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TOWER EQUIVALENCE AND LUSZTIG'S TRUNCATED FOURIER TRANSFORM

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ABSTRACT. If f denotes the truncated Lusztig Fourier transform, we show that the image by f of the normalized characteristic function of a Coxeter element is the alternate sum of the exterior powers of the reflection representation, and that any class function is tower equivalent to its image by f. In particular this gives a proof of the results of Chapuy and Douvropoulos on "Coxeter factorizations with generalized Jucys-Murphy weights and matrix tree theorems for reflection groups" for irreducible spetsial reflection groups, based on Deligne-Lusztig combinatorics.

Introduction. This paper is a kind of follow-up to [20], which gives a uniform proof of the results of Chapuy and Stump [7] for Weyl groups using Deligne-Lusztig combinatorics. Here we extend this to all spetsial reflection groups and to the results of Chapuy and Douvropoulos [6].

In the current paper we extend the results of [20] to all irreducible spetsial groups (which coincide with the well-generated irreducible complex reflection groups except for 6 well-generated primitive groups of rank 2 which are not spetsial, see for example [17, Corollary 8.3]). In particular, proposition 2 (valid for any complex reflection group) extends [20, Lemma 3] and proposition 4 extends [20, Lemma 5].

Let us recall some definitions of Chapuy and Douvropoulos [6]. Let $W \subset \mathrm{GL}(V)$ be a well-generated finite irreducible reflection group on the complex vector space V of dimension n. A *tower* is a maximal chain $\{1\} = W_0 \subsetneq W_1 \subsetneq \dots W_n = W$ of parabolic subgroups of W; a parabolic subgroup is the fixator of a flat (an intersection of some reflecting hyperplanes for W), and is a reflection subgroup by a theorem of Steinberg. The chain being maximal means that W_i is the fixator of a flat of codimension i.

Let $\mathrm{refl}(W)$ be the set of reflections of W. To a tower $T = W_0 \subsetneq \dots \subsetneq W_n$, Chapuy and Douvropoulos [6] associate a set $J_T = \{J_T^i\}_{i=1,\dots,n}$ of elements of the group algebra given by

$$J_T^i = \sum_{s \in \mathrm{refl}(W_i) - \mathrm{refl}(W_{i-1})} s,$$

and associate the commutative subalgebra $\mathbb{C}[J_T] \subset \mathbb{C}[W]$. Two class functions on W are said to be T-equivalent if they have the same restriction to $\mathbb{C}[J_T]$. Two class functions χ, χ' are said to be tower equivalent if they are T-equivalent for all towers T. We will simply write $\chi \equiv \chi'$ in this case.

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The main result of [6] (from which they derive uniformly the others) is that

$$\sum_{\chi\in\mathrm{Irr}(W)}\chi(c^{-1})\chi\equiv\sum_i(-1)^i\Lambda^i(V)$$

where c is a Coxeter element of W and where on the right appear the exterior powers of the reflection representation of W.

We derive this formula of Chapuy-Douvropoulos from Deligne-Lusztig combinatorics, using specifically Lusztig's truncated Fourier transform. Let us recall what is this Fourier transform in the Weyl group case. Let $\mathbf{G}^F$ be a split finite reductive group with Frobenius F and Weyl group W, and let U_χ for $\chi\in\mathrm{Irr}(W)$ be the principal series unipotent character indexed by χ. We define the "truncated Lusztig Fourier transform" (truncated because it is projected to the principal series) as the linear operator which maps χ to the class function $f(\chi)$ on W defined by

$$\text{for } w\in W,\ f(\chi)(w)=\langle U_\chi,R^{\mathbf{G}}_{\mathbf{T}_w}(\mathbf{1})\rangle_{\mathbf{G}^F}.$$

where $R^{\mathbf{G}}_{\mathbf{T}_w}(\mathbf{1})$ is the Deligne-Lusztig character induced from the trivial character of a torus of type w. The definition is extended by linearity to all class functions on W.

