Generalized possibilistic Theories: the tensor product problem

Eric Buffenoir

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Eric Buffenoir*

Université de la Côte d'Azur, CNRS, InPhyNi, FRANCE

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Abstract

Inspired by the operational quantum logic program, we have the contention that probabilities can be viewed as a derived concept, even in a reconstruction program of Quantum Mechanics. We propose an operational description of physical theories where probabilities are replaced by counterfactual statements belonging to a three-valued (i.e. possibilistic) semantic domain. The space of states and the space of effects are then built as posets put in duality through a Chu_3 space. The convexity requirements on the spaces of states and effects, addressed basically in Generalized Probabilistic Theories, are then replaced by semi-lattice structures on these spaces. The pure states are also easily constructed as completely meet-irreducible elements which generate the whole space of states. The channels (i.e. symmetries) of the theory are then naturally built as Chu morphisms. An axiomatic can then be summarized for what can be called "Generalized possibilistic Theory" based on this States/Effects Chu space's category. The problem of bipartite experiment is then addressed as the main skill of this paper. An axiomatic for the tensor product of the space of states is then given and a solution is explicitly constructed. The relations/differences between this tensor product and the tensor product of semi-lattices present in the mathematical literature are then analyzed. This new proposal for the tensor product of semi-lattices can be considered as an interesting byproduct of this work.

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*Email: eric.buffenoir@cnrs.fr

1 Introduction

General Probabilistic Theory (GPT) is a framework developed within the foundations of physics (see [22] for a recent review of the abundant literature and an axiomatic construction of GPTs). Promoters of GPT intent to answer the question: what is a physical theory? This study appeared initially in the context of axiomatizations of quantum theory, as many researchers were attempting to derive quantum theory from a set of reasonably motivated axioms.
In the current days, research on GPT is oriented towards operational properties of GPTs, the main skill being to identify what structure is needed to realize certain protocols or constructions known from quantum information theory or classical information theory. One uses GPTs to get better understanding of what makes different things in quantum information theory work.
Despite the indeterministic character of quantum theory, it is an empirical fact that the distinct outcomes of measurements, operated on a large collection of samples of a quantum object, prepared according to the same experimental procedure, have reproducible relative frequencies. This fundamental fact has led physicists to consider large collections of statistically independent experimental sequences as the basic objects of physical description, rather than a single experiment on a singular realization of the object under study. According to GPT, a physical state (corresponding to a class of operationally equivalent preparation procedures) is then defined by a vector of probabilities associated with the outcomes of a maximal and irredundant set of fiducial tests that can be effectuated on collections of samples produced by any of these preparation procedures. In other words, two distinct collections of prepared samples will be considered as operationally equivalent if they lead to the same probabilities for the outcomes of any test on them. The physical description consists, therefore, in a set of prescriptions that allows sophisticated constructs to be defined from elementary ones. In particular, combination rules are defined for the concrete mixtures of states and for the allowed operations/tests.
It is a basic fact in GPT that this approach is the same as starting with an abstract state space, but instead of using vectors we would describe states in terms of all of the probabilities they can produce. In GPTs, ensembles of objects, conditional probabilities and conditional states can be represented by their respective state spaces and so we can treat them as any other state space and we can use known results, instead of having to prove them *ab initio*. Representing all transformations by "channels" allows us to use the constructions from frameworks based on category theory, since one can interpret state spaces as objects and channels as morphisms.

Although this probabilistic approach is now accepted as a standard conceptual framework for the reconstruction of quantum theory, the adopted perspective appears puzzling for different reasons.
First of all, the observer contributes fundamentally to give an intuitive meaning to the notions of preparation, operation and measurement on physical systems. However, the concrete process of 'acquisition of information' (by the observer / on the system) has no real place in this description. Secondly, the definition of the state has definitively lost its meaning for a singular prepared sample, and the physical state is now intrinsically attached to large collections of similarly prepared samples. The GPT approach adopts the probabilistic description of quantum phenomena without any discussion or attempt to explain why it is necessary. Thirdly, in order to clarify the requirements of the basic set of fiducial tests necessary and sufficient to define the space of states, this approach must proceed along a technical analysis which fundamentally limits this description to 'finite dimensional' systems (finite dimensional Hilbert spaces of states). Lastly, the axioms chosen to characterize quantum theory, among other theories encompassed by the GPT formalism, must exhibit a 'naturality' that - from our point of view - is still missing in the existing proposals.

Alternative research programs have tried to overcome some of these conceptual problems. Adopting another perspective, the *operational quantum logic* approach tries to avoid the introduction of probabilities and explores the relevant categorical structures underlying the space of states and the set of properties of a quantum system. In this description, probabilities appear only as a derived concept. Following G. Birkhoff and J. Von Neumann [9] and G. W. Mackey [19], this approach focuses on the structured space

of 'testable properties' of a physical system. The mathematical structure associated with the set of quantum propositions defined by the closed subspaces of a Hilbert space is not a Boolean algebra (contrary to the case encountered in classical mechanics). By shifting the attention to the set of closed subspaces instead of the Hilbert space itself, the possibility is open to build an operational approach to quantum mechanics, because the basic elements of this description are yes/no tests. G.W. Mackey identified axioms on the set of yes/no tests sufficient to relate this set to the set of closed subspaces of a complex Hilbert space. Later, C. Piron [20, 21] proposed a set of axioms that (almost) lead back to the general framework of quantum mechanics (see [11] for a historical perspective on the abundant literature). Piron's framework has been developed into a full operational approach and the categories underlying this approach were analyzed. It must be noted that these constructions are established in reference to some general results of projective geometry and are not restricted to a finite-dimensional perspective.
Despite some beautiful results (in particular the restriction of the division ring associated to the Hilbert space from Piron's propositional lattices [17]) and the attractiveness of a completely categorical approach (see [26] for an analysis of the main results on propositional systems), this approach has encountered several problems. Among these problems, we may cite the difficulty of building a consistent description of compound systems due to no-go results related to the existence of a tensor product of Piron's propositional systems [24][6, 7]. These problems have cast doubts on the adequacy of Piron's choice of an "orthomodular complete lattice" structure for the set of properties of the system.

Other categorical formalisms, adapted to the axiomatic study of quantum theory, have been developed more recently [3] and their relation with the 'operational approach' has been partly explored [1, 2, 4]. In [1, Theorem 3.15], S. Abramsky makes explicit the fact that the *Projective quantum symmetry groupoid* $PSymmH$[1] is fully and faithfully represented by the category $bmChu_{[0,1]}$, i.e., by the sub-category of the category of bi-extensional Chu spaces associated with the evaluation set $[0,1]$ obtained by restricting it to Chu morphisms (f_*, f^*) for which f_* is injective. This result suggests that Chu categories could have a central role in the construction of axiomatic quantum mechanics as they provide a natural characterization of the automorphisms of the theory. More surprisingly, and interestingly for us, S. Abramsky shows that the aforementioned representation of *PSymmH* is 'already' full and faithfull if we replace the evaluation space of the Chu category by a three-element set, where the three values represent "definitely yes", "definitely no" and "maybe" [1, Theorem 4.4]. S. Abramsky did not affirm that a three valued semantic is sufficient to found a complete axiomatic quantum theory, close to Piron's program or alternative to it, and allowing a complete reconstruction of the usual Hilbert formalism, although its result was clearly leading to this prospect. It was the purpose of our last paper [10] to explore this question for the first time. This paper was devoted to present the basic elements of this 'possibilistic'[2] semantic formalism.

In the present paper, we begin to develop an analog of GPT based on this three-valued Chu space operational description of physical systems. As a sort of word game, we will designate this attempt as Generalized possibilistic Theory (GpT). To allow for the same degree of generality as GPT, we present in Section 2 a set of axioms for the spaces of states and the spaces of effects of single systems which appear more general than in [10]. In Section 3, we intentionally focus our study on the problem of bipartite experiments (this question had been led untouched in [10]). To complete this description, we exhibit a construction of the tensor product of complete semi-lattices which necessarily differs from the traditional construction of this tensor product, present in mathematical literature. This can be considered as a significant byproduct of the present paper, which deserves further investigations.

[1] The objects of this category are the natural space of states in quantum mechanics, i.e., the Hilbert spaces of dimension greater than two, and the morphisms are the orbits on semi-unitary maps (i.e. unitary or anti-unitary) under the $U(1)$ group action, which are the relevant symmetries of Hilbert spaces from the point of view of quantum mechanics.

[2] In the rest of this paper we refer to this construction, based on a three-valued Chu space, as a 'possibilistic' approach to distinguish it from the 'probabilistic' one.

2 Generalized possibilistic Theories (GpT) for a single system

Adopting the operational perspective on quantum experiments, we will introduce the following definitions.

A *preparation process* is an objectively defined, and thus 'repeatable', experimental sequence that allows singular samples of a certain physical system to be produced, in such a way that we are able to submit them to tests. We will denote by $\mathfrak{P}$ the set of preparation processes (each element of $\mathfrak{P}$ can be equivalently considered as the collection of samples produced through this preparation procedure). [3]

For each *property*, that the observer aims to test macroscopically on *any particular sample* of the considered micro-system, it will be assumed that the observer is able to define (i) some detailed 'procedure', in reference to the modes of use of some experimental apparatuses chosen to perform the operation/test, and (ii) a 'rule' allowing the answer 'yes' to be extracted if the macroscopic outcome of the experiment conforms with the expectation of the observer, when the test is performed on any input sample (as soon as this experimental procedure can be opportunely applied to this particular sample). These operations/tests, designed to determine the occurrence of a given property for a given sample, will be called *yes/no tests associated with this property*. The set of 'yes/no tests' at the disposal of the observer will be denoted by $\mathfrak{T}$. [4]

A yes/no test $\mathfrak{t} \in \mathfrak{T}$ will be said to be *positive with certainty* (resp. *negative with certainty*) relatively to a preparation process $\mathfrak{p} \in \mathfrak{P}$ iff the observer is led to affirm that the result of this test, realized on any of the particular samples that could be prepared according to this preparation process, would be 'positive with certainty' (resp. would be 'negative with certainty'), 'should' this test be effectuated. If the yes/no test can not be stated as 'certain', this yes/no test will be said to be *indeterminate*. Concretely, the observer can establish the 'certainty' of the result of a given yes/no test on any given sample issued from a given preparation procedure, by running the same test on a sufficiently large (but finite) collection of samples issued from this same preparation process: if the outcome is always the same, the observer will be led to claim that similarly prepared 'new' samples would also produce the same result, if the experiment was effectuated. To summarize, for any preparation process $\mathfrak{p}$ and any yes/no test $\mathfrak{t}$, the element $\mathfrak{e}(\mathfrak{p},\mathfrak{t}) \in \mathfrak{B} := \{\bot, \mathbf{Y}, \mathbf{N}\}$ will be defined to be $\bot$ (alternatively, $\mathbf{Y}$ or $\mathbf{N}$) if the outcome of the yes/no test $\mathfrak{t}$ on any sample prepared according to the preparation procedure $\mathfrak{p}$ is judged as 'indeterminate' ('positive with certainty' or 'negative with certainty', respectively) by the observer.

$$\begin{array}{rccl} \mathfrak{e} & : & \mathfrak{P}\times\mathfrak{T} & \longrightarrow & \mathfrak{B} := \{\bot, \mathbf{Y}, \mathbf{N}\} \\ & & (\mathfrak{p},\mathfrak{t}) & \mapsto & \mathfrak{e}(\mathfrak{p},\mathfrak{t}). \end{array} \tag{1}$$

When the determinacy of a yes/no test is established for an observer, we can consider that this observer possesses some elementary 'information' about the state of the system, whereas, in the 'indeterminate case', the observer has none (relatively to the occurrence of the considered property).

The set $\mathfrak{B}$ will then be equipped with the following poset structure, characterizing the 'information' gathered by the observer:

$$\forall u, v \in \mathfrak{B}, \qquad (u \leq v) \ :\Leftrightarrow \ (u = \bot \text{ or } u = v). \tag{2}$$

$(\mathfrak{B}, \leq)$ is also an Inf semi-lattice which infima will be denoted $\bigwedge$. We have

$$\forall x, y \in \mathfrak{B}, \qquad x \wedge y = \begin{cases} x & \text{if } x = y \\ \bot & \text{if } x \neq y \end{cases} \tag{3}$$

We will also introduce a commutative monoid law denoted $\bullet$ and defined by

$$\forall x \in \mathfrak{B}, \qquad x \bullet \mathbf{Y} = x, \quad x \bullet \mathbf{N} = \mathbf{N}, \quad \bot \bullet \bot = \bot. \tag{4}$$

[3] The information corresponding to macroscopic events/operations describing the procedure depend on an observer O. If this dependence has to be made explicit, we will adopt the notation $\mathfrak{P}^{(O)}$ to denote the set of preparation processes defined by the observer O. This mention of the observer will be also attached to the different quotients associated to the space of preparations.

[4] If the dependence with respect to the observer O has to be made explicit, we will adopt the notation $\mathfrak{T}^{(O)}$ to denote the set of tests defined by the observer O. This mention of the observer will be also attached to the different quotients associated to the space of yes/no tests.

$x \bullet y$ will be called *the product of the determinations x and y*.
This law verifies the following properties

$$\forall x \in \mathfrak{B}, \forall B \subseteq \mathfrak{B} \qquad x \bullet \bigwedge B = \bigwedge_{b \in B} (x \bullet b), \tag{5}$$

$$\forall x \in \mathfrak{B}, \forall C \subseteq_{Chain} \mathfrak{B} \qquad x \bullet \bigvee B = \bigvee_{b \in B} (x \bullet b). \tag{6}$$

$(\mathfrak{B}, \leq)$ will be also equipped with the following involution map :

$$\overline{\top} := \bot \qquad \overline{\mathbf{Y}} := \mathbf{N} \qquad \overline{\mathbf{N}} := \mathbf{Y}. \tag{7}$$

2.1 The space of states

A pre-order relation can be defined on the set $\mathfrak{P}$ of preparation processes. A preparation process $\mathfrak{p}_2 \in \mathfrak{P}$ is said to be *sharper* than another preparation process $\mathfrak{p}_1 \in \mathfrak{P}$ (this fact will be denoted $\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2$) iff any yes/no test $\mathfrak{t} \in \mathfrak{T}$ that is 'determinate' for the samples prepared through $\mathfrak{p}_1$ is also necessarily 'determinate' with the same determination for the samples prepared through $\mathfrak{p}_2$, i.e.,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \qquad (\ \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2\ ) \ :\Leftrightarrow\ (\ \forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}(\mathfrak{p}_1, \mathfrak{t}) \leq \mathfrak{e}(\mathfrak{p}_2, \mathfrak{t})\ ), \tag{8}$$

If $\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2$ (i.e., $\mathfrak{p}_2$ is 'sharper' than $\mathfrak{p}_1$), $\mathfrak{p}_1$ is said to be 'coarser' than $\mathfrak{p}_2$.

An equivalence relation, denoted $\sim_{\mathfrak{P}}$, is defined on the set of preparations $\mathfrak{P}$ from this pre-order relation. Two preparation processes are identified iff the statements established by the observer about the corresponding prepared samples are identical. A *state* of the physical system is an equivalence class of preparation processes corresponding to the same informational content. The set of equivalence classes, modulo $\sim_{\mathfrak{P}}$, will be called *space of states* and denoted $\mathfrak{S}$. In other words,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P},\ (\mathfrak{p}_1 \sim_{\mathfrak{P}} \mathfrak{p}_2) \ :\Leftrightarrow\ (\ \forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}(\mathfrak{p}_1, \mathfrak{t}) = \mathfrak{e}(\mathfrak{p}_2, \mathfrak{t})\ ) \Leftrightarrow (\ \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2 \text{ and } \mathfrak{p}_1 \sqsupseteq_{\mathfrak{P}} \mathfrak{p}_2\ ), \tag{9}$$

$$\lceil \mathfrak{p} \rceil \ := \ \{ \mathfrak{p}' \in \mathfrak{P} \mid \mathfrak{p}' \sim_{\mathfrak{P}} \mathfrak{p} \}, \tag{10}$$

$$\mathfrak{S} \ := \ \{ \lceil \mathfrak{p} \rceil \mid \mathfrak{p} \in \mathfrak{P} \}. \tag{11}$$

The space of states $\mathfrak{S}$ is partially ordered. Explicitly

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\ \sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2\ ) \ :\Leftrightarrow\ (\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P},\ (\ \sigma_1 = \lceil \mathfrak{p}_1 \rceil, \sigma_2 = \lceil \mathfrak{p}_2 \rceil\ )\ \Rightarrow\ (\ \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2\ )). \tag{12}$$

We will derive a map ε according to the following definition :

$$\begin{array}{rcl} \varepsilon \ : \ \mathfrak{T} & \to & \mathfrak{B}^{\mathfrak{S}} \\ \mathfrak{t} & \mapsto & \varepsilon_{\mathfrak{t}} \quad | \quad \varepsilon_{\mathfrak{t}}(\lceil \mathfrak{p} \rceil) := \mathfrak{e}(\mathfrak{p}, \mathfrak{t}),\ \forall \mathfrak{p} \in \mathfrak{P}. \end{array} \tag{13}$$

For any $\mathfrak{t} \in \mathfrak{T}$, $\varepsilon_{\mathfrak{t}}$ is an order-preserving map on $\mathfrak{S}$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \qquad (\ \sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2\ ) \ :\Leftrightarrow\ (\ \forall \mathfrak{t} \in \mathfrak{T},\ \varepsilon_{\mathfrak{t}}(\sigma_1) \leq \varepsilon_{\mathfrak{t}}(\sigma_2)\ ), \tag{14}$$

If we consider a collection of preparation processes $P \subseteq \mathfrak{P}$, we can define a new preparation procedure, called *mixture* and denoted $\bigsqcap^{\mathfrak{P}} P$, as follows. The samples produced from the preparation procedure $\bigsqcap^{\mathfrak{P}} P$ are obtained by a random mixing of the samples issued from the preparation processes of the collection P indiscriminately. As a consequence, the statements that the observer can establish after a sequence of tests $\mathfrak{t} \in \mathfrak{T}$ on these samples produced through the procedure $\bigsqcap^{\mathfrak{P}} P$ is given as the infimum of the statements that the observer can establish for the elements of P separately. In other words,

$$\forall P \subseteq \mathfrak{P},\ \exists \bigsqcap^{\mathfrak{P}} P \in \mathfrak{P} \ \mid \ (\forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}(\bigsqcap^{\mathfrak{P}} P, \mathfrak{t}) = \bigwedge\nolimits_{\mathfrak{p} \in P} \mathfrak{e}(\mathfrak{p}, \mathfrak{t})\ ). \tag{15}$$

The space of states inherits a notion of *mixed states* by defining

$$\forall P \subseteq \mathfrak{P}, \qquad \bigsqcap\nolimits^{\mathfrak{S}}_{\mathfrak{p} \in P} \lceil \mathfrak{p} \rceil := \lceil \bigsqcap^{\mathfrak{P}} P \rceil. \tag{16}$$

As a result, the space of states inherits a structure of *down-complete Inf semi-lattice*. In other words,

$$\textbf{(A1)} \qquad \forall S \subseteq \mathfrak{S},\ (\bigsqcap^{\mathfrak{S}} S) \text{ exists in } \mathfrak{S},\ \text{and } \forall \mathfrak{t} \in \mathfrak{T},\ \varepsilon_{\mathfrak{t}}(\bigsqcap^{\mathfrak{S}} S) = \bigwedge_{\sigma \in S} \varepsilon_{\mathfrak{t}}(\sigma). \tag{17}$$

As a direct consequence, the space of states is then also *bounded-complete*, i.e.

$$\forall S \subseteq \mathfrak{S} \mid \widehat{S}^{\mathfrak{S}}, \qquad (\bigsqcup^{\mathfrak{S}} S) \text{ exists in } \mathfrak{S}. \tag{18}$$

where

$$\forall \mathfrak{S}' \subseteq \mathfrak{S}, \forall S \subseteq \mathfrak{S}', \qquad \widehat{S}^{\mathfrak{S}'} :\Leftrightarrow \exists \sigma' \in \mathfrak{S}' \mid \sigma \sqsubseteq_{\mathfrak{S}} \sigma', \forall \sigma \in S. \tag{19}$$

We will adopt the shortened notation $\forall \sigma, \sigma' \in \mathfrak{S}, \widehat{\sigma\sigma'}^{\mathfrak{S}} := \widehat{\{\sigma, \sigma'\}}^{\mathfrak{S}}$.

We will also assume that there exists a preparation process, unique from the point of view of the statements that can be produced about it, that can be interpreted as a 'randomly-selected' collection of 'un-prepared samples'. This element leads to complete indeterminacy for any yes/no test realized on it.

$$\exists\, \mathfrak{p}_{\perp} \in \mathfrak{P} \ \mid\ (\forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}(\mathfrak{p}_{\perp}, \mathfrak{t}) = \perp). \tag{20}$$

Hence, the partial order $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ admits *a bottom element*, denoted $\perp_{\mathfrak{S}} := \lceil \mathfrak{p}_{\perp} \rceil$. In other words,

$$\textbf{(A2)} \qquad \exists\, \perp_{\mathfrak{S}} \in \mathfrak{S} \mid \forall \sigma \in \mathfrak{S},\ \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} \sigma, \tag{21}$$

2.2 The space of effects

We can introduce a pre-order relation on the space of yes/no tests $\mathfrak{T}$ as well :

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}, \qquad (\ \mathfrak{t}_1 \sqsubseteq_{\mathfrak{T}} \mathfrak{t}_2\ ) \ :\Leftrightarrow\ (\ \forall \sigma \in \mathfrak{S},\ \varepsilon_{\mathfrak{t}_1}(\sigma) \leq \varepsilon_{\mathfrak{t}_2}(\sigma)\ ), \tag{22}$$

and an equivalence relation, denoted $\sim_{\mathfrak{T}}$, can be derived from this pre-order on the set of yes/no tests $\mathfrak{T}$, i.e. $\mathfrak{t}_1 \sim_{\mathfrak{T}} \mathfrak{t}_2$ is equivalent to ($\mathfrak{t}_1 \sqsubseteq_{\mathfrak{T}} \mathfrak{t}_2$ and $\mathfrak{t}_1 \sqsupseteq_{\mathfrak{T}} \mathfrak{t}_2$). An *effect* of the physical system is an equivalence class of yes/no tests, i.e., a class of yes/no tests that are not distinguished from the point of view of the statements that the observer can produce by using these yes/no tests on finite collections of samples. The set of equivalence classes of yes/no tests, modulo the relation $\sim_{\mathfrak{T}}$, will be denoted $\mathfrak{E}$. In other words,

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T},\ (\ \mathfrak{t}_1 \sim_{\mathfrak{T}} \mathfrak{t}_2\ ) \ :\Leftrightarrow\ (\ \forall \sigma \in \mathfrak{S},\ \varepsilon_{\mathfrak{t}_1}(\sigma) = \varepsilon_{\mathfrak{t}_2}(\sigma)\ ), \tag{23}$$

$$\lfloor \mathfrak{t} \rfloor \ := \ \{ \mathfrak{t}' \in \mathfrak{T} \mid \mathfrak{t}' \sim_{\mathfrak{T}} \mathfrak{t} \}, \tag{24}$$

$$\mathfrak{E} \ := \ \{ \lfloor \mathfrak{t} \rfloor \mid \mathfrak{t} \in \mathfrak{T} \}. \tag{25}$$

The set of effects $\mathfrak{E}$ is then equipped naturally with a partial order denoted $\sqsubseteq_{\mathfrak{E}}$.
We will adopt the following abuse of notation $\varepsilon_{\lfloor \mathfrak{t} \rfloor} := \varepsilon_{\mathfrak{t}}$, $\forall \mathfrak{t} \in \mathfrak{T}$.
We have by construction

$$\forall \mathfrak{l}, \mathfrak{l}' \in \mathfrak{E}, \qquad (\forall \sigma \in \mathfrak{S},\ \varepsilon_{\mathfrak{l}}(\sigma) = \varepsilon_{\mathfrak{l}'}(\sigma)) \ \Leftrightarrow\ (\mathfrak{l} = \mathfrak{l}'), \tag{26}$$

$$\forall \sigma, \sigma' \in \mathfrak{S}, \qquad (\forall \mathfrak{l} \in \mathfrak{E},\ \varepsilon_{\mathfrak{l}}(\sigma) = \varepsilon_{\mathfrak{l}}(\sigma')) \ \Leftrightarrow\ (\sigma = \sigma'). \tag{27}$$

We note that $(\mathfrak{S}, \mathfrak{E}, \varepsilon)$ forms a bi-extensional Chu space [23].

