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From particular polynomials to rational solutions to the KPI equation

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Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI) from particular polynomials. We obtain rational solutions written as a second derivative with respect to the variable x of a logarithm of a determinant of order n. So we get with this method an infinite hierarchy of rational solutions to the KPI equation.
We give explicitly the expressions of these solutions for the first five orders.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the Kadomtsev-Petviashvili equation which can be written in the form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1}$$

where the subscripts x, y and t denote partial derivatives.
This equation first appeared in 1970, in a paper written by Kadomtsev and Petviashvili [1]. This equation is considered as a model for surface and internal water waves or in nonlinear optics. Zakharov [2] used the inverse scattering method to solve this equation, and obtained several exact solutions to the KPI equation.
Dubrovin [3] constructed for the first time in 1981 the solutions to the KPI equation given in terms of Riemann theta functions.
There is a lot of papers which deal with solutions to the KPI equation, in particular we can cite the works of Krichever [4], Matveev [5], Freeman [6], Boiti [7], Kodama [8], Ma [9].

In the following, we consider particular polynomials and we construct rational solutions to the KPI equation by using the Hirota bilinear method. Then we obtain rational solutions to the KPI equation as a second derivative with respect to the variable x of a logarithm of a determinant of order n. That provides an effective method to construct an infinite hierarchy of rational solutions of order n. We present rational solutions explicitly for the first simplest orders.

2 Rational solutions to the KPI equation

We consider the particular polynomials p_n defined by

$$\begin{cases} p_n(x,y,t) = \sum_{k_1=0}^{n} x^{k_1} \sum_{k_2=0}^{n-k_1} \dfrac{y^{\left(\frac{k_2}{2}\right)} \left[1 - \frac{1}{2}\left(k_2 - 2\left[\frac{k_2}{2}\right]\right)\right]}{\left(\frac{k_2}{2}\right)!} \times \dfrac{t^{\left(\frac{n-k_1-k_2}{3}\right)}}{\left(\frac{n-k_1-k_2}{3}\right)!} \\ \times \left(1 - \left[\frac{1}{2}\left(n - k_1 - k_2 + 1 - 3\left[\frac{n-k_1-k_2}{3}\right]\right)\right]\right), \text{ for } n \geq 0, \\ \\ p_n(x,y,t) = 0, \text{ for } n < 0. \end{cases} \tag{2}$$

In the previous definition of p_n, $[x]$ means the greater integer less or equal to x. We consider the following determinant

$$\det(p_{n-i+j}(x,t))_{\{1\leq i\leq n,\, 1\leq j\leq n\}}, \tag{3}$$

or in other words, it can be written as

$$\begin{vmatrix} p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_n \end{vmatrix}.$$

With these polynomials, we construct a type of rational solutions to the KPI equation. We prove this result by using the Hirota method. This method gives some rational solutions of the KPI equation.
With these notations and $\partial_x^2 = \frac{\partial^2}{\partial x^2}$, we have the following result

Theorem 2.1 *The function $v_n(x,y,t)$ defined by*

$$v_n(x,y,t) = 2\partial_x^2(\det(p_{n-i+j}(x,t))_{\{1\leq i\leq n,\, 1\leq j\leq n\}}) \tag{4}$$

is a rational solution to the (KPI) equation (1)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \tag{5}$$

3 Explicit rational solutions to the KPI equation for the first orders

In this section we will give some explicit examples of rational solutions to the KPI equation.
We present solutions in the form

$$v(x,y,t) = \frac{n(x,y,t)}{d(x,y,t)}. \tag{6}$$

The solutions constructed by this method have singularities. As a result, the module of these solutions becomes infinite for certain values of x and y. The representation of the modules of these solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions. However I give some of them.

3.1 Rational solutions of order 1 to the KPI equation

Example 3.1 *The function $v(x,y,t)$ defined by*

$$v(x,y,t) = -\frac{2}{x^2} \tag{7}$$

is a rational to the KPI equation.

This solution is not interesting since it does not depend on t and y variables.

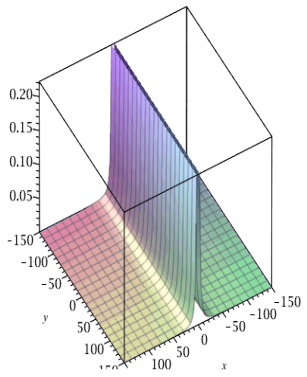

Figure 1. Solution of order 1 to the KPI equation for $t = 0$.

