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The mathematical background of proving processes in discrete optimization— Exemplification with Research Situations for the Classroom

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Abstract. Discrete mathematics brings interesting problems for teaching and learning proof, with accessible objects such as integers (arithmetic), graphs (modeling, order) or polyominoes (geometry). Many problems that are still open can be explained to a large public. The objects can be manipulated by simple dynamic operations (removing, adding, ‘gluing’, contracting, splitting, decomposing, etc.). All these operations can be seen as tools for proving. In this paper we particularly explore the field of ‘discrete optimization’. A theoretical background is defined by taking two main axes into account, namely, the epistemological analysis of discrete problems studied by contemporary researchers in discrete optimization and the design of didactical situations for classrooms in the frame of the Theory of Didactical Situations. Two problems coming from ongoing research in discrete optimization (the Pentamino Exclusion and the Eight Queens problems) are developed and transposed for the classroom. They underscore the learning potentialities of discrete mathematics and epistemological obstacles concerning proving processes. They emphasize the understanding of a necessary condition and a sufficient condition and

problematize the difference between optimal and optimum. They provide proofs involving partitioning strategies, greedy algorithms but also primal-dual methods leading to the concept of duality. The way such problems can be implemented in the classroom is described in a collaborative study by mathematicians and mathematics education researchers (*Maths à Modeler* Research Federation) through the Research Situations for the Classroom project.

Keywords: discrete mathematics, optimization, proving processes, research situations for the classroom, polyominoes, eight queens problem.

1. Introduction

The literature in mathematics education concerning proof has produced fundamental findings regarding the epistemological, philosophical and didactical aspects of proof (e.g., Hanna & de Villiers, 2012; Hanna et al., 2010; Mariotti et al., 2018; Stylianides, 2016). In all the research, the fields of arithmetic, number theory and combinatorics are often used. Such discrete mathematics clearly favors an “authentic mathematical activity” (Lampert, 1990) with work on heuristics in non-routine problem solving (Polyá, 1954). It also brings a new beginning for students and teachers (Goldin, 2010). The concepts involved (mainly integers) are not an obstacle to the solving process, the students “come to the problem fresh” (Maher et al., 2010), and the proof is therefore central.

All the research dealing specifically with the teaching of discrete mathematics at different school levels has pointed out the advantages of discrete mathematics for the teaching and learning of proof (e.g., Grenier & Payan, 1998, 1999; Ouvrier-Buffer, 2020; Rosenstein et al., 1997; Heinze et al., 2004; Hart & Sandefur, 2018). The resolution of various kinds of problems (existence, characterization, recognition, optimization and extremal problems) mobilizes notions and skills such as implication, induction, and several proving processes, as follows: proofs by exhaustion, proofs by induction, algorithmic proofs, generic proofs, but also probabilistic methods, decomposition techniques, and structuration of objects (such as coloration in graph theory). We choose the field of discrete optimization¹, less studied in itself in mathematics education than other fields of discrete mathematics

¹ Discrete optimization is a branch of optimization in applied mathematics and computer science that deals mainly with problems where one has to construct an optimal solution from a finite (or countable) number of possibilities.

(see for instance the main view of the teaching and learning of discrete mathematics worldwide by Hart & Sandefur, 2018 or Heinze et al., 2004). It is also an ongoing contemporary field with proving processes and open problems in constant evolution.

Few of the above-quoted researchers have characterized precisely the proving processes that are specific to discrete optimization from the epistemological point of view. Moreover, the seeming accessibility of discrete concepts does not mean that solving discrete problems and searching proofs is easy. Thus, the didactical transposition of such proving processes in classrooms must be investigated.

Therefore, we addressed two unexplored research questions, as follows: What are the features of proving processes in discrete optimization? Are these proving processes accessible for students from the perspective of didactical transposition?

In this paper, we present the background to the study, designed collaboratively by discrete mathematicians and didacticians in the *Maths à Modeler* Research Federation, involving the didactical features of Research Situations for the Classroom and the Theory of Didactical Situations (Section 2). Then, we develop two problems in discrete optimization (derived from the *Pentamino Exclusion* problem and the *Eight Queens* problem) (Section 3) and their transpositions for the classrooms (Section 4). From an epistemological point of view, this allows us to deal with an existence and a non-existence problem and to highlight and characterize proving processes which are transversal to discrete mathematics and specific to discrete optimization, namely, framing optimal values, using necessary and sufficient proof, reasoning with local *versus* global optimality, developing algorithms or partitioning strategies, and using duality and primal-dual methods (Sections 5, 6, and 7). This then provides a frame for these proving processes and the epistemological obstacles they imply for an educational perspective. We use students' work coming from several experiments designed with a common methodology, in order to highlight the accessibility of such proving processes. The conclusion opens perspectives regarding the teaching and learning of proof through discrete optimization and the perspective of Research Situations for the Classroom.

2. Definition, design and implementation of Research Situations for the Classroom with the Theory of Didactical Situations

Since 1999, the French *Maths à Modeler* Research Federation has promoted collaborative work between mathematicians in discrete mathematics and mathematics education researchers. We design innovative Research Situations for the Classroom from real problems coming from ongoing mathematical research (e.g., Grenier & Payan, 1999, 2003; Ouvrier-Bufferet, 2009). The aims of Research Situations for the Classroom are to put students in the role of researchers and to engage them in a proving process, or more generally in an inquiry process, including several transversal skills such as conjecturing, modeling, defining, etc. The main idea is also to foster a mathematical community in which perspectives are close to the sociomathematical norms where the criteria for acceptability of arguments are negotiated in the classroom (e.g., Lampert, 1990; Yackel & Cobb, 1996). Besides this, the social dimension of exploratory proving remains fundamental (e.g., Balacheff, 2010; Stylianides, 2016).

Research Situations for the Classroom must fulfill several criteria. Some of them are close to problem solving and to the French tradition of open problems (Arsac & Mante, 2007). The Theory of Didactical Situations (Brousseau, 1997) is used for the design, the analysis and the implementation of Research Situations in the Classroom. We present the main components of this theoretical background before defining the criteria of such situations.

2.1 The use of the Theory of Didactical Situations

An epistemological study of mathematical problems in discrete mathematics gives us a great variety of techniques and proving processes. Therefore, in a didactical perspective, we must make choices about certain points. In particular, we have to identify the variables that are characteristic of the problems, to choose values of such variables and to predict the effects of these choices on students' reasoning and proving processes. Brousseau (1997) uses the term *didactical variables* "to designate the command variables which will be shown to have an important qualitative effect on the evolution of the procedures" (p. 176). The characterization of the potential proving processes and the identification of the variables of the mathematical problems studied allow us to select values for some didactical

variables in order to engage students in certain proving processes and to influence some changes in their proving processes. We can also let students search for the variables of the problem they propose: we call such variables *research variables*. Their choice can lead to a difficult or unsolvable problem or, on the contrary, it can facilitate the *devolution* of the problem. The devolution happens when the teacher makes the students accept the responsibility for a problem. In the case of RSCs, a specific *research contract* is implemented to avoid the bias of the usual *didactical contract*.

In Brousseau's engineering, the situation is divided into three didactical phases (*action, formulation, validation*) that we follow when implementing Research Situations in the Classroom to "foster the discovery and demonstration (...)" (ibid., p. 4). The interactions between the student(s) and the environment (*milieu*) constitutes the *dialectic of action*. The *action* phase engages students in the exploration of the problem, often with manipulatives. This production of a set of experiences and results should be rich enough to nourish the collective discussion in the next phase. The *dialectic of formulation* consists in "progressively establishing a shared language", making "possible the explanation of actions and modes of action". It has an explicit social dimension. During the *validation* phase, students "have to put forward propositions and to prove to an opponent that they are either true or false" (p. 4).

