



HAL
open science

From particular polynomials to rational solutions to the KPI equation

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. From particular polynomials to rational solutions to the KPI equation. 2022. hal-03725302

HAL Id: hal-03725302

<https://hal.science/hal-03725302v1>

Preprint submitted on 16 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

From particular polynomials to rational solutions to the KPI equation

Pierre Gaillard,

Université de Bourgogne Franche Comté,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI) from particular polynomials. We obtain rational solutions written as a second derivative with respect to the variable x of a logarithm of a determinant of order n . So we get with this method an infinite hierarchy of rational solutions to the KPI equation.

We give explicitly the expressions of these solutions for the first five orders.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the Kadomtsev-Petviashvili equation which can be written in the form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where the subscripts x , y and t denote partial derivatives.

This equation first appeared in 1970, in a paper written by Kadomtsev and Petviashvili [1]. This equation is considered as a model for surface and internal water waves or in nonlinear optics. Zakharov [2] used the inverse scattering method to solve this equation, and obtained several exact solutions to the KPI equation.

Dubrovin [3] constructed for the first time in 1981 the solutions to the KPI equation given in terms of Riemann theta functions.

There is a lot of papers which deal with solutions to the KPI equation, in particular we can cite the works of Krichever [4], Matveev [5], Freeman [6], Boiti [7], Kodama [8], Ma [9].

In the following, we consider particular polynomials and we construct rational solutions to the KPI equation by using the Hirota bilinear method. Then we obtain rational solutions to the KPI equation as a second derivative with respect to the variable x of a logarithm of a determinant of order n . That provides an effective method to construct an infinite hierarchy of rational solutions of order n . We present rational solutions explicitly for the first simplest orders.

2 Rational solutions to the KPI equation

We consider the particular polynomials p_n defined by

$$\left\{ \begin{array}{l} p_n(x, y, t) = \sum_{k_1=0}^n x^{k_1} \sum_{k_2=0}^{n-k_1} \frac{y^{\binom{k_2}{2}} \left[1 - \frac{1}{2} \left(k_2 - 2 \left[\frac{k_2}{2} \right] \right) \right]}{\left(\frac{k_2}{2} \right)!} \times \frac{t^{\left(\frac{n-k_1-k_2}{3} \right)}}{\left(\frac{n-k_1-k_2}{3} \right)!} \\ \times \left(1 - \left[\frac{1}{2} \left(n - k_1 - k_2 + 1 - 3 \left[\frac{n-k_1-k_2}{3} \right] \right) \right] \right), \text{ for } n \geq 0, \\ p_n(x, y, t) = 0, \text{ for } n < 0. \end{array} \right. \quad (2)$$

In the previous definition of p_n , $[x]$ means the greater integer less or equal to x . We consider the following determinant

$$\det(p_{n-i+j}(x, t))_{\{1 \leq i \leq n, 1 \leq j \leq n\}}, \quad (3)$$

or in other words, it can be written as

$$\begin{vmatrix} p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_n \end{vmatrix}.$$

With these polynomials, we construct a type of rational solutions to the KPI equation. We prove this result by using the Hirota method. This method gives some rational solutions of the KPI equation.

With these notations and $\partial_x^2 = \frac{\partial^2}{\partial x^2}$, we have the following result

Theorem 2.1 *The function $v_n(x, y, t)$ defined by*

$$v_n(x, y, t) = 2\partial_x^2(\det(p_{n-i+j}(x, t))_{\{1 \leq i \leq n, 1 \leq j \leq n\}}) \quad (4)$$

is a rational solution to the (KPI) equation (1)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \quad (5)$$

3 Explicit rational solutions to the KPI equation for the first orders

In this section we will give some explicit examples of rational solutions to the KPI equation.

We present solutions in the form

$$v(x, y, t) = \frac{n(x, y, t)}{d(x, y, t)}. \quad (6)$$

The solutions constructed by this method have singularities. As a result, the module of these solutions becomes infinite for certain values of x and y . The representation of the modules of these solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions. However I give some of them.