If we consider a collection of tests $T \subseteq \mathfrak{T}$, we can define a new test, called *mixture* and denoted $\bigsqcap^{\mathfrak{T}} T$, as follows. The result obtained for the test $\bigsqcap^{\mathfrak{T}} T$ is obtained by a random mixing of the results issued from the tests of the collection T indiscriminately. As a consequence, the statements that the observer can establish after a sequence of tests is given as the infimum of the statements that the observer can establish for each test separately. In other words,

$$\forall T \subseteq \mathfrak{T},\ \exists \bigsqcap^{\mathfrak{T}} T \in \mathfrak{T} \ \mid\ (\forall \sigma \in \mathfrak{S},\ \varepsilon_{\bigsqcap^{\mathfrak{T}} T}(\sigma) = \bigwedge_{\mathfrak{t} \in T} \varepsilon_{\mathfrak{t}}(\sigma)). \tag{28}$$

The space of effects inherits a notion of *mixed effects* by defining

$$\forall T \subseteq \mathfrak{T}, \qquad \left\lceil \prod^{\mathfrak{E}}_{\mathfrak{t}\in T} \mathfrak{t} \right\rceil := \left\lceil \prod^{\mathfrak{T}} T \right\rceil. \tag{29}$$

As a result, the space of effects inherits a structure of *down-complete Inf semi-lattice*. In other words,

$$\textbf{(A3)} \qquad \forall E \subseteq \mathfrak{E},\ \left(\prod^{\mathfrak{E}} E\right) \text{ exists in } \mathfrak{E}, \text{ and } \forall \sigma \in \mathfrak{S},\ \varepsilon_{\prod^{\mathfrak{E}} E}(\sigma) = \bigwedge\nolimits_{\mathfrak{l}\in E} \varepsilon_{\mathfrak{l}}(\sigma). \tag{30}$$

The conjugate of a yes/no test $\mathfrak{t} \in \mathfrak{T}$ is the yes/no test denoted $\overline{\mathfrak{t}}$ and defined from $\mathfrak{t}$ by exchanging the roles of $\mathbf{Y}$ and $\mathbf{N}$ in every result obtained by applying $\mathfrak{t}$ to any given input sample. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \sigma \in \mathfrak{S}, \qquad \varepsilon_{\overline{\mathfrak{t}}}(\sigma) := \overline{\varepsilon_{\mathfrak{t}}(\sigma)}. \tag{31}$$

We note the following definition of the conjugate of an effect

$$\forall \mathfrak{l} \in \mathfrak{E}, \qquad \overline{\mathfrak{l}} = \{\, \overline{\mathfrak{t}} \mid \mathfrak{l} = \lfloor \mathfrak{t} \rfloor \,\}. \tag{32}$$

We will sometimes use a particular effect called "partial trace", denoted $\mathfrak{Y}_{\mathfrak{E}}$ and defined by

$$\forall \sigma \in \mathfrak{S}, \qquad \varepsilon_{\mathfrak{Y}_{\mathfrak{E}}}(\sigma) := \mathbf{Y}. \tag{33}$$

An effect $\mathfrak{l} \in \mathfrak{E}$ will be said to be *testable* iff it can be revealed as 'certain' at least for some collections of prepared samples. In other words, '$\mathfrak{l}$ is testable' means ${\varepsilon_{\mathfrak{l}}}^{-1}(\mathbf{Y}) \neq \varnothing$.

Lemma 1. For any testable effect $\mathfrak{l}$, there exists an element $\Sigma_{\mathfrak{l}} := \prod^{\mathfrak{S}} {\varepsilon_{\mathfrak{l}}}^{-1}(\mathbf{Y}) \in \mathfrak{S}$, called *effect-state*, such that the filter ${\varepsilon_{\mathfrak{l}}}^{-1}(\mathbf{Y})$ is the principal filter $(\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}})$. ■

We will allow for a generalized definition of effects. Let us consider $\Sigma, \Sigma' \in \mathfrak{S}$ such that $\neg \widehat{\Sigma\Sigma'}^{\mathfrak{S}}$. We define $\mathfrak{l}_{(\Sigma,\Sigma')}$ according to ${\varepsilon_{\mathfrak{l}_{(\Sigma,\Sigma')}}}^{-1}(\mathbf{Y}) := \uparrow^{\mathfrak{S}} \Sigma$ and ${\varepsilon_{\mathfrak{l}_{(\Sigma,\Sigma')}}}^{-1}(\mathbf{N}) := \uparrow^{\mathfrak{S}} \Sigma'$.

Theorem 1. Let us consider $B := \{\mathfrak{b}_{\mathfrak{l}} \mid \mathfrak{l} \in \mathfrak{E}\}$ a family of elements of $\mathfrak{B}$ satisfying

$$\forall \mathfrak{l}, \mathfrak{l}' \in \mathfrak{E}, \qquad (\mathfrak{l} \sqsubseteq_{\mathfrak{E}} \mathfrak{l}') \Rightarrow (\mathfrak{b}_{\mathfrak{l}} \leq \mathfrak{b}_{\mathfrak{l}'}), \tag{34}$$

$$\forall \{\mathfrak{l}_i \mid i \in I\} \subseteq \mathfrak{E}, \qquad \mathfrak{b}_{\prod^{\mathfrak{E}}_{i\in I} \mathfrak{l}_i} = \bigwedge\nolimits_{i\in I} \mathfrak{b}_{\mathfrak{l}_i}, \tag{35}$$

$$\forall \mathfrak{l} \in \mathfrak{E}, \qquad \mathfrak{b}_{\overline{\mathfrak{l}}} = \overline{\mathfrak{b}_{\mathfrak{l}}}. \tag{36}$$

Then, we have

$$\exists! \, \sigma \in \mathfrak{S} \quad | \quad \forall \mathfrak{l} \in \mathfrak{E},\ \varepsilon_{\mathfrak{l}}(\sigma) = \mathfrak{b}_{\mathfrak{l}}. \tag{37}$$

■

Proof. Straightforward. It suffices to define $\mathfrak{l}_B := \prod^{\mathfrak{E}} \{\, \mathfrak{l} \mid \mathfrak{b}_{\mathfrak{l}} = \mathbf{Y} \,\}$ and $\sigma := \Sigma_{\mathfrak{l}_B} = \prod^{\mathfrak{S}} {\varepsilon_{\mathfrak{l}_B}}^{-1}(\mathbf{Y})$. □

Corollary 1.

$$\forall \{\sigma_i \mid i \in I\} \subseteq_{Chain} \mathfrak{S},\ \exists \sigma \in \mathfrak{S} \quad | \quad \forall \mathfrak{l} \in \mathfrak{E},\ \varepsilon_{\mathfrak{l}}(\sigma) = \bigvee\nolimits_{i\in I} \varepsilon_{\mathfrak{l}}(\sigma_i), \tag{38}$$

$$\sigma \quad = \quad \bigsqcup\nolimits^{\mathfrak{S}}_{i\in I} \sigma_i. \tag{39}$$

■

Proof. First of all, we note that $\{\sigma_i \mid i \in I\} \subseteq_{Chain} \mathfrak{S}$ and property (22) implies that $\{\varepsilon_{\mathfrak{l}}(\sigma_i) \mid i \in I\} \subseteq_{Chain} \mathfrak{B}$ for any $\mathfrak{l} \in \mathfrak{E}$ and then $\bigvee_{i\in I} \varepsilon_{\mathfrak{l}}(\sigma_i)$ exists for any $\mathfrak{l} \in \mathfrak{E}$ due to the chain-completeness of $\mathfrak{B}$.
Using the properties (22)(28)(31) of the map ε and the complete-distributivity properties satisfied by $\mathfrak{B}$, we can check easily that $\{\bigvee_{i\in I} \varepsilon_{\mathfrak{l}}(\sigma_i) \mid \mathfrak{l} \in \mathfrak{E}\}$ satisfies properties (34) (35) (36). As a consequence, the property (38) is a direct consequence of Theorem 1.
By definition of the poset structure (14), we deduce, from the property $(\forall \mathfrak{l} \in \mathfrak{E},\ \varepsilon_{\mathfrak{l}}(\sigma) = \bigvee_{i\in I} \varepsilon_{\mathfrak{l}}(\sigma_i))$, that $\sigma \sqsupseteq_{\mathfrak{S}} \sigma_i$, $\forall i \in I$ and $(\sigma' \sqsupseteq_{\mathfrak{S}} \sigma_i,\ \forall i \in I) \Rightarrow (\sigma \sqsupseteq_{\mathfrak{S}} \sigma')$. In other words, $\sigma = \bigsqcup^{\mathfrak{S}}_{i\in I} \sigma_i$. □

2.3 Pure states

A state is said to be *a pure state* iff it cannot be built as a mixture of other states (the set of pure states will be denoted $\mathfrak{S}^{pure}$). More explicitly,

$$\sigma \in \mathfrak{S}^{pure} \quad :\Leftrightarrow \quad (\forall S \subseteq^{\neq \varnothing} \mathfrak{S},\ (\sigma = \bigsqcap^{\mathfrak{S}} S) \ \Rightarrow\ (\sigma \in S)). \tag{40}$$

In other words, pure states are associated with completely meet-irreducible elements in $\mathfrak{S}$. [5]

We will moreover assume that every state can be written as a mixture of pure states. In other words,

$$\textbf{(A4)} \qquad \forall \sigma \in \mathfrak{S},\ \sigma = \bigsqcap^{\mathfrak{S}} \underline{\sigma},\ \text{where } \underline{\sigma} = (\mathfrak{S}^{pure} \cap (\uparrow^{\mathfrak{S}} \sigma)). \tag{42}$$

Remark 1. If $\mathfrak{S}$ is a bounded-complete algebraic domain (here, $\mathfrak{S}$ is already assumed to be a bounded-complete and chain-complete Inf semi-lattice), previous property is a direct consequence of [15, Theorem I-4.26 p.126].

Remark 2. We note that $\mathfrak{S}^{pure} = \sqcap - Irr(\mathfrak{S})$ is the unique smallest subset of $\mathfrak{S}$ satisfying property (42). This point is mentioned in [15, Remark I-4.22 p.125].

A simple characterization of completely meet-irreducible elements within posets is given in [15, Definition I-4.21] :

$$\sigma \in \mathfrak{S}^{pure} \quad \Leftrightarrow \quad \begin{cases} \sigma \in Max(\mathfrak{S}) & \text{(Type 1)} \\ (\uparrow^{\mathfrak{S}} \sigma) \setminus \{\sigma\} \text{ admits a minimum element} & \text{(Type 2)} \end{cases} \tag{43}$$

This characterization is equivalent to the basic definition (40) for a bounded-complete Inf semi-lattice like $\mathfrak{S}$.
From Corollary 1, using Zorn's Lemma, we deduce that

$$\forall \sigma \in \mathfrak{S}, \qquad \exists \sigma' \in Max(\mathfrak{S}) \mid \sigma \sqsubseteq_{\mathfrak{S}} \sigma'. \tag{44}$$

From that remark, we can decide to eliminate Type 2 pure states. Indeed, it is clear that 'Type 2' pure states have no physical meaning. Indeed, for any 'Type 2' pure states, it exists some 'Type 1' pure states sharper than it (and, then, containing more information than it). The existence of 'Type 2' pure states in the space of states leads then to a redundant description of the system. We will then require that $\mathfrak{S}^{pure}$, i.e. the set of completely meet-irreducible elements $\sqcap - Irr(\mathfrak{S})$, be constituted exclusively of maximal elements of $\mathfrak{S}$. In other words, we require the space of states to be such that

$$\textbf{(A5)} \qquad \sqcap - Irr(\mathfrak{S}) = Max(\mathfrak{S}). \tag{45}$$

From now on, Chu spaces $(\mathfrak{S}, \mathfrak{E}, \varepsilon)$ which elements satisfy the axioms **(A1)** $-$ **(A5)** will be called *States/Effects Chu spaces*.

2.4 Remarkable properties of spaces of states

According to [16, definition p.117 and Section 11 Lemma 1 p.118], we introduce the following notion.

Definition 1. A space of states $\mathfrak{S}$ is said to be *distributive* iff

$$\forall \sigma, \sigma_1, \sigma_2 \in \mathfrak{S},\ \ (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqsubseteq_{\mathfrak{S}} \sigma \ \ \Rightarrow\ \ \exists \sigma_1', \sigma_2' \in \mathfrak{S} \mid (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_1',\ \ \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma_2' \ \text{ and } \ \sigma = \sigma_1' \sqcap_{\mathfrak{S}} \sigma_2'). \tag{46}$$

When $\mathfrak{S}$ is distributive, we have the following standard properties satisfied, as soon as the implied

[5] We note that complete meet-irreducibility implies meet-irreducibility. In other words,

$$\sigma \in \mathfrak{S}_{pure} \quad \Rightarrow \quad (\forall \sigma_1, \sigma_2 \in \mathfrak{S},\ (\sigma = \sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \ \Rightarrow\ (\sigma = \sigma_1 \ \text{ or } \ \sigma = \sigma_2)). \tag{41}$$

suprema are well defined

$$\sigma_1 \sqcap_{\mathfrak{S}} (\sigma_2 \sqcup_{\mathfrak{S}} \sigma_3) = (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqcup_{\mathfrak{S}} (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_3) \quad (47)$$

$$\sigma_1 \sqcup_{\mathfrak{S}} (\sigma_2 \sqcap_{\mathfrak{S}} \sigma_3) = (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \sqcap_{\mathfrak{S}} (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_3). \quad (48)$$

According with [10, Section 3.5], we introduce the following notion.

Definition 2. Let us introduce the following binary relation, denoted $\breve{\bowtie}_{\mathfrak{S}}$ and defined on $\mathfrak{S}$ by

$$\forall(\sigma,\sigma') \in \mathfrak{S}^{\times 2}, \quad \sigma \breve{\bowtie}_{\mathfrak{S}} \sigma' \quad :\Leftrightarrow \quad (\forall \sigma'' \sqsubset_{\mathfrak{S}} \sigma', \widehat{\sigma\sigma''}^{\mathfrak{S}} \text{ and } \forall \sigma'' \sqsubset_{\mathfrak{S}} \sigma, \widehat{\sigma'\sigma''}^{\mathfrak{S}} \text{ and not } \widehat{\sigma\sigma'}^{\mathfrak{S}}). \quad (49)$$

(here we have used the notation introduced in (19)).
The space of states $\mathfrak{S}$ is said to be *orthocomplemented* iff there exists a map $\star : \mathfrak{S} \setminus \{\bot_{\mathfrak{S}}\} \to \mathfrak{S} \setminus \{\bot_{\mathfrak{S}}\}$ such that

$$\forall \sigma \in \mathfrak{S}, \quad \sigma^{\star\star} = \sigma, \quad (50)$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \quad \sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2 \Rightarrow \sigma_1^{\star} \sqsupseteq_{\mathfrak{S}} \sigma_2^{\star}, \quad (51)$$

$$\forall \sigma \in \mathfrak{S}, \quad \sigma \breve{\bowtie}_{\mathfrak{S}} \sigma^{\star}. \quad (52)$$

2.5 Symmetries of the system

Observer O_1 has prepared a state $\sigma_1 \in \mathfrak{S}^{(O_1)}$ and intends to describe it to observer O_2. Observer O_2 is able to interpret the macroscopic data defining σ_1 in terms of the elements of $\mathfrak{S}^{(O_2)}$ using a map $f_{(12)} : \mathfrak{S}^{(O_1)} \to \mathfrak{S}^{(O_2)}$ (i.e., O_2 knows how to identify a state $f_{(12)}(\sigma_1)$ corresponding to any σ_1). Observer O_2 has selected an effect $\mathfrak{l}_2 \in \mathfrak{E}^{(O_2)}$ and intends to address the corresponding question to O_1. Observer O_1 is able to interpret the macroscopic data defining $\mathfrak{l}_2$ in terms of the elements of $\mathfrak{E}^{(O_1)}$ using a map $f^{(21)} : \mathfrak{E}^{(O_2)} \to \mathfrak{E}^{(O_1)}$ (i.e., O_1 knows how to fix an effect $f^{(21)}(\mathfrak{l}_2)$ corresponding to any $\mathfrak{l}_2$). The pair of maps $(f_{(12)}, f^{(21)})$ where $f_{(12)} : \mathfrak{S}^{(O_1)} \to \mathfrak{S}^{(O_2)}$ and $f^{(21)} : \mathfrak{E}^{(O_2)} \to \mathfrak{E}^{(O_1)}$ defines *a dictionary* formalizing the transaction from O_1 to O_2. The main task these observers want to accomplish is to confront their knowledge, i.e., to compare their 'statements' about the system. As soon as the transaction is formalized using a dictionary, the two observers can formulate their statements and each confront them with the statements of the other. First, observer O_1 can interpret the macroscopic data defining $\mathfrak{l}_2$ using the map $f^{(21)}$. Then, he produces the statement $\varepsilon^{(O_1)}_{f^{(21)}(\mathfrak{l}_2)}(\sigma_1)$ concerning the results associated to this effect on the chosen state. Secondly, observer O_2 can interpret the macroscopic data defining σ_1 using the map $f_{(12)}$. Then, observer O_2 pronounces her statement $\varepsilon^{(O_2)}_{\mathfrak{l}_2}(f_{(12)}(\sigma_1))$ concerning the results associated to the effect $\mathfrak{l}_2$ on the correspondingly prepared state. The two observers, O_1 and O_2, are said to *agree about all their statements* iff

$$\forall \sigma_1 \in \mathfrak{S}^{(O_1)}, \forall \mathfrak{l}_2 \in \mathfrak{E}^{(O_2)}, \quad \varepsilon^{(O_2)}_{\mathfrak{l}_2}(f_{(12)}(\sigma_1)) = \varepsilon^{(O_1)}_{f^{(21)}(\mathfrak{l}_2)}(\sigma_1). \quad (53)$$

To summarize, we will define symmetries of the system as follows.

Definition 3. The symmetries of the system are defined as Chu morphisms [23] from a States/Effects Chu space $(\mathfrak{S}^{(O_1)}, \mathfrak{E}^{(O_1)}, \varepsilon^{(O_1)})$ defining the space of states and effects associated to the observer O_1, to another States/Effects Chu space $(\mathfrak{S}^{(O_2)}, \mathfrak{E}^{(O_2)}, \varepsilon^{(O_2)})$ associated to the observer O_2, i.e. as pairs of bijective maps $f_{(12)} : \mathfrak{S}^{(O_1)} \to \mathfrak{S}^{(O_2)}$ and $f^{(21)} : \mathfrak{E}^{(O_2)} \to \mathfrak{E}^{(O_1)}$ satisfying property (53).

Lemma 2. Let us consider $(f_{(12)}, f^{(21)})$ a symmetry from $(\mathfrak{S}^{(O_1)}, \mathfrak{E}^{(O_1)}, \varepsilon^{(O_1)})$ to $(\mathfrak{S}^{(O_2)}, \mathfrak{E}^{(O_2)}, \varepsilon^{(O_2)})$,

we have

$$f^{(21)}(\mathfrak{E}^{(O_2)+}) \subseteq \mathfrak{E}^{(O_1)+} \tag{54}$$

■

Proof. Immediate consequence of the defining property (53). □

Definition 4. The composition of a symmetry $(f_{(12)}, f^{(21)})$ from $(\mathfrak{S}^{(O_1)}, \mathfrak{E}^{(O_1)}, \varepsilon^{(O_1)})$ to $(\mathfrak{S}^{(O_2)}, \mathfrak{E}^{(O_2)}, \varepsilon^{(O_2)})$ by another symmetry $(g_{(23)}, g^{(32)})$ defined from $(\mathfrak{S}^{(O_2)}, \mathfrak{E}^{(O_2)}, \varepsilon^{(O_2)})$ to $(\mathfrak{S}^{(O_3)}, \mathfrak{E}^{(O_3)}, \varepsilon^{(O_3)})$ is given by the pair of bijective maps $(g_{(23)} \circ f_{(12)}, f^{(21)} \circ g^{(32)})$ defining a valid symmetry from $(\mathfrak{S}^{(O_1)}, \mathfrak{E}^{(O_1)}, \varepsilon^{(O_1)})$ to $(\mathfrak{S}^{(O_3)}, \mathfrak{E}^{(O_3)}, \varepsilon^{(O_3)})$.

As noted in [10], the duality property (53) suffices to deduce the following properties.