2.2 The definition of Research Situations for the Classroom

Taking into account the above developments and the frame of *Maths à Modeler*, Research Situations for the Classroom must fulfill the eight following criteria (derived from Grenier & Payan, 1999, 2003; Ouvrier-Bufferet, 2009, 2020).

- Firstly, the research field should be 'huge', and have roots in mathematical ongoing research: a large part of it should be accessible to the students.
- Secondly, the mathematical problem should be easily understood.
- Thirdly, there should be few notional traps to start the research. Initial strategies without prerequisites should exist.

These first three criteria foster the *devolution* of the mathematical problem and the dialectic of *action*, as well as the fact that nobody (neither the students nor the teacher) knows the solution.

- Fourthly, the students should manage their research themselves: among the *didactical variables*, at least one of them, a *research variable*, is left to the

students, while the others can be set by the teacher. These variables are chosen depending on the didactical interest of the questions they can generate for the mathematical activity.

- Fifthly, many strategies should be available to put the research forward, and several developments should be possible to enable the research process and the emergence of mathematical skills (such as proof, modeling etc.) and knowledge.
- Sixthly, there are criteria of local resolution (local ending criteria) and possibly no final ending: an answered question often leads to a new question.
- Seventhly, at the end of the research process, students are invited to share their results, e.g., through posters in their school or a presentation at university (in front of mathematicians and other school students, who are also there to present their results).
- Eighthly, we underline that research is an activity that requires a large amount of time: a long-term situation is essential if one expects students to do research.

The implementation of Research Situations in the Classroom should preserve the dialectics of *action*, *formulation* and *validation* (Grenier & Payan, 2003). The closure of a Research Situation with a presentation by students of their results ends the research process of the students and contributes to the *institutionalization* process. For this paper, we focus on the proving processes in discrete optimization, the design aspect, and on the accessibility of such proving processes for students.

3. Two mathematical problems in discrete optimization

We chose a minimization problem and a maximization problem, derived from the *Pentamino Exclusion* problem and the *Eight Queens* problem. We focus specifically on the following topics when identifying the proving processes:

- Concepts of bound, framing optimal values, and in particular, lower and upper bounds *versus* necessary conditions and sufficient conditions;
- Minimal, minimum; local *versus* global optimality;
- Dual problems; and
- Algorithmic approach and heuristics.

3.1 Hunting the Beast! based on the Pentamino Exclusion problem

The *Hunting the beast!* problem involves polyominoes popularized by the game Tetris and the puzzles of Martin Gardner. A polyomino is a shape “made by connecting certain numbers of equal-sized squares, each joined together with at least 1 other square along an edge” (Golomb, 1966, p. 19).

Here is the usual way to present the problem to students (Figure 1) in *Maths à Modeler*:

Your garden is a collection of adjacent squares and a beast is a collection of squares. Your objective is to prevent a beast from entering your garden. To do this, you can buy traps. A trap is represented by a single square that can be placed on any square of the garden. The question is: what is the minimum number of traps you need to place so that no beast can land on your garden? These rules allow the beast to be rotated (90,180 or 270 degrees) or reflected (flipped over) at will.

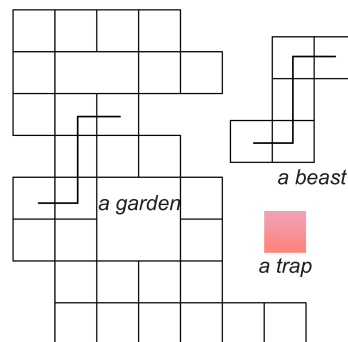


Fig. 1. A garden, a beast and a trap

In this problem, placing a trap on each square is clearly not optimal, except if the beast is reduced to a single square (monomino). The difficulty thus lies in finding a configuration with the *smallest* number of traps. In the literature, this problem belongs to *covering* problems and can be seen as a variation of the *Pentomino Exclusion* problem introduced by Golomb (1966). In the latter problem, the garden is a $k \times n$ rectangular board and the aim is to minimize the number of traps so that no pentomino can be placed on it. For both problems, it turns out that the computation of the minimum number of traps is NP-complete² in terms of algorithmic complexity (Gravier et al., 2007). However, for some special cases of the board and the beast, results exist concerning the minimum number of traps (Dorbec, 2007; Gravier & Payan, 2001). Such a problem leads to new research questionings in

² That is, “nondeterministic polynomial-time complete” (See Garey & Johnson, 1979).

graph theory (e.g., Gravier et al., 2007), such as new integer programming models that are still in progress (e.g., Kartak & Fabarisova, 2019).

3.2 Fireworks based on the *Eight Queens* problem

Here is the usual way to present the problem to students in *Maths à Modeler*:

In a warehouse, we try to store the maximum number of boxes of fireworks under the following constraint: if one box explodes, it does not damage any other box. For our purpose, the warehouse will be a subset of the grid. A box of fireworks consists of a single cell of the grid to which is attached a security zone. A box of fireworks placed in the warehouse can therefore prohibit all cells in its security zone (Figure 2).

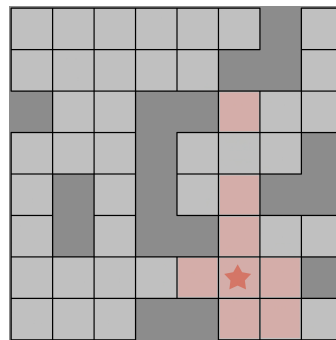


Fig. 2. Example of a warehouse, a box of fireworks (star) and a security zone (red colored squares)

If the warehouse is the 8×8 chessboard and the security zone of a box of fireworks placed on a cell is the set of cells in the same row, column and its two diagonals, then we get the so-called well-known *Eight Queens* problem. One attributes this problem to the German chessman Bezzel (1848). It was generalized for n queens by the mathematician Nauck (for historical references see Campbell, 1977). Lucas (1882) published the first complete solutions of this problem. Since then, several extensions of this problem have been studied (e.g., Simkin, 2021) such as dominating and stable sets in grid-like graphs, generalized Latin squares, etc.

3.3. Existence and non-existence problems to be addressed

Many problems of discrete optimization consist of finding a subset of a given *ground set* satisfying some given *constraints* with maximum or minimum cardinality. A *feasible solution* is simply a set satisfying the required constraints.

A feasible solution of *Hunting the beast!* is a set of cells of the garden (the ground set) where one places traps with the following constraint: each beast in the garden contains at least one cell with a trap (Figure 3).

For *Fireworks*, a feasible solution consists of storing a set of cells of the warehouse (the ground set) where one places boxes of fireworks with the following constraint: no placed box of fireworks belongs to the security zone of another (Figure 4).