3.2 Rational solutions of order 2 to the KP equation

Example 3.2 *The function $v(x,y,t)$ defined by*

$$v(x,y,t) = \frac{-8\,x^6 - 96\,tx^3 + 288\,y^2x^2 - 288\,t^2}{(-x^4 + 12\,tx - 12\,y^2)^2} \tag{8}$$

is a rational to the KPI equation.

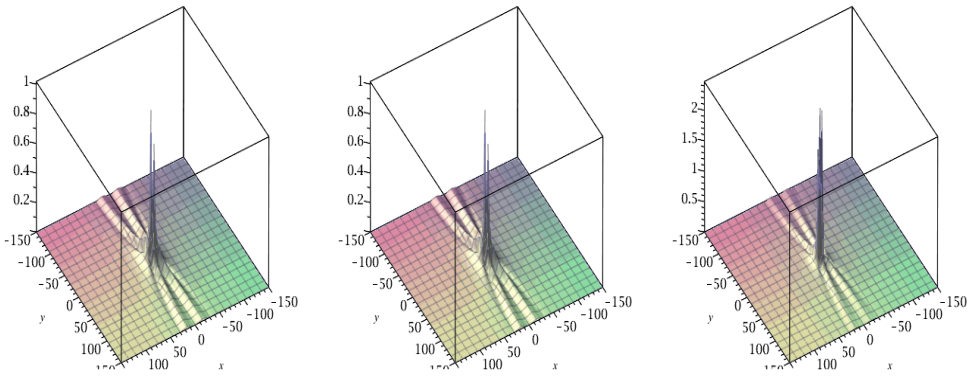

Figure 2. Solution of order 2 to the KPI equation, on the left for $t = 0$; in the center for $t = 0.01$; on the right for $t = 10$.

3.3 Rational solutions of order 3 to the KPI equation

Example 3.3 *The function $v(x, y, t)$ defined by*

$$v(x, y, t) = \frac{n(x, y, t)}{d(x, y, t)} \tag{9}$$

with
$n(x, y, t) = -18\,x^{16}+864\,tx^{13}+288\,y^2x^{12}-62208\,t^2x^{10}+432000\,y^2tx^9-285120\,y^4x^8+1244160\,t^3x^7-4976640\,y^2t^2x^6+6842880\,y^4tx^5-3110400\,y^6x^4-37324800\,t^4x^4+24883200\,y^2t^3x^3-74649600\,y^4t^2x^2+37324800\,y^6tx+149299200\,t^4y^2-9331200\,y^8$
and
$d(x, y, t) = \left(x^9 - 72\,tx^6 + 72\,y^2x^5 + 4320\,ty^2x^2 - 2160\,y^4x + 8640\,t^3\right)^2$
is a rational to the KPI equation.

In this case $N = 3$, the solutions constructed by this method have singularities. The representation of the modules of these solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions.

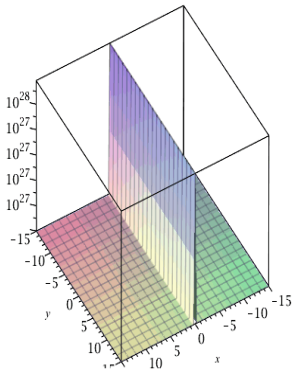

Figure 3. Solution of order 3 to the KPI equation $t = 0$.

3.4 Rational solutions of order 4 to the KPI equation

The full expression of the solution is given in the appendix.

In this case $N = 4$, there is singularity in $x = 0$ and $y = 0$ for any t. The module of this solution becomes infinite and the representation of its module is therefore distorted by this singularity and does not give an exact picture of the structure of this solution.

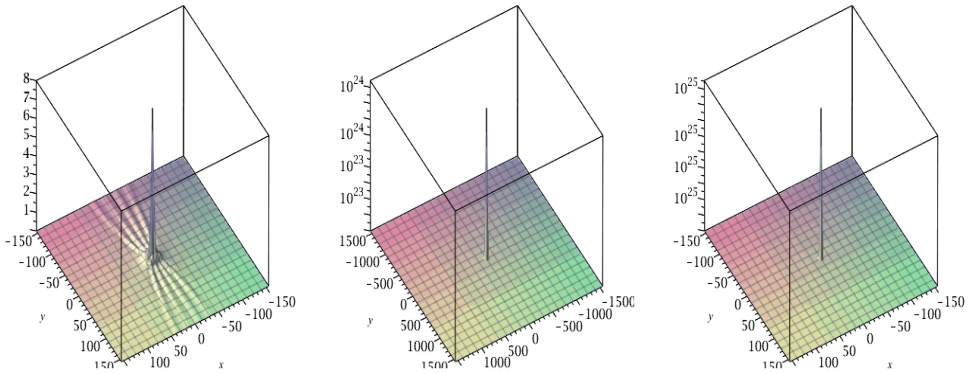

Figure 4. Solution of order 4 to the KPI equation, on the left for $t = 0$; in the center for $t = 1$; on the right for $t = 10$.

