



HAL
open science

ALGEBRAIC PROPERTIES OF THE GROUP OF GERMS OF DIFFEOMORPHISMS

Dominique Cerveau, Julie Déserti

► **To cite this version:**

Dominique Cerveau, Julie Déserti. ALGEBRAIC PROPERTIES OF THE GROUP OF GERMS OF DIFFEOMORPHISMS. 2022. hal-03725182

HAL Id: hal-03725182

<https://hal.science/hal-03725182v1>

Preprint submitted on 15 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ALGEBRAIC PROPERTIES OF THE GROUP OF GERMS OF DIFFEOMORPHISMS

DOMINIQUE CERVEAU AND JULIE DÉSERTI

ABSTRACT. We establish some algebraic properties of the group $\text{Diff}(\mathbb{C}^n, 0)$ of germs of analytic diffeomorphisms of \mathbb{C}^n , and its formal completion $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. For instance we describe the commutator of $\text{Diff}(\mathbb{C}^n, 0)$, but also prove that any finitely generated subgroup of $\text{Diff}(\mathbb{C}^n, 0)$ is residually finite; we thus obtain some constraints of groups that embed into $\text{Diff}(\mathbb{C}^n, 0)$. We show that $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ is an Hopfian group, and that $\text{Diff}(\mathbb{C}^n, 0)$ and $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ are not co-Hopfian. We end by the description of the automorphisms groups of $\widehat{\text{Diff}}(\mathbb{C}, 0)$, and $\text{Diff}(\mathbb{C}, 0)$.

INTRODUCTION

Let \mathcal{F} be a codimension p holomorphic foliation on the complex manifold M^n of dimension n . Suppose that $N \subset M$ is a submanifold of dimension $n - p$ invariant by \mathcal{F} . Then there exists a natural morphism

$$\Pi_1(N \setminus \text{Sing}(\mathcal{F}), *) \rightarrow \text{Diff}(\mathbb{C}^p, 0)$$

the so-called holonomy representation. As a consequence the study of such representations is an interesting problem, and requires the knowledge of algebraic properties of the groups $\text{Diff}(\mathbb{C}^p, 0)$. It is the purpose of this note in the same spirit of the description of the classical groups $\text{GL}(n, \mathbb{k})$ (see [Die55]).

Structure of the paper. If n is an integer, then G_0^n denotes $\text{Diff}(\mathbb{C}^n, 0)$, and \widehat{G}_0^n its formal completion. Furthermore, G_k^n denotes the set of elements of G_0^n tangent to the identity at order k , and \widehat{G}_k^n its formal completion. In §2 we establish some consequences of Poincaré and Siegel theorems:

Theorem A. *For any $k \geq 1$ we have*

$$[G_0^n, G_k^n] = G_k^n, \quad \text{and} \quad [\widehat{G}_0^n, \widehat{G}_1^n] = \widehat{G}_1^n.$$

As a consequence, we obtain:

$$[G_0^n, G_0^n] = \{f \in G_0^n \mid \det Df_{(0)} = 1\}.$$

Finding the finitely generated subgroups that embed into G_0^n is a problem related to the foliations theory; in §3 we deal with this question and get:

Theorem B. *Any finitely generated subgroup of G_0^n (resp. \widehat{G}_0^n) is residually finite.*

Hence, if H is a finitely generated and non residually finite group, then H does not embed into G_0^n .

But any finitely generated residually finite group is a Hopfian group ([Mal40, Mal65]); as a result any finitely generated subgroup of G_0^n (resp. \widehat{G}_0^n) is a Hopfian group. In §4 we refine this result, and also look at the co-Hopfian notion:

Theorem C. *The group \widehat{G}_0^1 is a Hopfian group.*

The groups G_1^n and \widehat{G}_1^n are not co-Hopfian groups.

Inspired by [D06] we study in §6 the automorphisms groups of G_0^1 and \widehat{G}_0^1 :

Theorem D. *The group $\text{Aut}(\widehat{G}_0^1)$ is generated by the inner automorphisms and the automorphisms of the field \mathbb{C} . In other words*

$$\text{Out}(\widehat{G}_0^1) \simeq \text{Aut}(\mathbb{C}, +, \cdot)$$

where $\text{Out}(\widehat{G}_0^1)$ denotes the non-inner automorphisms of \widehat{G}_0^1 .

The group $\text{Out}(G_0^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Finally we try to generalize this result in higher dimension, and give some partial results (§7).

1. NOTATIONS AND DEFINITIONS

Let n be an integer; consider

$$G_0^n = \text{Diff}(\mathbb{C}^n, 0) = \{f: \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ holomorphic} \mid Df_{(0)} \in \text{GL}(\mathbb{C}^n)\}.$$

Denote by \widehat{G}_0^n the formal completion of G_0^n ; in other words \widehat{G}_0^n is the set of formal diffeomorphisms $\widehat{f} = (\widehat{f}_1, \widehat{f}_2, \dots, \widehat{f}_n)$ where

- \widehat{f}_i is a formal serie,
- $\widehat{f}(0) = 0$
- and $D\widehat{f}_{(0)}$ belongs to $\text{GL}(\mathbb{C}^n)$.