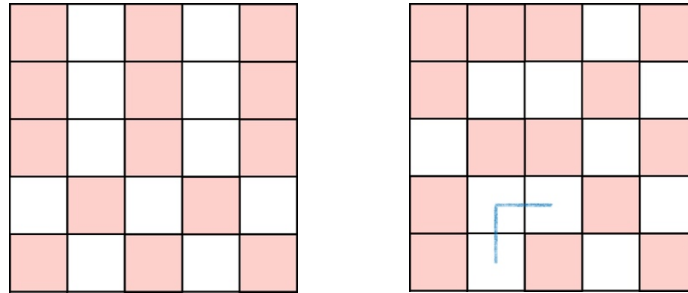


Fig. 3. A feasible solution [left] and a non-feasible solution [right] for *Hunting the beast!*

(L-tromino)

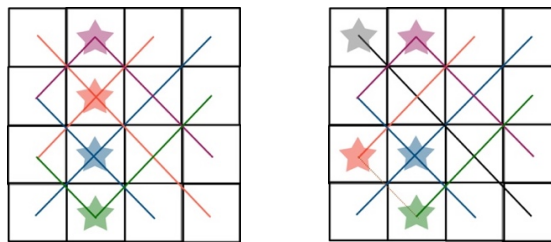


Fig. 4. A feasible solution [left] and a non-feasible solution [right] for *Fireworks*

(security zone defined by the bishop's move)

A *solution* of the optimization problem is a feasible solution with *optimal cardinality*. To solve our discrete optimization problems means to ‘find’ the *best* solution from all feasible solutions, i.e. a global optimum. Therefore, we have two distinct problems, as explicitly stated below³: finding a ‘good’ solution and proving its optimality.

Finding a feasible solution of cardinality k is an *existence* problem (called $\mathbf{P}_{\text{existence}, k}$):

- It means that there exists a configuration of k traps that is a feasible solution; one can exhibit a way of placing k traps in the garden so as to exclude the given polyomino. The goal is to find the smallest cardinality of the feasible

³ Their proofs are distinct, whereas in continuous optimization problems, their resolutions are often put together in an asymptotic argument.

solutions. Hence, for the minimization problem *Hunting the beast!*, finding a feasible solution with k traps gives an *upper bound* of that cardinality.

- Analogously, it gives a *lower bound* for a maximization problem (*Fireworks*).

Proving that a feasible solution of cardinality k is optimal is often a difficult *non-existence* problem (called $\mathbf{P}_{\text{non-existence, } k-1}$ i.e. every configuration of $k-1$ traps is not a feasible solution) which involves a universal quantifier (\forall). Indeed, in *Hunting the beast!* we have to prove that any set of size n , with $n < k$, is not a feasible solution. It means proving that no fewer traps could have been used for the same purpose. This leads to the *lower bound* (the optimal value is greater than or equal to k).

We now justify our choices for the design of Research Situations for the Classroom.

4. Towards Research Situations for the Classroom

An epistemological and mathematical study of the problem helps us to define variables and to predict proving processes. But an empirical and cognitive adjustment of the values of these variables is always necessary. Most of the PhD theses defended in *Maths à Modeler* and international publications (e.g., Grenier-Payan, 1998; Ouvrier-Buffer, 2011, 2020) offer Research Situations for the Classroom with *didactic engineering* (Artigue, 2014) in the Theory of Didactical Situations and an epistemological background in discrete mathematics. The topics of these research projects deal with combinatorial games (Colipan, 2014), implication and necessary and sufficient conditions (Deloustal-Jorrand, 2004), arithmetic and geometry (Dissa, 2020), modeling (Giroud, 2011), open-problems and research situations (Godot, 2005), algorithms (Modeste, 2012), and defining and proving processes (Ouvrier-Buffer, 2003). In all these works, several Research Situations for the Classroom were designed, analyzed, tested and implemented with a common methodology in different contexts, namely, elementary and secondary levels, university level, pre-service and in-service teacher training, popularization events, and also with autistic children in a medical environment (Coffin et al., 2006). We use students' work from these experiments to illustrate the accessibility of specific proving processes in discrete optimization. Although the mathematical concepts and reasoning are complex, we choose to illustrate how students from Grade 4 to Grade 7 approach them without prerequisites in discrete optimization or specific courses on proof.

4.1. The Research Situation *Hunting the beast!* for the Classroom

Presenting the problem in a general context leads to discussions to reduce the space of research. One may simplify the problem by setting the following didactical variables:

- The number of squares used to define the beast: naturally, hunting small beasts appears to be more accessible. A beast is connected but not necessarily convex.
- The shape of the beast plays an important part; for example, hunting a rectangle (whatever its size) is a problem that is solvable through elementary considerations depending on the size of the garden (it is easier to conceptualize such beasts in a squared board rather than beasts such as in

Figure 5). On the contrary, the problems may become harder if the beast is a non-connected set of squares. Several beasts can be given considering the topology of the beasts (Figure 5).

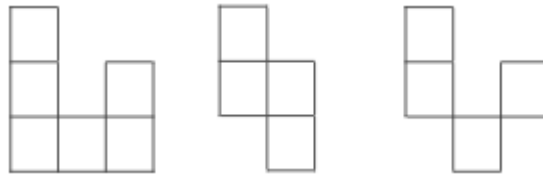


Fig. 5. A connected but not convex beast, a convex beast, a non-connected beast

- The number of types of beasts that are simultaneously excluded makes the research more complex; in the initial case, we consider only one type of beast. In the *Pentomino Exclusion Problem*, all the polyominoes of size 5 are hunted, which is very complex. After having considered each type of polyomino separately, excluding all of them simultaneously is also interesting.
- The size and the topology of the board are also important: it may be relevant to let the students work on larger sizes, or also play on rectangular boards.

During experiments (from elementary to university levels), we set the didactical variables in order to facilitate the devolution of the problem and an effective search of optimal configurations and proofs. It appears that playing on a rectangular board is reasonable. A first possible step may consist in letting the students choose a reasonable sized beast (consisting of less than 6 squares) and an 8×8 or a 7×7 board. The shape of the beast is then a research variable. After this first step, three kinds of polyominoes are chosen for the experiments in the classrooms to make the proving process evolve (see Sections 5 & 6). We usually choose a 5×5 garden, a monomino for a trap and three kinds of beast (Figure 6). We use this configuration in the following analysis of the proving processes at stake. Considering a 5×5 board is already enough to effect a nontrivial effective search of optimal configurations and to formulate arguments and proofs. It is also possible to open the problem with other values of the didactical variables and then to question the generalization of the results. For instance, starting from the 5×5 garden, we can generalize the results for rectangular gardens $n \times m$ with the straight tromino: the whole part of $nm/3$ is the optimal value (Dorbec, 2007).

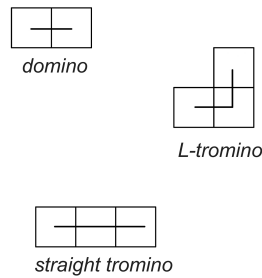


Fig. 6. The beasts

4.2. The Research Situation *Fireworks* for the Classroom

We propose simpler formulations of the problem so that it is accessible at elementary school. Even in the simplified formulations, interesting proving processes arise (Section 7). In a similar manner as for *Hunting the beast!*, the didactical variables of *Fireworks* concern the shape and the size of the warehouse and the characterization of the security zone. The topology of a box of fireworks can also be a didactical variable, but here we consider only a single cell.

From now on, the warehouse will be a $n \times n$ grid. To build a Research Situation for the Classroom based on this problem, we leave n as a research variable available to the students.

This situation is usually scheduled with a first step which enables the devolution. We ask students to set a (reasonable) security zone. Then, the problem become a *Fireworks*' problem on a warehouse (generally an 8×8 or 7×7 grid) with the security zone defined by students. During this step, symmetric properties on the security zone are discussed: for example, if the cell $(i, j+t)$ is in the security zone of cell (i, j) , then one may assume that cell $(i, j-t)$ also belongs to its security zone. Indeed, in a feasible solution, cells $(i, j-t)$ and (i, j) cannot occur simultaneously since, by definition of that security zone, (i, j) belongs to the security zone of $(i, j-t)$. This first step mainly allows us to identify the questions which are the existence problem and the non-existence problem (Section 3.3). As it appeared that finding answers for these problems seemed difficult with the original security zones, we proposed to study special security zones such as those of chess moves.

