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Geodesic motion in Schwarzschild black hole with quintessence and string cloud background space-times

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Abstract

This paper analyzes the effective potential of time like and null observers in the vicinity of Schwarzschild black holes with quintessence field parameter $q$ and string cloud parameter $a$. Based on the effective potential, geodesics are mapped.

1 Introduction

The Schwarzschild metric [1], also referred to as the Schwarzschild solution, is an exact solution to the Einstein field equations [2] that describes the gravitational field outside a spherical mass under the supposition that the mass’s electric charge, angular momentum, and universal cosmological constant are all equal to zero. It is a part of Einstein’s theory of general relativity. For characterising slowly rotating astronomical objects, such as numerous stars and planets, including Earth and the Sun, the solution is a suitable approximation. Karl Schwarzschild discovered it in 1916. Test particle motion in the gravitational field of a central fixed mass $M$, or motion in the Schwarzschild metric, is described by Schwarzschild geodesics. Einstein’s general theory of relativity has been successfully validated in large part thanks to Schwarzschild geodesics. We comprehend the structure of the space-time surrounding any black hole by using null and timelike geodesics. Particle motion provides a basis for comparison with observational data and aids in our understanding of the gravitational fields of black holes from an experimental perspective. The study of the inner architecture of black holes can only be theoretical because the event horizon prevents information from black holes from reaching us. Because gravitational waves are so challenging to measure, geodesic studies are the only practical method left to physicists. Therefore, look into the null and timelike geodesic motion around the Schwarzschild black hole when the string cloud parameter $a$ and the quintessence field parameter $q$ are present.

String cloud parameter $a$ [3], is a contribution to the usual the Schwarzschild metric by String theory. String theory predicts the existence of string clouds which can be defined as a collection strings created due to symmetry breaking in the early stages of the universe. Although entropy is unaffected, a string cloud background has a significant impact on horizon structure, geodesics, and thermodynamic variables.

The Lambda-CDM model [4] is a parameterization of the Big Bang cosmological theory, according to which the universe is made up of three main elements: ordinary matter, cold dark matter, and the cosmological constant Lambda ($\Lambda$), which is related with dark energy. Due to its simplicity and ability to fairly explain the following characteristics of the cosmos, it is commonly referred to as the standard model of Big Bang cosmology. Quintessence is a hypothesised type of dark energy—more specifically, a scalar field—that has been proposed as a possible explanation for the universe’s observed acceleration of expansion. Ratra and Peebles [5] offered the first illustration of this situation (1988) since then, the quintessence field has been incorporated into space-time metrics for more accurate studies. Here it has been incorporated into the Schwarzschild metric as $\frac{q}{r^{3}}$. 

\[ \frac{q}{r^{3}} \]
Therefore, spherically symmetric and static spacetime in the background of quintessence and the cloud of strings is defined as:

\[ ds^2 = - \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right) dt^2 + \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]  

where \( \omega_q \) is the the equation of state parameter (EoS) for quintessence field, \( a \) is the string cloud parameter, \( M \) is the mass of the black hole, and \( q \) is the quintessence parameter. The equation of state parameter for the quintessence field is limited to the values, \(-1 < \omega_q < -\frac{1}{3}\). This work considers only single case when \( \omega_q = -\frac{2}{3} \).

In the absence of \( a \) and \( q \), the above spacetime can be reduced to the Schwarzschild spacetime. With \( \omega_q = -\frac{2}{3} \),

\[ ds^2 = - \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right) dt^2 + \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

For the metric given by eq.(2), the lagrangian is defined as

\[ L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = - \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right) \dot{t}^2 + \frac{r^2}{(1 - a - \frac{2M}{r} - qr)} + r^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \]

where the four-momentum \( p^\mu = m u^\mu = m \dot{x}^\mu \). For the given lagrangian, we have two conserved quantities, the specific energy \( E \) and the specific angular momentum \( L_z \) of the neutral particle in the surrounding spacetime:

\[ E = -\dot{t} \left( 1 - a - \frac{2M}{r} - qr \right); \quad L_z = \dot{\phi} r^2 \sin^2 \theta \]

Effective potential for any particle in the equitorial plane in the vicinity of this black hole is given by \( \dot{\theta} = 0 \), \( \theta = \frac{\pi}{2} \). Using the normalization condition 4-velocity \( u_\mu u^\mu = -\zeta \), where \( \zeta = 0, 1 \) is used to define the null and timelike geodesics respectively, we obtain [7]

\[ U_{eff} = (1 - a - \frac{2M}{r} - qr) \left( \zeta + \frac{L_z^2}{r^2} \right) \]