Let k be an integer; consider G_k^n the set of elements of G_0^n tangent to the identity at order k (i.e. f belongs to G_k^n if and only if $f = \text{id} +$ terms of order $\geq k+1$). Denote by \widehat{G}_k^n the formal completion of G_k^n .

The G_k^n 's (resp. \widehat{G}_k^n 's) are normal subgroups of G_0^n (resp. \widehat{G}_0^n). The quotients G_0^n/G_k^n and $\widehat{G}_0^n/\widehat{G}_k^n$ are isomorphic, and can be identified with the group of polynomial maps

$$\text{Pol}_k^n = \{f: \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ polynomial} \mid f(0) = 0, \deg f \leq k, Df_{(0)} \in \text{GL}(\mathbb{C}^n)\}$$

whose the law group is the law of composition truncated to order $k+1$.

Remark that G_0^n/G_k^n acts faithfully on the space $\mathbb{C}[x_1, x_2, \dots, x_n]_k$ of polynomials of degree less or equal than k (still by truncated composition). Therefore the G_0^n/G_k^n 's can be identified with subgroups of $\text{GL}(\mathbb{C}[x_1, x_2, \dots, x_n]_k)$.

The group G_0^1 is quite amazing: it is not solvable, contains free subgroups ([BCLN96]) and is "limit" of solvable groups; indeed, the quotients G_0^1/G_k^1 are solvable.

Denote by $p_k: G_0^n \rightarrow G_0^n/G_k^n$ the projection.

Let \mathfrak{M}_n be the maximal ideal of $O(\mathbb{C}^n, 0)$ given by

$$\mathfrak{M}_n = \{f \in O(\mathbb{C}^n, 0) \mid f(0) = 0\}.$$

Denote by χ_0^n the set of germs of holomorphic vector fields at the origin of \mathbb{C}^n , and by $\widehat{\chi}_0^n$ its formal completion. Let χ_k^n be the subspace of χ_0^n made up of vector fields that vanish at 0 at order $k-1$, that is $\chi_k^n = \mathfrak{M}_n^k \chi_0^n$. Finally, let us denote by $\widehat{\chi}_k^n$ the formal completion of χ_k^n .

2. POINCARÉ AND SIEGEL THEOREMS AND THEIR CONSEQUENCES

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a n -uplet of non-zero complex numbers. We say that λ is **without resonance** if the equality $\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_n^{p_n} = \lambda_j$, $p_\ell \in \mathbb{N}$, implies $p_j = 1$, and $p_\ell = 0$ for $\ell \neq j$.

Theorem 2.1 (formal Poincaré Theorem). *Let ϕ be an element of \widehat{G}_0^n . Set $A = D\phi_{(0)}$.*

Assume that $\text{Spec}(A)$ is without resonance.

Then ϕ is formally conjugate to A , i.e. there exists $\varphi \in \widehat{G}_1^n$ such that $\phi = \varphi A \varphi^{-1}$.

Theorem 2.2 (holomorphic Poincaré Theorem). *Let ϕ be an element of G_0^n . Set $A = D\phi_{(0)}$.*

Assume that $\text{Spec}(A)$ is without resonance, and that $\text{Spec}(A)$ is contained either in the open unit disk $\mathbb{D}(0, 1)$, or in the complement $\mathbb{C}\mathbb{D}(0, 1)$ of $\mathbb{D}(0, 1)$.

Then ϕ is holomorphically conjugate to A , i.e. there exists $\varphi \in G_1^n$ such that $\phi = \varphi A \varphi^{-1}$.

We say that ϕ is **formally linearizable** (resp. **holomorphically linearizable**) in the formal (resp. holomorphic) case. In both cases the diffeomorphism φ is called the **linearizing map**.

Remark 2.3. We do not use explicitly the Siegel linearization Theorem when $\text{Spec}(A) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is contained neither in $\mathbb{D}(0, 1)$, nor in $\mathbb{C}\mathbb{D}(0, 1)$; it requires diophantine conditions controlling $|\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_n^{p_n} - \lambda_j|$ that produce the convergence of the linearizing maps ([Sie42, Sie52]).

Remark 2.4. In the Poincaré Theorems assume that ϕ can be written as Ah where $h = \text{id} + \dots \in G_k^n$; then the linearizing map φ can be chosen in G_k^n (resp. \widehat{G}_k^n in the formal case).

Curiously while the proof of Poincaré Theorem comes from analysis (in its holomorphic version) we deduce from it algebraic properties. Let us recall that if H is a group, the **derived subgroup** (or **commutator group**) of H is the subgroup of H generated by all the commutators; we denote it by $[H, H]$.

Theorem 2.5. *We have $[G_0^n, G_1^n] = G_1^n$, and similarly $[\widehat{G}_0^n, \widehat{G}_1^n] = \widehat{G}_1^n$.*

Remark 2.6. Such a result has been proved in [CCD13] for $n = 1$.