From now on, we consider the security zone defined by the bishop's move in chess, that is the set of cells belonging to the same diagonals of the place where the box of fireworks (bishop) is. Students often used a graphical register instead of a formal one to represent the security zone or the diagonals (Figure 7).

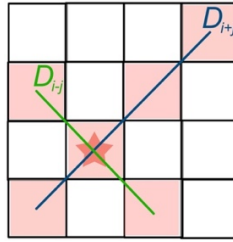


Fig. 7. The diagonals

4.3. Extensions of *Hunting the beast!* and *Fireworks* problems

For these two optimization problems, we chose certain values of the didactical variables in order to facilitate the devolution of the Research Situations to the students and to have an effect on the evolution of the students proving processes. We illustrate the details in the next sections.

We underscore here some other interesting values of the didactical variable (or research variable for the reader). For the *Hunting the beast!* problem, another perspective is to exclude both straight and L-trominoes.

For the *Fireworks* problem, we could propose other chess moves to investigate.

For both problems, we can try to generalize the results to a rectangular $n \times m$ board ($m \neq n$) (starting with a fixed small n). Here, reasoning by induction can be explored. We remind the reader that the general problems (i.e. without fixed values of the didactical variables) are still open in ongoing mathematical research.

In the following sections, we describe the main phases which occur in proofs when solving combinatorial optimization problems for the above-mentioned values of the didactical variables (Sections 4.1 and 4.2). We analyze the construction of *feasible solutions* ($\mathbf{P}_{\text{existence}}$) and the proof of the optimality of our best feasible solution or the proof of the bound of the optimal value ($\mathbf{P}_{\text{non-existence}}$) for *Hunting the beast!* (Sections 5 & 6) and *Fireworks* (Section 7). The described strategies are illustrated with the work of students. The aim is twofold, as follows: to reach an overview of the mathematical potentialities from the proof perspective of such optimization problems, and to point out the accessibility of proving processes in the classrooms. Of course, other interesting mathematical proofs exist, such as exploring rows and columns arguments (Ouvrier-Buffet et al., 2017) or covering numbers, but we focus on processes of reasoning that are transversal to discrete mathematics and often used in discrete optimization.

5. Existence problem: Construction of feasible solutions—*Hunting the beast!*

Local arguments and difference between minimal and minimum

To find a solution with n traps means to solve $\mathbf{P}_{\text{existence},n}$ but it is not necessarily an optimal solution of the problem. We need to show that the statements $\mathbf{P}_{\text{existence},n}$ and $\mathbf{P}_{\text{non-existence},n-1}$ are true in order to prove that n is the optimal value.

We remind the reader that we construct feasible solutions, i.e., we exhibit a way of placing a certain number of traps in the garden so as to exclude the given polyomino (existence problem).

One can develop local strategies of optimality when searching for the use of a minimum of traps. Additionally, the symmetry of the garden (*ground set*) suggests finding a symmetrical feasible solution as well. We give an example of such a strategy (which often occurs in the classrooms) with the straight tromino.

In a 5×5 garden, to exclude a beast from a single row, we need at least one trap. This trap must be placed on the middle square. The same is true for columns. Consider the first row and the last row by symmetry, on the one hand, and the first and the last columns on the other hand: we obtain the partial selection of the traps in Figure 8a. Still by symmetric argument, we choose the midpoint of the 5×5 garden (Figure 8b).

The cell (2;2) is one of the remaining cells of the garden which contains the greatest number of straight trominoes (4 as marked in Figure 8c), then we select this cell by a local optimality argument. Finally, by symmetry, we obtain the solution with 9 traps in Figure 8d.

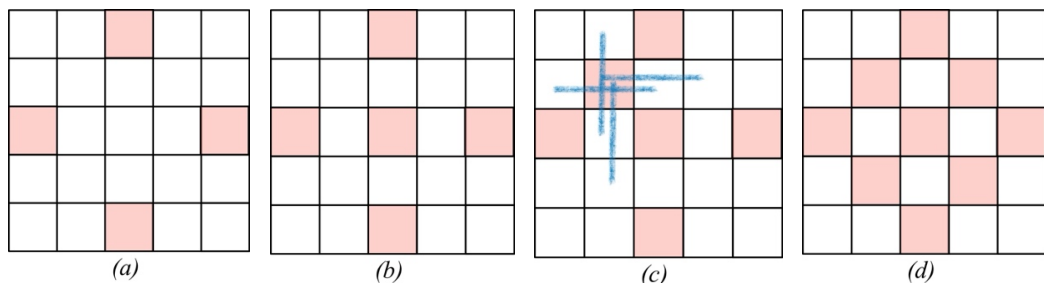


Fig. 8. Strategy based on symmetrical and local arguments

Such a strategy is usable for any shape of the beast. It generates a feasible solution which can have the following property of *minimality*: if one removes any trap from such a feasible solution, then one gets a set of traps which is not a feasible solution (Figure 9).

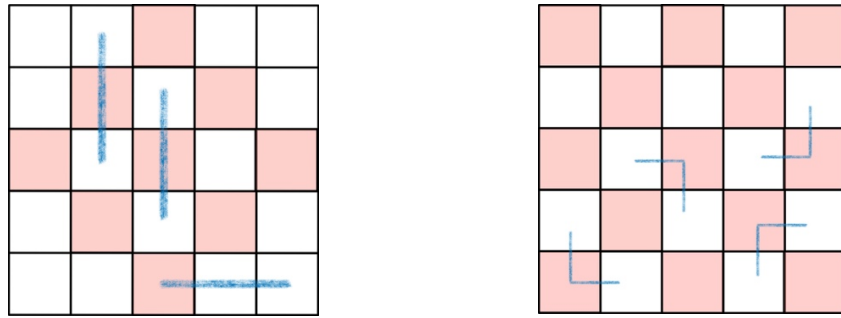


Fig. 9. Discussion of minimality of feasible solutions for the straight tromino (left) and the L-tromino (right): minimal but not minimum

This kind of argument is used by students to claim the optimality of a solution: “If I remove any trap, a beast can land in the garden. Therefore, my value is optimal.” But it only proves that the feasible solution cannot be improved by removing certain traps: it means that this feasible solution is (sometimes called *local optimum*) *minimal* but not necessarily *minimum*. Yet this does not prove it is optimal. To show that this reasoning is wrong, a counterexample built with another locally minimal placement of the traps, with more traps, is enough (Figure 10): such a counterexample may emerge during students’ discussions or be introduced by the teacher. It is unlikely that any student would think that the placement on the right (with 13 traps) is the best possible. This is an interesting way to discuss the difference between *minimal* and *minimum*.

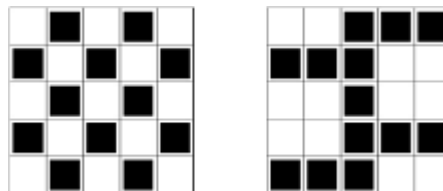


Fig. 10. Straight tromino is excluded with two locally minimal but non-optimal placements of the traps

Students may also be convinced that the above-mentioned argument fails because they get a better solution after a few trials. That is the case with the domino, when 7th graders found a feasible solution with 13 traps and the optimal solution with 12 traps (Figure 11). During their research, notice that their representation of the problem evolved, trying a packing (with some mistakes) (Figure 12).

