). \end{aligned}$$

Hence, condition (3) of Corollary 6.2 is of the form

$$\begin{aligned} &[-\partial_{\xi_i}, x^{-1}\delta\partial_x] \\ &= v_3^{(1)} - v_3^{(0)} + [t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), v']. \end{aligned}$$

Therefore,

$$-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, take

$$v' = \sum_{1 \leq k \leq 4} \left(A_k \xi_k (x\partial_x + \nabla) + B_k \xi_k \partial_x \right), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} &[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \\ &= \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - B_k x^{-2}\delta\partial_{\xi_k} - B_k x^{-1}\delta_k\partial_x - A_k \delta_k\partial_x \right) \\ &\sim \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - 2B_k x^{-1}\delta_k\partial_x \right). \end{aligned}$$

But then, for $B_i = -\frac{1}{2t}$, and $B_j = 0$ for $j \in \{1, \dots, 4 \mid j \neq i\}$, and $A_k = 0$ for $k = 1, \dots, 4$, we have

$$\begin{aligned} &-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \\ &\sim -x^{-1}\delta_i\partial_x + t \sum_{1 \leq k \leq 4} \left(2A_k x^{-1}\delta\partial_{\xi_k} - 2B_k x^{-1}\delta_k\partial_x \right) \sim 0. \end{aligned}$$

Thus, all fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Hence, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.7. Theorem. *Let supermanifold $(\mathbb{CP}^1, \mathcal{O})$ be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x.$$

Then (recall notation (2)),

$$\begin{aligned} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\overline{0}} &= \langle -\partial_x, x^2\partial_x + x\nabla - t\omega - \delta\partial_x, 2x\partial_x + \nabla, \\ &\xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \\ &\xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} \rangle, \text{ where } \omega = \sum_{1 \leq i \leq 4} \delta_i\partial_{\xi_i}; \\ \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\overline{1}} &= \langle -\partial_{\xi_i} - \frac{1}{2t}\xi_i\partial_x, i = 1, \dots, 4, \\ &-x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_4} - \xi_2\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x, \\ &-x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x, \\ &-x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \frac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x, \\ &-x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \frac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x \rangle. \end{aligned}$$

6.8 Theorem (Summary). *The supermanifold isomorphic to one of the supermanifolds of cases 1 – 5 or 10 is homogeneous; it is not homogeneous in cases 6 – 9.*

7 Description of $\mathfrak{v}(\mathcal{M})$ for homogeneous superstrings \mathcal{M} with retract $\mathcal{CP}^{1|4}$

The cases are numbered as in Theorem 5.3. We reproduce here the bases found in §2.3, and multiplication tables. In all cases (recall notation (2)),

$$h := [e, f] = 2x\partial_x + \nabla \text{ and } [h, e] = 2e, [h, f] = -2f.$$

1. Fact (well-known): $\mathfrak{v}(\mathcal{CP}^{1|4}) \simeq \mathfrak{pgl}_{\mathbb{C}}(4|2)$.
2. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 12 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1}, f = -\partial_x, a_1 = -x\partial_x - \xi_2\partial_{\xi_2}, \\ a_2 &= -x\partial_x - \xi_3\partial_{\xi_3}, a_3 = -x\partial_x - \xi_4\partial_{\xi_4}, a_4 = -\xi_2\partial_{\xi_1}, a_5 = -\xi_2\partial_{\xi_3}, \\ a_6 &= -\xi_2\partial_{\xi_4}, a_7 = -\xi_3\partial_{\xi_2}, a_8 = -\xi_3\partial_{\xi_1}, a_9 = -\xi_3\partial_{\xi_4}, a_{10} = -\xi_4\partial_{\xi_2}, \\ a_{11} &= -\xi_4\partial_{\xi_3}, a_{12} = -\xi_4\partial_{\xi_1}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i, i = 1, \dots, 14 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, z_2 = -\partial_{\xi_2}, z_3 = -\partial_{\xi_3}, z_4 = -\partial_{\xi_4}, z_5 = -\xi_2\partial_x, \\ z_6 &= -\xi_3\partial_x, z_7 = -\xi_4\partial_x, z_8 = -x\partial_{\xi_1}, z_9 = -\xi_2(x\partial_x + \nabla), \\ z_{10} &= -\xi_3(x\partial_x + \nabla), z_{11} = -\xi_4(x\partial_x + \nabla), z_{12} = -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1}, \\ z_{13} &= -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1}, z_{14} = -x\partial_{\xi_3} + \xi_2\xi_3\partial_{\xi_1}. \end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0 \text{ for all } i = 1, \dots, 12.$$