Proof. Let A be an element of $\mathrm{GL}(n, \mathbb{C})$ that satisfies the assumptions of Poincaré Theorem, for instance $A = \lambda \mathrm{id}$ with $0 < |\lambda| < 1$. If h belongs to G_1^n , then hA is linearizable, that is there exists $\varphi \in G_1^n$ such that

$$\varphi A \varphi^{-1} = hA.$$

As a result $[\varphi, A] = h$, and we get the result.

A similar proof works in the formal case. \square

Corollary 2.7. *We can describe the commutator of G_0^n :*

$$[G_0^n, G_0^n] = \{f \in G_0^n \mid \det Df_{(0)} = 1\}.$$

Proof. Let f be an element of G_0^n ; denote by $A \in \mathrm{SL}(n, \mathbb{C})$ its linear part. We can write A as $[\alpha, \beta]$ with $\alpha, \beta \in \mathrm{GL}(n, \mathbb{C})$. Note that one can choose β without resonance, and $\mathrm{Spec}(\beta) \subset \mathbb{D}(0, 1)$. Let h be the element of G_1^n defined by: $h = \alpha\beta^{-1}\alpha^{-1}f\beta$; in particular $f = \alpha\beta\alpha^{-1}h\beta^{-1}$. According to Poincaré Theorem there exists $\varphi \in G_1^n$ such that

$$\alpha\beta\alpha^{-1}h = \varphi\beta\varphi^{-1};$$

then $f = [\varphi, \beta]$. \square

Theorem 2.5 can be generalized as follows:

Theorem 2.8. *Any element of G_k^n is the commutator of an element of G_0^n and an element of G_k^n , i.e. $[G_0^n, G_k^n] = G_k^n$.*

We deduce from it the following statement that can be useful:

Proposition 2.9. *Let H be a group, and let $\tau: G_0^n \rightarrow H$ be a group homomorphism.*

Assume that there exists f in $\ker \tau$ such that $Df_{(0)} = A$ satisfies the assumptions of Poincaré (resp. Siegel) Theorem. Then G_1^n is contained in $\ker \tau$.

Proof. Let h be an element of G_1^n . Since A satisfies the assumptions of Poincaré (resp. Siegel) Theorem, then hf and f are conjugate: there exists φ in G_1^n such that $hf = \varphi f \varphi^{-1}$. As a consequence, $h = [\varphi, f]$ and h belongs to $\ker \tau$. \square

Remark 2.10. Since A and $f \in \ker \tau$ are conjugate, A belongs to $\ker \tau$, and so does the normal subgroup of $\mathrm{GL}(n, \mathbb{C})$ generated by A . The quotient $G_0^n /_{\ker \tau} \simeq \mathrm{im} \tau$ can thus be identified with the quotient $\mathrm{GL}(\mathbb{C}^n) /_{\mathrm{GL}(\mathbb{C}^n) \cap \ker \tau}$.

3. RESIDUALLY FINITE GROUPS

There are a number of equivalent definitions of residually finite groups; we will use the following one:

Definition 3.1. A group H is **residually finite** if for every element h in $H \setminus \{\text{id}_H\}$ there exists a group morphism $\varphi: H \rightarrow F$ from H to a finite group F such that $\varphi(h) \neq \text{id}_F$.

The groups \mathbb{Z} and $\text{SL}(n, \mathbb{Z})$ are residually finite (by reduction modulo p). Subgroups of residually finite groups are residually finite. Conversely, a non-finite simple group is not residually finite, and the Baumslag-Solitar group

$$\text{BS}(2, 3) = \langle a, b \mid a^{-1}b^2a = b^3 \rangle$$

is not residually finite ([BS62]).

A group is **linear** if it is isomorphic to a subgroup of $\text{GL}(n, \mathbb{k})$ where \mathbb{k} is a field.

Theorem 3.2 ([Mal40, Mal65]). *A finitely generated linear group is residually finite.*

In particular there is no faithful linear representation of a non-residually finite, finitely generated group.

Remark 3.3. The assumption "finitely generated" turns out to be essential. For instance \mathbb{Q} is not residually finite. If ξ is a positive transcendental number, then $\xi^{\mathbb{Q}} = \{\xi^{\frac{p}{q}} \mid \frac{p}{q} \in \mathbb{Q}\}$ is isomorphic to \mathbb{Q} . As a subgroup of a residually finite group is residually finite, we thus get that $\text{GL}(1, \mathbb{C}) \simeq \mathbb{C}^*$, $\text{GL}(n, \mathbb{C})$, G_0^n and \widehat{G}_0^n are not residually finite.