Theorem 2. $f_{(12)}$ and $f^{(21)}$ are maps satisfying

$$\forall S_1 \subseteq \mathfrak{S}^{(O_1)}, \qquad f_{(12)}(\bigsqcap^{\mathfrak{S}^{(O_1)}} S_1) = \bigsqcap_{\sigma_1 \in S_1}^{\mathfrak{S}^{(O_2)}} f_{(12)}(\sigma_1) \tag{55}$$

$$\forall C_1 \subseteq_{Chain} \mathfrak{S}^{(O_1)}, \qquad f_{(12)}(\bigsqcup^{\mathfrak{S}^{(O_1)}} C_1) = \bigsqcup_{\sigma_1 \in C_1}^{\mathfrak{S}^{(O_2)}} f_{(12)}(\sigma_1) \tag{56}$$

$$f_{(12)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot_{\mathfrak{S}^{(O_2)}} \tag{57}$$

and

$$\forall E_2 \subseteq \mathfrak{E}^{(O_2)}, \qquad f^{(21)}(\bigsqcap^{\mathfrak{E}^{(O_2)}} E_2) = \bigsqcap_{\mathfrak{l}_2 \in E_2}^{\mathfrak{E}^{(O_1)}} f^{(21)}(\mathfrak{l}_2) \tag{58}$$

$$\forall \mathfrak{l}_2 \in \mathfrak{E}^{(O_2)}, \qquad f^{(21)}(\,\overline{\mathfrak{l}_2}\,) = \overline{f^{(21)}(\mathfrak{l}_2)} \tag{59}$$

$$f^{(21)}(\,\mathfrak{Y}_{\mathfrak{E}}^{(O_2)}\,) = \mathfrak{Y}_{\mathfrak{E}}^{(O_1)}. \tag{60}$$

Note that, due to properties (55) (57) and (58), as long as $\mathfrak{S}^{(O_1)}$ satisfies axioms **(A1) (A2) (A3)**, $\mathfrak{S}^{(O_2)}$ satisfies axioms **(A1) (A2) (A3)** as well. ■

Proof. All proofs follow the same trick. For example, for any $S_1 \subseteq \mathfrak{S}^{(O_1)}$ and any $\mathfrak{l}_2 \in \mathfrak{E}^{(O_2)}$, we have, using (53) and (30) :

$$\begin{aligned}
\varepsilon_{\mathfrak{l}_2}^{(O_2)}(f_{(12)}(\bigsqcap^{\mathfrak{S}^{(O_1)}} S_1)) &= \varepsilon_{f^{(21)}(\mathfrak{l}_2)}^{(O_1)}(\bigsqcap^{\mathfrak{S}^{(O_1)}} S_1) \\
&= \bigwedge_{\sigma_1 \in S_1} \varepsilon_{f^{(21)}(\mathfrak{l}_2)}^{(O_1)}(\sigma_1) \\
&= \bigwedge_{\sigma_1 \in S_1} \varepsilon_{\mathfrak{l}_2}^{(O_2)}(f_{(12)}(\sigma_1)) \\
&= \varepsilon_{\mathfrak{l}_2}^{(O_2)}(\bigsqcap_{\sigma_1 \in S_1}^{\mathfrak{S}^{(O_2)}} f_{(12)}(\sigma_1))
\end{aligned} \tag{61}$$

We now use the property (27) to conclude on (55).
To give another example, we justify the property (57) :

$$\forall \mathfrak{l}_2 \in \mathfrak{E}^{(O_2)}, \quad \varepsilon_{\mathfrak{l}_2}^{(O_2)}(f_{(12)}(\bot_{\mathfrak{S}^{(O_1)}})) = \varepsilon_{f^{(21)}(\mathfrak{l}_2)}^{(O_1)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot. \tag{62}$$

implies $f_{(12)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot_{\mathfrak{S}^{(O_2)}}$. □

Theorem 3. Pure states in $\mathfrak{S}^{(O_2)}$ are exactly the direct images by $f_{(12)}$ of pure states in $\mathfrak{S}^{(O_1)}$. Moreover, as long as $\mathfrak{S}^{(O_1)}$ satisfies axiom **(A4)**, $\mathfrak{S}^{(O_2)}$ satisfies axiom **(A4)** as well. ■

Proof. Let us consider a state σ_2 in $\mathfrak{S}^{(O_2)}$ such that $f_{(12)}^{-1}(\sigma_2)$ is a pure state in $\mathfrak{S}^{(O_1)}$. For any $S_2 \subseteq \mathfrak{S}^{(O_2)}$ satisfying $\sigma_2 = \bigwedge^{\mathfrak{S}^{(O_2)}} S_2$, we have $f_{(12)}^{-1}(\sigma_2) = f_{(12)}^{-1}(\bigwedge^{\mathfrak{S}^{(O_2)}} S_2) = \bigwedge_{\sigma_2' \in S_2}^{\mathfrak{S}^{(O_1)}} f_{(12)}^{-1}(\sigma_2')$ using (55), and then $f_{(12)}^{-1}(\sigma_2) \in f_{(12)}^{-1}(S_2)$ (due to complete irreducibility of $f_{(12)}^{-1}(\sigma_2)$), and then $\sigma_2 \in S_2$. As a conclusion, σ_2 is completely meet-irreducible in $\mathfrak{S}^{(O_2)}$, i.e. it is a pure state of $\mathfrak{S}^{(O_2)}$.
Conversely, let us consider σ_2 a pure state in $\mathfrak{S}^{(O_2)}$ and let us consider $S_1 \subseteq \mathfrak{S}^{(O_1)}$ such that $f_{(12)}^{-1}(\sigma_2) = \bigwedge^{\mathfrak{S}^{(O_1)}} S_1$, we have $\sigma_2 = f_{(12)}(\bigwedge^{\mathfrak{S}^{(O_1)}} S_1) = \bigwedge_{\sigma_1' \in S_1}^{\mathfrak{S}^{(O_1)}} f_{(12)}(\sigma_1')$ using (55). Now, using complete irreducibility of σ_2, we deduce that there exists $\sigma_1 \in S_1$ such that $\sigma_2 = f_{(12)}(\sigma_1)$, i.e. $f_{(12)}^{-1}(\sigma_2) \in S_1$. Hence, $f_{(12)}^{-1}(\sigma_2)$ is a pure state in $\mathfrak{S}^{(O_1)}$.
Secondly, let us consider that $\mathfrak{S}^{(O_1)}$ satisfies axiom **(A4)**. We note that, due to the property $f_{(12)}(\mathfrak{S}_{pure}^{(O_1)}) = \mathfrak{S}_{pure}^{(O_2)}$ and the monotonic character of the map $f_{(12)}$, we have $f_{(12)}(\underline{\sigma}) = \underline{f_{(12)}(\sigma)}$. Using this result, axiom **(A4)** and property (55), we obtain for any $\sigma_1 \in \mathfrak{S}^{(O_1)}$, $f_{(12)}(\sigma_1) = f_{(12)}(\bigwedge^{\mathfrak{S}^{(O_1)}} \underline{\sigma_1}) = \bigwedge^{\mathfrak{S}^{(O_1)}} f_{(12)}(\underline{\sigma_1}) = \bigwedge^{\mathfrak{S}^{(O_1)}} \underline{f_{(12)}(\sigma_1)}$. In other words, $\mathfrak{S}^{(O_2)} = f_{(12)}(\mathfrak{S}^{(O_1)})$ satisfies axiom **(A4)**. □

Theorem 4. As long as $\mathfrak{S}^{(O_1)}$ satisfies axiom **(A5)**, $\mathfrak{S}^{(O_2)}$ satisfies axiom **(A5)** as well and $f_{(12)}(Max(\mathfrak{S}^{(O_1)})) = Max(\mathfrak{S}^{(O_2)})$. ■

Proof. For any σ_2 completely meet-irreducible element in $\mathfrak{S}^{(O_2)}$, $f_{(12)}^{-1}(\sigma_2)$ is a completely meet-irreducible element in $\mathfrak{S}^{(O_1)}$ and then $f_{(12)}^{-1}(\sigma_2) \in Max(\mathfrak{S}^{(O_1)})$ because $\mathfrak{S}^{(O_1)}$ satisfies axiom **(A5)**. Let us imagine that there exists $\sigma_2' \sqsupseteq_{\mathfrak{S}^{(O_2)}} \sigma_2$, we have necessarily $f_{(12)}^{-1}(\sigma_2') \sqsupseteq_{\mathfrak{S}^{(O_2)}} f_{(12)}^{-1}(\sigma_2)$ because $f_{(12)}$ is bijective and order-preserving, and then $f_{(12)}^{-1}(\sigma_2) = f_{(12)}^{-1}(\sigma_2')$ because $f_{(12)}^{-1}(\sigma_2) \in Max(\mathfrak{S}^{(O_1)})$. As a result, $\sigma_2 \in Max(\mathfrak{S}^{(O_2)})$. We conclude that $\mathfrak{S}^{(O_2)}$ satisfies axiom **(A5)**.
Let us consider $\sigma_1 \in Max(\mathfrak{S}^{(O_1)})$ and let us consider that there exists $\sigma_2 \sqsupseteq_{\mathfrak{S}^{(O_2)}} f_{(12)}(\sigma_1)$. We have necessarily $f_{(12)}^{-1}(\sigma_2) \sqsupseteq_{\mathfrak{S}^{(O_2)}} \sigma_1$ because $f_{(12)}$ is bijective and order-preserving, and then $f_{(12)}^{-1}(\sigma_2) = \sigma_1$ because $\sigma_1 \in Max(\mathfrak{S}^{(O_1)})$. As a result, $f_{(12)}(\sigma_1) \in Max(\mathfrak{S}^{(O_2)})$. We conclude that $f_{(12)}(Max(\mathfrak{S}^{(O_1)})) \subseteq Max(\mathfrak{S}^{(O_2)})$. On another part, for any $\sigma_2 \in Max(\mathfrak{S}^{(O_2)})$, $f_{(12)}^{-1}(\sigma_2)$ is a completely meet-irreducible element of $\mathfrak{S}^{(O_1)}$, i.e. an element of $Max(\mathfrak{S}^{(O_1)})$, and then $\sigma_2 = f_{(12)}(f_{(12)}^{-1}(\sigma_2)) \in f_{(12)}(Max(\mathfrak{S}^{(O_1)}))$. As a final conclusion, $f_{(12)}(Max(\mathfrak{S}^{(O_1)})) = Max(\mathfrak{S}^{(O_2)})$. □

As a conclusion of all results of this subsection, the defined symmetries relate fully and faithfully the States/Effects Chu spaces.

Remark 3. We note that the identity map $(id_{\mathfrak{S}}, id_{\mathfrak{E}})$ is a symmetry from the States/Effects Chu space $(\mathfrak{S}, \mathfrak{E}, \varepsilon)$ to itself.

From now on, we will consider this category of States/Effects Chu spaces equipped with Chu morphisms and denote it $Chu_{\mathfrak{B}}^{S/E}$.
We will also have to consider the category $\mathscr{S}$ which objects are spaces of states (Axioms **A1-A5**) and which morphisms are bijective order-preserving maps satisfying (55)(56)(57).

3 Multipartite experiments

3.1 An axiomatic proposal

We now intent to describe an experiment on compound systems, implying two parties : Alice and Bob. The bipartite state space will be formed from two given spaces of states $\mathfrak{S}_A$ and $\mathfrak{S}_B$. It will be clear later on that this notion of bipartite space of states is ambiguous and different constructions can be proposed.

We now begin with a basic axiomatic proposal for the description of bipartite experiments (see [22, Section 5] for an analogue proposal in GPT's perspective). We will denote by $\mathfrak{S}_{AB} = \mathfrak{S}_A \boxtimes \mathfrak{S}_B$ the corresponding space of states.[6] We will also denote by $\mathfrak{E}_{AB} = \mathfrak{E}_A \boxtimes \mathfrak{E}_B$ the bipartite effect space formed from two given effect spaces $\mathfrak{E}_A$ and $\mathfrak{E}_B$. We will denote ε^{AB} the corresponding bipartite evaluation map from $\mathfrak{E}_{AB}$ to $\mathfrak{B}^{\mathfrak{S}_{AB}}$. We will assume the following requirements about these elements.

First of all, we have to build $(\mathfrak{S}_{AB}, \mathfrak{E}_{AB}, \varepsilon^{AB})$ as a valid Spaces/Effects Chu space.

In particular, we will assume that $\mathfrak{S}_{AB}$ admits mixed bipartite states. In other words,

$$\textbf{(B1)} \qquad \begin{array}{l} \forall \{ \sigma_{i,AB} \mid i \in I \} \subseteq \mathfrak{S}_{AB}, \quad \prod_{i\in I}^{\mathfrak{S}_{AB}} \sigma_{i,AB} \text{ exists in } \mathfrak{S}_{AB}, \\ \forall \{ \sigma_{i,AB} \mid i \in I \} \subseteq \mathfrak{S}_{AB}, \forall \mathfrak{l}_{AB} \in \mathfrak{E}_{AB}, \ \varepsilon^{AB}_{\mathfrak{l}_{AB}}(\prod_{i\in I}^{\mathfrak{S}_{AB}} \sigma_{i,AB}) = \bigwedge_{i\in I} \varepsilon^{AB}_{\mathfrak{l}_{AB}}(\sigma_{i,AB}). \end{array} \tag{63}$$

In the same logic, we will assume that $\mathfrak{E}_{AB}$ admits mixed bipartite effects. In other words,

$$\textbf{(B2)} \qquad \begin{array}{l} \forall \{ \mathfrak{l}_{i,AB} \mid i \in I \} \subseteq \mathfrak{E}_{AB}, \quad \prod_{i\in I}^{\mathfrak{E}_{AB}} \mathfrak{l}_{i,AB} \text{ exists in } \mathfrak{E}_{AB}, \\ \forall \{ \mathfrak{l}_{i,AB} \mid i \in I \} \subseteq \mathfrak{E}_{AB}, \forall \sigma_{AB} \in \mathfrak{S}_{AB}, \ \varepsilon^{AB}_{\prod_{i\in I}^{\mathfrak{E}_{AB}} \mathfrak{l}_{i,AB}}(\sigma_{AB}) = \bigwedge_{i\in I} \varepsilon^{AB}_{\mathfrak{l}_{i,AB}}(\sigma_{AB}). \end{array} \tag{64}$$

Secondly, for every effects $\mathfrak{l}_A$ and $\mathfrak{l}_B$ realized independently by Alice and Bob respectively, we will assume that there must exist a unique associated bipartite effect in $\mathfrak{E}_{AB}$. As a consequence, we will assume that there are maps $\iota^{\mathfrak{E}}_{AB} : \mathfrak{E}_A \times \mathfrak{E}_B \longrightarrow \mathfrak{E}_{AB}$ which describe the inclusion of 'pure tensors' in $\mathfrak{E}_{AB}$ (for readability, we shall write $\mathfrak{l}_A \boxtimes \mathfrak{l}_B$ rather than $\iota^{\mathfrak{E}}_{AB}(\mathfrak{l}_A, \mathfrak{l}_B)$). This axiom will be denoted **(B3)**.
In the same logic, for every states $\sigma_A \in \mathfrak{S}_A$ and $\sigma_B \in \mathfrak{S}_B$, prepared independently by Alice and Bob, we will assume that there must exist a unique associated bipartite state in $\mathfrak{S}_{AB}$. As a consequence, we will assume that there are maps $\iota^{\mathfrak{S}}_{AB} : \mathfrak{S}_A \times \mathfrak{S}_B \longrightarrow \mathfrak{S}_{AB}$ which describe the inclusion of 'pure tensors' in $\mathfrak{S}_{AB}$ (for readability, we shall write $\sigma_A \boxtimes \sigma_B$ rather than $\iota^{\mathfrak{S}}_{AB}(\sigma_A, \sigma_B)$). This axiom will be denoted **(B4)**.

Thirdly, for every $\sigma_{AB}, \sigma'_{AB} \in \mathfrak{S}_{AB}$ such that $\sigma_{AB} \neq \sigma'_{AB}$, we will assume that there must exist effects $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$ such that when Alice and Bob prepare σ_{AB} and apply $\mathfrak{l}_A$ and $\mathfrak{l}_B$ respectively, the resulting determination is different from the experiment where Alice and Bob prepare σ'_{AB} and apply $\mathfrak{l}_A$ and $\mathfrak{l}_B$ respectively. As a summary, applying effects locally is sufficient to distinguish all of the states in $\mathfrak{S}_{AB}$ (this principle is called "tomographic locality"), i.e.

$$\textbf{(B5)} \qquad \forall \sigma_{AB}, \sigma'_{AB} \in \mathfrak{S}_{AB}, \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \mathfrak{l}_B \in \mathfrak{E}_B, \ \varepsilon^{AB}_{\mathfrak{l}_A \boxtimes \mathfrak{l}_B}(\sigma_{AB}) = \varepsilon^{AB}_{\mathfrak{l}_A \boxtimes \mathfrak{l}_B}(\sigma'_{AB})) \ \Leftrightarrow \ (\sigma_{AB} = \sigma'_{AB}). \tag{65}$$

Endly, let us consider that Alice and Bob realize their experiments on a pure tensor state. In the simplest scenario, Alice applies $\mathfrak{l}_A \in \mathfrak{E}_A$ and Bob applies $\mathfrak{l}_B \in \mathfrak{E}_B$ independently. Since these two experiments are independent, the resulting determination has to be the 'product' of the respective determinations, i.e.

$$\textbf{(B6)} \qquad \forall \sigma_A \in \mathfrak{S}_A, \forall \sigma_B \in \mathfrak{S}_B, \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \quad \varepsilon^{AB}_{\mathfrak{l}_A \boxtimes \mathfrak{l}_B}(\sigma_A \boxtimes \sigma_B) = \varepsilon^{A}_{\mathfrak{l}_A}(\sigma_A) \bullet \varepsilon^{B}_{\mathfrak{l}_B}(\sigma_B). \tag{66}$$

[6] Throughout this short axiomatic introduction, we adopt the inadequate notation $\boxtimes$ for the tensor product in order to allow for different candidates for this tensor product. These different candidates will be denoted $\otimes$, $\widetilde{\otimes}$,...

It is essential to note that our identification of the bipartite states space $\mathfrak{S}_{AB}$ according to previous axioms is such that, if Alice (or Bob) prepares a mixture of states, then this results in a mixture of the respective bipartite states. More explicitly,

$$(\bigsqcap_{i\in I}^{\mathfrak{S}_A}\sigma_{i,A})\boxtimes\sigma_B = \bigsqcap_{i\in I}^{\mathfrak{S}_{AB}}(\sigma_{i,A}\boxtimes\sigma_B), \quad (67)$$

$$\sigma_A\boxtimes(\bigsqcap_{i\in I}^{\mathfrak{S}_B}\sigma_{i,B}) = \bigsqcap_{i\in I}^{\mathfrak{S}_{AB}}(\sigma_A\boxtimes\sigma_{i,B}). \quad (68)$$

Indeed, using properties (66) (15) (5) (63), we deduce that, for any $\mathfrak{l}_A\in\mathfrak{E}_A,\mathfrak{l}_B\in\mathfrak{E}_B$, $\{\sigma_{i,A}\mid i\in I\}\subseteq\mathfrak{S}_A$ and $\sigma_B\in\mathfrak{S}_B$

$$\begin{aligned}
\varepsilon^{AB}_{\mathfrak{l}_A\boxtimes\mathfrak{l}_B}((\bigsqcap_{i\in I}^{\mathfrak{S}_A}\sigma_{i,A})\boxtimes\sigma_B) &= \varepsilon^{A}_{\mathfrak{l}_A}(\bigsqcap_{i\in I}^{\mathfrak{S}_A}\sigma_{i,A})\bullet\varepsilon^{B}_{\mathfrak{l}_B}(\sigma_B)\\
&= (\bigwedge_{i\in I}\varepsilon^{A}_{\mathfrak{l}_A}(\sigma_{i,A}))\bullet\varepsilon^{B}_{\mathfrak{l}_B}(\sigma_B)\\
&= \bigwedge_{i\in I}(\varepsilon^{A}_{\mathfrak{l}_A}(\sigma_{i,A})\bullet\varepsilon^{B}_{\mathfrak{l}_B}(\sigma_B))\\
&= \bigwedge_{i\in I}\varepsilon^{AB}_{\mathfrak{l}_A\boxtimes\mathfrak{l}_B}(\sigma_{i,A}\boxtimes\sigma_B)\\
&= \varepsilon^{AB}_{\mathfrak{l}_A\boxtimes\mathfrak{l}_B}(\bigsqcap_{i\in I}^{\mathfrak{S}_{AB}}(\sigma_{i,A}\boxtimes\sigma_B)), \quad (69)
\end{aligned}$$

and then, using property (65), we obtain the property (67). We obtain the property (68) along the same lines of proof.
In the following, we intent to identify potential candidates for this bipartite space of states $\mathfrak{S}_{AB}$ and space of effects $\mathfrak{E}_{AB}$ and posit it with respect to the standard construction of tensor products of Inf semi-lattices. But before that, we complete the previous axiomatic by a discussion of the symmetries of the multipartite experiments.