$[,]$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	a_7	0	0	a_{10}	0	0
a_2	0	0	0	0	a_5	0	$-a_7$	$-a_8$	$-a_9$	0	a_{11}	0
a_3	0	0	0	0	0	a_6	0	0	a_9	$-a_{10}$	$-a_{11}$	$-a_{12}$
a_4	a_4	0	0	0	0	0	a_8	0	0	a_{12}	0	0
a_5	a_5	$-a_5$	0	0	0	0	$a_2 - a_1$	$-a_4$	$-a_6$	a_{11}	0	0
a_6	a_6	0	$-a_6$	0	0	0	a_9	0	0	$a_3 - a_1$	$-a_5$	$-a_4$
a_7	$-a_7$	a_7	0	$-a_8$	$a_1 - a_2$	$-a_9$	0	0	0	0	a_{12}	0
a_8	0	a_8	0	0	a_4	0	0	0	0	0	a_{10}	0
a_9	0	a_9	$-a_9$	0	a_6	0	0	0	0	$-a_7$	$a_3 - a_2$	$-a_8$
a_{10}	$-a_{10}$	0	$-a_{10}$	$-a_{12}$	$-a_{11}$	$a_1 - a_3$	0	0	a_7	0	0	0
a_{11}	0	$-a_{11}$	a_{11}	0	0	a_5	$-a_{12}$	$-a_{10}$	$a_2 - a_3$	0	0	0
a_{12}	0	0	a_{12}	0	0	a_4	0	0	a_8	0	0	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
e	$-z_8$	$-z_{12}$	$-z_{13}$	$-z_{14}$	$-z_9$	$-z_{10}$	$-z_{11}$	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	$-z_1$	$-z_5$	$-z_6$	$-z_7$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	$-z_7$	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
a_1	0	z_2	0	0	0	z_6	z_7	$-z_8$	$-z_9$	0	0	0	$-z_{13}$	$-z_{14}$
a_2	0	0	z_3	0	z_5	0	z_7	$-z_8$	0	$-z_{10}$	0	$-z_{12}$	0	$-z_{14}$
a_3	0	0	0	z_4	z_5	z_6	0	$-z_8$	0	0	$-z_{11}$	$-z_{12}$	$-z_{13}$	0
a_4	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0	0
a_5	0	z_3	0	0	0	$-z_5$	0	0	0	$-z_9$	0	$-z_{12}$	0	0
a_6	0	z_4	0	0	0	0	$-z_5$	0	0	0	$-z_9$	z_{14}	0	0
a_7	0	0	z_2	0	z_6	0	0	0	$-z_{10}$	0	0	0	z_{12}	0
a_8	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0
a_9	0	0	z_4	0	0	0	$-z_6$	0	0	0	$-z_{10}$	0	z_4	0
a_{10}	0	0	0	z_2	$-z_7$	0	0	0	$-z_{11}$	0	0	0	0	z_{12}
a_{11}	0	0	0	z_3	0	$-z_7$	0	0	0	$-z_{11}$	0	0	0	z_{13}
a_{12}	0	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
z_1	0	0	0	0	0	0	0	0	a_4	a_8	a_{12}	0	0	0
z_2	0	0	0	0	$-f$	0	0	0	a_1+h	a_7	a_{10}	0	$-a_{12}$	a_8
z_3	0	0	0	0	0	$-f$	0	0	a_5	a_2+h	a_{11}	a_{12}	0	$-a_4$
z_4	0	0	0	0	0	0	$-f$	0	a_8	a_9	a_3+h	$-a_8$	a_4	0
z_5	0	$-f$	0	0	0	0	0	$-a_4$	0	0	0	$-a_1$	$-a_5$	$-a_6$
z_6	0	0	$-f$	0	0	0	0	$-a_8$	0	0	0	$-a_7$	$-a_2$	$-a_9$
z_7	0	0	0	$-f$	0	0	0	$-a_{12}$	0	0	0	$-a_{10}$	$-a_{11}$	$-a_3$
z_8	0	0	0	0	$-a_4$	$-a_8$	$-a_{12}$	0	0	0	0	0	0	0
z_9	a_4	a_1+h	a_5	a_6	0	0	0	0	0	0	0	e	0	0
z_{10}	a_8	a_7	a_2+h	a_9	0	0	0	0	0	0	0	0	e	0
z_{11}	a_{12}	a_{10}	a_{11}	a_3+h	0	0	0	0	0	0	0	0	0	e
z_{12}	0	0	a_{12}	$-a_8$	$-a_1$	$-a_7$	$-a_{10}$	0	e	0	0	0	0	0
z_{13}	0	$-a_{12}$	0	a_4	$-a_5$	$-a_2$	$-a_{11}$	0	0	e	0	0	0	0
z_{14}	0	a_8	$-a_4$	0	$-a_6$	$-a_9$	$-a_3$	0	0	0	e	0	0	0