The following statement, that is mentioned in [CLPT19] without detail in the 1-dimensional and formal case, is a direct consequence of Theorem 3.2:

Theorem 3.4. *Any finitely generated subgroup of G_0^n (resp. \widehat{G}_0^n) is residually finite.*

Proof. Let H be a finitely generated subgroup of G_0^n . Let h be an element of $G_0^n \setminus \{\text{id}\}$. There exists an integer k such that $p_k(h)$ is non-trivial in the quotient group G_0^n/G_k^n . Let us recall that G_0^n/G_k^n is isomorphic to a subgroup of a linear group; Theorem 3.2 applied to the group $p_k(H)$ asserts the existence of a morphism $\varphi_k: p_k(H) \rightarrow F_k$ from $p_k(H)$ to a finite group F_k such that $\varphi_k(p_k(h)) \neq \text{id}_{F_k}$. Then, the morphism $\varphi_k \circ p_{k|H}: H \rightarrow F_k$ suits. \square

Finding the finitely generated subgroups that embed into G_0^n is an important problem, in particular related to the theory of foliations (representations of holonomy, and cycles in leaves). For instance in [CCGS20] one can find:

Theorem 3.5 ([CCGS20]). *The fundamental group of the compact surface of genus g embeds into G_0^1 , and so into G_0^n .*

Conversely we get the following statement:

Corollary 3.6. *The Baumslag-Solitar group $BS(2,3)$ does not embed into G_0^n .*

More generally, if H is a finitely generated and non residually finite group, then H does not embed into G_0^n .

Theorem 3.4 has direct applications in the theory of holomorphic foliations. Toledo constructs smooth complex projective varieties with fundamental groups which are not residually finite, answering to Serre's question ([Tol93]). Assume that \mathcal{F} is, for instance, a codimension one holomorphic foliation on the complex manifold M having an invariant variety $N \subset M$ satisfying Toledo's property. Then the holonomy representation ([CCD13])

$$\text{Hol}: \Pi_1(N, *) \rightarrow \text{Diff}(\mathbb{C}, 0)$$

is not faithful. As a consequence, there exist families of cycles in the leaves of \mathcal{F} near the invariant manifold N .

In [DS05] Drutu and Sapir construct residually finite groups that are not linear. One of their examples is the group $\mathcal{G} = \langle a, b \mid b^2ab^{-2} = a^2 \rangle$; we prove that \mathcal{G} can not be embedded into G_0^1 (or \widehat{G}_0^1):

Proposition 3.7. *There is no faithful representation of $\mathcal{G} = \langle a, b \mid b^2ab^{-2} = a^2 \rangle$ into \widehat{G}_0^1 (resp. G_0^1).*

Proof. Assume by contradiction that there exists a faithful representation φ of \mathcal{G} into \widehat{G}_0^1 . Set $A = \varphi(a)$, and $B = \varphi(b)$. From $B^2AB^{-2} = A^2$ we get that A and A^2 are conjugate (by B^2). In particular A is tangent to the identity, that is A belongs to \widehat{G}_1^1 ; so there exists a formal vector field X of order at least 2 such that $A = \exp X$. Since $\exp X$ and $\exp 2X$ are conjugate, the vector fields X and $2X$ are conjugate (by B^2). One can assume, up to conjugacy, that $X = \frac{x^{v+1}}{1-\lambda x} \frac{\partial}{\partial x}$ with $v \geq 1$ and $\lambda \in \mathbb{C}^*$. Let μx be the linear part of B^2 ; note that μx has to conjugate the first non-zero jet $x^{v+1} \frac{\partial}{\partial x}$ of X to the first non-zero jet $2x^{v+1} \frac{\partial}{\partial x}$ of $2X$. Hence $\mu^v = 2$, and the linear part of B is a $2v$ -th root of 2; we thus can linearize B , i.e. assume that $B = \alpha x$ where $\alpha = 2^{\frac{1}{2v}}$. Let $Y = a(x) \frac{\partial}{\partial x}$ be a vector field of order $v+1$ such that $B_*^2 Y = 2Y$, that is such that $\alpha^{-2} a(\alpha^2 x) = 2a$, i.e. in other words

$$a(\alpha^2 x) = 2\alpha^2 a. \quad (3.1)$$

Write a as $a = \sum_{\ell \geq v+1} a_\ell x^\ell$; then (3.1) yields to $\alpha^{2\ell} a_\ell = 2\alpha^2 a_\ell$, i.e. $\alpha^{2(\ell-1)} a_\ell = 2a_\ell$. For any $a_\ell \neq 0$ we get $\alpha^{2(\ell-1)} = 2$; in other words $2^{\frac{\ell-1}{v}-1} = 1$, and so $2^{\frac{\ell-(v+1)}{v}} = 1$; as a consequence, $\ell = v+1$. As a result, in the linearizing coordinate for B , we have: $B = \alpha x$ and $A = \exp cx^{v+1} \frac{\partial}{\partial x}$ for some c . In particular the group generated by A and B is linear, whereas \mathcal{G} is not ([DS05]): contradiction. \square

In [CL98] the authors prove the following curious result. Let γ be an irreducible curve in $\mathbb{P}_{\mathbb{C}}^2$ of degree p^s with p prime number. If $\varphi: \Pi_1(\mathbb{P}_{\mathbb{C}}^2 \setminus \gamma, *) \rightarrow G_0^1$ or $\varphi: \Pi_1(\mathbb{P}_{\mathbb{C}}^2 \setminus \gamma, *) \rightarrow \widehat{G}_0^1$ is a morphism, then the image of φ is a finite group (conjugate to a group of linear rotations); moreover, there are some γ such that $\Pi_1(\mathbb{P}_{\mathbb{C}}^2 \setminus \gamma, *)$ contains a free group of rank 2.