We extend below this definition to spetsial groups, using the results of [4], [5] and [15]. Then the three steps of our proof are

- For any $w\in W$, we have $\sum_{\chi\in\mathrm{Irr}(W)}\chi(w^{-1})\chi\equiv\sum_{\chi\in\mathrm{Irr}(W)}\chi(w)\chi$.
- $f(\sum_{\chi\in\mathrm{Irr}(W)}\chi(c)\chi)=\sum_i(-1)^i\Lambda^i(V)$.
- For any class function χ on W, we have $f(\chi)\equiv\chi$.

The third step implies that the image of $\mathrm{Id}-f$ consists of functions which are tower equivalent to 0. A natural question is whether the image of $\mathrm{Id}-f$ spans the space of all such functions. It turns out that in all cases we could compute, except surprisingly for the primitive spetsial group G_{32}, these two spaces coincide.

There are two features of any current work on spetses which should be mentioned.

The first is that we use the papers [5], [16] and [18], which assume the trace conjecture for cyclotomic Hecke algebras ([4, Theorem-Assumption 2 (1)]). This conjecture is currently established for finite Coxeter groups, and all irreducible spetsial groups excepted some primitive groups of rank ≥ 3; see [2] for a recent paper on the topic.

The second is that some properties of spetses are based on the fact that they satisfy a certain number of "axioms" stated in [5, Chapter 4]. Here the situation is the opposite; these axioms have been checked for finite Coxeter groups and primitive irreducible spetsial groups, but only some of them have been checked in [14] and [15] for imprimitive spetsial groups. We point out where we use such axioms.

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First step. This step, valid for any finite complex reflection group, results from

Lemma 1. *For any $\chi\in\mathrm{Irr}(W)$, we have $\chi\equiv\overline{\chi}$.*

Proof. Let i be the anti-involution of $\mathbb{C}[W]$ induced by $w\mapsto w^{-1}$ for $w\in W$. The effect of i on $\chi\in\mathrm{Irr}(W)$ is to send it to $\overline{\chi}$. For any tower T, the algebra $\mathbb{C}[J_T]$ is fixed pointwise by i since its generators are fixed by i and it is a commutative algebra. The lemma follows. □

Coxeter numbers. In this section $W \subset \mathrm{GL}(V)$ is any finite complex reflection group, not necessarily irreducible or well-generated. As in [13], we define the *Coxeter number* of $\chi \in \mathrm{Irr}(W)$ as $\mathrm{cox}_\chi := \omega_\chi(\mathcal{R})$, the scalar by which the central element $\mathcal{R} = \sum_{s \in \mathrm{refl}(W)}(1-s)$ in the group algebra acts on the representation underlying χ. We denote by $\deg \chi \in \mathbb{N}[q]$ the fake degree of χ, the graded degree of χ in the coinvariant algebra of W, and define $N(\chi) = (\frac{d}{dq} \deg \chi)_{q=1}$.

Let $\zeta_e := \exp(2i\pi/e) \in \mathbb{C}$. The spetsial Hecke algebra $\mathcal{H}$ of W (see [5, 6.4]) is the Hecke algebra over $\mathbb{C}[q^{\pm 1}]$ with parameters given by $q, \zeta_e, \zeta_e^2, \ldots, \zeta_e^{e-1}$ for a reflection hyperplane H of W such that $|C_W(H)| = e$. In the following, we assume that $\mathcal{H}$ satisfies the trace conjecture since we use results of [4], [16] and [18]..