3.1 Rational solutions of order 1 to the KPI equation

Example 3.1 *The function $v(x, y, t)$ defined by*

$$v(x, y, t) = -\frac{2}{x^2} \quad (7)$$

is a rational to the KPI equation.

This solution is not interesting since it does not depend on t and y variables.

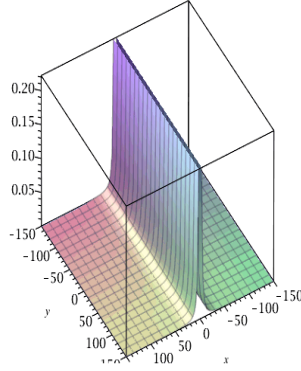


Figure 1. Solution of order 1 to the KPI equation for $t = 0$.

3.2 Rational solutions of order 2 to the KP equation

Example 3.2 *The function $v(x, y, t)$ defined by*

$$v(x, y, t) = \frac{-8x^6 - 96tx^3 + 288y^2x^2 - 288t^2}{(-x^4 + 12tx - 12y^2)^2} \quad (8)$$

is a rational to the KPI equation.

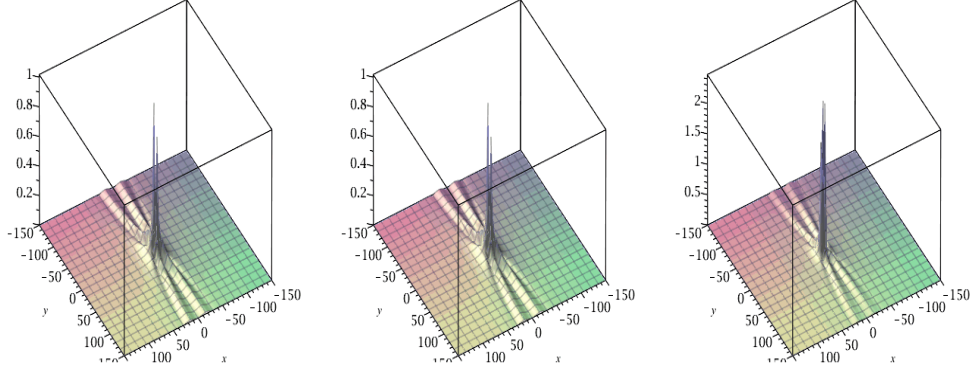


Figure 2. Solution of order 2 to the KPI equation, on the left for $t = 0$; in the center for $t = 0.01$; on the right for $t = 10$.

3.3 Rational solutions of order 3 to the KPI equation

Example 3.3 The function $v(x, y, t)$ defined by

$$v(x, y, t) = \frac{n(x, y, t)}{d(x, y, t)} \quad (9)$$

with

$$n(x, y, t) = -18x^{16} + 864tx^{13} + 288y^2x^{12} - 62208t^2x^{10} + 432000y^2tx^9 - 285120y^4x^8 + 1244160t^3x^7 - 4976640y^2t^2x^6 + 6842880y^4tx^5 - 3110400y^6x^4 - 37324800t^4x^4 + 24883200y^2t^3x^3 - 74649600y^4t^2x^2 + 37324800y^6tx + 149299200t^4y^2 - 9331200y^8$$

and

$$d(x, y, t) = (x^9 - 72tx^6 + 72y^2x^5 + 4320ty^2x^2 - 2160y^4x + 8640t^3)^2$$

is a rational to the KPI equation.

In this case $N = 3$, the solutions constructed by this method have singularities. The representation of the modules of these solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions.

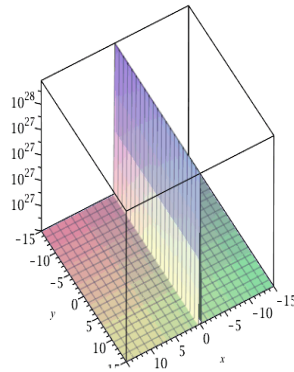


Figure 3. Solution of order 3 to the KPI equation $t = 0$.