Problem 3.8. Let γ be a curve in $\mathbb{P}_{\mathbb{C}}^2$; are the groups $\Pi_1(\mathbb{P}_{\mathbb{C}}^2 \setminus \gamma, *)$ linear groups ? are the groups $\Pi_1(\mathbb{P}_{\mathbb{C}}^2 \setminus \gamma, *)$ residually finite groups ?

4. HOPFIAN AND CO-HOPFIAN GROUPS

Definition 4.1. A group G is **Hopfian** if every surjective homomorphism group from G to G is an isomorphism.

Equivalently, a group is Hopfian if and only if it is not isomorphic to any of its proper quotients.

Definition 4.2. A group G is **co-Hopfian** if every injective homomorphism group from G to G is an isomorphism.

Equivalently, a group is co-Hopfian if and only if it is not isomorphic to any of its proper subgroups.

Every finite group and every simple group are Hopfian groups. The group \mathbb{Z} of integers and the group \mathbb{Q} of rationals are Hopfian groups. However, \mathbb{C}^* is not a Hopfian group (the morphisms $\mathbb{C}^* \rightarrow \mathbb{C}^*$, $x \mapsto x^p$ are not injective), and \mathbb{R}^* is not a Hopfian group (the morphisms $\mathbb{R}^* \rightarrow \mathbb{R}^*$, $x \mapsto x^p$, p even, are not injective). In [DÓ7] the author shows that the group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ of birational self-maps of the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$ is Hopfian.

Let us mention an other statement due to Malcev:

Theorem 4.3 ([Mal40, Mal65]). *Any finitely generated residually finite group is a Hopfian group.*

Corollary 4.4. *Any finitely generated subgroup of G_0^n (resp. \widehat{G}_0^n) is a Hopfian group.*

Let us now establish the following statement:

Theorem 4.5. *The group \widehat{G}_0^1 is a Hopfian group.*

To prove it we will use the following result of finite determination, result specific to the 1-dimensional and formal case:

Lemma 4.6. *Let h be an element of \widehat{G}_1^1 . There exists an integer ℓ such that if g belongs to \widehat{G}_ℓ^1 , then h and hg are conjugate in the group \widehat{G}_1^1 .*

In other words, if two elements of \widehat{G}_1^1 coincide up to a sufficiently large order, then they are conjugate.

The proof is part of folklore and it is difficult to determine the "first author", maybe F. Takens. It is based on the following two facts:

- If $X = a(x) \frac{\partial}{\partial x}$ is a formal vector field such that $a(0) = a'(0) = 0$, then X is formally conjugate to a vector field of the type $X_{p,\lambda} = \frac{x^{p+1}}{1+\lambda x^p} \frac{\partial}{\partial x}$ for some $\lambda \in \mathbb{C}$ and some p . It is a smart consequence of the local inversion theorem ([CCD13]). Such vector fields are thus finitely "determined": if Y is of order sufficiently large, then X and $X + Y$ are conjugated.

- If h belongs to \widehat{G}_1^1 , then there exists a formal vector field $X = a(x)\frac{\partial}{\partial x}$ with $a(0) = a'(0) = 0$ such that $h = \exp X$, i.e. h is the flow at time 1 of the vector field X .

Remark 4.7. The first fact is still true in the holomorphic case whereas the second is not ([É75]): if h belongs to G_1^1 and $h = \exp X$, then X is most of the time divergent.

Proof of Theorem 4.5. Let $\varphi: \widehat{G}_0^1 \rightarrow \widehat{G}_0^1$ be a surjective morphism. Assume that φ is not injective. Let $f \in \widehat{G}_0^1 \setminus \{\text{id}\}$ such that $\varphi(f) = \text{id}$. Replacing f by a non-trivial commutator $[g, f]$ (that also belongs to $\ker \varphi$) if needed we can assume that f belongs to $\widehat{G}_1^1 \setminus \{\text{id}\}$. Consider h in \widehat{G}_ℓ^1 for ℓ sufficiently large. According to Lemma 4.6 the elements f and hf are conjugate, i.e. there exists $g \in \widehat{G}_1^1$ such that $gfg^{-1} = hf$. As a consequence, $h = [g, f]$ belongs to $\ker \varphi$. Hence, $\ker \varphi$ contains \widehat{G}_ℓ^1 for ℓ sufficiently large. Since φ is surjective, \widehat{G}_0^1 and $\widehat{G}_0^1 / \ker \varphi$ are isomorphic. As \widehat{G}_1^ℓ is contained in $\ker \varphi$ the morphism $\widehat{G}_0^1 / \widehat{G}_\ell^1 \rightarrow \widehat{G}_0^1 / \ker \varphi$ is surjective. The group $\widehat{G}_0^1 / \widehat{G}_\ell^1$ is solvable, so does $\widehat{G}_0^1 / \ker \varphi \simeq \widehat{G}_0^1$: contradiction with the fact that \widehat{G}_0^1 contains free subgroups ([BCLN96]). \square

Problems 4.8. 1) Is the group G_0^1 a Hopfian group? One way to answer to this question is to show that if $\tau: G_0^1 \rightarrow G_0^1$ is surjective, then τ can be extended to a morphism $\tilde{\tau}: \widehat{G}_0^1 \rightarrow \widehat{G}_0^1$ still surjective.