The algebra $\mathcal{H}$ splits over some extension $\mathbb{C}[t^{\pm 1}]$ of $\mathbb{C}[q^{\pm 1}]$ where $t^m = q$ for some $m \in \mathbb{N}$ (one can take $m = |Z(W)|$, see [16, 7.2(a)]). The algebra $\mathbb{C}[W]$ is a deformation of $\mathcal{H}$ for $t \mapsto 1$, which induces a bijection $\chi_t \mapsto \chi : \mathrm{Irr}(\mathcal{H}) \to \mathrm{Irr}(W)$. Through this bijection the Galois action $t \mapsto e^{2i\pi/m}t$ on $\mathrm{Irr}(\mathcal{H})$ induces a permutation on $\mathrm{Irr}(W)$ that we will denote δ. The map $\chi \mapsto \delta(\overline{\chi})$ is an involution, called Opdam's involution. We will denote by S_χ the Schur element of $\mathcal{H}$ for the character of $\mathcal{H}$ which specializes to χ, and we denote by a_χ and A_χ, the valuation and the degree in q of $S_{\mathbf{1}}/S_\chi$ (they may be rational numbers, not integral, but the next proposition shows that their sum is integral).

Proposition 2. *For any complex reflection group W and any $\chi \in \mathrm{Irr}(W)$ we have $\mathrm{cox}_\chi = (N(\chi) + N(\overline{\chi}))/\chi(1) = a_\chi + A_\chi$.*

Proof. The second equality is a direct consequence of the first one and of [4, Corollary 6.9]. Let us prove the first one.

We can reduce to the case where W is irreducible, since both sides of the first equality are additive for an external tensor product of characters.

As pointed out in [20, Remark 2], it results from [16, 6.5] that we have $\mathrm{cox}_\chi = (N(\chi)+N(i(\chi)))/\chi(1)$ where i is Opdam's involution. Thus it is sufficient to prove that $N(\overline{\chi}) = N(i(\chi))$, or equivalently to prove that for any $\chi \in \mathrm{Irr}(W)$ we have $N(\chi) = N(\delta(\chi))$. Let $T_w \in \mathcal{H}$ be an element which specializes to $w \in W$. We note that if for $\chi_t \in \mathrm{Irr}(\mathcal{H})$ we have $\chi_t(T_w) \in \mathbb{C}[q]$ then $\delta(\chi)(w) = \chi(w)$, since the Galois action which defined δ leaves q invariant. We apply this to a braid reflection $\mathbf{s}$ in the braid group of W of image $s \in \mathrm{refl}(W)$ in W, and whose image in $\mathcal{H}$ we denote by T_s. Then if s is of order e, by definition of $\mathcal{H}$ the eigenvalues of T_s in a representation of character χ_t are in the set $\{q, \zeta_e, \ldots, \zeta_e^{e-1}\}$, in particular $\chi_t(T_s) \in \mathbb{C}[q]$. The same considerations apply to powers of $\mathbf{s}$, so we get that for any $s \in \mathrm{refl}(W)$ and any $\chi \in \mathrm{Irr}(W)$ we have $\chi(s) = \delta(\chi)(s)$. We conclude by using [4, formula 1.10] which states that $N(\chi) = |\mathrm{refl}(W)|/2 - \sum_{s \in \mathrm{refl}(W)} \chi(s)/(1 - \overline{\det(s)})$. □

Corollary 3. *For any complex reflection group, cox_χ is constant on Rouquier families of irreducible characters.*

Proof. This is an immediate consequence of the second equality in proposition 2 and [18, Lemme 2.8]. □

This is the occasion to give an erratum to [20]. The last sentence of this paper claims that Corollary 3 fails for G_6 and G_8, but this was due to a programming error.