3.4 Rational solutions of order 4 to the KPI equation

The full expression of the solution is given in the appendix.

In this case $N = 4$, there is singularity in $x = 0$ and $y = 0$ for any t . The module of this solution becomes infinite and the representation of its module is therefore distorted by this singularity and does not give an exact picture of the structure of this solution.

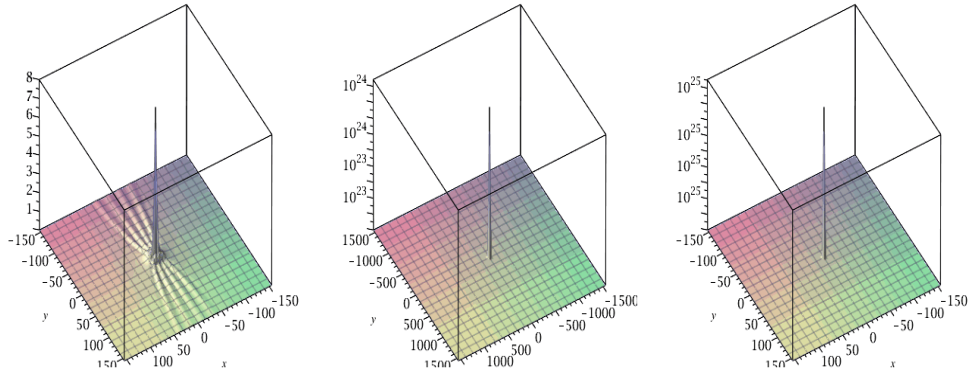


Figure 4. Solution of order 4 to the KPI equation, on the left for $t = 0$; in the center for $t = 1$; on the right for $t = 10$.

3.5 Rational solutions of order 5 to the KPI equation

In order not to weigh down the text, we give explicit rational solution in this case in the appendix.

For $N = 5$, the solution constructed by this method has singularities and the module becomes infinite for certain values of x and y . The representation of the modules of this solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions.

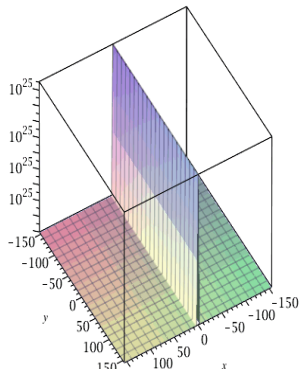


Figure 5. Solution of order 5 to KPI for $t = 10$.

Explicit expressions of rational solutions are not given for n greater than 5. For example, in the case of order 6, the explicit expression of the rational solution takes about 3 pages in tiny characters. In this case, the numerator is a polynomial of degree 70 in x , 34 in y , 22 in t ; the denominator is a polynomial of degree 72 in x , 36 in y , and 24 in t . The numerator contains 227 terms, the denominator 240 terms with big coefficients. In the case of order 10, the numerator is a polynomial of degree 198 in x , 98 in y , 66 in t ; the denominator is a polynomial of degree 200 in x , 98 in y , and 66 in t . We cannot give these explicit expressions in this paper.

4 Conclusion

In this article, rational solutions to the KPI equation have been explicitly built. Other approaches to building solutions to the KPI equation have been conducted and the most significant can be mentioned. In 1990, Hirota and Ohta [10] built solutions as a particular case of a hierarchy of coupled bilinear equations given in terms of Pfaffians. In 1993, Oevel [11] used the Darboux transformations to obtain the solutions of the multi-component KPI hierarchy. No explicit solutions were given. In the article published in 1996 [12], the same author gave explicit solutions in terms of wronskians of order 2. In 2014, using iterated Darboux transformations, in particular, solutions to the KPI equation were constructed in terms of reduced multi-component wronskian solutions [13]. In the later study, an explicit solution at order 1 was given. Only one asymptotic study has been carried out for order higher than 2.