And the equation of motion is

\[ \dot{r}^2 = E^2 - U_{eff} \]

2. Nature of effective potential and timelike geodesics of test particles

Effective potential given in equation (5) for neutral particles moving along the timelike geodesics for \( \zeta = 1 \),

\[ U_{eff}(r) = \left( 1 - a - \frac{2M}{r} - \frac{q}{r^{3 \omega_q + 1}} \right) \left( 1 + \frac{L_z^2}{r^2} \right) \]

2.1 Non-radial motion of test particle

The Effective Potential for the given case:

\[ U_{eff} = \left( 1 - a - \frac{2M}{r} - qr \right) \left( 1 + \frac{L_z^2}{r^2} \right) \]

The timelike geodesics for arriving test particles with nonzero angular momentum are being studied in this case. Let’s use the effective potential to try to understand the orbits of the particles in the stringcloud quintessence black holes. \( U_{eff} \) given by equation (8).

By analysing the effective potential, we can see that the quintessence parameter \( q \), which when varied, has a
significant impact on the shape of the graph and observer orbits, while the stringcloud parameter $a$, which is a constant in the metric, needs to be sufficiently small for the metric to make sense. But aside from that, the variation in $a$ does not give the spacetime surrounding the black hole any other unusual characteristics, thus for the time being we can round $a$ to a sufficiently low value, say $a = 0.1$. But the existence of stable orbits and geodesic depends on the variation of the parameter $q$.

2.2 Analysis of effective potential and geodesics

$U_{\text{eff}} = 3.0104$ (maximum value) at $r = 3.468$ at $D$, for unstable critical orbit.
$U_{\text{eff}} = 0.8857$ (local minima) at $r = 103.65$, for stable critical orbit at $F$.
Marginally stable orbits are allowed between $r = 3.468$ and $r = 103.656$ (that is, between $D$ and $F$).
$r = 188.345$ is a local maxima anomaly due to the presence of quintessence field.

Figure 1: Effective Potential of a unit mass black hole for Non-Radial timelike geodesic motion for particle with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$. Blue dotted lines denote different energy levels for particle, with points $A$ to $F$ denoting intersections between those lines and the effective potential.
The following orbits are allowed depending on the values of the constant $E$ (i.e. energy of the incoming test particle) as plotted in Figure (1).

The geodesics are defined using the following relation obtained from Eqs.(4) and (6). [6]

$$\left( \frac{dr}{d\phi} \right)^2 = \frac{\mathcal{E}^2 - (1 - a - \frac{2M}{r} - qr) \left( 1 + \frac{L^2}{r^4} \right)}{L^2/r^4}$$

(9)

- $\mathcal{E} = E_C$: The test particle starting from the point $B$, falls into the singularity. Hence it is a terminating bound orbit. $\mathcal{E} = E_C$ also implies a circular orbit if particle is at $F$ as shown in figure (2).

![Figure 2: Terminating bound orbit for Non-Radial timelike geodesic motion for particle around a unit mass black hole with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$ with the solid line representing the orbit for test particle starting from a distance at $r = r_B$. $r_H$ is the horizon radii.](image)

**Circular Orbit:** From Eqs.(4) and (6), we know that at $r = r_0 = r_C$, when particle is at $F$.

$$\frac{\mathcal{E}_0^2}{L_{z0}^2} = \frac{2r_C}{r^2_C (q - \frac{2M}{r_C})} = \frac{\dot{t}^2 (1 - a - \frac{2M}{r_C} - qr_C)^2}{\dot{\phi}^2 r_C^4}$$

(10)

In case of circular orbit, $r = r_C = constant$, where $r_C$ is the distance of the circular orbit from the singularity and hence $\dot{t} = 0$. In order to calculate the time periods for circular orbits, we can use the result $\Delta t = T_t$ and $\Delta \phi = 2\pi$ to obtain

$$\frac{dt}{d\phi} = (1 - a - \frac{2M}{r_C} - qr_C) \left( \frac{2r_C}{q - \frac{2M}{r_C}} \right)^{1/2}$$

(11)

$$T_t = 2\pi \left( 1 - a - \frac{2M}{r_C} - qr_C \right) \left( \frac{2r_C}{q - \frac{2M}{r_C}} \right)^{1/2}$$