2) Are the groups G_0^n and \widehat{G}_0^n Hopfian groups?

Unfortunately the method used for the proof of Theorem 4.5 turns out to be ineffective for Problems 4.8.

Let us now deal with the notion of co-Hopfian group.

Note that if $\tau: \mathbb{C} \rightarrow \mathbb{C}$ is a non-surjective morphism of the field \mathbb{C} , then τ induces an injective and non-surjective homomorphism from \widehat{G}_0^n into itself defined by

$$\sum A_I x^I \mapsto \sum \tau(A_I) x^I$$

where A_I belongs to \mathbb{C}^n . In particular \widehat{G}_0^n is not co-Hopfian.

Theorem 4.9. *The groups G_1^n and \widehat{G}_1^n are not co-Hopfian groups.*

Proof. Let us first assume that $n = 1$. The morphism $\tau_1: f \mapsto \tau_1(f)$ defined by $\tau_1(f)(x) = (f(x^2))^{1/2}$ is injective but not surjective; indeed any $\tau_1(f)$ commutes with the involution $x \mapsto -x$ (we choose the determination $\sqrt{1} = 1$).

Suppose now that $n > 1$. We will use a similar idea considering the application

$$E: (x_1, x_2, \dots, x_n) \mapsto (x_1^2, x_1 x_2, x_1 x_3, \dots, x_1 x_n)$$

whose inverse is

$$E^{-1}: (x_1, x_2, \dots, x_n) \mapsto \left(\sqrt{x_1}, \frac{x_2}{\sqrt{x_1}}, \frac{x_3}{\sqrt{x_1}}, \dots, \frac{x_n}{\sqrt{x_1}} \right).$$

Let us choose the determination of E^{-1} associated to the principal determination of $\sqrt{\cdot}$; the application τ_n defined by

$$\tau_n(f) = \tau_n(f_1, f_2, \dots, f_n)(x) = E^{-1}(f \circ E)(x)$$

is an injective morphism that is not surjective; indeed the $\tau_n(f)$ commute with the involution $x \mapsto -x$. \square

Problem 4.10. Is the group G_0^n a co-Hopfian group ?

5. TITS ALTERNATIVE

A group G satisfies **Tits alternative** if for every finitely generated subgroup H of G

- either H is virtually solvable (*i.e.* H contains a solvable subgroup of finite index),
- or H contains a non-abelian free subgroup.

Tits proved in [Tit72] that linear groups satisfy Tits alternative.

Problem 5.1. Do the groups G_0^n and \widehat{G}_0^n satisfy Tits alternative ?

This important question is related to the Galois theory of holomorphic foliations. Note that, as it can be seen in [BCLN96], the group G_0^n contain free subgroups of rank ≥ 2 .

6. AUTOMORPHISMS GROUPS OF \widehat{G}_0^1 AND G_0^1

In [Whi63] Whittaker proves the following statement: *let X and Y be compact manifolds, with or without boundary, and φ be a group isomorphism between the group $\text{Hom}(X)$ of all homeomorphisms of X into itself and $\text{Hom}(Y)$, then there exists an homeomorphism ψ of X onto Y such that $\varphi(f) = \psi f \psi^{-1}$ for all $f \in \text{Hom}(X)$.* When $X = Y$ we get that every automorphism of $\text{Hom}(X)$ is an inner one. In [Fil82] Filipkiewicz gives a similar result in the context of differentiable manifolds: let M and N be smooth manifolds without boundary, and let $\text{Diff}^p(M)$ denote the group of C^p -diffeomorphisms of M . The author proves that if $\text{Diff}^p(M)$ and $\text{Diff}^q(N)$ are isomorphic as abstract groups, then $p = q$, and the isomorphism induced by a C^p -diffeomorphism from M to N . Let us mention that there are similar results in different contexts: *see for instance* [Ban86, Ban97, D06]... in particular in [D06] the author proves that any automorphism of the Cremona group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ of birational self-maps of the complex projective plane is the composition of an inner automorphism and an automorphism of the field of complex numbers. The proof is based on the study of maximal abelian uncountable subgroups of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. Let G_1, G_2, \dots, G_ℓ be some of such subgroups, and let φ be an automorphism of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$; the author proves that up to inner conjugacy and the action of an automorphism of the field \mathbb{C} we have $\varphi|_{G_k} = \text{id}$ for $1 \leq k \leq \ell$, and deduce from it that $\varphi|_{\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)} = \text{id}$. A similar strategy will be used in this section.