3. We have $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 9 \rangle$, where

$$\begin{aligned}
e &= x^2 \partial_x + x \nabla - \delta_1 \partial_{\xi_1} - \delta_2 \partial_{\xi_2}, \quad f = -\partial_x, \quad a_1 = -x \partial_x - \xi_3 \partial_{\xi_3}, \\
a_2 &= -x \partial_x - \xi_4 \partial_{\xi_4}, \quad a_3 = \xi_1 \partial_{\xi_2} - \xi_2 \partial_{\xi_1}, \quad a_4 = -\xi_3 \partial_{\xi_1}, \quad a_5 = -\xi_3 \partial_{\xi_2}, \\
a_6 &= -\xi_3 \partial_{\xi_4}, \quad a_7 = -\xi_4 \partial_{\xi_1}, \quad a_8 = -\xi_4 \partial_{\xi_2}, \quad a_9 = -\xi_4 \partial_{\xi_3};
\end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 12 \rangle$, where

$$\begin{aligned}
z_1 &= -\partial_{\xi_1}, \quad z_2 = -\partial_{\xi_2}, \quad z_3 = -\partial_{\xi_3}, \quad z_4 = -\partial_{\xi_4}, \quad z_5 = -\xi_3 \partial_x, \\
z_6 &= -\xi_4 \partial_x, \quad z_7 = -x \partial_{\xi_1} - \xi_3 \xi_4 \partial_{\xi_2}, \quad z_8 = -x \partial_{\xi_2} + \xi_3 \xi_4 \partial_{\xi_1}, \\
z_9 &= -x \partial_{\xi_3} - \xi_2 \xi_4 \partial_{\xi_1} + \xi_1 \xi_4 \partial_{\xi_2}, \quad z_{10} = -x \partial_{\xi_4} + \xi_2 \xi_3 \partial_{\xi_1} - \xi_1 \xi_3 \partial_{\xi_2}, \\
z_{11} &= -\xi_3 (x \partial_x + \nabla), \quad z_{12} = -\xi_4 (x \partial_x + \nabla).
\end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0, \quad i = 1, \dots, 9.$$

$[,]$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	0	0	a_9
a_2	0	0	0	0	0	a_6	$-a_7$	$-a_8$	$-a_9$
a_3	0	0	0	$-a_5$	a_4	0	$-a_8$	a_7	0
a_4	a_4	0	a_5	0	0	0	0	0	a_7
a_5	a_5	0	$-a_4$	0	0	0	0	0	a_8
a_6	a_6	$-a_6$	0	0	0	0	$-a_4$	$-a_5$	$a_2 - a_1$
a_7	0	a_7	a_8	0	0	a_4	0	0	0
a_8	0	a_8	$-a_7$	0	0	a_5	0	0	0
a_9	$-a_9$	a_9	0	$-a_7$	$-a_8$	$a_1 - a_2$	0	0	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
e	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	$-z_{11}$	$-z_{12}$	0	0	0	0	0	0
f	0	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
a_1	0	0	z_3	0	0	z_6	$-z_7$	$-z_8$	0	$-z_{10}$	$-z_{11}$	0
a_2	0	0	0	z_4	z_5	0	$-z_7$	$-z_8$	$-z_9$	0	0	z_{12}
a_3	$-z_2$	z_1	0	0	0	0	$-z_8$	z_7	0	0	0	0
a_4	0	0	z_1	0	0	0	0	0	z_7	0	0	0
a_5	0	0	z_2	0	0	0	0	0	z_8	0	0	0
a_6	0	0	z_4	0	0	$-z_5$	0	0	z_{10}	0	0	$-z_{11}$
a_7	0	0	0	z_1	0	0	0	0	0	z_7	0	0
a_8	0	0	0	z_2	0	0	0	0	0	z_8	0	0
a_9	0	0	0	z_3	$-z_6$	0	0	0	0	z_9	$-z_{12}$	0