Similarly,

$$T_r = 2\pi \left( \frac{r_C^2 \left( -\frac{2}{r_C(a+qr_C-1)+2M} - q \right) + 2M}{q - \frac{2M}{r_C}} \right)^{1/2}$$

(12)

- $\mathcal{E} = E_1$: The test particle starts from the point $A$ and falls into the singularity, similar to particle starting from point $B$. Hence it is a terminating bound orbit.
- \( E = E_2 \): The test particle starting from the point \( C \), falls into the singularity, similar to particles starting from \( A \) and \( B \). Hence it is a terminating bound orbit as shown in figure (3). However, if particle is positioned at \( E \), it is a bound planetary orbit between \( E \) and \( F \) as shown in figure (4).

Figure 3: Terminating bound orbit for Non-Radial timelike geodesic motion for particle around a unit mass black hole with \( \mathcal{L}_z = 10 \) for \( q = 0.00003 \) and \( a = 0.1 \). \( r_H \) is the horizon radii.

Figure 4: Planetary orbit between \( E \) and \( F \) for Non-Radial timelike geodesic motion for particle around a unit mass black hole with \( \mathcal{L}_z = 10 \) for \( q = 0.00003 \) and \( a = 0.1 \). \( r_H \) is the horizon radii.
• $\mathcal{E} = E_D$: The possible orbits are unstable circular orbit at point $D$. The particle starting from point $D$ can either crash into the singularity after crossing the horizon as shown in figure (5) or escape the gravitational pull of the black hole. Particles coming from infinity can jump into critical unstable orbit at $r_D$ as shown in figure (6).

Figure 5: Terminating bound orbit for Non-Radial timelike geodesic motion for particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$. $r_H$ is the horizon radii

Figure 6: Critical unstable orbit at $r_D$ for non-radial timelike geodesic motion for particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$. $r_H$ is the horizon radii
• $E = E_4$: The test particle starting at any point with $E = E_4$, or any value greater than $E_3$ falls into the singularity. Hence it is a terminating bound orbit as shown in figure (7).

![Terminating bound orbit for Non-Radial timelike geodesic motion for particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$. $r_H$ is the horizon radii](image)

Figure 7: Terminating bound orbit for Non-Radial timelike geodesic motion for particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$. $r_H$ is the horizon radii

### 3 Nature of effective potential and null geodesics of massless particles and photons

Effective potential given in equation (5) for particles moving along the null geodesics for $\zeta = 0$ or $u^\mu u_\mu = 0$

$$U_{eff}(r) = \left(1 - a - \frac{2M}{r} - qr\right) \frac{\mathcal{L}_z^2}{r^2}$$

(13)

And the equation of motion is again

$$\dot{r}^2 = E^2 - U_{eff}$$

(14)

#### 3.1 Non-radial motion

$$U_{eff}(r) = \left(1 - a - \frac{2M}{r} - qr\right) \frac{\mathcal{L}_z^2}{r^2}$$

(15)

The radii of the photon orbit can be derived by taking $\frac{dU_{eff}}{dr} = 0$

$$\Rightarrow \left(\frac{2M}{r^2} - q\right) \frac{\mathcal{L}_z^2}{r^2} - \left(1 - a - \frac{2M}{r} - qr\right) \frac{2\mathcal{L}_z^2}{r^3} = 0$$

$$r_q^N = \frac{1 - a - \sqrt{a^2 - 2a - 6Mq + 1}}{q} \quad r_c^N = \frac{1 - a + \sqrt{a^2 - 2a - 6Mq + 1}}{q}$$

(16)
In the required limit $r_c^N$, reduces to the photon orbit for the Schwarzschild black hole for $a = q = 0$. The $r_c^N$, is a result of the quintessence field’s existence. There is no stable critical orbit, only an unstable critical orbit.

### 3.2 Analysis of effective potential and geodesics

![Figure 8: Effective Potential of a unit mass black hole for non-radial null geodesic motion for massless particle with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$, where $r = 3.3335$ and $U_{eff} = 2.699$, represented by point B is the maxima which also represents unstable critical photon radii. Blue dotted lines denote different energy levels for particle, with points A to C denoting intersections between those lines and the effective potential.](image)

$U_{eff} = 2.699$ (maximum value) at $r = 3.3335$, for unstable critical photon orbit.

$U_{eff} = 1.5$ at $r = 2.475$, 6.2047.
The following orbits are allowed depending on the values of the constant $E$ (i.e. energy of the incoming test particle) as plotted in figure (8).