3.2 Symmetries of the bipartite experiments

Definition 5. Let us consider a symmetry $(f_{(12)},f^{(21)})$ from a States/Effects Chu space $(\mathfrak{S}_{A_1},\mathfrak{E}_{A_1},\varepsilon^{A_1})$ associated a first observer, to another States/Effects Chu space $(\mathfrak{S}_{A_2},\mathfrak{E}_{A_2},\varepsilon^{A_2})$ associated to another observer. Let us also consider a symmetry $(g_{(12)},g^{(21)})$ from the Chu space $(\mathfrak{S}_{B_1},\mathfrak{E}_{B_1},\varepsilon^{B_1})$ to the Chu space $(\mathfrak{S}_{B_2},\mathfrak{E}_{B_2},\varepsilon^{B_2})$. We define the pair of maps $((f\boxtimes g)_{(12)},(f\boxtimes g)^{(21)})$ from the Chu space $(\mathfrak{S}_{A_1B_1},\mathfrak{E}_{A_1B_1},\varepsilon^{A_1B_1})$ to the Chu space $(\mathfrak{S}_{A_2B_2},\mathfrak{E}_{A_2B_2},\varepsilon^{A_2B_2})$ by

$$(f\boxtimes g)_{(12)}(\bigsqcap_{i\in I}^{\mathfrak{S}_{A_1B_1}}\sigma_{i,A_1}\boxtimes\sigma_{i,B_1}) := \bigsqcap_{i\in I}^{\mathfrak{S}_{A_2B_2}}f_{(12)}(\sigma_{i,A_1})\boxtimes g_{(12)}(\sigma_{i,B_1}) \quad (70)$$

$$(f\boxtimes g)^{(21)}(\bigsqcap_{j\in J}^{\mathfrak{E}_{A_2B_2}}\mathfrak{l}_{j,A_2}\boxtimes\mathfrak{l}_{j,B_2}) := \bigsqcap_{j\in J}^{\mathfrak{E}_{A_1B_1}}f^{(21)}(\mathfrak{l}_{j,A_1})\boxtimes g^{(21)}(\mathfrak{l}_{j,B_1}) \quad (71)$$

Theorem 5. The pair of maps $((f\boxtimes g)_{(12)},(f\boxtimes g)^{(21)})$ is a well defined symmetry, i.e. a Chu morphism from the Chu space $(\mathfrak{S}_{A_1B_1},\mathfrak{E}_{A_1B_1},\varepsilon^{A_1B_1})$ to the Chu space $(\mathfrak{S}_{A_2B_2},\mathfrak{E}_{A_2B_2},\varepsilon^{A_2B_2})$. ■

Proof.

$$\begin{aligned}
\varepsilon^{A_2B_2}_{\bigsqcap_{j\in J}^{\mathfrak{E}_{A_2B_2}}\mathfrak{l}_{j,A_2}\boxtimes\mathfrak{l}_{j,B_2}}\Big((f\boxtimes g)_{(12)}(\bigsqcap_{i\in I}^{\mathfrak{S}_{A_1B_1}}\sigma_{i,A_1}\boxtimes\sigma_{i,B_1})\Big) &= \varepsilon^{A_2B_2}_{\bigsqcap_{j\in J}^{\mathfrak{E}_{A_2B_2}}\mathfrak{l}_{j,A_2}\boxtimes\mathfrak{l}_{j,B_2}}\Big(\bigsqcap_{i\in I}^{\mathfrak{S}_{A_2B_2}}f_{(12)}(\sigma_{i,A_1})\boxtimes g_{(12)}(\sigma_{i,B_1})\Big)\\
&= \bigwedge_{j\in J}\bigwedge_{i\in I}\varepsilon^{A_2}_{\mathfrak{l}_{j,A_2}}(f_{(12)}(\sigma_{i,A_1}))\bullet\varepsilon^{B_2}_{\mathfrak{l}_{j,B_2}}(g_{(12)}(\sigma_{i,B_1}))\\
&= \bigwedge_{j\in J}\bigwedge_{i\in I}\varepsilon^{A_1}_{f^{(21)}(\mathfrak{l}_{j,A_2})}(\sigma_{i,A_1})\bullet\varepsilon^{B_1}_{g^{(21)}(\mathfrak{l}_{j,B_2})}(\sigma_{i,B_1})\\
&= \varepsilon^{A_1B_1}_{\bigsqcap_{j\in J}^{\mathfrak{E}_{A_1B_1}}f^{(21)}(\mathfrak{l}_{j,A_2})\boxtimes g^{(21)}(\mathfrak{l}_{j,B_2})}\Big(\bigsqcap_{i\in I}^{\mathfrak{S}_{A_2B_2}}\sigma_{i,A_1}\boxtimes\sigma_{i,B_1}\Big)\\
&= \varepsilon^{A_1B_1}_{(f\boxtimes g)^{(21)}\left(\bigsqcap_{j\in J}^{\mathfrak{E}_{A_2B_2}}\mathfrak{l}_{j,A_2}\boxtimes\mathfrak{l}_{j,B_2}\right)}\Big(\bigsqcap_{i\in I}^{\mathfrak{S}_{A_2B_2}}\sigma_{i,A_1}\boxtimes\sigma_{i,B_1})\Big). \quad (72)
\end{aligned}$$

□

From this result, we deduce that the categories $Chu_{\mathfrak{B}}^{S/E}$ and $\mathscr{S}$ are equipped with a tensor product.

3.3 Towards a monoïdal sub-category of spaces of states

In order to equip $\mathscr{S}$ with a full monoïdal structure, we have to check that there exists an object I (called unit object) in this category and two natural isomorphisms ensuring $I \boxtimes X \cong X$ and $X \boxtimes I \cong X$. In this subsection, we are going to check in fact that there exist generic objects X such that $\mathfrak{B} \boxtimes X = X$ and $X \boxtimes \mathfrak{B} = X$. This will lead to identify a sub-category of $\mathscr{S}$, which will be a monoïdal category of space of states.

Let us consider S a given object in the category of spaces of states $\mathscr{S}$. We define the *co-slice category* denoted $S \downarrow \mathscr{S}$ as follows. The objects in $S \downarrow \mathscr{S}$ are couples (A,a) where A is an object in $\mathscr{S}$ and $a : S \to A$ is a morphism in $\mathscr{S}$. The morphisms from (A,a) to (A',a') in $S \downarrow \mathscr{S}$ are given by morphisms $f : A \to A'$ in $\mathscr{S}$ such that

$$\begin{array}{ccc} S & \xrightarrow{a} & A \\ & \searrow^{a'} & \downarrow f \\ & & A' \end{array} \quad \text{commutes.} \tag{73}$$

We note that the *initial object* of this co-slice category denoted $\perp_{S\downarrow\mathscr{S}}$ is given by

$$\perp_{S\downarrow\mathscr{S}} \quad = \quad (S, id_S). \tag{74}$$

The unique morphism, from the initial object $\perp_{S\downarrow\mathscr{S}}$ to the object (A,a), denoted $\perp_{(A,a)}$, is given by

$$\perp_{(A,a)} \quad = \quad a. \tag{75}$$

For any endo-functor F of the category $\mathscr{S}$ and any morphim $\pi : S \to F(S)$ in $\mathscr{S}$, we can define an endo-functor of the category $S \downarrow \mathscr{S}$, denoted F_π, as follows. For any object (A,a) and any morphism f :

$$F_\pi(A,a) := (F(A), F(a) \circ \pi) \quad \text{and} \quad F_\pi(f) := F(f). \tag{76}$$

Using (76) and (75), we deduce

$$\perp_{F_\pi(\perp_{S\downarrow\mathscr{S}})} \quad = \quad \pi. \tag{77}$$

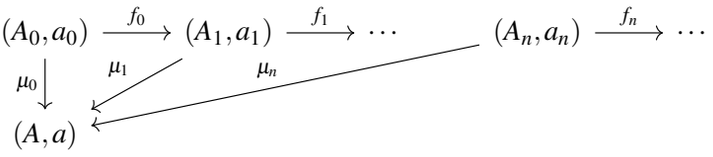
A $F_\pi-$*algebra* is a couple $((A,a),\alpha)$ where (A,a) is an object in $S \downarrow \mathscr{S}$ and $\alpha : F(A) \to A$ is a morphism in $S \downarrow \mathscr{S}$ such that

$$\begin{array}{ccccc} S & \xrightarrow{\pi} & F(S) & \xrightarrow{F(a)} & F(A) \\ & \searrow^{a} & & & \downarrow \alpha \\ & & & & A \end{array} \quad \text{commutes.} \tag{78}$$

$F_\pi-$algebras are objects of a category where the morphisms $f : ((A,a),\alpha) \to (A',a'),\alpha')$ are given by morphisms $f : A \to A'$ in $\mathscr{S}$ such that

$$\begin{array}{ccc} S & \xrightarrow{a} & A \\ & \searrow^{a'} & \downarrow f \\ & & A' \end{array} \quad \text{and} \quad \begin{array}{ccc} F(A) & \xrightarrow{\alpha} & A \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle f} \\ F(A') & \xrightarrow[\alpha']{} & A' \end{array} \quad \text{commute.} \tag{79}$$

An $\omega-$*chain* being given by $\Delta : (A_0,a_0) \xrightarrow{f_0} (A_1,a_1) \xrightarrow{f_1} \cdots (A_n,a_n) \xrightarrow{f_n} \cdots$, a *co-cone* $\mu : \Delta \to (A,a)$ is defined as long as



commutes. (80)

This co-cone is said to be *co-limiting* iff, for any other co-cone $\nu : \Delta \to (A',a')$, there exists a unique mediating morphism f such that $\forall n, \nu_n = f \circ \mu_n$.

Let us consider the following particular $\omega-$chain associated to F_π :

$$\forall n \geq 0, \qquad \begin{array}{rcl} (A_n,a_n) & := & F_\pi^n(\bot_{S\downarrow\mathscr{S}}) \\ f_n & := & F_\pi^n(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) \end{array} \tag{81}$$

Using (76) and (77) we note that

$$F_\pi^n(\bot_{S\downarrow\mathscr{S}}) = (F^n(S), F^{n-1}(\pi)\circ\cdots\circ\pi) \quad \text{and} \quad F_\pi^n(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) = F^n(\pi). \tag{82}$$

Lemma 3. Let $((A',a'),\alpha')$ be any $F_\pi-$algebra, we can define a co-cone $\nu : \Delta \to (A',d')$ by

$$\nu_0 := \bot_{(A',a')} = a', \qquad \nu_1 := \alpha' \circ F_\pi(\nu_0) = \alpha' \circ F(a'), \quad \cdots \quad \nu_{n+1} := \alpha' \circ F_\pi(\nu_n). \tag{83}$$

Indeed, we can check by recursion that

$$\nu_{n+1} \circ F_\pi^n(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) \quad = \quad \nu_n. \tag{84}$$

■

Proof. The equality $\nu_1 \circ F_\pi^0(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) = \alpha' \circ F(a') \circ \pi = a' = \nu_0$ is a consequence of (78). We have also $\nu_{n+1} \circ F_\pi^n(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) = \nu_n$ implies $\nu_{n+2} \circ F_\pi^{n+1}(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})}) = \alpha' \circ F_\pi(\nu_{n+1} \circ F_\pi^n(\bot_{F_\pi(\bot_{S\downarrow\mathscr{S}})})) = \alpha' \circ F_\pi(\nu_n) = \nu_{n+1}$. □

Theorem 6. Let us suppose that the co-cones $\mu : \Delta \to (A,a)$ and $F_\pi(\mu) = F_\pi(\Delta) \to F_\pi(A,a)$ are co-limiting. Then the initial $F_\pi-$algebra exists and is equal to $((A,a),\alpha)$ where α is the mediating morphism from $F_\pi(\mu)$ to μ^-, i.e. $\alpha \circ F_\pi(\mu_n) = \mu_{n+1}$. As a conclusion, $((A,a),\alpha)$ is an initial fixed point of the equation $F_\pi(X) \cong X$. ■

Proof. Let $((A',a'),\alpha')$ be any $F_\pi-$algebra and let us consider a co-cone $\nu : \Delta \to (A',a')$ defined according to (83). Let f denote the unique mediating morphism from μ to ν, i.e. $\forall n, \nu_n = f \circ \mu_n$ and α denote the unique mediating morphism from $F_\pi(\mu)$ to μ^-, i.e. $\alpha \circ F_\pi(\mu_n) = \mu_{n+1}$.
We have $(f \circ \alpha) \circ F_\pi(\mu_n) = f \circ (\alpha \circ F_\pi(\mu_n)) = f \circ \mu_{n+1} = \nu_{n+1}$ and $(\alpha' \circ F_\pi(f)) \circ F_\pi(\mu_n) = \alpha' \circ F_\pi(f \circ \mu_n) = \alpha' \circ F_\pi(\nu_n) = \nu_{n+1}$. The unicity of the mediating morphism from $F_\pi(\mu)$ to ν^- implies then $f \circ \alpha = \alpha' \circ F(f)$, i.e. there exists a homomorphism $f : ((A,a),\alpha) \to ((A',a'),\alpha')$ of $F_\pi-$algebras.
Let us now suppose that $f : ((A,a),\alpha) \to ((A',a'),\alpha')$ is a homomorphism of $F_\pi-$algebra. We check by recursion that f is unambiguously fixed by $\nu_n = f \circ \mu_n$. Indeed, we have first of all $\nu_0 = \bot_{(A',a')} = d' = f \circ a = f \circ \bot_{(A,a)} = f \circ \mu_0$ and $f \circ \mu_{n+1} = f \circ \alpha \circ F_\pi(\mu_n) = \alpha' \circ F_\pi(f) \circ F_\pi(\mu_n) = \alpha' \circ F_\pi(f \circ \mu_n) = \alpha' \circ F_\pi(\nu_n) = \nu_{n+1}$. Hence, the homomorphism of $F_\pi-$algebras f is uniquely defined.
As a conclusion of the two previous results, we obtain that $((A,a),\alpha)$ is the initial $F_\pi-$algebra.
Let us now suppose that $((A,a),\alpha)$ is an initial $F_\pi-$algebra. $(F_\pi(A,a),F_\pi(\alpha))$ is also a $F_\pi-$algebra

and there exists a unique homomorphism of $F_\pi-$algebra $f:((A,a),\alpha)\to(F_\pi(A,a),F_\pi(\alpha))$. Note that $\alpha:(F_\pi(A,a),F_\pi(\alpha))\to((A,a),\alpha)$ is also a homorphism of $F_\pi-$algebra. Then, $\alpha\circ f:((A,a),\alpha)\to((A,a),\alpha)$ is also a homomorphism of $F_\pi-$algebra, equal to $id_{(A,a)}$ as $((A,a),\alpha)$ is initial. At the same time, we have $f\circ\alpha=F_\pi(\alpha)\circ F_\pi(f)=F_\pi(\alpha\circ f)=F_\pi(id_{(A,a)})=id_{F_\pi(A,a)}$. Then, α is an isomorphism. As a final conclusion, $((A,a),\alpha)$ is an initial fixed point. □

Let us now particularize these results in order to find simultaneous solutions of the equations $\mathfrak{B}\boxtimes X\cong X$ and $X\boxtimes\mathfrak{B}\cong X$ in the category $\mathscr{S}$.
Let us now fix an object $\mathfrak{S}$ in $\mathscr{S}$ and let us consider the co-slice category $\mathfrak{S}\downarrow\mathscr{S}$. Let us then consider the endo-functor F of the category $\mathscr{S}$ defined by $F(X):=\mathfrak{B}\boxtimes X$ for any object X of $\mathscr{S}$ and $F(f):=id_{\mathfrak{B}}\boxtimes f$ for any morphism f in the category $\mathscr{S}$.

We define the morphism $\pi_X:X\to F(X)$ as follows

$$\begin{array}{rccc}\pi_X & : & X & \longrightarrow & \mathfrak{B}\boxtimes X\\ & & u & \mapsto & \bot\boxtimes u\end{array}\tag{85}$$

We note that the embedding π_X forms an embedding-projection pair with the projection π_X^* defined by

$$\begin{array}{rccc}\pi_X^* & : & \mathfrak{B}\boxtimes X & \longrightarrow & X\\ & & \bigsqcap_{\sigma\in\mathfrak{B}}^{\mathfrak{B}\boxtimes X}\sigma\boxtimes u_\sigma & \mapsto & \bigsqcap_{\sigma\in\mathfrak{B}}^{X}u_\sigma\end{array}\tag{86}$$

We check easily the following embedding-projection relations

$$\pi_X^*\circ\pi_X=id_X,\qquad \pi_X\circ\pi_X^*\sqsubseteq id_{\mathfrak{B}\boxtimes X}.\tag{87}$$

We check moreover the following homomorphic properties

$$\forall\{u_i\,|\,i\in I\}\subseteq X,\ \ \pi_X(\bigsqcap_{i\in I}^{X}u_i)=\bigsqcap_{i\in I}^{\mathfrak{B}\boxtimes X}\pi_X(u_i),\tag{88}$$

$$\forall\{u_i\,|\,i\in I\}\subseteq_{Chain}X,\ \ \pi_X(\bigsqcup_{i\in I}^{X}u_i)=\bigsqcup_{i\in I}^{\mathfrak{B}\boxtimes X}\pi_X(u_i),\tag{89}$$

$$\pi_X(\bot_X)=\bot_{\mathfrak{B}\boxtimes X}.\tag{90}$$

Let us then compute the components of the $\omega-$chain $\Delta:(A_0,a_0)\underset{f_0^*}{\overset{f_0}{\rightleftarrows}}(A_1,a_1)\underset{f_1^*}{\overset{f_1}{\rightleftarrows}}\cdots(A_n,a_n)\underset{f_n^*}{\overset{f_n}{\rightleftarrows}}\cdots$ defined in reference to (82) and (81) (we denote by f_k^* the projection associated to the embedding f_k) :

$$A_0:=\mathfrak{S},\ \text{ and }\ \forall n\geq 1,\ A_n:=F^n(\mathfrak{S})=\mathfrak{B}^{\boxtimes n}\ \boxtimes\ \mathfrak{S}\tag{91}$$

$$a_0:=id_{\mathfrak{S}}\ \text{ and }\ \forall n\geq 1,\ a_n:=F^{n-1}(\pi_{\mathfrak{S}})\circ\cdots\circ\pi_{\mathfrak{S}}=\bot^{\boxtimes n}\ \boxtimes\ id_{\mathfrak{S}}\tag{92}$$

$$\begin{array}{rccc}f_0:=\pi_{\mathfrak{S}}\ \text{ and }\ \forall n\geq 1,\ f_n=F^n(\pi_{\mathfrak{S}}) & : & A_n & \longrightarrow & A_{n+1}\\ & & \bigsqcap_{i\in I}^{A_n}u_i^{(1)}\boxtimes\cdots\boxtimes u_i^{(n)}\boxtimes v_i^{(\mathfrak{S})} & \mapsto & \bigsqcap_{i\in I}^{A_{n+1}}u_i^{(1)}\boxtimes\cdots\boxtimes u_i^{(n)}\boxtimes\bot\boxtimes v_i^{(\mathfrak{S})}\end{array}\tag{93}$$

$$\begin{array}{rccc}f_0^*:=\pi_{\mathfrak{S}}^*\ \text{ and }\ \forall n\geq 1,\ f_n^* & : & A_{n+1} & \longrightarrow & A_n\\ & & \bigsqcap_{i\in I}^{A_{n+1}}u_i^{(1)}\boxtimes\cdots\boxtimes u_i^{(n)}\boxtimes u_i^{(n+1)}\boxtimes v_i^{(\mathfrak{S})} & \mapsto & \bigsqcap_{i\in I}^{A_n}u_i^{(1)}\boxtimes\cdots\boxtimes u_i^{(n)}\boxtimes v_i^{(\mathfrak{S})}\end{array}\tag{94}$$

We also define the *co-cone* $\mu:\Delta\to(A,a)$ as follows (we denote by μ_k^* the projection associated to the embedding μ_k)

$$A:=\overbrace{\mathfrak{B}\boxtimes\cdots}^{\omega}\boxtimes\ \mathfrak{S},\qquad\qquad a:=\overbrace{\bot\boxtimes\cdots}^{\omega}\boxtimes\ id_{\mathfrak{S}},\tag{95}$$

$$\begin{array}{rccc}\mu_n & : & A_n & \longrightarrow & A\\ & & \bigsqcap_{i\in I}^{A_n}(\widetilde{\bigotimes}_{1\leq j\leq n}u_i^{(j)})\boxtimes v_i^{(\mathfrak{S})} & \mapsto & \bigsqcap_{i\in I}^{A}(\widetilde{\bigotimes}_{1\leq j\leq n}u_i^{(j)})\boxtimes\bot^{\boxtimes\omega}\boxtimes v_i^{(\mathfrak{S})}\end{array}\tag{96}$$

$$\begin{array}{rccc}\mu_n^* & : & A & \longrightarrow & A_n\\ & & \bigsqcap_{i\in I}^{A}(\widetilde{\bigotimes}_{1\leq j}u_i^{(j)})\boxtimes v_i^{(\mathfrak{S})} & \mapsto & \bigsqcap_{i\in I}^{A_n}(\widetilde{\bigotimes}_{1\leq j\leq n}u_i^{(j)})\boxtimes v_i^{(\mathfrak{S})}\end{array}\tag{97}$$

We note immediately the following embedding-projection relations

$$\forall n\geq 0,\qquad \mu_n^*\circ\mu_n=id_{A_n},\qquad \mu_n\circ\mu_n^*\sqsubseteq id_A.\tag{98}$$

We also check the following result.

Lemma 4.

$$(\mu_n \circ \mu_n^*)_{n\geq 0} \text{ is an increasing sequence of elements of } Hom(A,A) \tag{99}$$
$$\bigsqcup_n \mu_n \circ \mu_n^* = id_A. \tag{100}$$

As a conclusion, μ is an $O-$colimit of Δ (see [25, Definition 7]). ■

Proof. Straightforward. □

Lemma 5. μ is a colimit of Δ. ■

Proof. Direct consequence of Lemma 4.
Let $\mu' : \Delta \to A'$ be another co-cone and let us consider $f : A \to A'$ a mediating morphism from μ to μ' (i.e. $\mu'_n = f \circ \mu_n$). We check that f is defined uniquely by $f = f \circ \bigsqcup_n \mu_n \circ \mu_n^* = \bigsqcup_n (f \circ \mu_n) \circ \mu_n^* = \bigsqcup_n \mu'_n \circ \mu_n^*$ (we have used property (100)).
Concerning the existence of such a mediating morphism, we have to use property (99) to conclude that $f = \bigsqcup_n \mu'_n \circ \mu_n^*$ is a solution. Indeed, we have $f \circ \mu_m = (\bigsqcup_{n\geq m} \mu'_n \circ \mu_n^*) \circ \mu_m = \bigsqcup_{n\geq m} \mu'_n \circ \mu_n^* \circ \mu_n \circ f_{n-1} \circ \cdots \circ f_m = \bigsqcup_{n\geq m} \mu'_n \circ f_{n-1} \circ \cdots \circ f_m = \mu'_m$. □

Lemma 6. $F_{\pi_{\mathfrak{S}}}(\mu) = \mu^-$, i.e. $\forall n \geq 0,\ F_{\pi_{\mathfrak{S}}}(\mu_n) = \mu_{n+1}$, ■

Proof. Straightforward. □

Theorem 7. $((A,a),id)$ is an initial fixed point of the equation $F_{\pi_{\mathfrak{S}}}(X) \cong X$. In other words, we have $\mathfrak{B} \boxtimes A = A$. ■

Proof. Consequence of Lemma 5 and Lemma 6 using Theorem 6. □

In order to treat simultaneously the two recursion relations, we have to replace $\mathfrak{S}$ by $A = \mathfrak{B}^{\boxtimes\omega} \boxtimes \mathfrak{S}$ from the beginning of the construction, to consider the co-slice category $A \downarrow \mathscr{S}$ and to choose the endo-functor G defined by $G(X) := X \boxtimes \mathfrak{B}$ in place of F. The proof follows the same lines as before and we obtain :

Theorem 8. For any space of states $\mathfrak{S}$, the space of states defined as $\overbrace{\mathfrak{B} \boxtimes \cdots}^{\omega} \boxtimes \mathfrak{S} \boxtimes \overbrace{\cdots \boxtimes \mathfrak{B}}^{\omega}$ is a simultaneous solution of the recursion equations $\mathfrak{B} \boxtimes X = X$ and $X \boxtimes \mathfrak{B} = X$. ■

We then intent to consider the sub-category of $\mathscr{S}$, denoted $\ulcorner\mathscr{S}\urcorner$, and formed by objects of the form $\overbrace{\mathfrak{B} \boxtimes \cdots}^{\omega} \boxtimes \mathfrak{S} \boxtimes \overbrace{\cdots \boxtimes \mathfrak{B}}^{\omega}$ where $\mathfrak{S}$ is an object of $\mathscr{S}$. The morphisms from $\overbrace{\mathfrak{B} \boxtimes \cdots}^{\omega} \boxtimes \mathfrak{S} \boxtimes \overbrace{\cdots \boxtimes \mathfrak{B}}^{\omega}$ to $\overbrace{\mathfrak{B} \boxtimes \cdots}^{\omega} \boxtimes \mathfrak{S}' \boxtimes \overbrace{\cdots \boxtimes \mathfrak{B}}^{\omega}$ are given by maps of the form $\overbrace{f_1 \boxtimes f_2 \boxtimes \cdots}^{\omega} \boxtimes f_0 \boxtimes \overbrace{\cdots \boxtimes f_{-2} \boxtimes f_{-1}}^{\omega}$ where f_i and f_{-i} are bijective order-preserving maps from $\mathfrak{B}$ to $\mathfrak{B}$ satisfying (55)(56)(57), for any $i \geq 1$, and where f_0 is a bijective order-preserving map from $\mathfrak{S}$ to $\mathfrak{S}'$ satisfying (55)(56)(57).