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
z_1	0	0	0	0	0	0	0	0	a_8	$-a_5$	a_4	a_7
z_2	0	0	0	0	0	0	0	0	$-a_7$	a_4	a_5	a_8
z_3	0	0	0	0	$-f$	0	$-a_8$	a_7	0	$-a_3$	a_1+h	a_9
z_4	0	0	0	0	0	$-f$	a_5	$-a_4$	a_3	0	a_6	a_2+h
z_5	0	0	$-f$	0	0	0	$-a_4$	$-a_5$	$-a_1$	$-a_6$	0	0
z_6	0	0	0	$-f$	0	0	$-a_7$	$-a_8$	$-a_9$	$-a_2$	0	0
z_7	0	0	$-a_8$	a_7	$-a_4$	$-a_7$	0	0	0	0	0	0
z_8	0	0	a_7	$-a_4$	$-a_5$	$-a_8$	0	0	0	0	0	0
z_9	a_8	$-a_7$	0	a_3	$-a_1$	$-a_9$	0	0	0	0	e	0
z_{10}	$-a_5$	a_4	$-a_3$	0	$-a_6$	$-a_2$	0	0	0	0	0	e
z_{11}	a_4	a_5	a_1+h	a_6	0	0	0	0	e	0	0	0
z_{12}	a_7	a_8	a_9	a_2+h	0	0	0	0	0	e	0	0

4. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2} - \delta_3\partial_{\xi_3}, \quad f = -\partial_x, \quad a = -x\partial_x - \xi_4\partial_{\xi_4}, \\ a_1 &= \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \quad a_2 = \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \quad a_3 = \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \\ a_4 &= -\xi_4\partial_{\xi_1}, \quad a_5 = -\xi_4\partial_{\xi_2}, \quad a_6 = -\xi_4\partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 10 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, z_2 = -\partial_{\xi_2}, z_3 = -\partial_{\xi_3}, z_4 = -\xi_4 \partial_x, z_5 = -\partial_{\xi_4}, \\ z_6 &= -x \partial_{\xi_1} - \xi_3 \xi_4 \partial_{\xi_2} + \xi_2 \xi_4 \partial_{\xi_3}, z_7 = -x \partial_{\xi_2} + \xi_3 \xi_4 \partial_{\xi_1} - \xi_1 \xi_4 \partial_{\xi_3}, \\ z_8 &= -x \partial_{\xi_3} - \xi_2 \xi_4 \partial_{\xi_1} + \xi_1 \xi_4 \partial_{\xi_2}, z_9 = -\xi_4 (x \partial_x + \nabla), \\ z_{10} &= -x \partial_{\xi_4} + \xi_2 \xi_3 \partial_{\xi_1} - \xi_1 \xi_3 \partial_{\xi_2} + \xi_1 \xi_2 \partial_{\xi_3}. \end{aligned}$$

$$[h, v] = [e, v] = [f, v] = 0, \text{ where } v \in \{a, a_i \mid i = 1, \dots, 6\}.$$