3.5 Rational solutions of order 5 to the KPI equation

In order not to weigh down the text, we give explicit rational solution in this case in the appendix.

For $N = 5$, the solution constructed by this method has singularities and the module becomes infinite for certain values of x and y. The representation of the modules of this solutions is therefore distorted by these singularities and does not give an exact picture of the structure of these solutions.

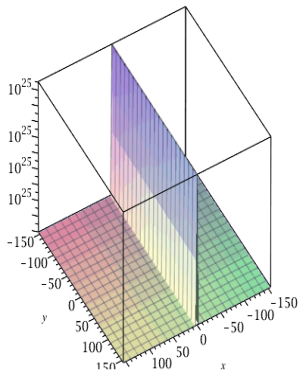

Figure 5. Solution of order 5 to KPI for $t = 10$.

Explicit expressions of rational solutions are not given for n greater than 5. For example, in the case of order 6, the explicit expression of the rational solution takes about 3 pages in tiny characters. In this case, the numerator is a polynomial of degree 70 in x, 34 in y, 22 in t; the denominator is a polynomial of degree 72 in x, 36 in y, and 24 in t. The numerator contains 227 terms, the denominator 240 terms with big coefficients. In the case of order 10, the numerator is a polynomial of degree 198 in x, 98 in y, 66 in t; the denominator is a polynomial of degree 200 in x, 98 in y, and 66 in t. We cannot give these explicit expressions in this paper.

4 Conclusion

In this article, rational solutions to the KPI equation have been explicitly built. Other approaches to building solutions to the KPI equation have been conducted and the most significant can be mentioned. In 1990, Hirota and Ohta [10] built solutions as a particular case of a hierarchy of coupled bilinear equations given in terms of Pfaffians. In 1993, Oevel [11] used the Darboux transformations to obtain the solutions of the multi-component KPI hierarchy. No explicit solutions were given. In the article published in 1996 [12], the same author gave explicit solutions in terms of wronskians of order 2. In 2014, using iterated Darboux transformations, in particular, solutions to the KPI equation were constructed in terms of reduced multi-component wronskian solutions [13]. In the later study, an explicit solution at order 1 was given. Only one asymptotic study has been carried out for order higher than 2.

Here, using special polynomials, we have given an approach to find explicit rational solutions for higher orders and tried to describe the structure of these rational solutions.

We get polynomials in x, y and t; for the order n, the numerator is of degree $2n^2 - 2$ in x and the denominator of degree $2n^2$ in x. The structure in t and y appears more complicated. It will be relevant to study in details the structure

of these polynomials.

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Appendix

In thecase of order 4, the function $v(x,y,t)$ defined by

$$v(x,y,t) = \frac{n(x,y,t)}{d(x,y,t)} \tag{10}$$

with

$n(x,y,t) = -32\,x^{30} + 9600\,tx^{27} - 5760\,y^2x^{26} - 1296000\,t^2x^{24} + 3456000\,y^2tx^{23} -$
$1958400\,y^4x^{22} + 47692800\,t^3x^{21} + 520473600\,y^2t^2x^{20} - 1278720000\,y^4tx^{19} + 513792000\,y^6x^{18} +$
$3794688000\,t^4x^{18} - 79750656000\,y^2t^3x^{17} + 176110848000\,y^4t^2x^{16} - 438939648000\,t^5x^{15} -$
$161574912000\,y^6tx^{15} + 48522240000\,y^8x^{14} + 3508531200000\,y^2t^4x^{14} - 6026711040000\,y^4t^3x^{13} +$
$4075868160000\,y^6t^2x^{12} - 5957038080000\,t^6x^{12} - 1358622720000\,y^8tx^{11} - 36369285120000\,y^2t^5x^{11} -$
$174112727040000\,y^4t^4x^{10} + 267079680000\,y^{10}x^{10} + 319798886400000\,t^7x^9 + 460538265600000\,y^6t^3x^9 +$
$94058496000000\,y^2t^6x^8 - 493807104000000\,y^8t^2x^8 + 221560012800000\,y^{10}tx^7 +$
$1881169920000000\,y^4t^5x^7 - 37391155200000\,y^{12}x^6 - 4740548198400000\,t^8x^6 -$