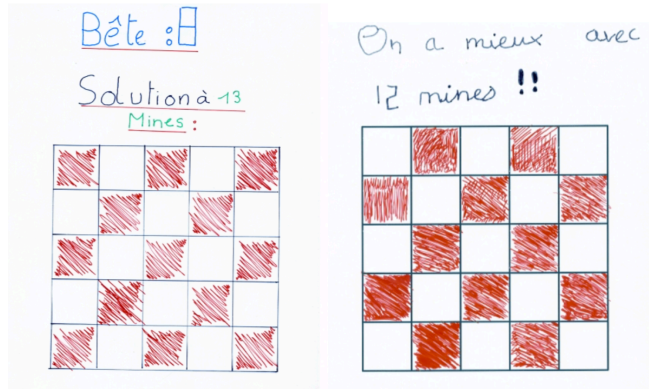


Fig. 11. 7th graders’ work on feasible solutions [domino]:
 “We found better with 12 traps!!”

1 mine par bête: $\boxed{\times}$ ou $\boxed{\times}$

| | | |
|---------------------|---------------------|---------------------|
| $\boxed{1 \times}$ | $\boxed{2 \times}$ | $\boxed{3 \times}$ |
| $\boxed{5 \times}$ | $\boxed{4 \times}$ | $\boxed{6 \times}$ |
| $\boxed{7 \times}$ | $\boxed{8 \times}$ | $\boxed{9 \times}$ |
| $\boxed{10 \times}$ | $\boxed{11 \times}$ | $\boxed{12 \times}$ |
| $\boxed{13 \times}$ | $\boxed{14 \times}$ | |

Fig.12. 7th graders’ packing attempt [domino]
 with some mistakes in the placement of the traps (“1 trap per beast”)

6. Non-existence problem: Proof of optimality— *Hunting the beast!*

In the previous section, we exhibited strategies to construct feasible solutions. Some of them are optimal (e.g., Figure 11, right). We now prove that no fewer traps could have been used for the same purpose “by whatever combinatorial reasoning or tricks suggest themselves” (Golomb, 1966, p. 42).

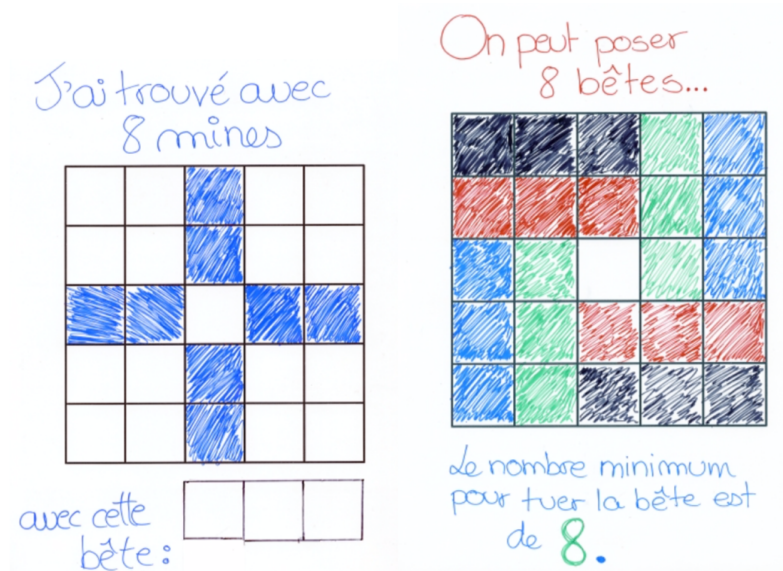
6.1. Proof by exhaustion (also called proof by cases, or by case analysis)

Examining all possible cases is always possible, mainly for small cases, but remains laborious and leads to a combinatorial explosion phenomenon in the general cases. It motivates the use of another kind of reasoning.

6.2. Necessary and sufficient proof—towards duality⁴

The manipulation of beasts brings the necessary proof, accessible in primary schools, and questions the sufficient proof. Observe that any beast in the garden forces us to place a trap on at least one of the cells occupied by it. This observation may suggest considering the *dual packing* problem, i.e., to fill all the garden with disjoint beasts.

For the domino (or, respectively, for the straight tromino), the packing of 12 dominoes (resp. 8 straight trominoes) in the garden proves that 12 traps (resp. 8) are necessary to cover all the nonoverlapping dominoes (resp. straight trominoes) of the garden. Then, a minimum of 12 traps (resp. 8) is necessary to exclude the domino (resp. straight tromino): it is a *lower bound*. To exhibit a placement of 12 traps (resp. 8) gives an *upper bound*. Since the two bounds coincide, then the optimal value is 12 (resp. 8). Domino (Figure 12) and straight tromino (Figure 13) can be seen as generic examples and can lead to a generic argument linked to the following proposition (which can be proved): ‘If one covers the garden with n non-overlapping beasts, then at least n traps are needed to exclude the beast’. For the L-tromino, this kind of proof is not enough. Indeed, only 8 L-trominoes cover the 5×5 garden, but 10 traps are needed to exclude the L-tromino. Then, the optimal value is framed between 8 and 10 (Figure 14).



⁴ We can point out that the word duality has the same meaning in geometry, algebra and combinatorics: it is linked to the notions of sphere covering, generating set and transversal on the one hand, and sphere packing, independent set and matching on the other hand.

Fig. 13. 7th graders' solution and packing [straight tromino]: "I found 8 traps with this beast." "One can put 8 beasts ... The minimum number to exclude the beast is 8."

In some experiments, it appears that after solving the problem with the straight tromino, the students started to solve the problem with the L-tromino by placing beasts in the garden. This is the case with 7th graders who have explored the problem intensively with manipulatives, and realized the interest of studying the dual problem when dealing with the trominoes, the L-tromino in particular. They claimed "The minimum of traps is 8 since there are 8 beasts. We have a solution with 10 traps and a tiling with 8 beasts. No luck! Best solution with 8, 9 or 10?" (Figure 14). Clearly, there is a confusion here between *necessary* and *sufficient* conditions: this is an epistemological obstacle. Then, one may ask students to give a solution with 8 traps. So, studying the L-tromino enables students to work on the differences between necessary and sufficient conditions.

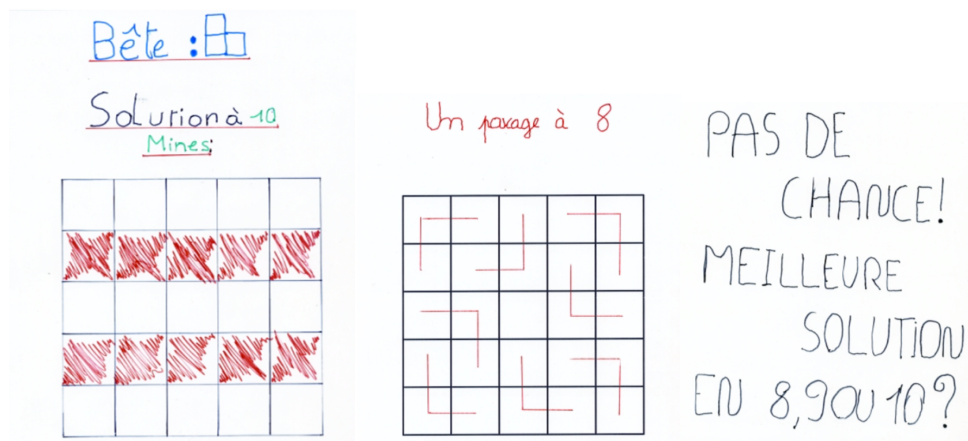


Fig. 14. 7th graders' frame of the optimal value [L-tromino] at the end of an experiment

It is particularly interesting to study clearly how the notions of upper and lower bounds are related to sufficient and necessary conditions. Indeed, determining that the optimal value is less than or equal to k shows that k elements are sufficient. Moreover, proving $P_{\text{non-existence}}$ shows that k elements are necessary. Indeed, for minimizing optimization problems, the lower bound (the optimal value is greater than or equal to k) shows that k elements are necessary to satisfy all *constraints*. Moreover, exhibiting a feasible solution of cardinality k shows that k elements are sufficient to satisfy all constraints.