Fourier transform. Let now W be an irreducible finite spetsial complex reflection group. We recall that W is called a spetsial group if for any $\chi \in \mathrm{Irr}(W)$ we have $S_{\mathbf{1}}/S_\chi \in \mathbb{C}[q]$ (it is a priori only an element of $\mathbb{C}(t)$). We have defined in [5] for the spetses $\mathbb{G} = (V, W)$ a set $\mathrm{Uch}(\mathbb{G})$ of "unipotent characters", which contains a principal series $\{U_\chi\}_{\chi\in\mathrm{Irr}(W)}$; a unipotent characters ρ has a "generic degree" $\mathrm{Deg}(\rho)$, a polynomial in q, which for the principal series is $\mathrm{Deg}(U_\chi) = S_{\mathbf{1}}/S_\chi$. We also defined "virtual characters" $R^{\mathbb{G}}_{\mathbb{T}_w}(\mathbf{1})$ (elements of $\mathbb{Z}\,\mathrm{Uch}(\mathbb{G})$), so the definition we gave of f by $f(\chi)(w) = \langle U_\chi, R^{\mathbb{G}}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}}$ still makes sense for spetsial complex reflection groups. Since $R^{\mathbb{G}}_{\mathbb{T}_w}(\mathbf{1})$ is a virtual character, $f(\chi)$ takes rational integral values for $\chi \in \mathrm{Irr}(W)$.

The spetsial complex reflection groups are well-generated, so they have a unique largest reflection degree h, called the Coxeter number. An element $c \in W$ is called a Coxeter element if it is a regular element in the sense of Springer for the eigenvalue $e^{2i\pi/h}$.

Proposition 4. *Let c be a Coxeter element of the irreducible spetsial complex reflection group W. For $\chi \in \mathrm{Irr}(W)$ we have*

$$f(\chi)(c) = \begin{cases} (-1)^i & \text{if } \chi = \Lambda^i(V), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For Weyl groups, this is a reformulation of [20, Lemma 5].

For the irreducible spetsial primitive complex reflection groups, we checked the property by computer. Here is a `Chevie` [19] program which returns `true` for a spetsial group W if and only if Proposition 4 holds:

```
proposition3:=function(W)local un,c,ex;
  un:=UnipotentCharacters(W).harishChandra[1].charNumbers;
  c:=DeligneLusztigCharacter(W,PositionRegularClass(W,
                              Maximum(ReflectionDegrees(W)))).v{un};
  ex:=ChevieCharInfo(W).extRefl;
  return PositionsProperty(c,x->x<>0)=Permuted(ex,SortingPerm(ex))
    and c{ex}=List([0..Length(ex)-1],i->(-1)^i);
end;
```

It remains to deal with the infinite series $G(e, 1, n)$ and $G(e, e, n)$. We use the Φ-Harish-Chandra theory in spetsial groups. Let Φ be a K-cyclotomic polynomial, where K is the field of definition of W, and $(\mathbb{L}, \lambda)$ a Φ-cuspidal pair of the spets $\mathbb{G} = (V, W)$ (see [5, 4.31]). Then we have the following decomposition in unipotent characters indexed by characters of the relative Weyl group:

$$R^{\mathbb{G}}_{\mathbb{L}}(\lambda) = \sum_{\varphi\in\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))} \varepsilon_\varphi \varphi(1)\rho_\varphi$$

where the degree of the unipotent character ρ_φ is given by

$$\mathrm{Deg}(\rho_\varphi) = \varepsilon_\varphi \,\mathrm{Deg}(\lambda)\frac{(|\mathbb{G}|/|\mathbb{L}|)_{q'}}{S_\varphi}$$

where $(|\mathbb{G}|/|\mathbb{L}|)_{q'}$ is the quotient of the polynomial orders stripped of any factors of q that divide it and S_φ is the Schur element associated to φ in the relative Hecke algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$. If ζ is a root of Φ, the relative Hecke algebra is

a ζ-cyclotomic deformation of the group algebra of $W_{\mathbb{G}}(\mathbb{L},\lambda)$, which implies that $S_\varphi(\zeta) = |W_{\mathbb{G}}(\mathbb{L},\lambda)|/\varphi(1)$. Thus

$$\varepsilon_\varphi \varphi(1) = \frac{|W_{\mathbf{G}}(\mathbb{L},\lambda)|\,\mathrm{Deg}(\rho_\varphi)(\zeta)}{\mathrm{Deg}(\lambda)(\zeta)(|\mathbb{G}|/|\mathbb{L}|)_{q'}(\zeta)}.$$