6.1. **Automorphisms groups of \widehat{G}_0^1 .** The study of the maximal abelian subgroups of \widehat{G}_0^n (resp. G_0^n) is essential for the understanding of automorphism groups of \widehat{G}_0^n (resp. G_0^n). Unfortunately the comprehension of these groups can only be apprehended in the case \widehat{G}_0^1 . Once again the essential argument is the following one: if f belongs to $\widehat{G}_k^1 \setminus \widehat{G}_{k+1}^1$, then f is conjugate to $\exp X_{k,\lambda} = \exp \frac{x^{k+1}}{1+\lambda x^k} \frac{\partial}{\partial x}$ for a certain λ . A computation ([CM88, CCD13]) shows that the centralizer

$$\text{Cent}(\exp X_{k,\lambda}, \widehat{G}_0^1) = \{f \in \widehat{G}_0^1 \mid f \exp X_{k,\lambda} = \exp X_{k,\lambda} f\}$$

of $\exp X_{k,\lambda}$ in \widehat{G}_0^1 coincides with the group

$$A_{k,\lambda} = \{ \exp t X_{k,\lambda} \mid t \in \mathbb{C} \} \times \{ x \mapsto \xi x \mid \xi^k = 1 \}.$$

This group, which is abelian and so maximal abelian, contains exactly $(k-1)$ non-trivial torsion elements.

Let λ be in \mathbb{C}^* ; denote by $\underline{\lambda}: x \mapsto \lambda x$ the homothety of ratio λ . If λ is not a root of unity, then

$$\text{Cent}(\underline{\lambda}, \widehat{G}_0^1) = A_0 = \{ \underline{\mu} \mid \mu \in \mathbb{C}^* \}$$

which is also a maximal abelian subgroup. A subgroup of \widehat{G}_0^1 whose all elements are periodic is abelian and conjugate to a subgroup of A_0 (see [CCD13, Corollary 7.21]); in particular a maximal abelian subgroup of \widehat{G}_0^1 contains a non-periodic element. As a consequence we get:

Theorem 6.1. *The maximal abelian subgroups of \widehat{G}_0^1 are the conjugate of the groups A_0 and $A_{k,\lambda}$ where $k \geq 1$ is an integer and λ an element of \mathbb{C} .*

Let us now consider $\sigma: \widehat{G}_0^1 \rightarrow \widehat{G}_0^1$ an automorphism of \widehat{G}_0^1 . For instance the inner automorphisms of \widehat{G}_0^1

$$f \mapsto \tau(f) = \varphi f \varphi^{-1}$$

are such examples. Let $\text{Aut}(\mathbb{C}, +, \cdot)$ be the automorphisms group of the field \mathbb{C} . Any element $\tau \in \text{Aut}(\mathbb{C}, +, \cdot)$ is an automorphism of the field \mathbb{C} , then τ induces an automorphism σ^τ that sends $f = \sum_{\ell \geq 1} a_\ell x^\ell$ to

$$\sigma^\tau(f) = \sum_{\ell \geq 1} \tau(a_\ell) x^\ell.$$

Note that the image of a maximal abelian subgroup of \widehat{G}_0^1 by σ is still a maximal abelian subgroup of \widehat{G}_0^1 .

Theorem 6.2. *Let σ be an automorphism of \widehat{G}_0^1 . Then, up to a suitable conjugacy, $\sigma(A_0) = A_0$ and $\sigma(A_{k,0}) = A_{k,0}$ for any integer $k \geq 1$.*

Furthermore, if λ is non-zero, then $\sigma(A_{k,\lambda})$ is conjugate to $A_{k,\mu}$ for some μ in \mathbb{C}^ .*

Proof. The group A_0 has an infinite number of torsion elements whereas the $A_{k,\lambda}$'s don't; this gives the first assertion. We can thus assume that $\sigma(A_0) = A_0$. Let us note that A_0 acts by conjugacy on the groups $A_{k,0}$: if $\underline{\mu}$ belongs to A_0 , then

$$\underline{\mu} \exp t X_{k,0} \underline{\mu}^{-1} = \exp \mu^{-k} t X_{k,0}.$$

However, the conjugate of $A_{k,\lambda}$ by $\underline{\mu}$ is $A_{k,\mu^{-k}\lambda}$, and A_0 does not act on the $A_{k,\lambda}$. As a result $A_0 = \sigma(A_0)$ acts by conjugacy on $\sigma(A_{k,0})$. Counting the torsion elements we get that $\sigma(A_{k,0})$ is conjugate to $A_{k,\lambda}$ for some λ in \mathbb{C} . In particular

$$\sigma\left(\left\{\exp t x^{k+1} \frac{\partial}{\partial x}\right\}\right) = \left\{\exp t X \mid t \in \mathbb{C}\right\}$$

where X is a formal vector field conjugate to $X_{k,\lambda}$ for some λ in \mathbb{C} .