Here, using special polynomials, we have given an approach to find explicit rational solutions for higher orders and tried to describe the structure of these rational solutions.

We get polynomials in x , y and t ; for the order n , the numerator is of degree $2n^2 - 2$ in x and the denominator of degree $2n^2$ in x . The structure in t and y appears more complicated. It will be relevant to study in details the structure

of these polynomials.

References

- [1] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.* V. 15 N. 6 (1970) 539-541.
- [2] V.E. Zakharov, A.B. Shabat A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. *Funct. Anal. And Its Appl.* V. 8 N. 3 (1974) 226-235.
- [3] B.A. Dubrovin Theta functions and non-linear equations, *Russian Math. Surveys.* V.36 N.2 (1981) 11-92.
- [4] I. Krichever, Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of n particles on a line, *Funct. Anal. and Appl.* V. 12 N. 1 (1978) 76-78.
- [5] V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtsev-Petviashvili equation depending on functional parameters, *Letters in Mathematical Physics.* V. 3 (1979) 213-216.
- [6] N. C Freeman, J.J.C. Nimmo Rational solutions of the KdV equation in wronskian form, *Phys. Letters.* V. 96 A N. 9 (1983) 443-446.
- [7] M. Boiti, F. Pempinelli, A. Pogrebkov Properties of the solutions of the Kadomtsev-Petviashvili equation, *J. Math. Phys.* V. 35 (1994)
- [8] Y. Kodama, Young diagrams and N solitons solutions to the KP equation, *J. Phys. A: Math. Gen.*, V. 37 (2004) 1116911190.
- [9] V.X. Ma, Lump solutions to the KP equation, *Phys. Lett. A.* V. 379 (2015) 1975.
- [10] R. Hirota, Y. Ohta Hierarchies of coupled soliton equations, *Phys. Soc. Japan.* V. 60 (1991) 798-809.
- [11] W. Oevel, Darboux theorems and wronskian formulas for integrable systems I : constrained KP flows, *Physica A.* V. 195 (1993) 533-576.
- [12] W. Oevel, W. Stramp, Wronskian solutions of the constrained KP hierarchy, *J. Math. Phys.* V. 37 (1996) 6213-6219
- [13] T. Xu, F.W. Sun, Y. Zhang, J. Li, Multi-component wronskian solution to the Kadomtsev-Petviashvili equation, *Comp. Math. Math. Phys. B*, V. 54 (2014) 97-113.
- [14] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. **161**, 2002

- [15] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [16] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [17] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009
- [18] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [19] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [20] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [21] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [22] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [23] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013
- [24] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [25] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [26] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [27] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [28] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014

- [29] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [30] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [31] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [32] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [33] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [34] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [35] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [36] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [37] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015
- [38] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [39] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [40] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [41] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [42] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016

- [43] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [44] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [45] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 3, 130-139, 2017
- [46] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N^2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. 3, 60-70, 2017
- [47] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [48] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 2, 92-98, 2017
- [49] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 9, 19-37, 2017
- [50] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 11, 24-30, 2017
- [51] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. 1, 14-19, 2018
- [52] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [53] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [54] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [55] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [56] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019

- [57] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [58] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [59] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [60] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021
- [61] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [62] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. 3, 14-21, 2021
- [63] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. **11**, N. 4, 2102-2113, 2021
- [64] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. **17**, N. 6, 27-34, 2021
- [65] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. **107**, 102824-1-11, 2021
- [66] P. Gaillard, Multiparametric Rational Solutions of Order N to the KPI Equation and the Explicit Case of Order 3, *Arch. Of cur. Res. Int.*, V. **21**, N . 6, 58-71, 2021