$1931334451200000\,y^6t^4x^6-2370274099200000\,y^8t^3x^5+41084751052800000\,y^2t^7x^5-13168189440000000\,y^4t^6x^4+3950456832000000\,y^{10}t^2x^4+31603654656000000\,t^9x^3-1853300736000000\,y^{12}tx^3-31603654656000000\,y^6t^5x^3+292626432000000\,y^{14}x^2+7900913664000000\,t^4y^8x^2+94810963968000000\,t^8y^2x^2-1755758592000000\,t^3y^{10}x-221225582592000000\,t^7y^4x-292626432000000\,t^2y^{12}+52672757760000000\,t^6y^6-94810963968000000\,t^{10}$
and
$d(x,y,t)=(x^{16}-240\,tx^{13}+240\,y^2x^{12}+10080\,t^2x^{10}+14400\,y^2tx^9-7200\,y^4x^8-172800\,t^3x^7+3628800\,y^2t^2x^6-6048000\,y^4tx^5+2016000\,y^6x^4+18144000\,t^4x^4-72576000\,y^2t^3x^3+36288000\,y^4t^2x^2-24192000\,y^6tx+217728000\,t^5x+217728000\,t^4y^2+6048000\,y^8)^2$
is a rational to the KPI equation.

In the case of order 5, the function The function $v(x,y,t)$ defined by

$$v(x,y,t)=\frac{n(x,y,t)}{d(x,y,t)} \tag{11}$$

with
$n(x,y,t)=-50\,x^{48}+45600\,tx^{45}-36000\,y^2x^{44}-17452800\,t^2x^{42}+29635200\,y^2tx^{41}-14011200\,y^4x^{40}+3314304000\,t^3x^{39}-2322432000\,y^2t^2x^{38}-4705344000\,y^4tx^{37}-348945408000\,t^4x^{36}+2326464000\,y^6x^{36}+986743296000\,y^2t^3x^{35}-4358914560000\,y^4t^2x^{34}+37492761600000\,t^5x^{33}+4709456640000\,y^6tx^{33}-649264896000000\,y^2t^4x^{32}-1299291840000\,y^8x^{32}+2748249907200000\,y^4t^3x^{31}-6455339089920000\,t^6x^{30}-4239181578240000\,y^6t^2x^{30}+2811042662400000\,y^8tx^{29}+127907013427200000\,y^2t^5x^{29}-470514040012800000\,y^4t^4x^{28}-639104256000000\,y^{10}x^{28}+688325647564800000\,y^6t^3x^{27}+463344692428800000\,t^7x^{27}-9977273774899200000\,y^2t^6x^{26}-524488985395200000\,y^8t^2x^{26}+267351116562432000000\,y^4t^5x^{25}+224656627507200000\,y^{10}tx^{25}-41152624128000000\,y^{12}x^{24}-205739791810560000000\,y^6t^4x^{24}-26464110317568000000\,t^8x^{24}+327856313401344000000\,y^2t^7x^{23}-2931238969344000000\,y^8t^3x^{23}-160300057146163200000\,t^6y^4x^{22}+114545690542080000000\,y^{10}t^2x^{22}-4602379751424000000\,y^{12}tx^{21}+5110753409040384000000\,t^5y^6x^{21}+2782354152259584000000\,t^9x^{21}+509023678464000000\,y^{14}x^{20}-5616696316428288000000\,t^8y^2x^{20}-1213466565633638400000\,t^4y^8x^{20}+1169796635629977600000000\,t^7y^4x^{19}+1525608488484864000000\,y^{10}t^3x^{19}-4522470339811737600000000\,t^6y^6x^{18}-1891744201860710400000000\,t^{10}x^{18}-999148956696576000000\,y^{12}t^2x^{18}+7064772612400074240000000\,t^5y^8x^{17}+1147590011649392640000000\,t^9y^2x^{17}+3245563651276800000000\,y^{14}tx^{17}-4616314231947264000000000\,y^{10}t^4x^{16}-4168796386406400000000\,y^{16}x^{16}-5532097441389281280000000\,t^8y^4x^{16}+1288467347546308608000000\,t^7y^6x^{15}+3599145550259159040000000\,t^{11}x^{15}+4236224100433920000000\,y^{12}t^3x^{15}+9043701816360960000000\,y^{14}t^2x^{14}-2157480371670220800000000\,t^6y^8x^{14}+2983678281441607680000000\,t^{10}y^2x^{14}+3837378955640832000000000\,y^{10}t^5x^{13}+2442914467353722880000000\,t^9y^4x^{13}-4164387237642240000000\,y^{16}tx^{13}+563080559247360000000\,y^{18}x^{12}+10161899797107179520000000\,t^{12}x^{12}-4020272339085950976000000\,t^8y^6x^{12}-5303532331385487360000000\,y^{12}t^4x^{12}+4447890771408322560000000\,y^{14}t^3x^{11}+847137176634418790400000\,t^7y^8x^{11}-$