Remark 4. We note that a bijective order-preserving map from $\mathfrak{B}$ to $\mathfrak{B}$ satisfying (55)(56)(57) is either the identity map $id_{\mathfrak{B}}$ or the involutive map $\overline{\cdot}$ on $\mathfrak{B}$.

4 Candidates for a tensor product

4.1 The basic tensor product construction

We begin to introduce the classical construction of G.A. Fraser for the tensor product of semi-lattices [13, 14]. As it will be clarified in the next subsection new proposals for the tensor product of semi-lattices have to be made in order to complete our work. In this subsection, it will be assumed that $(\mathfrak{S}_A, \varepsilon^A)$ satisfy Axiom **(A1)**.

Definition 6. Let A, B and C be semilattices. A function $f : A \times B \longrightarrow C$ is a bi-homomorphism if the functions $g_a : B \longrightarrow C$ defined by $g_a(b) = f(a,b)$ and $h_b : A \longrightarrow C$ defined by $h_b(a) = f(a,b)$ are homomorphisms for all $a \in A$ and $b \in B$.

Theorem 9. [13, Definition 2.2 and Theorem 2.3]
The tensor product $S_{AB} := \mathfrak{S}_A \otimes \mathfrak{S}_B$ of the two Inf semi-lattices $\mathfrak{S}_A$ and $\mathfrak{S}_B$ is obtained as a solution of the following universal problem : there exists a bi-homomorphism, denoted $\imath$ from $\mathfrak{S}_A \times \mathfrak{S}_B$ to S_{AB}, such that, for any Inf semi-lattice $\mathfrak{S}$ and any bi-homomorphism f from $\mathfrak{S}_A \times \mathfrak{S}_B$ to $\mathfrak{S}$, there is a unique homomorphism g from S_{AB} to $\mathfrak{S}$ with $f = g \circ \imath$. We denote $\imath(\sigma, \sigma') = \sigma \otimes \sigma'$ for any $\sigma \in \mathfrak{S}_A$ and $\sigma' \in \mathfrak{S}_B$.
The tensor product S_{AB} exists and is unique up to isomorphism, it is built as the homomorphic image of the free $\sqcap$ semi-lattice generated by the set $\mathfrak{S}_A \times \mathfrak{S}_B$ under the congruence relation determined by identifying $(\sigma_1 \sqcap_{\mathfrak{S}_A} \sigma_2, \sigma')$ with $(\sigma_1, \sigma') \sqcap (\sigma_2, \sigma')$ for all $\sigma_1, \sigma_2 \in \mathfrak{S}_A, \sigma' \in \mathfrak{S}_B$ and identifying $(\sigma, \sigma'_1 \sqcap_{\mathfrak{S}_B} \sigma'_2)$ with $(\sigma, \sigma'_1) \sqcap (\sigma, \sigma'_2)$ for all $\sigma \in \mathfrak{S}_A, \sigma'_1, \sigma'_2 \in \mathfrak{S}_B$.
In other words, S_{AB} is the Inf semi-lattice (the infimum of $S \subseteq S_{AB}$ will be denoted $\bigsqcap^{S_{AB}} S$) generated by the elements $\sigma_A \otimes \sigma_B$ with $\sigma_A \in \mathfrak{S}_A, \sigma_B \in \mathfrak{S}_B$ and subject to the conditions

$$(\sigma_A \sqcap_{\mathfrak{S}_A} \sigma'_A) \otimes \sigma_B = (\sigma_A \otimes \sigma_B) \sqcap_{S_{AB}} (\sigma'_A \otimes \sigma_B), \qquad \sigma_A \otimes (\sigma_B \sqcap_{\mathfrak{S}_B} \sigma'_B) = (\sigma_A \otimes \sigma_B) \sqcap_{S_{AB}} (\sigma_A \otimes \sigma'_B). \quad (101)$$

The elements of S_{AB} can be written $(\bigsqcap_{i \in I}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B})$ with I finite and $\sigma_{i,A} \in \mathfrak{S}_A, \sigma_{i,B} \in \mathfrak{S}_B$, for any $i \in I$. ■

Definition 7. The space $S_{AB} = \mathfrak{S}_A \otimes \mathfrak{S}_B$ is turned into a partially ordered set with the following binary relation

$$\forall \sigma_{AB}, \sigma'_{AB} \in S_{AB}, \quad (\sigma_{AB} \sqsubseteq_{S_{AB}} \sigma'_{AB}) \quad :\Leftrightarrow \quad (\sigma_{AB} \sqcap_{S_{AB}} \sigma'_{AB} = \sigma_{AB}). \quad (102)$$

Definition 8. A non-empty subset $\mathfrak{R}$ of $\mathfrak{S}_A \times \mathfrak{S}_B$ is called a bi-filter of $\mathfrak{S}_A \times \mathfrak{S}_B$ iff

$$\begin{aligned}
&\forall \sigma_A, \sigma_{1,A}, \sigma_{2,A} \in \mathfrak{S}_A, \forall \sigma_B, \sigma_{1,B}, \sigma_{2,B} \in \mathfrak{S}_B, \\
&((\sigma_{1,A}, \sigma_{1,B}) \leq (\sigma_{2,A}, \sigma_{2,B}) \text{ and } (\sigma_{1,A}, \sigma_{1,B}) \in \mathfrak{R}) \ \Rightarrow \ (\sigma_{2,A}, \sigma_{2,B}) \in \mathfrak{R}, && (103)\\
&(\sigma_{1,A}, \sigma_B), (\sigma_{2,A}, \sigma_B) \in \mathfrak{R} \ \Rightarrow \ (\sigma_{1,A} \sqcap_{\mathfrak{S}_A} \sigma_{2,A}, \sigma_B) \in \mathfrak{R}, && (104)\\
&(\sigma_A, \sigma_{1,B}), (\sigma_A, \sigma_{2,B}) \in \mathfrak{R} \ \Rightarrow \ (\sigma_A, \sigma_{1,B} \sqcap_{\mathfrak{S}_B} \sigma_{2,B}) \in \mathfrak{R}. && (105)
\end{aligned}$$

Definition 9. If $\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\}$ is a non-empty finite subset of $\mathfrak{S}_A \times \mathfrak{S}_B$, then the intersection of the collection of all bi-filters of $\mathfrak{S}_A \times \mathfrak{S}_B$ which contain $(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})$ is a bi-filter, which we denote by $\mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\}$.

Lemma 7. If F is a filter of $S_{AB} = \mathfrak{S}_A \otimes \mathfrak{S}_B$ then the set $\alpha(F) := \{ (\sigma_A, \sigma_B) \in \mathfrak{S}_A \times \mathfrak{S}_B \mid \sigma_A \otimes \sigma_B \in F \}$ is a bi-filter of $\mathfrak{S}_A \times \mathfrak{S}_B$. ■

Lemma 8. [14, Lemma 1] Let us choose $\sigma_A, \sigma_{1,A}, \cdots, \sigma_{n,A} \in \mathfrak{S}_A$ and $\sigma_B, \sigma_{1,B}, \cdots, \sigma_{n,B} \in \mathfrak{S}_B$. Then,

$$(\sigma_A, \sigma_B) \in \mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\} \quad \Leftrightarrow \quad \left(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B. \quad (106)$$

■

Proof. Let us suppose that $(\sigma_A, \sigma_B) \in \mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\}$. Let F be the principal filter in $\mathfrak{S}_A \otimes \mathfrak{S}_B$ generated by $\left(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right)$. Then $\sigma_{i,A} \otimes \sigma_{i,B} \in F$ for any $1 \leq i \leq n$, and then $(\sigma_{i,A}, \sigma_{i,B}) \in \alpha(F)$ for any $1 \leq i \leq n$. Hence, $\mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\} \subseteq \alpha(F)$, and then $(\sigma_A, \sigma_B) \in \alpha(F)$. As a result, $\sigma_A \otimes \sigma_B \in F$ and then $\left(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B$.
Let us now suppose that $\left(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B$. Let $u : \mathfrak{S}_A \times \mathfrak{S}_B \to \{0,1\}$ be such that

$$u(\sigma_A, \sigma_B) = 1 \Leftrightarrow (\sigma_A, \sigma_B) \in \mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\} \quad (107)$$

u is a bi-homomorhism. Then, there exists a homomorphism $v : \mathfrak{S}_A \otimes \mathfrak{S}_B \to \{0,1\}$ such that $u(\sigma_A, \sigma_B) = v(\sigma_A \otimes \sigma_B)$ for any $\sigma_A \in \mathfrak{S}_A$ and $\sigma_B \in \mathfrak{S}_B$. We have then $u(\sigma_A, \sigma_B) = v(\sigma_A \otimes \sigma_B) \geq v(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}) = \bigwedge_{1\leq i\leq n} v(\sigma_{i,A} \otimes \sigma_{i,B}) = \bigwedge_{1\leq i\leq n} u(\sigma_{i,A}, \sigma_{i,B})$. Since $u(\sigma_{i,A}, \sigma_{i,B}) = 1$ for any $1 \leq i \leq n$, we deduce that $u(\sigma_A, \sigma_B) = 1$ and then $(\sigma_A, \sigma_B) \in \mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})\}$. □

Lemma 9. [14, Theorem 1]
Let us choose $\sigma_A, \sigma_{1,A}, \cdots, \sigma_{n,A} \in \mathfrak{S}_A$ and $\sigma_B, \sigma_{1,B}, \cdots, \sigma_{n,B} \in \mathfrak{S}_B$. Then,

$$\left(\prod_{1\leq i\leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists a n}-\text{ary lattice polynomial } p \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A}) \\ \text{and } \sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma_{1,B}, \cdots, \sigma_{n,B}). \end{array} \quad (108)$$

where p^* denotes the lattice polynomial obtained from p by dualizing the lattice operations. ■

Proof. Let us fix $\sigma_{1,A}, \cdots, \sigma_{n,A} \in \mathfrak{S}_A$ and $\sigma_{1,B}, \cdots, \sigma_{n,B} \in \mathfrak{S}_B$ and let us consider

$$F := \{ (\sigma_A, \sigma_B) \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A}) \text{ and } \sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma_{1,B}, \cdots, \sigma_{n,B}) \text{ for some n}-\text{ary polynomial } p \} \quad (109)$$

It is obvious that F contains $(\sigma_{1,A}, \sigma_{1,B}), \cdots, (\sigma_{n,A}, \sigma_{n,B})$.
It is also easy to check that F is a bi-filter.
Endly, we can check that every bi-filter which contains $(\sigma_{1,A}, \sigma_{1,B})$, $\cdots$, $(\sigma_{n,A}, \sigma_{n,B})$ contains also F. This point can be checked by induction on the complexity of the polynomial p by using the following elementary result, consequence of the bi-filter character of F,

$$\forall \sigma_A, \sigma'_A \in \mathfrak{S}_A, \sigma_B, \sigma'_B \in \mathfrak{S}_B, \quad \left((\sigma_A, \sigma_B), (\sigma'_A, \sigma'_B) \in F\right) \quad \Rightarrow \quad \left\{ \begin{array}{l} (\sigma_A \sqcup_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcap_{\mathfrak{S}_B} \sigma'_B) \in F \\ (\sigma_A \sqcap_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcup_{\mathfrak{S}_B} \sigma'_B) \in F \end{array} \right.$$

□

Theorem 10. For any $\mathfrak{l}_A \in \mathfrak{E}_A, \mathfrak{l}_B \in \mathfrak{E}_B$ the map

$$\begin{array}{rccc} f^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B} : & \mathfrak{S}_A \times \mathfrak{S}_B & \to & \mathfrak{B} \\ & (\sigma_A, \sigma_B) & \mapsto & \varepsilon^A_{\mathfrak{l}_A}(\sigma_A) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_B) \end{array} \quad (110)$$

is a bi-homomorphism. It exists then a unique homomorphism from $S_{AB} = \mathfrak{S}_A \otimes \mathfrak{S}_B$ to $\mathfrak{B}$, denoted

$\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$, and satisfying $f^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B} = \nu^{AB}_{\mathfrak{t}_A,\mathfrak{t}_B} \circ \iota$. Explicitly, we have

$$\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\textstyle\bigsqcap^{S_{AB}}_{i\in I} \sigma_{i,A} \otimes \sigma_{i,B}) = \bigwedge_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}). \tag{111}$$

Anticipating the construction of the bipartite effect state, we may denote $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$ by $\varepsilon^{AB}_{\mathfrak{l}_A\otimes\mathfrak{l}_B}$. ■

Proof. The bi-homomorphic property is a direct consequence of (15) and (5). The existence of $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$ satisfying $f^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B} = \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B} \circ \iota$ is then obtained as a consequence of Theorem 9. □

Theorem 11.

$$\forall \sigma_{AB}, \sigma'_{AB} \in S_{AB}, \quad (\sigma_{AB} \sqsubseteq_{S_{AB}} \sigma'_{AB}) \;\Rightarrow\; (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B,\; \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{AB}) \leq \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma'_{AB})), \tag{112}$$

$$\forall \{\, \sigma_{i,AB} \mid i \in I \,\} \subseteq_{fin} S_{AB}, \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \quad \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\textstyle\bigsqcap^{S_{AB}}_{i\in I} \sigma_{i,AB}) = \bigwedge_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{i,AB}). \tag{113}$$

■

Remark 5. We have not managed to prove that S_{AB} is chain-complete when $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are chain-complete, or the fact that, for any $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$, the map $\varepsilon^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$ is chain-continuous. Nevertheless, we have a weaker result expressed as follows. We suppose that $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are chain-complete and that $\varepsilon^A_{\mathfrak{l}_A}$ and $\varepsilon^B_{\mathfrak{l}_B}$ are chain-continuous for any $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$. Then, we have

$$\begin{aligned} &\forall \{\, \sigma_{i,AB} \mid i \in I \,\} \subseteq_{Chain} S_{AB} \mid \sigma_{i,AB} := \sigma_{i,A} \otimes \sigma_{i,B}, \forall i \in I, \qquad \textstyle\bigsqcup^{S_{AB}}_{i\in I} \sigma_{i,AB} \text{ exists in } S_{AB}, \\ &\text{and} \quad \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \quad \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\textstyle\bigsqcup^{S_{AB}}_{i\in I} \sigma_{i,AB}) = \bigvee_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{i,AB}). \end{aligned} \tag{114}$$

Indeed, using Lemma 9, we know that $\{\, \sigma_{i,AB} \mid i \in I \,\} \subseteq_{Chain} S_{AB}$ implies immediately $\{\, \sigma_{i,A} \mid i \in I \,\} \subseteq_{Chain} \mathfrak{S}_A$ and $\{\, \sigma_{i,B} \mid i \in I \,\} \subseteq_{Chain} \mathfrak{S}_B$. Hence, $(\bigsqcup^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A})$ and $(\bigsqcup^{\mathfrak{S}_B}_{i\in I} \sigma_{i,B})$ exist in S_{AB}. As a consequence, the lowest upper-bound $\bigsqcup^{S_{AB}}_{i\in I} \sigma_{i,AB}$ exists in S_{AB} and we have explicitly

$$\textstyle\bigsqcup^{S_{AB}}_{i\in I} \sigma_{i,AB} = (\bigsqcup^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \otimes (\bigsqcup^{\mathfrak{S}_B}_{i\in I} \sigma_{i,B}). \tag{115}$$

Moreover, using (111), we have

$$\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{i,AB}) = \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}). \tag{116}$$

In reference to [5, Definition 2.1.10], we note that

$$\{\, \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i',B}) \mid (i,i') \in I \,\} \tag{117}$$

is a monotone net, and then, using (116), the relation [5, Proposition 2.1.12], the distributivity property (6), the chain continuity of $\varepsilon^A_{\mathfrak{l}_A}$ and $\varepsilon^B_{\mathfrak{l}_B}$, the homomorphic property (111) and the equality (115), we obtain

$$\begin{aligned} \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \qquad \textstyle\bigvee_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{i,AB}) &= \textstyle\bigvee_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}) \\ &= \textstyle\bigvee_{i\in I} \bigvee_{i'\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i',B}) \\ &= (\textstyle\bigvee_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A})) \bullet (\bigvee_{i'\in I} \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i',B})) \\ &= \varepsilon^A_{\mathfrak{l}_A}(\textstyle\bigsqcup^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\bigsqcup^{\mathfrak{S}_B}_{i'\in I} \sigma_{i',B}) \\ &= \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\textstyle\bigsqcup^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \otimes (\bigsqcup^{\mathfrak{S}_B}_{i'\in I} \sigma_{i',B})) \\ &= \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\textstyle\bigsqcup^{S_{AB}}_{i\in I} \sigma_{i,AB}). \end{aligned} \tag{118}$$

4.2 The maximal tensor-product

It is now possible to give a second definition of the tensor product of $\mathfrak{S}_A$ and $\mathfrak{S}_B$. This tensor product will be called *maximal tensor product* and denoted $\widetilde{S}_{AB}$. It will be defined in reference to the axiomatic relations **(B1)** − **(B2)** − **(B3)** − **(B4)** − **(B5)** − **(B6)**. It will be assumed that $(\mathfrak{S}_A, \varepsilon^A, \mathfrak{E}_A)$ and $(\mathfrak{S}_B, \varepsilon^B, \mathfrak{E}_B)$ satisfy the axioms of States/Effects Chu spaces.

Definition 10. The set $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ is equipped with the Inf semi-lattice structure $\cup$ and with the following Inf semi-lattice morphisms defined for any $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$,

$$\begin{array}{rccc} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B} & : & \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B) & \longrightarrow & \mathfrak{B} \\ & & \{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} & \mapsto & \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}) := \bigwedge_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}). \end{array} \tag{119}$$

Definition 11. $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ is equipped with a congruence relation defined between any two elements u_{AB} and u'_{AB} of $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ by

$$(u_{AB} \approx u'_{AB}) \quad :\Leftrightarrow \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B,\ \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(u_{AB}) = \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(u'_{AB})). \tag{120}$$

Definition 12. The space $\widetilde{S}_{AB} = \mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ is built as the quotient of $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ under the congruence relation $\approx$.

$$\forall \sigma_{AB} \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \qquad \widetilde{\sigma_{AB}} := \{u_{AB} \mid \sigma_{AB} \approx u_{AB}\}. \tag{121}$$

The map $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$ will be abusively defined as a map from $\widetilde{S}_{AB}$ to $\mathfrak{B}$ by $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\sigma_{AB}}) := \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\sigma_{AB})$ for any σ_{AB} in $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$.

Definition 13. $\widetilde{S}_{AB}$ is equipped with a partial order defined according to

$$\forall \widetilde{\sigma}_{AB}, \widetilde{\sigma}'_{AB} \in \widetilde{S}_{AB}, \quad (\widetilde{\sigma}_{AB} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{\sigma}'_{AB}) \quad :\Leftrightarrow \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B,\ \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\sigma}_{AB}) \leq \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\sigma}'_{AB})). \tag{122}$$

This poset structure can be "explicited" according to following lemma addressing the word problem in $\widetilde{S}_{AB}$.

Lemma 10. Let us consider $u_{AB} := \{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$ an element of $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$. We have explicitly, for any $\sigma_A \in \mathfrak{S}_A$ and $\sigma_B \in \mathfrak{S}_B$, the following equivalence

$$\begin{array}{rcl} \left(\widetilde{u_{AB}} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{(\sigma_A, \sigma_B)}\right) & \Leftrightarrow & \Big((\bigsqcap^{\mathfrak{S}_A}_{k\in I} \sigma_{k,A}) \sqsubseteq_{\mathfrak{S}_A} \sigma_A \text{ and } (\bigsqcap^{\mathfrak{S}_B}_{m\in I} \sigma_{m,B}) \sqsubseteq_{\mathfrak{S}_B} \sigma_B \text{ and} \\ & & \left(\forall \varnothing \subsetneq K \subsetneq I,\ (\bigsqcap^{\mathfrak{S}_A}_{k\in K} \sigma_{k,A}) \sqsubseteq_{\mathfrak{S}_A} \sigma_A \text{ or } (\bigsqcap^{\mathfrak{S}_B}_{m\in I-K} \sigma_{m,B}) \sqsubseteq_{\mathfrak{S}_B} \sigma_B\right)\Big). \end{array} \tag{123}$$

It is recalled that $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are down-complete Inf semi-lattice and then the infima in this formula are well-defined. ■

Proof. We intent to expand the inequality $\widetilde{u_{AB}} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{(\sigma_A, \sigma_B)}$. It is equivalent to

$$\forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \qquad \left(\bigwedge_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B})\right) \leq \varepsilon^A_{\mathfrak{l}_A}(\sigma_A) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_B). \tag{124}$$

We intent to choose a pertinent set of effects $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$ to reformulate this inequality.
Let us firstly choose $\mathfrak{l}_B = \mathfrak{Y}_{\mathfrak{E}_B}$. Using (4), we obtain

$$\varepsilon^A_{\mathfrak{l}_A}(\bigsqcap^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \leq \varepsilon^A_{\mathfrak{l}_A}(\sigma_A), \forall \mathfrak{l}_A \in \mathfrak{E}_A, \tag{125}$$

whcih leads immediately

$$\bigsqcap_{i\in I}^{\mathfrak{S}_A} \sigma_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma_A. \tag{126}$$

Choosing $\mathfrak{l}_A = \mathfrak{Y}_{\mathfrak{E}_A}$, we obtain along the same line

$$\bigsqcap_{i\in I}^{\mathfrak{S}_B} \sigma_{i,B} \sqsubseteq_{\mathfrak{S}_B} \sigma_B. \tag{127}$$

Let us now consider $\varnothing \subsetneq K \subsetneq I$ and let us choose $\mathfrak{l}_A$ and $\mathfrak{l}_B$ according to

$$\varepsilon^A_{\mathfrak{l}_A}(\sigma) := \mathbf{N}, \forall \sigma \sqsupseteq_{\mathfrak{S}_A} \bigsqcap_{k\in K}^{\mathfrak{S}_A} \sigma_{k,A} \quad \text{and} \quad \varepsilon^A_{\mathfrak{l}_A}(\sigma) := \bot, \quad \text{elsewhere}, \tag{128}$$

$$\varepsilon^B_{\mathfrak{l}_B}(\sigma) := \mathbf{N}, \forall \sigma \sqsupseteq_{\mathfrak{S}_B} \bigsqcap_{m\in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \quad \text{and} \quad \varepsilon^B_{\mathfrak{l}_B}(\sigma) := \bot, \quad \text{elsewhere}. \tag{129}$$