Appendix

In the case of order 4, the function $v(x, y, t)$ defined by

$$v(x, y, t) = \frac{n(x, y, t)}{d(x, y, t)} \quad (10)$$

with

$$\begin{aligned} n(x, y, t) = & -32x^{30} + 9600tx^{27} - 5760y^2x^{26} - 1296000t^2x^{24} + 3456000y^2tx^{23} - \\ & 1958400y^4x^{22} + 47692800t^3x^{21} + 520473600y^2t^2x^{20} - 1278720000y^4tx^{19} + 513792000y^6x^{18} + \\ & 3794688000t^4x^{18} - 79750656000y^2t^3x^{17} + 176110848000y^4t^2x^{16} - 438939648000t^5x^{15} - \\ & 161574912000y^6tx^{15} + 48522240000y^8x^{14} + 3508531200000y^2t^4x^{14} - 6026711040000y^4t^3x^{13} + \\ & 4075868160000y^6t^2x^{12} - 5957038080000t^6x^{12} - 1358622720000y^8tx^{11} - 36369285120000y^2t^5x^{11} - \\ & 174112727040000y^4t^4x^{10} + 267079680000y^{10}x^{10} + 319798886400000t^7x^9 + 460538265600000y^6t^3x^9 + \\ & 94058496000000y^2t^6x^8 - 493807104000000y^8t^2x^8 + 221560012800000y^{10}tx^7 + \\ & 1881169920000000y^4t^5x^7 - 37391155200000y^{12}x^6 - 4740548198400000t^8x^6 - \end{aligned}$$

$$\begin{aligned}
& 1931334451200000 y^6 t^4 x^6 - 2370274099200000 y^8 t^3 x^5 + 41084751052800000 y^2 t^7 x^5 - \\
& 13168189440000000 y^4 t^6 x^4 + 3950456832000000 y^{10} t^2 x^4 + 31603654656000000 t^9 x^3 - \\
& 1853300736000000 y^{12} t x^3 - 31603654656000000 y^6 t^5 x^3 + 292626432000000 y^{14} x^2 + \\
& 7900913664000000 t^4 y^8 x^2 + 94810963968000000 t^8 y^2 x^2 - 1755758592000000 t^3 y^{10} x - \\
& 221225582592000000 t^7 y^4 x - 292626432000000 t^2 y^{12} + 5267275776000000 t^6 y^6 - \\
& 94810963968000000 t^{10}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) = & (x^{16} - 240 t x^{13} + 240 y^2 x^{12} + 10080 t^2 x^{10} + 14400 y^2 t x^9 - 7200 y^4 x^8 - \\
& 172800 t^3 x^7 + 3628800 y^2 t^2 x^6 - 6048000 y^4 t x^5 + 2016000 y^6 x^4 + 18144000 t^4 x^4 - \\
& 72576000 y^2 t^3 x^3 + 36288000 y^4 t^2 x^2 - 24192000 y^6 t x + 217728000 t^5 x + 217728000 t^4 y^2 + \\
& 6048000 y^8)^2
\end{aligned}$$

is a rational to the KPI equation.