$$94969275523195207680000000\,t^{11}y^2x^{11}-21504498226554470400000000\,y^{16}t^2x^{10}-$$
$$12877983613845504000000000\,y^{10}t^6x^{10}-21503891183555837952000000\,t^{10}y^4x^{10}+$$
$$54977138239242240000000\,y^{18}tx^9+11895441160344698880000000\,y^{12}t^5x^9-$$
$$26692547854152499200000000\,t^{13}x^9+67168440594396610560000000\,t^9y^6x^9-$$
$$13286566912139329536000000000\,t^8y^8x^8\,-\,57723285137817600000000\,y^{20}x^8\,+$$
$$28237905887813959680000000000\,t^{12}y^2x^8-68058192189535027200000000\,y^{14}t^4x^8+$$
$$13393102958399324160000000000\,t^7y^{10}x^7-99464862530210365440000000000\,t^{11}y^4x^7+$$
$$29489281669280563200000000\,y^{16}t^3x^7-10198322377025126400000000\,y^{18}t^2x^6-$$
$$85811598286682849280000000\,y^{12}t^6x^6+15577208979307521638400000000\,t^{10}y^6x^6-$$
$$56644396288390987776000000000\,t^{14}x^6+23219394697691136000000000\,y^{20}tx^5+$$
$$48514876450705244160000000\,y^{14}t^5x^5-48973800957671374848000000000\,t^9y^8x^5-$$
$$24073868422566169804800000000\,t^{13}y^2x^5-22764112448716800000000\,y^{22}x^4+$$
$$43663388805634719744000000000\,t^{12}y^4x^4-34419338022459801600000000000\,t^8y^{10}x^4-$$
$$20760870553229721600000000\,y^{16}t^4x^4-28322198144195493888000000000\,t^{15}x^3+$$
$$15297483565537689600000000000\,t^7y^{12}x^3+67381772848201728000000000\,y^{18}t^3x^3-$$
$$33042564501561409536000000000\,t^{11}y^6x^3-14161099072097746944000000000\,t^{10}y^8x^2-$$
$$16390160963076096000000000\,y^{20}t^2x^2-26224257540921753600000000\,y^{14}t^6x^2+$$
$$16993318886517296332800000000\,t^{14}y^2x^2+14554462935211573248000000000\,y^{10}t^9x+$$
$$1639016096307609600000000000\,y^{16}t^5x+27316934938460160000000000\,y^{22}tx+$$
$$15577208979307521638400000000\,t^{13}y^4x-42483297216293240832000000000\,t^{16}-$$
$$22764112448716800000000000\,y^{24}\,-\,47203663573659156480000000000\,t^{12}y^6$$
$$-50481695766274375680000000000\,y^{12}t^8-21853547950768128000000000\,y^{18}t^4$$
and
$$d(x,y,t)=(-x^{25}+600\,tx^{22}-600\,y^2x^{21}-100800\,t^2x^{19}+50400\,y^2tx^{18}-25200\,y^4x^{17}+$$
$$6955200\,t^3x^{16}-29030400\,y^2t^2x^{15}+42336000\,y^4tx^{14}-254016000\,t^4x^{13}-14112000\,y^6x^{13}-$$
$$3556224000\,y^2t^3x^{12}+17273088000\,y^4t^2x^{11}-39626496000\,t^5x^{10}-16087680000\,y^6tx^{10}+$$
$$563915520000\,y^2t^4x^9+4021920000\,y^8x^9-1112590080000\,y^4t^3x^8-365783040000\,t^6x^7+$$
$$975421440000\,y^6t^2x^7-7681443840000\,y^2t^5x^6-533433600000\,y^8tx^6+106686720000\,y^{10}x^5+$$
$$14082647040000\,y^4t^4x^5-76814438400000\,t^7x^4+4267468800000\,y^6t^3x^4-12802406400000\,y^8t^2x^3-$$
$$153628876800000\,y^2t^6x^3+6401203200000\,y^{10}tx^2+76814438400000\,y^4t^5x^2+460886630400000\,t^8x-$$
$$128024064000000\,y^6t^4x-1066867200000\,y^{12}x-921773260800000\,y^2t^7+12802406400000\,y^8t^3)^2$$
is a rational to the KPI equation.