The geodesics are again defined using the following relation obtained from Eqs. (4) and (6).

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{E^2 - (1 - a - \frac{2M}{r} - qr^2)\left(0 + \frac{L^2}{r^2}\right)}{L^2/r^4}$$  \hspace{1cm} (17)

- $\mathcal{E} = E_C$: The massless test particle starting from the point B, falls into the singularity, hence forming terminating bound orbit as shown in figure (9). Or it can escape to infinity.

Or, $E = E_C$ also implies an unstable circular orbit if particle is at $r_B$ as shown in figure (10).

**Circular Orbit:** From equation (49), we know that at $r = r_e^N = \frac{1 - a + \sqrt{a^2 - 2a - 6Mq + 1}}{q}$, when massless particle is at $B$.

In case of circular orbit, $r = r_e^N = \text{constant}$, where $r_e^N$ is the distance of the circular orbit from the singularity and hence $\dot{r} = 0$. In order to calculate the time periods for circular orbits, we can use the result $\Delta t = T_t$ and $\Delta \phi = 2\pi$ to obtain

$$\frac{dt}{d\phi} = r_e^N \left(1 - a - \frac{2M}{r_e^N} - qr_e^N\right)^{-\frac{1}{2}}$$  \hspace{1cm} (18)

$$\therefore T_t = 2\pi r_e^N \left(1 - a - \frac{2M}{r_e^N} - qr_e^N\right)^{-\frac{1}{2}}$$  \hspace{1cm} (19)

Figure 9: Terminating bound orbit for non-radial null geodesic motion for massless particle around a unit mass black hole with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$. 

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Figure 10: Unstable critical photon orbit at $r_B$ for massless particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$.

- $E = E_1$: The massless test particle starts from the point A and falls into the singularity. Hence it is a terminating bound orbit as shown in figure (11).
  Or it is a fly-by orbit for the massless particle at $r_C$ as shown in figure (12).

Figure 11: Terminating bound orbit for non-radial timelike geodesic motion for particle around a unit mass black hole with $\mathcal{L}_z = 10$ for $q = 0.00003$ and $a = 0.1$. 
Figure 12: Fly-by orbit at $r_C$ for non-radial null geodesic motion for massless particle around a unit mass black hole with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$.

- $\mathcal{E} = E_2$: The massless test particle starting at any point with $\mathcal{E} = E_2$, or any value greater than $E_C$ falls into the singularity. Hence it is a *terminating bound orbit* as shown in figure (13).

Figure 13: Terminating bound orbit for non-radial null geodesic motion for particle around a unit mass black hole with $L_z = 10$ for $q = 0.00003$ and $a = 0.1$. 

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4 Conclusion

We have mapped various geodesics for Schwarzschild black holes with quintessence and string cloud background. Using the formulae for effective potential we have mapped the path that non-zero mass particles and massless particles or photons follow. According to their initial energy, entering massive test particles have a variety of orbits, including terminating bound, planetary, stable, and unstable circular orbits. In any event, no massive test particle or photon fly-by orbits are seen. In contrast to the situation of heavy test particles, fly-by and terminating escape orbits are present for massless test particles. A massless test particle has no stable circular orbits.

For the configuration of parameters $a$ and $q$ we can observe that is both stable and unstable critical orbits for timelike geodesics. We observed that low energy particles which started in the near vicinity of the black hole had no choice but fall into the singularity, resulting in terminating bound orbits. In contrast, planetary orbits exist when the initial energy is optimal. Particles with initial energy more than the optimal amount again results in the particle, crossing the horizon into the singularity.

For any configuration of parameters $a$ and $q$ we can observe stable critical orbits do not exist resulting in only unstable critical orbits for null geodesics. Similar to the previous case for timelike geodesics, for an approaching photon, if it originates in the close vicinity of the black hole, its fate is a terminating bound orbit where in it crosses the horizon and move towards the singularity. But photons with the exact optimal energy can jump into critical orbits, or a photon orbit. We also see a special case where the photon also has a fly-by orbit, when the energy of the photon is not enough to cross the highest point. It basically implies that the black hole bends the light around itself, which again is a well-known phenomena of black holes. In conclusion, for both null and timelike geodesics, we can observe that Schwarzschild black holes with quintessence and string cloud background, for the parameter configuration we have examined, behave like usual Schwarzschild black holes.

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