In the case of order 5, the function The function $v(x, y, t)$ defined by

$$v(x, y, t) = \frac{n(x, y, t)}{d(x, y, t)} \quad (11)$$

with

$$\begin{aligned}
n(x, y, t) = & -50 x^{48} + 45600 t x^{45} - 36000 y^2 x^{44} - 17452800 t^2 x^{42} + 29635200 y^2 t x^{41} - \\
& 14011200 y^4 x^{40} + 3314304000 t^3 x^{39} - 2322432000 y^2 t^2 x^{38} - 4705344000 y^4 t x^{37} - \\
& 348945408000 t^4 x^{36} + 2326464000 y^6 x^{36} + 986743296000 y^2 t^3 x^{35} - 4358914560000 y^4 t^2 x^{34} + \\
& 37492761600000 t^5 x^{33} + 4709456640000 y^6 t x^{33} - 649264896000000 y^2 t^4 x^{32} - \\
& 1299291840000 y^8 x^{32} + 2748249907200000 y^4 t^3 x^{31} - 6455339089920000 t^6 x^{30} - \\
& 4239181578240000 y^6 t^2 x^{30} + 2811042662400000 y^8 t x^{29} + 127907013427200000 y^2 t^5 x^{29} - \\
& 470514040012800000 y^4 t^4 x^{28} - 639104256000000 y^{10} x^{28} + 688325647564800000 y^6 t^3 x^{27} + \\
& 463344692428800000 t^7 x^{27} - 9977273774899200000 y^2 t^6 x^{26} - 524488985395200000 y^8 t^2 x^{26} + \\
& 26735111656243200000 y^4 t^5 x^{25} + 224656627507200000 y^{10} t x^{25} - 41152624128000000 y^{12} x^{24} - \\
& 20573979181056000000 y^6 t^4 x^{24} - 26464110317568000000 t^8 x^{24} + \\
& 327856313401344000000 y^2 t^7 x^{23} - 2931238969344000000 y^8 t^3 x^{23} - \\
& 1603000571461632000000 t^6 y^4 x^{22} + 11454569054208000000 y^{10} t^2 x^{22} - \\
& 4602379751424000000 y^{12} t x^{21} + 5110753409040384000000 t^5 y^6 x^{21} + \\
& 2782354152259584000000 t^9 x^{21} + 509023678464000000 y^{14} x^{20} - \\
& 5616696316428288000000 t^8 y^2 x^{20} - 12134665656336384000000 t^4 y^8 x^{20} + \\
& 116979663562997760000000 t^7 y^4 x^{19} + 15256084884848640000000 y^{10} t^3 x^{19} - \\
& 452247033981173760000000 t^6 y^6 x^{18} - 189174420186071040000000 t^{10} x^{18} - \\
& 9991489566965760000000 y^{12} t^2 x^{18} + 706477261240074240000000 t^5 y^8 x^{17} + \\
& 1147590011649392640000000 t^9 y^2 x^{17} + 3245563651276800000000 y^{14} t x^{17} - \\
& 461631423194726400000000 y^{10} t^4 x^{16} - 416879638640640000000 y^{16} x^{16} - \\
& 5532097441389281280000000 t^8 y^4 x^{16} + 12884673475463086080000000 t^7 y^6 x^{15} + \\
& 3599145550259159040000000 t^{11} x^{15} + 42362241004339200000000 y^{12} t^3 x^{15} + \\
& 90437018163609600000000 y^{14} t^2 x^{14} - 21574803716702208000000000 t^6 y^8 x^{14} + \\
& 29836782814416076800000000 t^{10} y^2 x^{14} + 38373789556408320000000000 y^{10} t^5 x^{13} \\
& + 24429144673537228800000000 t^9 y^4 x^{13} - 41643872376422400000000 y^{16} t x^{13} + \\
& 5630805592473600000000 y^{18} x^{12} + 101618997971071795200000000 t^{12} x^{12} - \\
& 402027233908595097600000000 t^8 y^6 x^{12} - 53035323313854873600000000 y^{12} t^4 x^{12} + \\
& 44478907714083225600000000 y^{14} t^3 x^{11} + 847137176634418790400000000 t^7 y^8 x^{11} -
\end{aligned}$$