We want to apply this to the case where $\mathbb{L} = \mathbb{T}$ is a Coxeter torus, that is when Φ is an irreducible factor of the h-th cyclotomic polynomial and $\zeta = \zeta_h$, in which case it simplifies somewhat: since $\mathbb{L}$ is the centralizer of a Φ-Sylow torus we have by [4, 5.3(ii) and 5.4] $|W_{\mathbb{G}}(\mathbb{L},\lambda)| = (|\mathbb{G}|/|\mathbb{L}|)_{q'}(\zeta_h)$, and since $\mathbb{L}$ is a torus we have $\mathrm{Deg}(\lambda) = 1$ so we get $\mathrm{Deg}(\rho_\varphi)(\zeta_h) = \varepsilon_\varphi\varphi(1)$. We finally get

$$R^{\mathbb{G}}_{\mathbb{T}}(\mathbf{1}) = \sum_{\varphi\in\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{T}))} \mathrm{Deg}(\rho_\varphi)(\zeta_h)\rho_\varphi.$$

The characters ρ_φ which appear on the right-hand side are the unipotent characters of the Φ-principal series — they are exactly the unipotent characters ρ such that $\mathrm{Deg}(\rho)(\zeta_h) \neq 0$. So to compute the Fourier transforms $f(\chi)(c)$ it remains to find which ρ_φ are unipotent characters U_χ in the principal series (which are similarly the unipotent characters ρ such that $\mathrm{Deg}(\rho)(1) \neq 0$), and for these compute $\mathrm{Deg}(U_\chi)(\zeta_h)$.

Malle describes in [14, 3.14 and 6.10] which characters are in the Φ-principal series. For $G(e,1,n)$ they are parameterized by the e-partitions whose corresponding symbol has a (n,ζ_e)-hook (here $h = en$). In the principal series, we get the characters U_{χ_k} where χ_k is parameterized by the e-partition $(n-k,1^k,\emptyset,\dots,\emptyset)$ for $k \in 0,\dots,n$, which is what we expect as $\chi_k = \Lambda^k(V)$.

For $G(e,e,n)$ characters are parameterized by e-partitions up to cyclic permutations (at least when the e-partition is not equal to one its $e-1$ distinct cyclic permutations, otherwise several characters are parameterized by the same e-partition) and similarly the Φ-principal series corresponds to e-partitions whose corresponding symbol has a $(n-1,\zeta_e)$-hook (here $h = e(n-1)$). For the principal series characters we find exactly the same partitions as in the $G(e,1,n)$ case, determining characters χ_k which correspond also to the $\Lambda^k(V)$.

It remains to compute $\mathrm{Deg}(U_{\chi_k})(\zeta_h)$. Since $U_{\chi_0} = \mathbf{1}$ it suffices to consider $k = 1,\dots,n$. For $G(e,1,n)$ we can use Chlouveraki's formula [8, 3.2] for the value of the corresponding Schur element S_{χ_k} of the Hecke algebra of $G(e,1,n)$ with parameters $((q,\zeta_e,\dots,\zeta_e^{e-1}),(q,-1),\dots,(q,-1))$, related to the degree by $\mathrm{Deg}(U_{\chi_k}) = S_{\chi_k}^{-1}\prod_{i=1}^n \frac{q^{ei}-1}{q-1}$, and after some transformations we get

$$S_{\chi_k} = \frac{e(q^k-1)(q^n-\zeta_e)\prod_{i=1}^{n-k}(q^{ei}-1)\prod_{i=1}^{k-1}(q^{ei}-1)}{q^{k+e\binom{k}{2}}(q-1)^n(q^{n-k}-\zeta_e)}$$

whence

$$\mathrm{Deg}(U_{\chi_k}) = \frac{q^{k+e\binom{k}{2}}(q^{n-k}-\zeta_e)\prod_{i=k}^{n}(q^{ei}-1)}{e(q^k-1)(q^n-\zeta_e)\prod_{i=1}^{n-k}(q^{ei}-1)}$$