We deduce, from the assumption (124), that for this $\varnothing \subsetneq K \subsetneq I$ we have

$$(\bigsqcap_{k\in K}^{\mathfrak{S}_A} \sigma_{k,A} \sqsubseteq_{\mathfrak{S}_A} \sigma_A) \text{ or } (\bigsqcap_{m\in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \sqsubseteq_{\mathfrak{S}_B} \sigma_B). \tag{130}$$

We let the reader check that we have obtained the whole set of independent inequalities reformulating the property (124). □

Definition 14. We will adopt the following definition

$$\begin{aligned} \forall \widetilde{\sigma} \in \widetilde{S}_{AB}, \quad \langle \widetilde{\sigma} \rangle &:= Max\{ u \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B) \mid \widetilde{u} \sqsupseteq_{\widetilde{S}_{AB}} \widetilde{\sigma} \} \\ &= \{ (\sigma_A, \sigma_B) \mid \widetilde{(\sigma_A, \sigma_B)} \sqsupseteq_{\widetilde{S}_{AB}} \widetilde{\sigma} \}, \end{aligned} \tag{131}$$

Lemma 11. We have the following Galois relation

$$\forall \widetilde{\sigma} \in \widetilde{S}_{AB}, \forall u \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \qquad \langle \widetilde{\sigma} \rangle \supseteq u \quad \Leftrightarrow \quad \widetilde{\sigma} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{u}. \tag{132}$$

■

Proof. Let us fix $u := \{ (\sigma_{i,A}, \sigma_{i,B}) \mid i \in I \}$. We derive straightforwardly the following equivalences

$$\begin{aligned} \langle \widetilde{\sigma} \rangle \supseteq u \quad &\Leftrightarrow \quad \forall i \in I, \widetilde{(\sigma_{i,A}, \sigma_{i,B})} \sqsupseteq_{\widetilde{S}_{AB}} \widetilde{\sigma} \\ &\Leftrightarrow \quad \forall i \in I, \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\sigma}) \leq \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}) \\ &\Leftrightarrow \quad \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\sigma}) \leq \bigwedge_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}) \\ &\Leftrightarrow \quad \widetilde{\sigma} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{u}. \end{aligned} \tag{133}$$

□

Theorem 12. $\widetilde{S}_{AB}$ is a down-complete Inf semi-lattice with

$$\forall \{ u_i \mid i \in I \} \subseteq \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \qquad \bigsqcap_{i\in I}^{\widetilde{S}_{AB}} \widetilde{u_i} = \widetilde{\bigcup_{i\in I} u_i}. \tag{134}$$

Moreover, for any $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$, we have

$$\forall \{ \widetilde{u}_i \mid i \in I \} \subseteq \widetilde{S}_{AB}, \qquad \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\bigsqcap_{i\in I}^{\widetilde{S}_{AB}} \widetilde{u}_i) = \bigwedge_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{u}_i) \tag{135}$$

■

Proof. The property (134) is a direct consequence of the Galois relation established in previous lemma. For any $\mathfrak{l}_A \in \mathfrak{E}_A$ and $\mathfrak{l}_B \in \mathfrak{E}_B$, using (134) and the homomorphic property for $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}$, we have

$$\begin{array}{rcl} \forall\{\, u_i \mid i \in I\,\} \subseteq \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \quad \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\prod^{\widetilde{S}_{AB}}_{i\in I} \widetilde{u_i}) & = & \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{\bigcup_{i\in I} u_i}) \\ & = & \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\bigcup_{i\in I} u_i) \\ & = & \bigwedge_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(u_i) \\ & = & \bigwedge_{i\in I} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\widetilde{u_i}) \end{array} \tag{136}$$

□

Definition 15. The element $\widetilde{u} \in \widetilde{S}_{AB}$ associated to the element $u := \{\, (\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\,\} \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ will be denoted $\prod^{\widetilde{S}_{AB}}_{i\in I} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}$.

Theorem 13.

$$\forall\{\, \sigma_{i,A} \mid i \in I\,\} \subseteq \mathfrak{S}_A, \forall \sigma_B \in \mathfrak{S}_B, \quad (\prod^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \widetilde{\otimes} \sigma_B = \prod^{\widetilde{S}_{AB}}_{i\in I} (\sigma_{i,A} \widetilde{\otimes} \sigma_B), \tag{137}$$

$$\forall\{\, \sigma_{i,B} \mid i \in I\,\} \subseteq \mathfrak{S}_B, \forall \sigma_A \in \mathfrak{S}_A, \quad \sigma_A \widetilde{\otimes} (\prod^{\mathfrak{S}_B}_{i\in I} \sigma_{i,B}) = \prod^{\widetilde{S}_{AB}}_{i\in I} (\sigma_A \widetilde{\otimes} \sigma_{i,B}). \tag{138}$$

■

Proof. Indeed, using successively properties (111) (15) (5) and (111) again, we deduce that, for any $\mathfrak{l}_A \in \mathfrak{E}_A, \mathfrak{l}_B \in \mathfrak{E}_B$,

$$\begin{array}{rcl} \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\prod^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}, \sigma_B)) & = & \varepsilon^A_{\mathfrak{l}_A}(\prod^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_B) \\ & = & (\bigwedge_{i\in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A})) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_B) \\ & = & \bigwedge_{i\in I} (\varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_B)) \\ & = & \nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\{\, (\sigma_{i,A}, \sigma_B) \mid i \in I\,\}), \end{array} \tag{139}$$

and then, by definition, we obtain the property

$$(\prod^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}, \sigma_B) \approx \{\, (\sigma_{i,A}, \sigma_B) \mid i \in I\,\} \tag{140}$$

and then

$$(\prod^{\mathfrak{S}_A}_{i\in I} \sigma_{i,A}) \widetilde{\otimes} \sigma_B = \prod^{\widetilde{S}_{AB}}_{i\in I} (\sigma_{i,A} \widetilde{\otimes} \sigma_B). \tag{141}$$

We obtain the second property along the same lines of proof. □

4.3 The bipartite construction for the States/Effects Chu space associated to the maximal tensor product

In this subsection, we will assume that $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^A)$ and $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^B)$ are valid States/Effects Chu spaces. In other words, they satisfy Axioms **(A1)**−**(A5)**.

Definition 16. The evaluation map will be defined as a map

$$\begin{array}{rccl} \varepsilon \quad : & \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B) & \longrightarrow & \mathfrak{B}^{\widetilde{S}_{AB}} \\ & \{(\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B}) \mid i \in I\} & \mapsto & \varepsilon^{AB}_{\{(\mathfrak{l}_{i,A},\mathfrak{l}_{i,B}) \mid i\in I\}} \mid \forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB},\ \varepsilon^{AB}_{\{(\mathfrak{l}_{i,A},\mathfrak{l}_{i,B}) \mid i\in I\}}(\widetilde{\sigma}_{AB}) = \bigwedge_{i\in I} \nu^{AB}_{\mathfrak{l}_{i,A},\mathfrak{l}_{i,B}}(\widetilde{\sigma}_{AB}). \end{array} \tag{142}$$

Definition 17. $\mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B)$ is equipped with a congruence relation defined between any two elements x_{AB} and x'_{AB} of $\mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B)$ by

$$(x_{AB} \simeq x'_{AB}) \quad :\Leftrightarrow \quad (\forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB},\ \varepsilon^{AB}_{x_{AB}}(\widetilde{\sigma}_{AB}) = \varepsilon^{AB}_{x'_{AB}}(\widetilde{\sigma}_{AB})). \tag{143}$$

Definition 18. The space $\widetilde{E}_{AB}$ is built as the quotient of $\mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B)$ under the congruence relation $\simeq$.

$$\forall \lambda_{AB} \in \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B), \qquad \widetilde{\lambda_{AB}} := \{\, x_{AB} \mid \lambda_{AB} \simeq x_{AB} \,\}. \tag{144}$$

The evaluation map will be defined as a map from $\widetilde{E}_{AB}$ to $\mathfrak{B}^{\widetilde{S}_{AB}}$ by $\varepsilon^{AB}_{\widetilde{\lambda_{AB}}} := \varepsilon^{AB}_{\lambda_{AB}}$ for any $\lambda_{AB} \in \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B)$,

Definition 19. $\widetilde{E}_{AB}$ is equipped with a partial order defined according to

$$\forall \widetilde{\lambda}_{AB}, \widetilde{\lambda}'_{AB} \in \widetilde{E}_{AB}, \quad (\widetilde{\lambda}_{AB} \sqsubseteq_{\widetilde{E}_{AB}} \widetilde{\lambda}'_{AB}) \quad :\Leftrightarrow \quad (\forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB},\ \varepsilon^{AB}_{\widetilde{\lambda}_{AB}}(\widetilde{\sigma}_{AB}) \leq \varepsilon^{AB}_{\widetilde{\lambda}'_{AB}}(\widetilde{\sigma}_{AB})). \tag{145}$$

Definition 20. We will adopt the following definition

$$\begin{aligned} \forall \widetilde{\lambda} \in \widetilde{E}_{AB}, \quad \langle \widetilde{\lambda} \rangle \ &:= \ Max\{\, x \in \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B) \mid \widetilde{x} \sqsupseteq_{\widetilde{E}_{AB}} \widetilde{\lambda} \,\} \\ &= \ \{\, (\mathfrak{l}_A, \mathfrak{l}_B) \mid \widetilde{(\mathfrak{l}_A, \mathfrak{l}_B)} \sqsupseteq_{\widetilde{E}_{AB}} \widetilde{\lambda} \,\}, \end{aligned} \tag{146}$$

Lemma 12. We have the following Galois relation

$$\forall \widetilde{\lambda} \in \widetilde{E}_{AB}, \forall x \in \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B), \qquad \langle \widetilde{\lambda} \rangle \supseteq x \ \Leftrightarrow \ \widetilde{\lambda} \sqsubseteq_{\widetilde{E}_{AB}} \widetilde{x}. \tag{147}$$

■

Proof. Let us fix $x := \{\, (\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B}) \mid i \in I \,\}$. We derive straightforwardly the following equivalences

$$\begin{aligned} \langle \widetilde{\lambda} \rangle \supseteq x \ &\Leftrightarrow \ \forall i \in I, \widetilde{(\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B})} \sqsupseteq_{\widetilde{E}_{AB}} \widetilde{\lambda} \\ &\Leftrightarrow \ \forall i \in I, \forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB}, \quad \varepsilon^{AB}_{\widetilde{\lambda}}(\widetilde{\sigma}_{AB}) \leq \nu^{AB}_{\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B}}(\widetilde{\sigma}_{AB}) \\ &\Leftrightarrow \ \forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB}, \quad \varepsilon^{AB}_{\widetilde{\lambda}}(\widetilde{\sigma}_{AB}) \leq \bigwedge_{i \in I} \nu^{AB}_{\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B}}(\widetilde{\sigma}_{AB}) = \varepsilon^{AB}_{x}(\widetilde{\sigma}_{AB}) \\ &\Leftrightarrow \ \widetilde{\lambda} \sqsubseteq_{\widetilde{E}_{AB}} \widetilde{x}. \end{aligned} \tag{148}$$

□

Theorem 14. $\widetilde{E}_{AB}$ is a down-complete Inf semi-lattice with

$$\forall \{\, x_i \mid i \in I \,\} \subseteq \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B), \qquad \textstyle\prod^{\widetilde{E}_{AB}}_{i \in I} \widetilde{x}_i = \widetilde{\bigcup_{i \in I} x_i}. \tag{149}$$

Moreover, we have

$$\forall \{\, \widetilde{\lambda}_i \mid i \in I \,\} \subseteq \widetilde{E}_{AB}, \forall \widetilde{\sigma}_{AB} \in \widetilde{S}_{AB}, \qquad \varepsilon^{AB}_{\prod^{\widetilde{E}_{AB}}_{i \in I} \widetilde{\lambda}_i}(\widetilde{\sigma}_{AB}) = \bigwedge_{i \in I} \varepsilon^{AB}_{\widetilde{\lambda}_i}(\widetilde{\sigma}_{AB}) \tag{150}$$

In other words, $\widetilde{E}_{AB}$ satisfies Axiom **(A3)**. ■

Proof. The property (149) is a direct consequence of the Galois relation established in previous lemma. For any $\widetilde{\sigma}_{AB} \in \widetilde{S}_{AB}$ we have

$$\begin{aligned} \forall \{ x_i \mid i \in I \} \subseteq \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B), \quad \varepsilon^{AB}_{\widetilde{\prod}_{i\in I}^{\widetilde{S}_{AB}} \widetilde{x_i}}(\widetilde{\sigma}_{AB}) &= \varepsilon^{AB}_{\widetilde{\bigcup_{i\in I} x_i}}(\widetilde{\sigma}_{AB}) \\ &= \varepsilon^{AB}_{\bigcup_{i\in I} x_i}(\widetilde{\sigma}_{AB}) \\ &= \bigwedge_{i\in I} \varepsilon^{AB}_{x_i}(\widetilde{\sigma}_{AB}) \\ &= \bigwedge_{i\in I} \varepsilon^{AB}_{\widetilde{x_i}}(\widetilde{\sigma}_{AB}) \end{aligned} \tag{151}$$

□

Definition 21. The element $\widetilde{\mathfrak{l}_{AB}} \in \widetilde{E}_{AB}$ associated to the element $\mathfrak{l}_{AB} := \{ (\mathfrak{l}_{i,A}, \mathfrak{l}_{i,B}) \mid i \in I \} \in \mathscr{P}(\mathfrak{E}_A \times \mathfrak{E}_B)$ will be denoted $\widetilde{\prod}_{i\in I}^{\widetilde{E}_{AB}} \mathfrak{l}_{i,A} \widetilde{\otimes} \mathfrak{l}_{i,B}$.

Theorem 15. $\widetilde{S}_{AB}$ satisfies Axiom **(A1)**. Explicitly, $\widetilde{S}_{AB}$ is a down-complete Inf semi-lattice. Moreover, we have

$$\forall \{ \widetilde{\sigma}_{i,AB} \mid i \in I \} \subseteq \widetilde{S}_{AB}, \forall \widetilde{\lambda}_{AB} \in \widetilde{E}_{AB}, \qquad \varepsilon^{AB}_{\widetilde{\lambda}_{AB}}(\widetilde{\prod}_{i\in I}^{\widetilde{S}_{AB}} \widetilde{\sigma}_{i,AB}) = \bigwedge_{i\in I} \varepsilon^{AB}_{\widetilde{\lambda}_{AB}}(\widetilde{\sigma}_{i,AB}) \tag{152}$$

■

Proof. Direct consequence of Theorem 12 with property (150). □

Theorem 16. If $\mathfrak{S}_A$ and $\mathfrak{S}_B$ satisfy the axiom **(A2)**, then $\widetilde{S}_{AB}$ satisfies the axiom **(A2)** as well : the bottom element of $\widetilde{S}_{AB}$ is explicitly given by $\perp_{\mathfrak{S}_A} \widetilde{\otimes} \perp_{\mathfrak{S}_B}$. ■

Proof. Trivial using the expansion (123). □

Theorem 17.

$$\widetilde{S}_{AB}^{\,pure} = \{ \sigma_A \widetilde{\otimes} \sigma_B \mid \sigma_A \in \mathfrak{S}_A^{\,pure}, \sigma_B \in \mathfrak{S}_B^{\,pure} \} \tag{153}$$

Moreover, $\widetilde{S}_{AB} = \mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ satisfies the axiom **(A5)**, i.e. $\widetilde{S}_{AB}^{\,pure} = Max(\widetilde{S}_{AB})$. ■

Proof. First of all, it is a trivial fact that the completely meet-irreducible elements of $\widetilde{S}_{AB}$ are necessarily pure tensors of $\widetilde{S}_{AB}$, i.e. elements of the form $\sigma_A \widetilde{\otimes} \sigma_B$.
Let us then consider $\sigma_A \widetilde{\otimes} \sigma_B$ a completely meet-irreducible element of $\widetilde{S}_{AB}$ and let us assume that $\sigma_A = \prod_{i\in I}^{\mathfrak{S}_A} \sigma_{i,A}$ for $\sigma_{i,A} \in \mathfrak{S}_A$ for any $i \in I$. We have then $(\sigma_A \widetilde{\otimes} \sigma_B) = ((\prod_{i\in I}^{\mathfrak{S}_A} \sigma_{i,A}) \widetilde{\otimes} \sigma_B) = \widetilde{\prod}_{i\in I}^{\widetilde{S}_{AB}} (\sigma_{i,A} \widetilde{\otimes} \sigma_B)$. On another part, $\sigma_A \widetilde{\otimes} \sigma_B$ being completely meet-irreducible in $\widetilde{S}_{AB}$, there exists $k \in I$ such that $\sigma_A \widetilde{\otimes} \sigma_B = \sigma_{k,A} \widetilde{\otimes} \sigma_B$, i.e, $\sigma_A = \sigma_{k,A}$. As a conclusion, σ_A is completely meet-irreducible. In the same way, σ_B is completely meet-irreducible. As a first result, pure states of $\widetilde{S}_{AB}$ are necessarily of the form $\sigma_A \widetilde{\otimes} \sigma_B$ with $\sigma_A \in \mathfrak{S}_A^{\,pure}, \sigma_B \in \mathfrak{S}_B^{\,pure}$.
Conversely, let us consider σ_A a pure state of $\mathfrak{S}_A$ and σ_B a pure state of $\mathfrak{S}_B$, and let us suppose that $(\widetilde{\prod}_{i\in I}^{\widetilde{S}_{AB}} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}) = (\sigma_A \widetilde{\otimes} \sigma_B)$ with $\sigma_{i,A} \in \mathfrak{S}_A$ and $\sigma_{i,B} \in \mathfrak{S}_B$ for any $i \in I$. We now exploit the expansion (123), and in particular the two conditions $(\prod_{k\in I}^{\mathfrak{S}_A} \sigma_{k,A}) = \sigma_A$ and $(\prod_{m\in I}^{\mathfrak{S}_B} \sigma_{m,B}) = \sigma_B$. From $\sigma_A \in Max(\mathfrak{S}_A)$ and $\sigma_B \in Max(\mathfrak{S}_B)$, we deduce that $\sigma_{i,A} = \sigma_A$ and $\sigma_{j,B} = \sigma_B$ for any $i, j \in I$. As a second result, we have then obtained that the state $(\sigma_A \widetilde{\otimes} \sigma_B)$, with σ_A a pure state of $\mathfrak{S}_A$ and σ_B a pure state of $\mathfrak{S}_B$, is completely meet-irreducible.
From the expansion (123), we deduce also immediately that $(\sigma_A \widetilde{\otimes} \sigma_B) \in Max(\widetilde{S}_{AB})$ as long as $\sigma_A \in Max(\mathfrak{S}_A)$ and $\sigma_B \in Max(\mathfrak{S}_B)$. □

Theorem 18. $\widetilde{S}_{AB} = \mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ satisfies the axiom **(A4)**. Explicitly,

$$\forall \sigma \in \widetilde{S}_{AB},\ \ \sigma = \prod\nolimits^{\widetilde{S}_{AB}} \underline{\sigma},\ \text{ where } \underline{\sigma} = (\widetilde{S}_{AB}^{pure} \cap (\uparrow^{\widetilde{S}_{AB}} \sigma)). \tag{154}$$

■

Proof. Let us fix $\sigma \in \widetilde{S}_{AB}$.
We note that $\sigma \sqsubseteq_{\widetilde{S}_{AB}} \sigma'$ for any $\sigma' \in (\widetilde{S}_{AB}^{pure} \cap (\uparrow^{\widetilde{S}_{AB}} \sigma))$ and then $\sigma \sqsubseteq_{\widetilde{S}_{AB}} \prod^{\widetilde{S}_{AB}} \underline{\sigma}$.
Secondly, denoting $\sigma := (\prod_{i\in I}^{\widetilde{S}_{AB}} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B})$, we note immediately that, for any $\sigma_A \in \mathfrak{S}_A^{pure}$ and $\sigma_B \in \mathfrak{S}_B^{pure}$, if $\sigma_A \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ and $\sigma_B \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}$, then $(\sigma_A \widetilde{\otimes} \sigma_B) \sqsupseteq_{\mathfrak{S}_{AB}} \sigma$, i.e. $(\sigma_A \widetilde{\otimes} \sigma_B) \in \underline{\sigma}$. As a consequence, we have

$$(\prod\nolimits_{i\in I}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_A \in \mathfrak{S}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_B \in \mathfrak{S}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}}^{\widetilde{S}_{AB}} \sigma_A \widetilde{\otimes} \sigma_B) \sqsupseteq_{\widetilde{S}_{AB}} \prod\nolimits^{\widetilde{S}_{AB}} \underline{\sigma}. \tag{155}$$

Endly, using Theorem 13,we have

$$\begin{aligned} \sigma = \prod\nolimits_{i\in I}^{\widetilde{S}_{AB}} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B} &= \prod\nolimits_{i\in I}^{\widetilde{S}_{AB}} (\prod\nolimits_{\sigma_A \in \mathfrak{S}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}}^{\widetilde{S}_{AB}} \sigma_A) \widetilde{\otimes} (\prod\nolimits_{\sigma_B \in \mathfrak{S}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}}^{\widetilde{S}_{AB}} \sigma_B) \\ &= \prod\nolimits_{i\in I}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_A \in \mathfrak{S}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_B \in \mathfrak{S}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}}^{\widetilde{S}_{AB}} \sigma_A \widetilde{\otimes} \sigma_B. \end{aligned} \tag{156}$$

As a final conclusion, we obtain

$$\sigma = (\prod\nolimits_{i\in I}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_A \in \mathfrak{S}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}}^{\widetilde{S}_{AB}} \prod\nolimits_{\sigma_B \in \mathfrak{S}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}}^{\widetilde{S}_{AB}} \sigma_A \widetilde{\otimes} \sigma_B) = \prod\nolimits^{\widetilde{S}_{AB}} \underline{\sigma}. \tag{157}$$

□

As a conclusion of previous theorems, we have also obtained that $\widetilde{S}_{AB}$ is a valid space of states and $\widetilde{E}_{AB}$ is a valid space of effects satisfying axioms **(A1)** $-$ **(A5)**. As a consequence, they satisfy axioms **(B1)** and **(B2)**.
Axioms **(B3)** and **(B4)** are also trivial by construction.
By construction of the maximal tensor product, it will also satisfy the axiom **(B5)**, i.e.

$$\forall \widetilde{\sigma}_{AB}, \widetilde{\sigma}'_{AB} \in \widetilde{S}_{AB}, \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B,\ \ \varepsilon^{AB}_{\mathfrak{l}_A \widetilde{\otimes} \mathfrak{l}_B}(\widetilde{\sigma}_{AB}) = \varepsilon^{AB}_{\mathfrak{l}_A \widetilde{\otimes} \mathfrak{l}_B}(\widetilde{\sigma}'_{AB})) \quad \Leftrightarrow \quad (\widetilde{\sigma}_{AB} = \widetilde{\sigma}'_{AB}). \tag{158}$$

Endly, Definition 10 has been chosen in such a way that we obtain trivially the axiom **(B6)**, i.e.

$$\forall \sigma_A \in \mathfrak{S}_A, \forall \sigma_B \in \mathfrak{S}_B, \forall \mathfrak{l}_A \in \mathfrak{E}_A, \forall \mathfrak{l}_B \in \mathfrak{E}_B, \quad \varepsilon^{AB}_{\mathfrak{l}_A \widetilde{\otimes} \mathfrak{l}_B}(\sigma_A \widetilde{\otimes} \sigma_B) = \varepsilon^{A}_{\mathfrak{l}_A}(\sigma_A) \bullet \varepsilon^{B}_{\mathfrak{l}_B}(\sigma_B). \tag{159}$$

4.4 Multipartite experiments defined by the maximal tensor product

Let $\mathfrak{S}_A, \mathfrak{S}_B, \mathfrak{S}_C$ be three spaces of states. We intent to define the tripartite state space $\mathfrak{S}_{ABC}$? Clearly one option is to first form the bipartite state space $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ and then tensor the result with $\mathfrak{S}_C$, so that we get $(\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B) \widetilde{\otimes} \mathfrak{S}_C$. Another way to build these tripartite experiments is to first form $\mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C$ and then tensor with $\mathfrak{S}_A$ to obtain $\mathfrak{S}_A \widetilde{\otimes} (\mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C)$. It is natural to require that both of these constructions yield the same result.