$$\begin{aligned}
& 949692755231952076800000000 t^{11} y^2 x^{11} - 21504498226554470400000000 y^{16} t^2 x^{10} - \\
& 1287798361384550400000000000 y^{10} t^6 x^{10} - 2150389118355583795200000000 t^{10} y^4 x^{10} + \\
& 5497713823924224000000000 y^{18} t x^9 + 1189544116034469888000000000 y^{12} t^5 x^9 - \\
& 2669254785415249920000000000 t^{13} x^9 + 6716844059439661056000000000 t^9 y^6 x^9 - \\
& 13286566912139329536000000000 t^8 y^8 x^8 - 577232851378176000000000 y^{20} x^8 + \\
& 2823790588781395968000000000 t^{12} y^2 x^8 - 680581921895350272000000000 y^{14} t^4 x^8 + \\
& 1339310295839932416000000000 t^7 y^{10} x^7 - 9946486253021036544000000000 t^{11} y^4 x^7 + \\
& 294892816692805632000000000 y^{16} t^3 x^7 - 101983223770251264000000000 y^{18} t^2 x^6 - \\
& 8581159828668284928000000000 y^{12} t^6 x^6 + 155772089793075216384000000000 t^{10} y^6 x^6 - \\
& 56644396288390987776000000000 t^{14} x^6 + 23219394697691136000000000 y^{20} t x^5 + \\
& 4851487645070524416000000000 y^{14} t^5 x^5 - 4897380095767137484800000000 t^9 y^8 x^5 - \\
& 240738684225661698048000000000 t^{13} y^2 x^5 - 2276411244871680000000000 y^{22} x^4 + \\
& 436633888056347197440000000000 t^{12} y^4 x^4 - 3441933802245980160000000000 t^8 y^{10} x^4 - \\
& 2076087055322972160000000000 y^{16} t^4 x^4 - 28322198144195493888000000000 t^{15} x^3 + \\
& 15297483565537689600000000000 t^7 y^{12} x^3 + 673817728482017280000000000 y^{18} t^3 x^3 - \\
& 330425645015614095360000000000 t^{11} y^6 x^3 - 14161099072097746944000000000 t^{10} y^8 x^2 - \\
& 1639016096307609600000000000 y^{20} t^2 x^2 - 2622425754092175360000000000 y^{14} t^6 x^2 + \\
& 1699331888651729633280000000000 t^{14} y^2 x^2 + 145544629352115732480000000000 y^{10} t^9 x + \\
& 1639016096307609600000000000 y^{16} t^5 x + 27316934938460160000000000 y^{22} t x + \\
& 1557720897930752163840000000000 t^{13} y^4 x - 424832972162932408320000000000 t^{16} - \\
& 2276411244871680000000000 y^{24} - 4720366357365915648000000000 t^{12} y^6 \\
& - 5048169576627437568000000000 y^{12} t^8 - 218535479507681280000000000 y^{18} t^4
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) = & (-x^{25} + 600 t x^{22} - 600 y^2 x^{21} - 100800 t^2 x^{19} + 50400 y^2 t x^{18} - 25200 y^4 x^{17} + \\
& 6955200 t^3 x^{16} - 29030400 y^2 t^2 x^{15} + 42336000 y^4 t x^{14} - 254016000 t^4 x^{13} - 14112000 y^6 x^{13} - \\
& 3556224000 y^2 t^3 x^{12} + 17273088000 y^4 t^2 x^{11} - 39626496000 t^5 x^{10} - 16087680000 y^6 t x^{10} + \\
& 563915520000 y^2 t^4 x^9 + 4021920000 y^8 x^9 - 1112590080000 y^4 t^3 x^8 - 365783040000 t^6 x^7 + \\
& 975421440000 y^6 t^2 x^7 - 7681443840000 y^2 t^5 x^6 - 533433600000 y^8 t x^6 + 106686720000 y^{10} x^5 + \\
& 14082647040000 y^4 t^4 x^5 - 76814438400000 t^7 x^4 + 4267468800000 y^6 t^3 x^4 - 12802406400000 y^8 t^2 x^3 - \\
& 153628876800000 y^2 t^6 x^3 + 6401203200000 y^{10} t x^2 + 76814438400000 y^4 t^5 x^2 + 460886630400000 t^8 x - \\
& 128024064000000 y^6 t^4 x - 1066867200000 y^{12} x - 921773260800000 y^2 t^7 + 12802406400000 y^8 t^3)^2
\end{aligned}$$

is a rational to the KPI equation.