From the combinatorial optimization point of view, the covering problem has a *dual* packing problem. This approach is usual when mobilizing duality theory in combinatorial optimization problems, which can be considered from two perspectives, called the *primal* and the *dual* problems.

No prerequisite in combinatorial optimization is required to understand the *weak duality* relationship that links the problem of placing the minimum number of traps (*primal* problem) so as to exclude all the beasts and the problem of placing the maximum number of non-overlapping beasts (*dual* problem), namely that any upper bound of the first provides a lower bound of the second.

Notice that methods to solve these two dual problems are rather different. In the *primal* problem (minimum number of traps so as to exclude all the beasts), we exhibit a configuration of n traps to exclude the beast to prove that the minimum number of traps is at most n . In the *dual* problem (maximum number of non-overlapping beasts), we exhibit a packing of m beasts to prove that the minimum number of traps is at least m . The non-existence problem is directly transformed, by duality, into an existence problem (find a packing of the ground set with beast)⁵. Thus, the universal quantifier is needed only to determine the constraints of the covering problem. For *Hunting the beast!* With the domino and the straight tromino, the bounds do coincide, the inequality is an equality. In that case, we say that the primal and dual problems have a *strong* duality relationship. Sometimes, these two bounds do not coincide: this is the case with the L-tromino (the minimum of traps is 10 whereas at most 8 beasts can be packed on the garden). In that case, we say that the primal and dual problems have a *weak* duality relationship. The L-tromino clearly leads to a shift in the proving process from the epistemological and didactical point of view (Ouvrier-Buffet, 2017 & 2021), thus the interest in choosing the L-tromino as a value of the didactical variable ‘shape of the beast’. The other values of this didactical variable are also justified. The domino allows the devolution of the problem, and the straight tromino fosters the reuse of actions and formulations produced when students explore dominoes: therefore, it helps students to stabilize their proving processes.

⁵ Packing and covering problems are existence problems for which we know effective procedures to get feasible solutions (see all the presented techniques for $P_{\text{existence}}$).

6.3. The L-tromino and the need to use another proving process: studying smaller cases—towards *cuts*

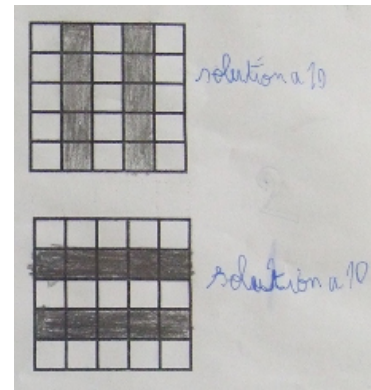
In any case, whenever the beast is the L-tromino, one may study smaller gardens. This allows us to remark that two traps are needed to keep the L-tromino off a 2×2 garden. Then, one trap is needed for the beast landed on this garden and one more trap on a cell not covered by a beast in order to keep another neighboring beast off the 2×2 garden. So, for each 2×2 garden, 2 traps are needed (Figure 15). Now, we consider a larger garden: if we focus on a 2×2 square included in this garden, then we may claim that we need at least 2 traps in this restricted 2×2 square. Therefore, we use the packing with 2×2 squares and L-trominoes (Figure 16). When one structures the 5×5 garden with 2×2 squares, four 2×2 squares ‘at worst’ appear and two other beasts can come: then, 4 times 2 traps and 2 traps are required, so 10 traps are sufficient in order to keep the beast off the 5×5 garden. We have solved *Hunting the beast!* on the 5×5 garden.

With the L-tromino, we here consider more restrictive constraints: 2 traps are necessary to exclude all the beasts off a 2×2 square. This technique of *adding new constraints* is classical in combinatorial optimization and can be seen in the case of linear programming as the *cutting plane method* introduced by Gomory (1960). The idea is to find a new problem for which the size of an optimal solution is equal to one in the initial problem. Therefore, we can link the previous reasoning to abstract reasoning, namely adding a set of valid inequalities using the *cut* given by the *constraints* on the 2×2 squares. Consider a 2×2 garden: at least two traps are required to exclude the L-tromino. Then, consider any packing of m L-tromino and m' 2×2 squares in the garden: at least $m+2m'$ traps are required to exclude the L-tromino (Figure 16).

This kind of reasoning is available even with young students (4–5th graders, see Ouvrier-Buffet et al., 2017). They claimed: “When we exclude the domino beasts, we find a solution with 12 traps, it is the best solution for now. The domino beasts, we can put 12 maximum; we need at least 1 trap per beast, so 12 dominoes, we need at least 12 traps. With the trominoes, 1 trap per beast does not work, you need 2 traps per beast.” So, they prove that 2 traps are necessary in the 2×2 garden, but they do not extend this argument to a larger garden. This extension of this argument appears to be an obstacle.

The L-tromino beast takes 2 lines. That allows us to find a solution with 10 traps: we have chosen one line out of two, avoiding the first one in order to have only two lines of five squares (and not 3×5 squares).

In a solution for L-trominoes, we need to use 2 traps for a beast.



Proof: for a four squares garden, we have:

(they draw this picture)

But sometimes, one trap is enough. We are sure that 8 is impossible because there are 8 beasts and sometimes 2 traps are required.



Fig. 15. 4–5th graders’ solution and arguments to exclude the L-tromino

The same kind of argument is extended by 7th graders to the 5×5 garden with a packing of L-trominos and 2×2 gardens (Figure 16): their representation highlights that 2 traps are required to exclude the L-tromino from a 2×2 square garden and 1 trap is required to exclude a L-tromino. They conclude that “At least 10 traps are required”.

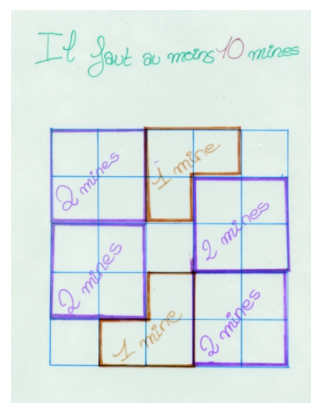


Fig. 16. 7th graders’ representation of the lower bound to exclude the L-tromino

6.4. A heuristic: the greedy algorithm

The *greedy algorithm* is a *heuristic*⁶. The definition we use here is a restrictive one: more general ones also exist (see for instance Bondy & Murty, 2008). A greedy

⁶ The word *heuristic* is often used in mathematics education as Polyá (1954) defined it. In problem-solving, it includes, for instance, analogy, generalization, induction, decomposing and

algorithm consists in building a feasible solution, element by element, following local choice at each step to try to reach the optimality of the solution obtained, i.e., always choosing the next element that offers an *optimal benefit* (it needs to be defined carefully for each problem). This approach never reconsiders the choices taken previously. To choose the next element, some heuristics may be used since there may be multiple choices that offer the same benefit. A greedy algorithm stops when we get a (feasible) solution with no more benefit. So, the validity of a greedy algorithm consists in proving that a solution with no benefit has to be a feasible solution. In many problems, it fails to give an optimal solution (for instance in problems like that of the travelling salesman).