$[,]$	a	a_1	a_2	a_3	a_4	a_5	a_6			
a	0	0	0	0	$-a_4$	$-a_5$	$-a_6$			
a_1	0	0	$-a_3$	a_2	$-a_5$	a_4	0			
a_2	0	a_3	0	$-a_1$	$-a_6$	0	a_4			
a_3	0	$-a_2$	a_1	0	0	$-a_6$	a_5			
a_4	a_4	a_5	a_6	0	0	0	0			
a_5	a_5	$-a_4$	0	a_6	0	0	0			
a_6	a_6	0	$-a_4$	$-a_5$	0	0	0			
$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
e	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$
f	$-z_6$	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	0	0	0	0	0
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	z_6	z_7	z_8	z_9	z_{10}
a	0	0	0	0	z_5	$-z_6$	$-z_7$	$-z_8$	$-z_9$	0
a_1	$-z_2$	z_1	0	0	0	$-z_7$	z_6	0	0	0
a_2	$-z_3$	0	z_1	0	0	$-z_8$	0	z_6	0	0
a_3	0	$-z_3$	z_2	0	0	0	$-z_8$	z_7	0	0
a_4	0	0	0	0	z_1	0	0	0	0	z_6
a_5	0	0	0	0	z_2	0	0	0	0	z_7
a_6	0	0	0	0	z_3	0	0	0	0	z_8
$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
z_1	0	0	0	0	0	0	$-a_6$	a_5	a_4	$-a_3$
z_2	0	0	0	0	0	a_6	0	$-a_4$	a_5	a_2
z_3	0	0	0	0	0	$-a_5$	a_4	0	a_6	$-a_1$
z_4	0	0	0	0	$-f$	$-a_4$	$-a_5$	$-a_6$	0	$-a$
z_5	0	0	0	$-f$	0	a_3	$-a_2$	a_1	$a + h$	0
z_6	0	a_6	$-a_5$	$-a_4$	a_3	0	0	0	0	0
z_7	$-a_6$	0	a_4	$-a_5$	$-a_2$	0	0	0	0	0
z_8	a_5	$-a_4$	0	$-a_6$	a_1	0	0	0	0	0
z_9	a_4	a_5	a_6	0	$a + h$	0	0	0	0	e
z_{10}	$-a_3$	a_2	$-a_1$	$-a$	0	0	0	0	e	0

5. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2 \partial_x + x \nabla - \omega, f = -\partial_x, a_1 = \xi_1 \partial_{\xi_2} - \xi_2 \partial_{\xi_1}, a_2 = \xi_1 \partial_{\xi_3} - \xi_3 \partial_{\xi_1}, \\ a_3 &= \xi_1 \partial_{\xi_4} - \xi_4 \partial_{\xi_1}, a_4 = \xi_2 \partial_{\xi_3} - \xi_3 \partial_{\xi_2}, a_5 = \xi_2 \partial_{\xi_4} - \xi_4 \partial_{\xi_2}, \\ a_6 &= \xi_3 \partial_{\xi_4} - \xi_4 \partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 8 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1}, \quad z_2 = -\partial_{\xi_2}, \quad z_3 = -\partial_{\xi_3}, \quad z_4 = -\partial_{\xi_4}, \\ z_5 &= -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}, \\ z_6 &= -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}, \\ z_7 &= -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}, \\ z_8 &= -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}. \end{aligned}$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0, \quad i = 1, \dots, 6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
e	$-z_5$	$-z_6$	$-z_7$	$-z_8$	0	0	0	0
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	z_5	z_6	z_7	z_8
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	0	0	0	0	0	$-a_6$	a_5	$-a_3$
z_2	0	0	0	0	a_6	0	$-a_4$	a_2
z_3	0	0	0	0	$-a_5$	a_4	0	$-a_1$
z_4	0	0	0	0	a_3	$-a_2$	a_1	0
z_5	0	a_6	$-a_5$	a_3	0	0	0	0
z_6	$-a_6$	0	a_4	$-a_2$	0	0	0	0
z_7	a_5	$-a_4$	0	a_1	0	0	0	0
z_8	$-a_3$	a_2	$-a_1$	0	0	0	0	0

Let $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0)$ and $\sigma_1 + \sigma_2 + \sigma_3 = 0$. In [BO1], the family of Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, discovered by Kaplansky [Kapp*], [Kap*], is described. Observe that the Lie superalgebra $\mathfrak{osp}(4|2; \alpha)$, where $\alpha = \frac{\sigma_i}{\sigma_j}$ for $\sigma_j \neq 0$, is simple except for $\alpha = 0$ or -1 .