Theorem 19. The maximal tensor product of state spaces is associative, i.e., we must have

$$(\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B) \widetilde{\otimes} \mathfrak{S}_C \quad = \quad \mathfrak{S}_A \widetilde{\otimes} (\mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C). \tag{160}$$

■

Proof. $(\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B) \widetilde{\otimes} \mathfrak{S}_C$ is defined as the quotient of $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C)$ by the congruence relation defined for any $u_{ABC}, u'_{ABC} \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C)$

$$(u_{ABC} \approx_{(AB)C} u'_{ABC}) \quad :\Leftrightarrow \quad (\forall \mathfrak{l}_{AB} \in \widetilde{E}_{AB}, \mathfrak{l}_C \in \mathfrak{E}_C, \quad \nu_{\mathfrak{l}_{AB},\mathfrak{l}_C}(u_{ABC}) = \nu_{\mathfrak{l}_{AB},\mathfrak{l}_C}(u'_{ABC})) \tag{161}$$

$$\Leftrightarrow \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \mathfrak{l}_B \in \mathfrak{E}_B, \mathfrak{l}_C \in \mathfrak{E}_C, \quad \nu_{\mathfrak{l}_A,\mathfrak{l}_B,\mathfrak{l}_C}(u_{ABC}) = \nu_{\mathfrak{l}_A,\mathfrak{l}_B,\mathfrak{l}_C}(u'_{ABC})) \tag{162}$$

where

$$\nu_{\mathfrak{l}_A,\mathfrak{l}_B,\mathfrak{l}_C}(\{(\sigma_{i,A}, \sigma_{i,B}, \sigma_{i,C}) \mid i \in I\}) \quad := \quad \bigwedge_{i \in I} \varepsilon^A_{\mathfrak{l}_A}(\sigma_{i,A}) \bullet \varepsilon^B_{\mathfrak{l}_B}(\sigma_{i,B}) \bullet \varepsilon^C_{\mathfrak{l}_C}(\sigma_{i,C}) \tag{163}$$

In the same way we have that $\mathfrak{S}_A \widetilde{\otimes} (\mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C)$ is defined as the quotient of $\mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C)$ by the congruence relation defined for any $u_{ABC}, u'_{ABC} \in \mathscr{P}(\mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C)$

$$(u_{ABC} \approx_{A(BC)} u'_{ABC}) \quad :\Leftrightarrow \quad (\forall \mathfrak{l}_{BC} \in \widetilde{E}_{BC}, \mathfrak{l}_A \in \mathfrak{E}_A, \quad \nu_{\mathfrak{l}_A,\mathfrak{l}_{BC}}(u_{ABC}) = \nu_{\mathfrak{l}_A,\mathfrak{l}_{BC}}(u'_{ABC})) \tag{164}$$

$$\Leftrightarrow \quad (\forall \mathfrak{l}_A \in \mathfrak{E}_A, \mathfrak{l}_B \in \mathfrak{E}_B, \mathfrak{l}_C \in \mathfrak{E}_C, \quad \nu_{\mathfrak{l}_A,\mathfrak{l}_B,\mathfrak{l}_C}(u_{ABC}) = \nu_{\mathfrak{l}_A,\mathfrak{l}_B,\mathfrak{l}_C}(u'_{ABC})). \tag{165}$$

The announced equality is then proved. □

We can then define a multiple tensor product of spaces of states.

Definition 22. The set $\mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)})$ is equipped with the Inf semi-lattice structure $\cup$ and with the following Inf semi-lattice morphisms defined for any $(\mathfrak{l}^{(j)})_{i \in J}$ with $\mathfrak{l}^{(j)} \in \mathfrak{E}^{(j)}$:

$$\begin{array}{rccc} \nu_{(\mathfrak{l}^{(j)})_{j \in J}} & : & \mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)}) & \longrightarrow & \mathfrak{B} \\ & & \{(\sigma_i^{(j)})_{j \in J} \mid i \in I\} & \mapsto & \bigwedge_{i \in I} \bigodot_{j \in J} \varepsilon_{\mathfrak{l}^{(j)}}(\sigma_i^{(j)}) \end{array} \tag{166}$$

where we have used the symbol $\bigodot$ to denote the multiple $\bullet$ product.

Definition 23. $\mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)})$ is equipped with a congruence relation defined between any two elements $u, u' \in \mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)})$ by

$$u \approx u' \quad :\Leftrightarrow \quad \forall (\mathfrak{l}^{(j)})_{j \in J} \in \prod_{j \in J} \mathfrak{E}^{(j)}, \ \nu_{(\mathfrak{l}^{(j)})_{j \in J}}(u) = \nu_{(\mathfrak{l}^{(j)})_{j \in J}}(u'). \tag{167}$$

Definition 24. The multiple tensor product $\widetilde{\bigotimes}_{j \in J} \mathfrak{S}^{(j)}$ is defined as the quotient of the set $\mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)})$ under the congruence relation $\approx$.

$$\forall \sigma \in \mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)}), \qquad \widetilde{\sigma} := \{u \mid \sigma \approx u\}. \tag{168}$$

The element $\widetilde{u} \in \widetilde{S} := \widetilde{\bigotimes}_{j \in J} \mathfrak{S}^{(j)}$ associated to the element $u := \{(\sigma_i^{(j)})_{j \in J} \mid i \in I\} \in \mathscr{P}(\prod_{j \in J} \mathfrak{S}^{(j)})$ will be denoted $\prod^{\tilde{S}}_{i \in I} \widetilde{\bigotimes}_{j \in J} \sigma_i^{(j)}$.

Theorem 20. Let us introduce the following notation

$$\mathscr{K}_I^{(J)} \quad := \quad \{(K^{(j)})_{j \in J} \mid (K^{(j)} \subseteq I, \ \forall j \in J) \text{ and } (K^{(j)} \cap K^{(j')} = \varnothing, \ \forall j, j' \in J) \text{ and } (\bigcup_{j \in J} K^{(j)} = I)\}. \tag{169}$$

The poset structure on $\widetilde{S} := \widetilde{\bigotimes}_{j \in J} \mathfrak{S}^{(j)}$ is defined according to

$$\begin{aligned} &(\prod^{\tilde{S}}_{i \in I} \widetilde{\bigotimes}_{j \in J} \sigma_i^{(j)}) \sqsubseteq_{\tilde{S}} \widetilde{\bigotimes}_{j \in J} \sigma^{(j)} \\ &\qquad \Leftrightarrow (\forall (K^{(j)})_{j \in J} \in \mathscr{K}_I^{(J)}, \ \exists j \in J \mid (K^{(j)} \neq \varnothing \text{ and } \sigma^{(j)} \sqsupseteq_{\mathfrak{S}^{(j)}} \prod^{\mathfrak{S}^{(j)}}_{k \in K^{(j)}} \sigma_k^{(j)})) \end{aligned} \tag{170}$$

■

Proof. The proof follows the same line as in Lemma 10. □

4.5 Comparison of the two tensor product constructions

Lemma 13. For any $\widetilde{\sigma}$ in $\widetilde{S}_{AB}$, $\langle\widetilde{\sigma}\rangle$ is a bi-filter of $\mathfrak{S}_A\times\mathfrak{S}_B$ and we have explicitly

$$\begin{aligned}\langle\textstyle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}(\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\rangle &= \{(\sigma_A,\sigma_B)\mid(\textstyle\bigsqcap_{k\in I}^{\mathfrak{S}_A}\sigma_{k,A})\sqsubseteq_{\mathfrak{S}_A}\sigma_A \text{ and } (\textstyle\bigsqcap_{m\in I}^{\mathfrak{S}_B}\sigma_{m,B})\sqsubseteq_{\mathfrak{S}_B}\sigma_B \text{ and}\\ &\quad \Big(\forall\varnothing\subsetneq K\subsetneq I,\ (\textstyle\bigsqcap_{k\in K}^{\mathfrak{S}_A}\sigma_{k,A})\sqsubseteq_{\mathfrak{S}_A}\sigma_A \text{ or } (\textstyle\bigsqcap_{m\in I-K}^{\mathfrak{S}_B}\sigma_{m,B})\sqsubseteq_{\mathfrak{S}_B}\sigma_B\Big)\}. \quad (171)\\ &= \Big\{(\sigma_A,\sigma_B)\mid\exists\mathscr{K},\mathscr{K}'\subseteq 2^I \text{ with } \mathscr{K}\cup\mathscr{K}'=2^I,\mathscr{K}\cap\mathscr{K}'=\varnothing,\{\varnothing\}\in\mathscr{K}',I\in\mathscr{K},\\ &\quad (\textstyle\bigsqcup_{K\in\mathscr{K}}^{\mathfrak{S}_A}\bigsqcap_{k\in K}^{\mathfrak{S}_A}\sigma_{k,A})\sqsubseteq_{\mathfrak{S}_A}\sigma_A \text{ and } (\textstyle\bigsqcup_{K'\in\mathscr{K}'}^{\mathfrak{S}_A}\bigsqcap_{m\in I-K'}^{\mathfrak{S}_B}\sigma_{m,B})\sqsubseteq_{\mathfrak{S}_B}\sigma_B\Big\}. \quad (172)\end{aligned}$$

We will also use the following notation $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}:=\langle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}(\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\rangle$. ■

Proof. From Definition 14 and Lemma 10 we deduce immediately the expression (171).
Let us now check the bi-filter properties.
The property (103) is trivially obtained from the expression (171).
Let us now consider that $(\sigma'_{1,A},\sigma'_B),(\sigma'_{2,A},\sigma'_B)\in\langle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}(\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\rangle$. In other words, we have for any $\mathfrak{l}_A\in\mathfrak{E}_A$ and $\mathfrak{l}_B\in\mathfrak{E}_B$: $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{1,A},\sigma'_B))\geq\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\})$ and $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{2,A},\sigma'_B))\geq\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\})$. Moreover, we have proved in (139) that $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{1,A},\sigma'_B))\wedge\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{2,A},\sigma'_B))=\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{1,A}\sqcap_{\mathfrak{S}_A}\sigma'_{2,A},\sigma'_B))$. As a consequence, we obtain $\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}((\sigma'_{1,A}\sqcap_{\mathfrak{S}_A}\sigma'_{2,A},\sigma'_B))\geq\nu^{AB}_{\mathfrak{l}_A,\mathfrak{l}_B}(\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\})$ for any $\mathfrak{l}_A\in\mathfrak{E}_A$ and $\mathfrak{l}_B\in\mathfrak{E}_B$. As a result, we obtain that $(\sigma'_{1,A}\sqcap_{\mathfrak{S}_A}\sigma'_{2,A},\sigma'_B)\in\langle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}(\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\rangle$.
We have then proved property (104).
The property (105) is proved along the same lines.
The expression (172) is a trivial reformulation of (171). □

Definition 25. We denote $\widetilde{S}^{fin}_{AB}$ the sub-poset of $\widetilde{S}_{AB}$ defined as follows :

$$\widetilde{S}^{fin}_{AB} := \{\widetilde{u}\mid u\subseteq_{fin}\mathfrak{S}_A\times\mathfrak{S}_B\}. \quad (173)$$

It is also a sub- Inf semi-lattice of $\widetilde{S}_{AB}$.

Theorem 21. We have the following obvious property relating the partial orders of $\widetilde{S}^{fin}_{AB}$ and S_{AB}. For any $\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}\subseteq_{fin}\mathfrak{S}_A\times\mathfrak{S}_B$,

$$(\textstyle\bigsqcap_{i\in I}^{S_{AB}}\sigma_{i,A}\otimes\sigma_{i,B})\sqsubseteq_{S_{AB}}\sigma'_A\otimes\sigma'_B \quad\Rightarrow\quad (\textstyle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\sqsubseteq_{\widetilde{S}_{AB}}\sigma'_A\widetilde{\otimes}\sigma'_B. \quad (174)$$

■

Proof. We intent to prove $\mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}\subseteq\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}$ for any $\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}\subseteq_{fin}\mathfrak{S}_A\times\mathfrak{S}_B$ (we recall that we have adopted the notation $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}:=\langle\bigsqcap_{i\in I}^{\widetilde{S}_{AB}}(\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})\rangle$).
First of all, it is recalled from Lemma 13 that $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}$ is a bi-filter.
Secondly, it is easy to check that $(\sigma_{k,A},\sigma_{k,B})\in\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}$ for any $k\in I$ using the expression (171). Indeed, for any $K\subseteq I$, if $k\in K$ we have $(\bigsqcap_{l\in K}^{\mathfrak{S}_A}\sigma_{l,A})\sqsubseteq_{\mathfrak{S}_A}\sigma_{k,A}$ and if $k\notin K$ we have $(\bigsqcap_{m\in I-K}^{\mathfrak{S}_B}\sigma_{m,B})\sqsubseteq_{\mathfrak{S}_B}\sigma_{k,B}$.
As a conclusion, and by definition of $\mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}$ as the intersection of all bi-filters containing $(\sigma_{i,A},\sigma_{i,B})$ for any $i\in I$, we have then $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}\supseteq\mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B})\mid i\in I\}$.
We now use Lemma 8 and Definition 14 to obtain the announced result. □

Theorem 22. If $\mathfrak{S}_A$ or $\mathfrak{S}_B$ are distributive, then $\widetilde{S}^{fin}_{AB}$ and S_{AB} are in fact isomorphic posets.
As shown in Remark 6, the distributivity of $\mathfrak{S}_A$ or $\mathfrak{S}_B$ is a key condition for this isomorphism to be valid. ■

Proof. We now suppose that $\mathfrak{S}_A$ or $\mathfrak{S}_B$ is distributive and we intent to prove that $\mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\} = \widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\}$ for any $\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\} \subseteq_{fin} \mathfrak{S}_A \times \mathfrak{S}_B$.
Let us prove the following fact : every bi-filter F which contains $(\sigma_{k,A},\sigma_{k,B})$ for any $k\in I$ contains also $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\}$. In fact, we can show that, for any bi-filter F we have

$$(\forall k\in I,\ (\sigma_{k,A},\sigma_{k,B})\in F) \ \Rightarrow\ (\textstyle\bigsqcup^{\mathfrak{S}_A}_{K\in\mathscr{K}} \bigsqcap^{\mathfrak{S}_A}_{k\in K} \sigma_{k,A}, \bigsqcup^{\mathfrak{S}_A}_{K'\in\mathscr{K}'} \bigsqcap^{\mathfrak{S}_B}_{m\in I-K'} \sigma_{m,B}) \in F,$$
$$\forall \mathscr{K},\mathscr{K}' \subseteq 2^I, \mathscr{K}\cup\mathscr{K}' = 2^I, \mathscr{K}\cap\mathscr{K}' = \varnothing, \{\varnothing\}\in\mathscr{K}', I\in\mathscr{K}. \quad (175)$$

The first step towards (175) is obtained by checking that $\forall \mathscr{K},\mathscr{K}' \subseteq 2^I, \mathscr{K}\cup\mathscr{K}' = 2^I, \mathscr{K}\cap\mathscr{K}' = \varnothing, \{\varnothing\}\in\mathscr{K}', I\in\mathscr{K}$,

$$(\textstyle\bigsqcup^{\mathfrak{S}}_{K'\in\mathscr{K}'} \bigsqcap^{\mathfrak{S}}_{m\in I-K'} \sigma_m) \sqsupseteq_{\mathfrak{S}} (\bigsqcap^{\mathfrak{S}}_{K\in\mathscr{K}} \bigsqcup^{\mathfrak{S}}_{k\in K} \sigma_k) \quad (176)$$

for any distributive $\mathfrak{S}$ and any collection of elements of $\mathfrak{S}$ denoted σ_k for $k\in I$ for which these two sides of inequality exist. To check this fact, we have to note that, using [8, Lemma 8 p. 50], we have first of all

$$(\textstyle\bigsqcap^{\mathfrak{S}}_{K\in\mathscr{K}} \bigsqcup^{\mathfrak{S}}_{k\in K} \sigma_k) = \bigsqcup^{\mathfrak{S}} \left\{ \bigsqcap^{\mathfrak{S}}_{K\in\mathscr{K}} \pi_K(A) \mid A\in \prod_{K\in\mathscr{K}} K \right\}, \quad (177)$$

where π_K denotes the projection of the component indexed by K in the cardinal product $\prod_{K\in\mathscr{K}} K$. Moreover, for any $A\in\prod_{K\in\mathscr{K}} K$, there exists $L\in\mathscr{K}'$ such that $\bigcup\{\pi_K(A) \mid K\in\mathscr{K}\} \supseteq (I\smallsetminus L)$ and then $(\bigsqcap^{\mathfrak{S}}_{K\in\mathscr{K}}\pi_K(A)) \sqsubseteq_{\mathfrak{S}} (\bigsqcap^{\mathfrak{S}}_{m\in I-L} \sigma_m) \sqsubseteq_{\mathfrak{S}} (\bigsqcup^{\mathfrak{S}}_{K'\in\mathscr{K}'} \bigsqcap^{\mathfrak{S}}_{m\in I-K'} \sigma_m)$. As a result, we obtain the property (176).

The second step towards (175) consists in showing that

$$(\forall k\in I,\ (\sigma_{k,A},\sigma_{k,B})\in F) \ \Rightarrow\ (\textstyle\bigsqcup^{\mathfrak{S}_A}_{K\in\mathscr{K}} \bigsqcap^{\mathfrak{S}_A}_{k\in K} \sigma_{k,A}, \bigsqcap^{\mathfrak{S}_B}_{K\in\mathscr{K}} \bigsqcup^{\mathfrak{S}_B}_{k\in K} \sigma_{k,B}) \in F \quad (178)$$

for any $\mathscr{K}\subseteq 2^I$. This intermediary result is obtained by induction on the complexity of the polynomial $(\bigsqcup^{\mathfrak{S}_A}_{K\in\mathscr{K}} \bigsqcap^{\mathfrak{S}_A}_{k\in K} \sigma_{k,A})$ by using the following elementary result

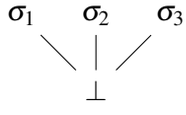
$$\forall \sigma_A,\sigma'_A\in\mathfrak{S}_A, \sigma_B,\sigma'_B\in\mathfrak{S}_B, \quad \big((\sigma_A,\sigma_B),(\sigma'_A,\sigma'_B)\in F\big) \ \Rightarrow\ \begin{cases} (\sigma_A \sqcup_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcap_{\mathfrak{S}_B} \sigma'_B)\in F \\ (\sigma_A \sqcap_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcup_{\mathfrak{S}_B} \sigma'_B)\in F \end{cases}$$

trivially deduced using the bi-filter character of F, i.e. properties (103)(104)(105).

As a final conclusion, using the explicit definition of $\mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\}$ as the intersection of all bi-ideals containing $(\sigma_{k,A},\sigma_{k,B})$ for any $k\in I$, we obtain $\widetilde{\mathfrak{F}}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\} = \mathfrak{F}\{(\sigma_{i,A},\sigma_{i,B}) \mid i\in I\}$.

$\widetilde{S}^{fin}_{AB}$ and S_{AB} are then isomorphic posets. □

Remark 6. We note that the distributivity property is a key condition to obtain previous isomorphism between $\widetilde{S}^{fin}_{AB}$ and S_{AB}. Indeed, let us consider that $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are both defined as the lattice associated to the following Hasse diagram:



According to (123), we have $(\bot_{\mathfrak{S}_A},\bot_{\mathfrak{S}_B}) \in \widetilde{\mathfrak{F}}\{(\sigma_1,\sigma_1),(\sigma_2,\sigma_2),(\sigma_3,\sigma_3)\}$. However, we have obviously $(\bot_{\mathfrak{S}_A},\bot_{\mathfrak{S}_B}) \notin \mathfrak{F}\{(\sigma_1,\sigma_1),(\sigma_2,\sigma_2),(\sigma_3,\sigma_3)\}$.