For instance, in *Hunting the beast!*, ‘to choose a trap which excludes at least one beast and the most number of beasts not excluded by the other selected traps’ is a greedy strategy. To choose the next element, some heuristics may be used. For instance, in the case of a minimization problem, if, at some step, one has chosen a subset X of elements, then one chooses a new element e such that the set $X+e$ satisfies a maximum number of constraints. One can still get a minimal feasible solution which is not necessarily optimal even if at each step the heuristic chooses an optimal element. The solution can then be locally optimal⁷ but not necessarily globally optimal. It is not natural at all and appears as an epistemological obstacle: ‘Doing the best at each step does not guarantee the best at the end’.

In *Hunting the beast!*, as the problem is posed, the students first hunt the beast by placing some traps on the board. The students can use a procedure to build, step by step, a solution, until they get a feasible solution by repeatedly choosing a trap which excludes a beast not excluded by the other selected traps. That is the case when they put one domino in the garden and one trap to keep this beast off the garden and so on (Figure 17). It leads to a solution with one trap for each two squares to exclude the domino, and then to an optimal solution (12 traps) or a non-optimal one (13 traps) (Figure 11). This in-action process emerges during the

recombining, working backward. We will keep in mind the definition of *heuristic* coming from operations research, which is non-contradictory with that of Polyá: “In general, for a given problem, a heuristic procedure is a collection of rules or steps that guide one to a solution that may or may not be the best (optimal) solution” (Laguna & Marti, 2013).

⁷ Sometimes the heuristic gives a minimal or a maximal solution (as described in *Fireworks*, Section 6).

manipulation of the beasts and the traps. Nevertheless, the verbalization and the conceptualization of such a process is not self-evident (Figure 12).



Fig. 17. A manipulation following a greedy algorithm leading to an optimal (12 traps) or a non-optimal (13 traps) solutions in Grades 4–5

It is worth pointing out that greedy algorithms are intensively studied because they are simple and practical. A first application of such algorithms was used by Dijkstra (1959) for a graph theoretical problem. Mathematicians are currently trying to characterize those problems for which a greedy algorithm is effective. Matroids theory has attempted to provide an axiomatic framework describing the problems for which a greedy algorithm applied with any order of selection of the elements guarantees the optimality of the result (Welsh, 2010).

7. Existence and non-existence problems: Construction of feasible solutions and proof of optimality—*Fireworks*

We explore the above-described proving processes for the maximization problem *Fireworks* and point out their features: in this problem, the partitioning strategy and the primal-dual method are emphasized. The aim is still to underscore the mathematical underlying proving processes in such discrete optimization problems in order to give to the reader mathematical clues to deal with another optimization problems, and to enrich *didactical a priori analysis* (following Brousseau's definition). In this section, we do not integrate pictures of students' works; we explain them.

The problems we deal with are the following.

$\mathbf{P}_{\text{existence},n}$: an arrangement of n boxes that does not overlap security zones (sufficient condition, existential quantifier) exists; the optimum is at least n .
And $\mathbf{P}_{\text{non-existence}, n+1}$ is for every arrangement of $n+1$ boxes; one box is in another's security zone (universal quantifier). The optimum is at most n .

7.1. Construction of feasible solutions—Difference between maximal and maximum—Towards partitioning strategy

To generate a feasible solution for *Fireworks*, we can use a *heuristic* (i.e. a *local strategy*) consisting in ‘placing one box of fireworks, removing its security zone, choosing another one and iterating this process until there are no more cells’. This strategy is often used by students even if it is not explicitly stated. The choice of the first element is not necessarily justified. Some geometrical arguments may be proposed (corner or center) without discussing a clear relationship with the problem.

As for *Hunting the beast!*, a greedy algorithm can be used in *Fireworks* to build a feasible solution: it means, for instance, at each step, choosing a new firework which does not damage another one if it explodes, while trying to optimize this local choice. Notice that the above-described heuristic can be seen as such a greedy algorithm with no optimization. Another way to optimize such a local choice should define a benefit: for instance, to choose a new firework that does not damage another one in such a way that its added security zone covers as few new spaces as possible.

An example of a local strategy for the security zone defined by the bishop’s move on an 8×8 board is given in Figure 18. Then, with the above-mentioned heuristic, we obtain a *maximal* solution with 9 boxes of fireworks. Here, *maximal* means that if we add another box of fireworks, then we get a non-feasible set, i.e., this new box of fireworks will damage another one if it explodes. So, we get the first bound: the optimal value is greater than or equal to 9.

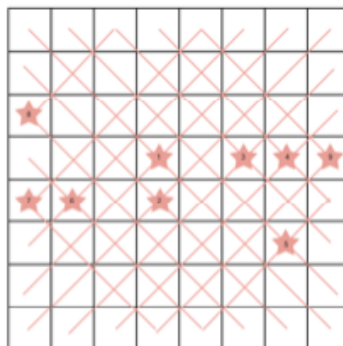


Fig. 18. Local strategy for bishop’s move on an 8×8 chessboard

A new investigation gives a solution with 14 bishops ($2n-2$ in the general case) in Figure 19. This allows us to discuss the difference between *maximal* and *maximum*.

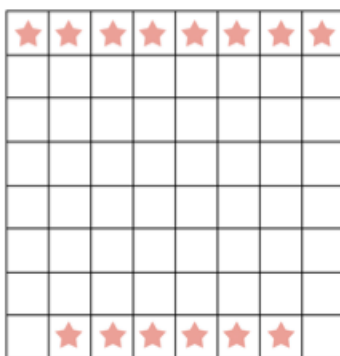


Fig. 19. Solution with 14 bishops

After more attempts, we fail to improve this solution, so the conjecture that we have reached the optimal value, 14, can be formulated.

7.2. The primal-dual method

We here present the *primal-dual method*, using *Fireworks* as an illustration. The primal-dual method, proposed by Egerváry in the early 1930s in combinatorial optimization and then by Dantzig, Ford, and Fulkerson in 1956 in linear programming, is nowadays modified to deal with NP-hard problems (Goemans & Williamson, 1996). From a didactical point of view, this approach should be carefully checked since it could reveal a confusion between necessary and sufficient conditions, as underscored above.

Remember that the primal problem consists in packing the maximum number of boxes in the ground set such that if one box explodes, it does not damage any other box. The dual problem consists in covering the ground set with regions (which may differ from the security zone of a box) such that at most one box can be placed on each region. Obviously, our optimum value cannot be larger than the number of regions. So, the dual is $\mathbf{P}^*_{\text{existence},n}$, i.e., there exists a covering of n regions that cannot contain more than one box. For the *Fireworks* problem defined by a king's move (e.g., Gandit et al., 2014), the regions are 2×2 squares; for a bishop's move, the regions are diagonals.

The study of different definitions of the security zone leads to the mobilization of several strategies. For example, when the security zone is bounded (not depending on n) such as for the king's move (left, right, up, down, diagonally), then it may induce a *partitioning strategy*. It consists in tiling the ground set (by translation) with tiles with one marked cell corresponding to the place of the king. Such a strategy can be described as follows: starting from a solution of the *dual* problem

(*covering*) which is also a *packing* (here, the tiling with 2×2 squares), one tries to find a solution of the *primal* problem where each region contains exactly one king. We have to make sure that this tiling provides a feasible solution. A way to ensure that we get a feasible solution is to consider the tiling defined by translations such that, for any case c and any translation used by the tiling, the case obtained by translation is not in the security zone of c . This strategy is illustrated for $n=8$ (Figure 20).