To evaluate this at ζ_{en} we first write $\frac{q^{en}-1}{q^n-\zeta_e} = \prod_{i\in0,\dots,e-1;i\neq1}(q^n-\zeta_e^i)$ whose value at ζ_{en} is $\prod_{i\in0,\dots,e-1;i\neq1}(\zeta_e-\zeta_e^i) = \zeta_e^{-1}\prod_{i=1}^{e-1}(1-\zeta_e^i) = e\zeta_e^{-1}$, the last equality by

taking the value at $q=1$ of $\frac{q^e-1}{q-1}$. We next evaluate

$$\frac{\prod_{i=k}^{n-1}(q^{ei}-1)}{\prod_{i=1}^{n-k}(q^{ei}-1)}\,|_{q=\zeta_{en}}=\frac{\prod_{i=k}^{n-1}(\zeta_n^i-1)}{\prod_{i=1}^{n-k}-\zeta_n^i(\zeta_n^{n-i}-1)}=\prod_{i=1}^{n-k}(-\zeta_n^{-i})=(-1)^{n-k}\zeta_n^{-\binom{n-k+1}{2}}$$

and finally get the expected value $(-1)^k$.

For $G(e,e,n)$ we use that the spetsial Hecke algebra $\mathcal{H}$ is a subalgebra of the Hecke algebra $\mathcal{H}'$ of $G(e,1,n)$ with parameters $((1,\zeta_e,\dots,\zeta_e^{e-1}),(q,-1),\dots,(q,-1))$, and the quotient $\mathcal{H}'/\mathcal{H}$ is of dimension e. For a partition which has no cyclic symmetry (the case of our partitions except χ_1 for $e=n=2$, which does not occur since $G(2,2,2)$ is not irreducible) we have $S_\chi=S'_\chi/e$ where S'_χ is the Schur element for $\mathcal{H}'$, since the restriction of χ to $G(e,e,n)$ is irreducible. After some transformations we get from the formula of [8] for S'_χ

$$S_{\chi_k}=\frac{a(q^{n-1}-\zeta_e)(q-\zeta_e^{-1})(q^{n-k}-1)(q^k-1)\prod_{i=1}^{n-k-1}(q^{ei}-1)\prod_{i=1}^{k-1}(q^{ei}-1)}{q^{1+e\binom{k}{2}}(q^{n-k-1}\zeta_e^{-1}-1)(\zeta_e-q^{k-1})(q-1)^n}$$

where the term (q^k-1) is absent if $k=0$ and the term $(q^{n-k}-1)$ absent if $k=n$, and further, we have $a=1$ if k is 0 or n and $a=e$ otherwise. Since $\mathrm{Deg}(U_{\chi_k})=S_{\chi_k}^{-1}\frac{q^n-1}{q-1}\prod_{i=1}^{n-1}\frac{q^{ei}-1}{q-1}$ we get

$$\mathrm{Deg}(U_{\chi_k})=\frac{q^{1+e\binom{k}{2}}(q^{n-k-1}\zeta_e^{-1}-1)(\zeta_e-q^{k-1})(q^n-1)\prod_{i=k}^{n-1}(q^{ei}-1)}{a(q^{n-1}-\zeta_e)(q-\zeta_e^{-1})(q^{n-k}-1)(q^k-1)\prod_{i=1}^{n-k-1}(q^{ei}-1)}.$$

To get the value at $\zeta_{e(n-1)}$ similarly to the $G(e,1,n)$ case we start by simplifying $\frac{q^{e(n-1)}-1}{q^{n-1}-\zeta_e}$, then $\frac{\prod_{i=k}^{n-2}(q^{ei}-1)}{\prod_{i=1}^{n-k-1}(q^{ei}-1)}\,|_{q=\zeta_{e(n-1)}}$, and get the expected value $(-1)^k$. □

The above proof for $G(e,1,n)$ and $G(e,e,n)$ is not really a proof in the sense that it has not yet been proven that these groups satisfy all the axioms for spetses, including the unicity of the Fourier matrix. Another more direct approach would be to check the result using the values given by Malle [14] for the Fourier matrix, which is feasible but takes several pages of not very enlightening computations. We have preferred the approach given above.