4.6 Remarkable properties of the tensor product

Theorem 23. Let $\widetilde{\sigma}_{AB}$ and $\widetilde{\sigma}'_{AB}$ be two elements of $\widetilde{S}_{AB}$ having a common upper-bound. Then the supremum of $\{\widetilde{\sigma}_{AB}, \widetilde{\sigma}'_{AB}\}$ exists in $\widetilde{S}_{AB}$ and its expression is given by

$$\widetilde{\sigma}_{AB} \sqcup_{\widetilde{S}_{AB}} \widetilde{\sigma}'_{AB} = \bigsqcap_{\widetilde{\sigma}\in \underline{\widetilde{\sigma}_{AB}}\cap\underline{\widetilde{\sigma}'_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma} \tag{179}$$

■

Proof. As long as $\widetilde{\sigma}_{AB}$ and $\widetilde{\sigma}'_{AB}$ have a common upper-bound, $\underline{\widetilde{\sigma}_{AB}} \cap \underline{\widetilde{\sigma}'_{AB}}$ is not empty.
Secondly, it is clear that $\widetilde{\sigma}_{AB} = (\bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma}) \sqsubseteq_{\widetilde{S}_{AB}} \bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}_{AB}}\cap\underline{\widetilde{\sigma}'_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma}$ and $\widetilde{\sigma}'_{AB} = (\bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}'_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma}) \sqsubseteq_{\widetilde{S}_{AB}} \bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}_{AB}}\cap\underline{\widetilde{\sigma}'_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma}$.
Then, if we suppose there exists $\widetilde{\sigma}''_{AB}$ such that $\widetilde{\sigma}_{AB}, \widetilde{\sigma}'_{AB} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{\sigma}''_{AB}$ we can use Theorem 18 to obtain the decomposition $\widetilde{\sigma}''_{AB} = (\bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}''_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma})$ with necessarily $\forall \widetilde{\sigma} \in \underline{\widetilde{\sigma}''_{AB}}$, $\widetilde{\sigma}_{AB} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{\sigma}$ and $\widetilde{\sigma}'_{AB} \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{\sigma}$, i.e. $\widetilde{\sigma} \in \underline{\widetilde{\sigma}_{AB}} \cap \underline{\widetilde{\sigma}'_{AB}}$, and then $(\bigsqcap_{\widetilde{\sigma}\in\underline{\widetilde{\sigma}_{AB}}\cap\underline{\widetilde{\sigma}'_{AB}}}^{\widetilde{S}_{AB}} \widetilde{\sigma}) \sqsubseteq_{\widetilde{S}_{AB}} \widetilde{\sigma}''_{AB}$. □

Theorem 24. If $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are distributive (cf. Definition 1), then $\widetilde{S}_{AB}$ is also distributive.
Note, using Theorem 22, that, in this situation, we have also $\widetilde{S}^{fin}_{AB} = S_{AB}$.
In that case, the explicit expression for the supremum of two elements in $\widetilde{S}^{fin}_{AB}$ is given by

$$(\bigsqcap_{i\in I}^{\widetilde{S}_{AB}} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}) \sqcup_{\widetilde{S}_{AB}} (\bigsqcap_{j\in J}^{\widetilde{S}_{AB}} \sigma'_{j,A} \widetilde{\otimes} \sigma'_{j,B}) = \bigsqcap_{i\in I,\, j\in J}^{\widetilde{S}_{AB}} (\sigma_{i,A} \sqcup_{\mathfrak{S}_A} \sigma'_{j,A}) \widetilde{\otimes} (\sigma_{i,B} \sqcup_{\mathfrak{S}_B} \sigma'_{j,B}). \tag{180}$$

■

Proof. First of all, using Theorem 22, we note that, as soon as $\mathfrak{S}_A$ or $\mathfrak{S}_B$ is distributive, we have $\widetilde{S}_{AB} = S_{AB}$ as Inf semi-lattices. We are then reduced to prove the distributivity of S_{AB}.
In reference to the definition of distributivity of an Inf semi-lattice given in Definition 1, we have then to prove that if $\bigsqcap_{1\le i\le n}^{S_{AB}} \sigma_{i,A}\otimes\sigma_{i,B} \sqsubseteq_{S_{AB}} \sigma_A\otimes\sigma_B$, then there exists $\sigma'_{i,A}\otimes\sigma'_{i,B} \sqsupseteq_{S_{AB}} \sigma_{i,A}\otimes\sigma_{i,B}$ for any $1 \le i \le n$ such that $\bigsqcap_{1\le i\le n}^{S_{AB}} \sigma'_{i,A}\otimes\sigma'_{i,B} = \sigma_A\otimes\sigma_B$. From Lemma 9, we conclude that it is sufficient to prove that, for any $n-$ary polynomial p, if $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A})$ and $\sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma_{1,B}, \cdots, \sigma_{n,B})$, then there exist $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ and $\sigma'_{i,B} \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}$ for $1 \le i \le n$ such that $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A})$ and $\sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma'_{1,B}, \cdots, \sigma'_{n,B})$, and $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ and $\sigma'_{i,B} \sqsupseteq_{\mathfrak{S}_B} \sigma_B$ for $1 \le i \le n$.
The proof of this fact is sketched in [14, Theorem 3], and we give here a developed version of it.

Let us prove the following statement for any $n-$ary polynomial p :

$$\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A}) \;\Rightarrow\; \exists \sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}, \forall 1 \le i \le n \mid \Big(\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A}) \text{ and } \sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A, \forall 1 \le i \le n\Big). \tag{181}$$

This statement is obviously true for $p(\sigma_{1,A}, \cdots, \sigma_{n,A}) := \sigma_{k,A}$, it suffices to chose $\sigma_{k,A} = \sigma_A$.

Let us assume that the induction statement is true for two $n-$ary polynomials p and q, and let us prove the statement is also true for $(p \sqcap q)$.
We will assume $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A}) \sqcap_{\mathfrak{S}_A} q(\sigma_{1,A}, \cdots, \sigma_{n,A})$. Then, there exist $\gamma_A, \delta_A \in \mathfrak{S}_A$ such that $\sigma_A \sqsupseteq_{\mathfrak{S}_A} (\gamma_A \sqcap_{\mathfrak{S}_A} \delta_A)$ and $\gamma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A})$ and $\delta_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \cdots, \sigma_{n,A})$.
From distributivity of $\mathfrak{S}_A$, we deduce that there exist γ'_A and δ'_A such that $\sigma_A = (\gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A)$ and $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} \gamma_A$ and $\delta'_A \sqsupseteq_{\mathfrak{S}_A} \delta_A$. As a result, we have $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A})$ and $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \cdots, \sigma_{n,A})$.
By assumption, there exist $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \le i \le n$ with $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A})$ and $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \cdots, \sigma''_{n,A})$, and with $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \delta'_A$ for $1 \le i \le n$.
Let us denote $\overline{\sigma}_{i,A} := \sigma'_{i,A} \sqcap_{\mathfrak{S}_A} \sigma''_{i,A}$.
We first note that $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \le i \le n$.

From $\overline{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma'_{i,A}$ and $\overline{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma''_{i,A}$ for any $1 \leq i \leq n$, and $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A})$ and $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \cdots, \sigma''_{n,A})$, we deduce $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$ and $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$. As a consequence, $\sigma_A = (\gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A) \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A}) \sqcap_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$.
From $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \delta'_A$ for $1 \leq i \leq n$, we deduce also $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A = \sigma_A$ for $1 \leq i \leq n$.
As a summary, there exist $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \leq i \leq n$, such that $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A}) \sqcap_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$, and $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ for $1 \leq i \leq n$. In other words, the $n-$ary polynomial $(p \sqcap q)$ satisfies also the induction assumption.

Let us assume that the induction statement is true for two $n-$ary polynomials p and q, and let us now prove the statement is also true for $(p \sqcup q)$.
We will assume $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A}) \sqcup_{\mathfrak{S}_A} q(\sigma_{1,A}, \cdots, \sigma_{n,A})$. Then, we have $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \cdots, \sigma_{n,A})$ and $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \cdots, \sigma_{n,A})$.
By assumption, there exist $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \leq i \leq n$ with $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A})$ and $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \cdots, \sigma''_{n,A})$, and with $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ for $1 \leq i \leq n$.
Let us denote $\overline{\sigma}_{i,A} := \sigma'_{i,A} \sqcap_{\mathfrak{S}_A} \sigma''_{i,A}$.
We first note that $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \leq i \leq n$.
From $\overline{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma'_{i,A}$ and $\overline{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma''_{i,A}$ for any $1 \leq i \leq n$, and $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \cdots, \sigma'_{n,A})$ and $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \cdots, \sigma''_{n,A})$, we deduce $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$ and $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$. As a consequence, $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A}) \sqcup_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$.
From $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ and $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ for $1 \leq i \leq n$, we deduce also $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ for $1 \leq i \leq n$.
As a summary, there exist $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$ for $1 \leq i \leq n$, such that $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A}) \sqcup_{\mathfrak{S}_A} q(\overline{\sigma}_{1,A}, \cdots, \overline{\sigma}_{n,A})$, and $\overline{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$ for $1 \leq i \leq n$. In other words, the $n-$ary polynomial $(p \sqcup q)$ satisfies also the induction assumption.

By induction on the complexity of the $n-$ary polynomial p we have then proved the statement. As a final consequence, S_{AB} and then also $\widetilde{S}_{AB}$ is a distributive Inf semi-lattice.

As a consequence of this distributivity property, we obtain the following simplification

$$\left(\prod_{i\in I}^{\widetilde{S}_{AB}} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}\right) \sqcup_{\widetilde{S}_{AB}} \left(\prod_{j\in J}^{\widetilde{S}_{AB}} \sigma'_{j,A} \widetilde{\otimes} \sigma'_{j,B}\right) \;=\; \prod_{i\in I}^{\widetilde{S}_{AB}} \prod_{j\in J}^{\widetilde{S}_{AB}} \left((\sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}) \sqcup_{\widetilde{S}_{AB}} (\sigma'_{j,A} \widetilde{\otimes} \sigma'_{j,B})\right). \tag{182}$$

Using the expansion (123), we know that

$$(\sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}) \sqcup_{\widetilde{S}_{AB}} (\sigma'_{j,A} \widetilde{\otimes} \sigma'_{j,B}) \;=\; (\sigma_{i,A} \sqcup_{\mathfrak{S}_A} \sigma'_{j,A}) \widetilde{\otimes} (\sigma_{i,B} \sqcup_{\mathfrak{S}_B} \sigma'_{j,B}) \tag{183}$$

This concludes the proof of the formula (180). □

Theorem 25. If $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are atomic, then $\widetilde{S}_{AB}$ is also atomic, i.e.

$$\exists \mathscr{A}_{\widetilde{S}_{AB}} \subseteq \widetilde{S}_{AB} \quad | \quad \forall \alpha_{AB} \in \mathscr{A}_{\widetilde{S}_{AB}},\ (\bot_{\mathfrak{S}_A} \widetilde{\otimes} \bot_{\mathfrak{S}_B}) \sqsubset_{\widetilde{S}_{AB}} \alpha_{AB}, \tag{184}$$

$$\forall \sigma_{AB} \in \widetilde{S}_{AB},\ \exists \alpha_{AB} \in \mathscr{A}_{\widetilde{S}_{AB}} \mid \alpha_{AB} \sqsubseteq_{\widetilde{S}_{AB}} \sigma_{AB}. \tag{185}$$

Here, we denote $\sigma \sqsubset \sigma'$ iff $(\sigma \sqsubset \sigma'$ and$(\sigma \sqsubseteq \sigma'' \sqsubseteq \sigma' \Leftrightarrow (\sigma = \sigma''$ or $\sigma'' = \sigma')))$.
The set of atoms of $\widetilde{S}_{AB}$ is indeed defined by

$$\mathscr{A}_{\widetilde{S}_{AB}} \;:=\; \{(\alpha_A \widetilde{\otimes} \bot_{\mathfrak{S}_B}) \sqcap_{\widetilde{S}_{AB}} (\bot_{\mathfrak{S}_A} \widetilde{\otimes} \alpha_B) \mid \alpha_A \in \mathscr{A}_{\mathfrak{S}_A},\ \alpha_B \in \mathscr{A}_{\mathfrak{S}_B}\}. \tag{186}$$

■

Proof. Using the expansion (123), we deduce immediately

$$\forall \alpha_A \in \mathscr{A}_{\mathfrak{S}_A}, \forall \alpha_B \in \mathscr{A}_{\mathfrak{S}_B}, \qquad (\alpha_A \widetilde{\otimes} \bot_{\mathfrak{S}_B}) \sqcap_{\widetilde{S}_{AB}} (\bot_{\mathfrak{S}_A} \widetilde{\otimes} \alpha_B) \not\sqsubseteq_{\widetilde{S}_{AB}} \bot_{\mathfrak{S}_A} \widetilde{\otimes} \bot_{\mathfrak{S}_B}. \tag{187}$$

In other words, $\bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B} \sqsubset_{\tilde{S}_{AB}} (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)$.

Secondly, let us show that, for any $\sigma_{AB} := (\prod_{i\in I}^{\tilde{S}_{AB}}\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})$ distinct from $\bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B}$, there exist $\alpha_A \in \mathscr{A}_{\mathfrak{S}_A}$ and $\alpha_B \in \mathscr{A}_{\mathfrak{S}_B}$ such that $((\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)) \sqsubseteq_{\tilde{S}_{AB}} \sigma_{AB}$. Using once again the expansion (123), we know that $\sigma_{AB} \sqsupset_{\tilde{S}_{AB}} \bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B}$ (or, in other words, $\sigma_{AB} \not\sqsubseteq_{\tilde{S}_{AB}} \bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B}$) implies that there exists $\varnothing \subseteq K \subseteq I$ such that $(\prod_{k\in K}^{\mathfrak{S}_A}\sigma_{k,A}) \sqsupset_{\mathfrak{S}_A} \bot_{\mathfrak{S}_A}$ and $(\prod_{m\in I-K}^{\mathfrak{S}_B}\sigma_{m,B}) \sqsupset_{\mathfrak{S}_B} \bot_{\mathfrak{S}_B}$. Let us fix such a K and let us choose $\alpha_A \in \mathscr{A}_{\mathfrak{S}_A}$ and $\alpha_B \in \mathscr{A}_{\mathfrak{S}_B}$ such that $(\prod_{k\in K}^{\mathfrak{S}_A}\sigma_{k,A}) \sqsupseteq_{\mathfrak{S}_A} \alpha_A$ and $(\prod_{m\in I-K}^{\mathfrak{S}_B}\sigma_{m,B}) \sqsupseteq_{\mathfrak{S}_B} \alpha_B$. We obtain $(\prod_{i\in K}^{\tilde{S}_{AB}}\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B}) \sqsupseteq_{\tilde{S}_{AB}} (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})$ and $(\prod_{i\in I-K}^{\tilde{S}_{AB}}\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B}) \sqsupseteq_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)$. As a first conclusion, we obtain $((\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)) \sqsubseteq_{\tilde{S}_{AB}} \sigma_{AB}$.

Thirdly, let us consider $\sigma_{AB} := (\prod_{i\in I}^{\tilde{S}_{AB}}\sigma_{i,A}\widetilde{\otimes}\sigma_{i,B})$ such that $\sigma_{AB} \sqsubseteq_{\tilde{S}_{AB}} (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)$. As a first case, we may have obviously $\sigma_{AB} = \bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B}$. If however $\sigma_{AB} \neq \bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B}$, the previous result implies that there exist $\alpha'_A \in \mathscr{A}_{\mathfrak{S}_A}$ and $\alpha'_B \in \mathscr{A}_{\mathfrak{S}_B}$ such that $((\alpha'_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha'_B)) \sqsubseteq_{\tilde{S}_{AB}} \sigma_{AB}$. Using once again the expansion (123), we deduce immediately that $\alpha_A = \alpha'_A$ and $\alpha_B = \alpha'_B$. As a result, we obtain

$$\sigma_{AB} \sqsubseteq_{\tilde{S}_{AB}} (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B) \quad\Rightarrow\quad \Big(\sigma_{AB} = \bot_{\mathfrak{S}_A}\widetilde{\otimes}\bot_{\mathfrak{S}_B} \text{ or } \sigma_{AB} = (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)\Big). \quad (188)$$

As a second conclusion, we then obtain $\bot_{\tilde{S}_{AB}} \overline{\sqsubset}_{\tilde{S}_{AB}} (\alpha_A\widetilde{\otimes}\bot_{\mathfrak{S}_B})\sqcap_{\tilde{S}_{AB}} (\bot_{\mathfrak{S}_A}\widetilde{\otimes}\alpha_B)$. □

Lemma 14. If the space of states $\mathfrak{S}$ is orthocomplemented, then the spaces of states $\mathfrak{B}\widetilde{\otimes}\mathfrak{S}$ and $\mathfrak{S}\widetilde{\otimes}\mathfrak{B}$ are orthocomplemented.
The star map defined on $\widetilde{S} := \mathfrak{B}\widetilde{\otimes}\mathfrak{S}$ will be denoted $\star$ as well. This star map is built according to

$$(z_1\widetilde{\otimes}z_2)^\star := z_1^\star\widetilde{\otimes}\bot\sqcap_{\tilde{S}}\bot\widetilde{\otimes}z_2^\star, \qquad \forall z_1\widetilde{\otimes}z_2 \in \widetilde{S}^{pure}, \quad (189)$$

$$(\prod^{\tilde{S}} U)^\star := \prod^{\tilde{S}}_{\sigma'\in\bigcap_{\sigma\in U}\underline{\sigma^\star}}\sigma', \qquad \forall U \subseteq \widetilde{S}^{pure}. \quad (190)$$

We have the same formulas for $\mathfrak{S}\widetilde{\otimes}\mathfrak{B}$. ■

Proof. The main point to check is the property (52) for the star map on $\widetilde{S}$. Precisely, we have to check that $u := z_1\widetilde{\otimes}z_2\sqcap_{\tilde{S}} z_1\widetilde{\otimes}z'_2$, $v := z_1\widetilde{\otimes}z_2\sqcap_{\tilde{S}} z_1^\star\widetilde{\otimes}z''_2$ and $w := z_1^\star\widetilde{\otimes}\bot\sqcap_{\tilde{S}}\bot\widetilde{\otimes}z_2^\star$ satisfy $\widehat{uw}^{\tilde{S}}$ and $\widehat{vw}^{\tilde{S}}$ for any $z_1 \in \mathfrak{B}$ and $z_2, z'_2, z''_2 \in \mathfrak{S}$ with $z_2 \neq z'_2$.
$\widehat{uw}^{\tilde{S}}$ is obtained quite easily as follows. We have (i) $z_1\widetilde{\otimes}z_2\sqcap_{\tilde{S}} z_1\widetilde{\otimes}z'_2 = z_1\widetilde{\otimes}(z_2\sqcap_{\mathfrak{S}} z'_2)$ and (ii) $(z_2\sqcap_{\mathfrak{S}} z'_2)$ and $z_2^\star$ have a common upper-bound denoted z (because of the property (52) applied to $\mathfrak{S}$). From (i) and (ii) we deduce that u and w have $z_1\widetilde{\otimes}z$ as common upper-bound.
$\widehat{vw}^{\tilde{S}}$ is obtained because $z_1^\star\widetilde{\otimes}z''_2$ is a common upper-bound of v and w. □

Theorem 26. If the space of states $\mathfrak{S}$ is orthocomplemented, then the space of states $\overbrace{\mathfrak{B}\widetilde{\otimes}\cdots\widetilde{\otimes}}^{\omega}\mathfrak{S}\overbrace{\widetilde{\otimes}\cdots\widetilde{\otimes}\mathfrak{B}}^{\omega}$ is also orthocomplemented. ■

Remark 7. Let us consider the following orthocomplemented space of states

$$S_4 := \quad \begin{array}{ccccc} \alpha_1 & \alpha_2 & & \alpha_1^\star & \alpha_2^\star \\ & \searrow & & \swarrow & \\ & & \bot & & \end{array} \quad (191)$$

It is easy to check that $\widetilde{S'} := S_4 \widetilde{\otimes} S_4$ is NOT orthocomplemented. Indeed, from the result above, the single candidate for $(\alpha_1 \widetilde{\otimes} \alpha_1)^\star$ is obviously $\alpha_1^\star \widetilde{\otimes} \perp \sqcap_{\widetilde{S'}} \perp \widetilde{\otimes} \alpha_1^\star$. However, we check immediately, using the expansion (123), that the two elements $\alpha_1^\star \widetilde{\otimes} \perp \sqcap_{\widetilde{S'}} \perp \widetilde{\otimes} \alpha_1^\star$ and $\alpha_1 \widetilde{\otimes} \alpha_1 \sqcap_{\widetilde{S'}} \alpha_2 \widetilde{\otimes} \alpha_2$ have no common upper-bound : this point contradicts the condition (52) for the definition of $\star$ on $\widetilde{S'}$.

Remark 8. According to [10, Axiom 9 and Lemma 40], we can introduce the following notion : the space of states $\mathfrak{S}$ is said to be *irreducible* iff

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^{pure}, \qquad \{\sigma_1, \sigma_2\} \subsetneq \underline{\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2}, \tag{192}$$

otherwise, $\mathfrak{S}$ is said to be *reducible*.
Then, it is important to remark that, even if $\mathfrak{S}_A$ and $\mathfrak{S}_B$ are both irreducible, the tensor product $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ appears to be always reducible. Indeed,

$$\forall \sigma_1, \sigma_1' \in \mathfrak{S}_A^{pure}, \forall \sigma_2, \sigma_2' \in \mathfrak{S}_B^{pure} \; | \sigma_1 \neq \sigma_1', \sigma_2 \neq \sigma_2', \qquad \underline{\sigma_1 \widetilde{\otimes} \sigma_2 \sqcap_{\widetilde{S}} \sigma_1' \widetilde{\otimes} \sigma_2'} = \{\sigma_1 \widetilde{\otimes} \sigma_2, \sigma_1' \widetilde{\otimes} \sigma_2'\}. \tag{193}$$

5 Conclusion

Inspired by the *operational quantum logic program*, we have the contention that probabilities can be viewed as a derived concept, even in a reconstruction program of Quantum Mechanics. The already cited remark of S. Abramsky [1, Theorem 4.4] can be viewed as another justification of this perspective on quantum mechanics. These two perspectives have stimulated our desire to build an operational description based on a possibilistic semantic (in a sense, the 'probabilities' are replaced by statements associated to a semantic domain made of three values 'indeterminate', 'definitely YES', 'definitely NO'). The present paper intents to develop such an operational formalism. It will be called Generalized possibilistic Theory (GpT) as it is partly inspired by the formalism of Generalized Probabilistic Theory (GPT). We note that we are also indebted to the work of Abramsky [1] for our choice to give to Chu duality a central role in our construction, in replacement of traditional duality between states and effects.
Section 2 is devoted to a brief summary of the axiomatic relative to the space of states (subsection 2.1), the space of effects (subsection 2.2), the set of pure states (subsection 2.3), and the notion of "channels" or symmetries for our theory (subsection 2.4). This section collects some elements already developed in our previous work [10]. The convexity requirements imposed traditionally in GPT on the space of states and space of effects are naturally replaced by Inf semi-lattice structures on these spaces in GpT, the set of pure states being naturally associated to completely meet-irreductible elements of the space of states. Our central point is the Chu duality imposed between the space of states and the space of effects, with an evaluation space given by the three elements domain associated to possibilistic statements of the observer. This Chu duality is sufficient to deduce the whole set of properties of the channels which are viewed as Chu morphisms. Section 3 and 4 are dedicated to the construction of bipartite experiments on compound systems. This point is central because it has been the main obstacle on the pathway towards a complete reconstruction of quantum mechanics along the operational quantum logic program. The central problem in our perspective is the construction of a tensor product for our space of states and space of effects. It is well known that this tensor product notion is ambiguous in GPT program [22, Section 5]. The traditional construction of tensor product of Inf semi-lattices should have been of some help for our work [13], it is succinctly recalled in subsection 4.1 and called *basic tensor product*. The tensor product, naturally build from the Chu construction [23], could also have played a role here. Surprisingly, the natural axiomatic for bipartite experiments, proposed in subsection 3.1, imposes a completely new construction for the tensor product of Inf semi-lattices, called *maximal tensor product* and presented in subsection 4.2. The comparison between maximal and basic tensor product is made in subsection 4.5 and some remarks concerning the specific properties of the maximal tensor product are made in subsection 4.6. The construction of the bipartite space of states and of the bipartite space of effects is achieved in subsection 4.3 and the construction of the symmetries associated to the bipartite space of states is completed in subsection 3.2

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