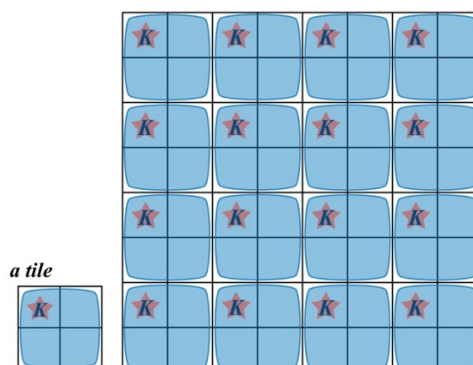


Fig. 20. Partitioning strategy to get a feasible solution with 16 kings

From this terminology, the partitioning strategy (for instance, the king's move above, or the bishop's move below) can be generalized: it leads to a *primal-dual method* by solving at the same time the *primal* and the *dual* problems. Such methods consist in using a dual solution to build a feasible solution of the primal problem which satisfies *optimally* each (or a maximum of) constraint(s) of the dual solution. We recall that for the bishop's move, we have built a solution with 14 bishops on the 8×8 board (Figure 19). After several attempts, we failed to improve this solution, so the conjecture that we reached the optimal value can be formulated for the *primal* problem. At this stage, the students tried a *case analysis*, but there were too many cases to consider, and they failed to find a full proof. One suggested studying the smallest warehouses. Therefore, a *partitioning strategy* could appear followed by work on the *dual covering* problem, that is, to place the smallest number of diagonals whose union covers the entire 8×8 board.

As for *Hunting the beast!* the students used *partitioning strategy* as a proof argument, but they did not see it as a new problem. Quickly, the students placed $2n-1$ such diagonals (Figure 21).

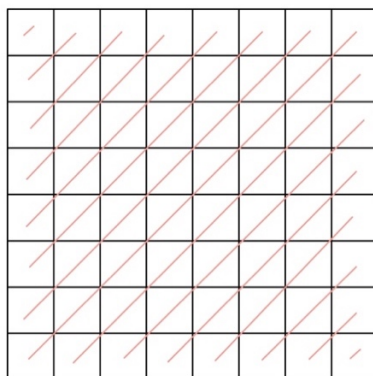


Fig. 21. $2n-1$ diagonals on a $n \times n$ board

So, we have the closest frame on optimal value with $2n-2$ and $2n-1$.

One can construct $2n-1$ non-overlapping diagonals to cover the board (Figure 21). Since one can put at most one box on each diagonal, it follows, by duality, that the highest number of boxes is $2n-1$. Since the two corner diagonals are just one square and the security zone of one contains that of the other, you cannot put a box on each, so the maximum is at most $2n-2$.

Finally, we get the desired upper bound $2n-2$, which corresponds to a necessary condition on the value for all solutions, therefore for the optimum.

Conclusion and perspectives

Discrete mathematics has huge potential for the design of learning situations involving proof and proving processes and, more generally, mathematical inquiry. Even if many discrete mathematics problems can be easily explained, most of them are really difficult to solve. A non-trivial selection of ‘good’ mathematical problems for classrooms has to be validated by epistemological and didactical *a priori* analysis, but also by several experiments in different classrooms in a long-term process.

Exploiting two contemporary problems, we have emphasized several proving processes used in discrete optimization. From the mathematical point of view, we have underscored specific proving processes and heuristics, including mainly the following: necessary and sufficient proof, partitioning strategies, greedy algorithms, primal-dual methods, weak and strong duality, and the cutting planes method. In such discrete problems, more classical proving processes such as proof by exhaustion or proof by induction can also be used. In all these proving processes, we found epistemological obstacles which lie in confusion or misunderstandings of necessary and sufficient conditions, of optimal and optimum, of local and global

optimality. To overcome these obstacles, we have pointed out solutions with illustrations involving the two optimization problems. In particular, necessary and sufficient conditions can make sense through the notions of lower and upper bound. Moreover, the definition of dual problems gives a general framework to obtain these bounds. The distinction between optimal and optimum appears during a problem of optimizing the cardinality of a set when the students are used to optimizing numerical functions. Again, the distinction between local and global optimality arises when the students produce feasible solutions with simple algorithms such as greedy ones.

Therefore, the field of discrete optimization is really promising for getting students to work on different proving processes and to overcome certain epistemological obstacles which are transversal to mathematics.

Using the Theory of Didactical Situations to transpose mathematical research activities developed in *Maths à Modeler*, we have designed two Research Situations for the Classroom (*Hunting the beast!* and *Fireworks*) derived from famous mathematical problems in discrete optimization (*Pentamino Exclusion* and the *Eight Queens* problems). Several experiments, mainly realized in action-research contexts but carefully planned and implemented with the help of the Theory of Didactical Situations, have strengthened our choices of didactical variables and revealed regularities in students' processes at all school levels (from primary school to university). Therefore, we have emphasized how students' proving processes are close to mathematical proofs with illustration of the work of 4th to 7th graders. In particular, the two Research Situations for the Classroom are particularly efficient for work on the distinction between necessary and sufficient conditions and on optimality (local *versus* global, optimal *versus* optimum). In the experiments, the proof of the sufficient condition which corresponds to solving an existence problem involved strategies, sometimes even algorithms, to produce a 'good' solution, i.e., a solution that students cannot improve. Greedy algorithms emerged in the students' work. They are also used by mathematicians. Students generally adopted a partitioning strategy to prove the necessary condition: this technique allowed them to prove that the previously obtained solution could not be improved. There are two clearly distinct ways to prove the sufficient condition and the necessary condition respectively. Nevertheless, the students did not see that their partitioning strategy could be seen as a new problem, i.e., the dual, and they did not detect this new

problem as the solution to the previous one. This points out the limits of didactical situations for proving processes and transversal processes and concepts such as primal-dual methods and duality. We also notice that the authors of some research studies concerning proof have pointed out the difficult shift in the validation process (e.g., Balacheff, 2010), the crucial role of the teacher's guidance (e.g., Mariotti et al., 2018; Stylianides, 2016), and the difficulties in *institutionalizing* (e.g., Stylianides, 2007). We have not developed all these aspects here. It leads us to point out new research questions. Indeed, in an educational context, and nowadays in the frame of *Inquiry Based Mathematics Education* (Artigue & Blomhøj, 2013), researchers in mathematics education must question the relevant skills at stake in a mathematical proving process and in a mathematical inquiry. Starting from the examples developed in this paper, the following new research questions emerge: How may one choose proving processes in order to teach for curricula in mathematics? How may one transpose such processes as are relevant in mathematical inquiry to the classroom? How may one institutionalize such processes, which *a priori* required a long-term process in the classroom? Finally, how may one redesign teacher training? Such questions are crucial at the international level both for discrete mathematics (Rosenstein, 2018) and mathematics in general.

It is obvious that collaborative work with research-active mathematicians can generate frameworks and tools for education at the interplay between mathematics education and mathematics (e.g., Alcock & Simpson, 2009). The research dealing with proof and ongoing mathematics using interviews with mathematicians (e.g., Lockwood et al., 2016; Ouvrier-Buffet, 2015; Weber, 2011) and the Research Situations for the Classroom endeavour, clearly open new perspectives for the teaching and learning of proving processes in discrete mathematics. The mathematical background developed in this paper allows the design of new didactic engineering in discrete optimization, including teacher training.

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