The referee pointed out to me the following facts. A more conceptual statement of the above result on the intersection of the Φ-principal and the principal series is that the only Φ-block of the spetsial Hecke algebra $\mathcal{H}=\mathcal{H}_{\mathbb{G}}(\mathbb{T},\mathbf{1})$ of defect 1 is the principal Φ-block (and all others have defect 0). This is [1, Theorem 6.6] for Weyl groups. For Weyl groups [12, Theorem 9.6] further shows that the corresponding Brauer tree algebra is of type A. Assuming this, in [9, page 286] it is shown that the Schur element ratios evaluated a $q=\zeta$ are Morita invariants (generalizing [3, §3.7]). Our result begs for a conceptual proof of these observations for spetsial Hecke algebras.

Proposition 5. *For any $w\in W$ we have*

$$f\left(\sum_{\chi\in\mathrm{Irr}(W)}\chi(w)\chi\right)=\sum_{\chi\in\mathrm{Irr}(W)}f(\chi)(w)\chi.$$

Proof. The equality is a consequence of the fact that the matrix of f is symmetric, that is for $\chi,\psi\in\mathrm{Irr}\,W$ we have $\langle f(\chi),\psi\rangle_W=\langle f(\psi),\chi\rangle_W$. For Weyl groups this is a consequence of the explicit determination of f, see for example [11, Theorem

14.2.3]. There is also a case-free proof based on Shintani descent [10, III, Corollaire 3.5(iii)]. For spetses it is a consequence of the axiom that the Fourier matrix $F_{\chi,\psi} = \langle U_\chi, |W|^{-1}\sum_w \psi(w) R^{\mathbf{G}}_{\mathbf{T}_w}\rangle_{\mathbf{G}^F}$ is symmetric (see [15], §5). It follows that for $\psi \in \mathrm{Irr}(W)$ we have

$$\begin{aligned}\langle f(\sum_{\chi\in\mathrm{Irr}(W)} \chi(w)\chi), \psi\rangle_W &= \sum_{\chi\in\mathrm{Irr}(W)} \chi(w)\langle f(\chi),\psi\rangle_W \\ &= \langle f(\psi), \sum_{\chi\in\mathrm{Irr}(W)} \chi(w^{-1})\chi\rangle_W \\ &= f(\psi)(w)\end{aligned}$$

the last equality since the function $\sum_{\chi\in\mathrm{Irr}(W)} \chi(w^{-1})\chi$ is the normalized characteristic function of the class of w. This shows the proposition. □

It is clear that Proposition 4 and Proposition 5 for $w = c$ imply the second step described at the end of the introduction.

Tower equivalence. We now prove the third step described at the end of the introduction.

Proposition 6. *For a spetsial reflection group W and a class function χ on W, we have $\chi \equiv f(\chi)$.*

Proof. We induct on the rank of W. Henceforth we assume the statement for every proper parabolic subgroup W' of W.

To deduce the statement for W from the statement for W', we use the nice criterion for tower equivalence given by [6, Lemma 7.2]. We say that a class function is Coxeter-isotypic if all its components in the basis of irreducible characters have the same Coxeter number. Lemma [6, 7.2] states that two class functions χ and χ' are tower equivalent if for any maximal proper parabolic subgroup W' of W, and for any Coxeter number, the restrictions of the Coxeter-isotypic components of χ and χ' of same Coxeter number to W' are tower equivalent.