Since $\Gamma(\sigma_1, \sigma_2, \sigma_3) \simeq \Gamma(\sigma'_1, \sigma'_2, \sigma'_3)$ if and only if $(\sigma'_1, \sigma'_2, \sigma'_3) = a(\sigma_1, \sigma_2, \sigma_3)$, where $a \in \mathbb{C}^\times$ and the triple σ differs from σ' by a permutation of its components, see [BGL*], it follows that $\mathfrak{v}(\mathcal{M})$ is isomorphic to $\Gamma(1, -1, 0) = \mathfrak{osp}(4|2; -1)$.

10. We have $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 6 \rangle$, where

$$\begin{aligned} e &= x^2\partial_x + x\nabla - t\omega - \delta\partial_x, \quad f = -\partial_x, \\ a_1 &= \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}, \quad a_2 = \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1}, \quad a_3 = \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1}, \\ a_4 &= \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}, \quad a_5 = \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2}, \quad a_6 = \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3}; \end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 8 \rangle$, where

$$\begin{aligned} z_1 &= -\partial_{\xi_1} - \frac{1}{2t}\xi_1\partial_x, & z_2 &= -\partial_{\xi_2} - \frac{1}{2t}\xi_2\partial_x, \\ z_3 &= -\partial_{\xi_3} - \frac{1}{2t}\xi_3\partial_x, & z_4 &= -\partial_{\xi_4} - \frac{1}{2t}\xi_4\partial_x, \\ z_5 &= -x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x, \\ z_6 &= -x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \frac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x, \\ z_7 &= -x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \frac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x, \\ z_8 &= -x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \frac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x. \end{aligned}$$

$$[e, a_i] = [f, a_i] = [h, a_i] = 0, \quad i = 1, \dots, 6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
e	z_5	z_6	z_7	z_8	0	0	0	0
h	z_1	z_2	z_3	z_4	$-z_5$	$-z_6$	$-z_7$	$-z_8$
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

$[,]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	$-\frac{1}{t}f$	0	0	0	$\frac{1}{2t}h$	$-ta_6 + \frac{1}{2t}a_1$	$ta_5 + \frac{1}{2t}a_2$	$-ta_4 + \frac{1}{2t}a_3$
z_2	0	$-\frac{1}{t}f$	0	0	$ta_6 - \frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3 + \frac{1}{2t}a_4$	$ta_2 + \frac{1}{2t}a_5$
z_3	0	0	$-\frac{1}{t}f$	0	$-ta_5 - \frac{1}{2t}a_2$	$ta_3 - \frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1 + \frac{1}{2t}a_6$
z_4	0	0	0	$-\frac{1}{t}f$	$ta_4 - \frac{1}{2t}a_3$	$-ta_2 - \frac{1}{2t}a_5$	$ta_1 - \frac{1}{2t}a_6$	$\frac{1}{2t}h$
z_5	$\frac{1}{2t}h$	$-ta_6 + \frac{1}{2t}a_1$	$ta_5 + \frac{1}{2t}a_2$	$-ta_4 + \frac{1}{2t}a_3$	$\frac{1}{t}e$	0	0	0
z_6	$ta_6 - \frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3 + \frac{1}{2t}a_4$	$ta_2 + \frac{1}{2t}a_5$	0	$\frac{1}{t}e$	0	0
z_7	$-ta_5 - \frac{1}{2t}a_2$	$ta_3 - \frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1 + \frac{1}{2t}a_6$	0	0	$\frac{1}{t}e$	0
z_8	$ta_4 - \frac{1}{2t}a_3$	$-ta_2 - \frac{1}{2t}a_5$	$ta_1 - \frac{1}{2t}a_6$	$\frac{1}{2t}h$	0	0	0	$\frac{1}{t}e$

Comparing the above table with the tables in [BO1] we deduce that this Lie superalgebra is

$$\Gamma\left(\frac{1}{2t}, -\frac{1}{2}\left(\frac{1}{2t} + t\right), -\frac{1}{2}\left(\frac{1}{2t} - t\right)\right) \simeq \Gamma(2, -(2t^2 + 1), 2t^2 - 1) \simeq \mathfrak{osp}(4|2; \frac{2t^2 + 1}{1 - 2t^2}).$$

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