For any irreducible character the class function $f(\chi)$ is Coxeter-isotypic; this is a consequence of Corollary 3, which says that the Coxeter number is constant on Rouquier families, and the fact that the blocks of the Fourier matrix (the Lusztig families) coincide with Rouquier families; see [21] for Weyl groups and [15, §5.2] in general. It is thus sufficient to prove for any irreducible character χ and for any maximal proper parabolic subgroup W' of W that we have $\mathrm{Res}^W_{W'} f(\chi) \equiv \mathrm{Res}^W_{W'}\chi$. Since by the inductive hypothesis we know that, if f' is the truncated Lusztig Fourier transform on W', we have $f'(\mathrm{Res}^W_{W'}\chi) \equiv \mathrm{Res}^W_{W'}(\chi)$, it suffices to prove

$$\mathrm{Res}^W_{W'} f(\chi) = f'(\mathrm{Res}^W_{W'}\chi).$$

Let us thus show that for $w \in W'$, we have $f(\chi)(w) = f'(\mathrm{Res}^W_{W'}\chi)(w)$, that is, if $\mathbb{G}'$ is a split Levi subgroup of $\mathbb{G}$ of Weyl group W', we have $\langle U_\chi, R^{\mathbb{G}}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}} = \langle U_{\mathrm{Res}^W_{W'}\chi}, R^{\mathbb{G}'}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}'}$. We use the facts:

- $U_{\mathrm{Res}^W_{W'}\chi} = {}^*R^{\mathbb{G}}_{\mathbb{G}'}U_\chi$ where ${}^*R$ is the Lusztig restriction functor. See for example [11, Lemma 7.2.11] for reductive groups. This is a general property of commuting algebras — the algebras involved here are $\mathcal{H}_{\mathbb{G}}(\mathbb{T},\mathbf{1})$ and $\mathcal{H}_{\mathbb{G}'}(\mathbb{T},\mathbf{1})$— so it still works for spetses, as suggested in [5] by Axiom 4.6 for $\zeta = 1$ or Axiom 4.16(ii) for $(\mathbb{L},\lambda) = (\mathbb{T},\mathbf{1})$.

- The transitivity of Deligne-Lusztig induction and the adjunction of Deligne-Lusztig induction and restriction; see for example [11, chapter 9] for reductive groups. For spetses adjunction is the definition of Lusztig restriction and for transitivity see for example [15, Theorem 4.3 (b)].

It follows that

$$\begin{aligned}\langle U_{\mathrm{Res}^W_{W'}\chi}, R^{\mathbb{G}'}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}'} &= \langle {}^*R^{\mathbb{G}}_{\mathbb{G}'}U_\chi, R^{\mathbb{G}'}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}'}\\ &= \langle U_\chi, R^{\mathbb{G}}_{\mathbb{G}'}\circ R^{\mathbb{G}'}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}}\\ &= \langle U_\chi, R^{\mathbb{G}}_{\mathbb{T}_w}(\mathbf{1})\rangle_{\mathbb{G}}\end{aligned}$$

the first equality by the first fact, the second by adjunction and the third by transitivity. This shows the proposition.

□

It follows from Proposition 6 that the image of $\mathrm{Id}-f$ is a vector space of class functions tower equivalent to 0. We define the kernel of the tower equivalence to be the space of all class functions tower equivalent to 0. Computer calculations suggest the following question:

Question 7. *Is it true that for any irreducible spetsial reflection group different from G_{32}, the image of $\mathrm{Id}-f$ is equal to the kernel of the tower equivalence?*

This question is based on the verification of the equality of the above two spaces for many spetsial reflection groups, in particular for all Weyl groups of rank ≤ 10, and for all primitive spetsial groups. The existence of one counter-example suggests the possibility of others and prevents making this question into a conjecture.

For G_{32} the space of class functions on W is of dimension 102, the dimension of the kernel of tower equivalence is 78, and that of the image of $\mathrm{Id}-f$ is 77. It is to note that the blocks of both matrices on the basis of irreducible characters coincide with the Rouquier families of characters. The discrepancy occurs in the 16th family (in the `Chevie` numbering), which contains 7 characters. In the space spanned by this family the image of $\mathrm{Id}-f$ is of dimension 5 while the kernel of tower equivalence is of dimension 6.

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