Lemma 6.3. *One has: $X = a_k x^{k+1} \frac{\partial}{\partial x}$ for some $a_k \in \mathbb{C}^*$.*

Proof of Lemma 6.3. The action of $\underline{\mu}$: $x \mapsto \mu x$ on $\exp t X$ can be written as

$$\underline{\mu} \exp t X \underline{\mu}^{-1} = \exp t \underline{\mu}_* X.$$

If $X = \sum_{\ell \geq k} a_\ell x^\ell \frac{\partial}{\partial x}$, $a_k \neq 0$, then

$$\underline{\mu}_* X = \sum_{\ell \geq k} a_\ell \mu^{1-\ell} x^\ell \frac{\partial}{\partial x}.$$

Let μ be a non-zero complex number; assume that μ is not a root of unity. As we have seen

$$\underline{\mu} \exp t X \underline{\mu}^{-1} = \exp t_0 X.$$

for some t_0 . As a consequence $\underline{\mu}_* X = t_0 X$ (see [CM88]) and

$$\begin{cases} a_\ell \mu^{1-\ell} = t_0 a_\ell \text{ for } \ell \geq k \\ t_0 = \mu^{1-k} \end{cases}$$

If for $\ell > k$ one of the a_ℓ is non-zero, then $\mu^{1-\ell} = t_0 = \mu^{1-k}$, and μ is a root of unity: contradiction. \square

As a result $\sigma(A_{k,0}) = A_{k,0}$ for any k ; the torsion elements can optionally be permuted and $\exp t x^{k+1} \frac{\partial}{\partial x}$ is sent onto $\exp t_{\sigma_k} x^{k+1} \frac{\partial}{\partial x}$. Let us remark that $\mathbb{C} \ni t \mapsto t_{\sigma_k} \in \mathbb{C}$ is an additive homomorphism group. \square

Hence an automorphism σ of \widehat{G}_0^1 induces a multiplicative isomorphism of \mathbb{C}^*

$$\sigma(\underline{\lambda}) = \lambda^\sigma$$

and an additive isomorphism σ_k of \mathbb{C} for any k

$$\sigma_k(t) = t_{\sigma_k}.$$

Let us come back to the action of A_0 on $A_{k,0}$

$$\left(\underline{\lambda}, \exp t x^{k+1} \frac{\partial}{\partial x}\right) \mapsto \exp \lambda^k t x^{k+1} \frac{\partial}{\partial x}$$

that corresponds to the action of \mathbb{C}^* on \mathbb{C}

$$(\lambda, t) \mapsto \lambda^k t.$$

The action is transformed by the automorphism σ into

$$\left(\lambda^\sigma, \exp t_{\sigma_k} x^{k+1} \frac{\partial}{\partial x}\right) \mapsto \exp (\lambda^\sigma)^k t_{\sigma_k} x^{k+1} \frac{\partial}{\partial x}$$

but also into

$$\left(\lambda^\sigma, \exp t_{\sigma_k} x^{k+1} \frac{\partial}{\partial x}\right) \mapsto \exp(\lambda^k t)_{\sigma_k} x^{k+1} \frac{\partial}{\partial x}.$$

Therefore

$$(\lambda^\sigma)^k t_{\sigma_k} = (\lambda^k t)_{\sigma_k};$$

in particular for $t = 1$ we get

$$s^\sigma 1_{\sigma_k} = s_{\sigma_k}.$$

Hence $s \mapsto s^\sigma$ is an automorphism of the field \mathbb{C} , and the additive morphisms $s \mapsto s_{\sigma_k}$ differ from s^σ only by a multiplicative constant 1_{σ_k} . Up to the action of the automorphism of \widehat{G}_0^1 associated to this field automorphism we can assume that $s \mapsto s^\sigma$ is the identity and that $s_{\sigma_k} = \varepsilon_k s$ where ε_k denotes a non-zero constant.

Lemma 6.4. *If σ belongs to $\text{Aut}(\widehat{G}_0^1)$, then $\sigma(\widehat{G}_k^1) = \widehat{G}_k^1$.*

Proof. Let h be an element of \widehat{G}_k^1 ; then there exist φ in \widehat{G}_0^1 , $p \geq k$, and λ in \mathbb{C} such that

$$h = \varphi \exp X_{p,\lambda} \varphi^{-1}.$$

Recall that $\sigma(\exp X_{p,\lambda}) = \exp a X_{p,\lambda'}$ for some $a \in \mathbb{C}^*$ and $\lambda' \in \mathbb{C}$. Hence

$$\sigma(h) = \sigma(\varphi) \exp a X_{p,\lambda'} \sigma(\varphi^{-1})$$

belongs to \widehat{G}_k^1 ; similarly $\sigma^{-1}(h)$ belongs to \widehat{G}_k^1 . □

Remark 6.5. From Lemma 6.4 we get that σ is a continuous automorphism of \widehat{G}_0^1 endowed with the Krull topology.

Let us recall the Baker-Campbell-Hausdorff formula applied to the formal vector fields $X = a(x) \frac{\partial}{\partial x}$ and $Y = b(x) \frac{\partial}{\partial x}$ of $\widehat{\mathcal{X}}_2^1$. If $Z \in \widehat{\mathcal{X}}_2^1$ is a solution of

$$\exp Z = \exp X \exp Y$$

then

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots$$

In particular

$$(\exp X)(\exp Y)(\exp -X)(\exp -Y) = \exp([X, Y] + \text{h.o.t.}) \quad (6.1)$$

where h.o.t. denotes terms of order ≥ 2 in the algebra generated by X and Y . We thus get

$$\left(\exp x^{k+1} \frac{\partial}{\partial x}\right) \left(\exp x^{\ell+1} \frac{\partial}{\partial x}\right) \left(\exp -x^{k+1} \frac{\partial}{\partial x}\right) \left(\exp -x^{\ell+1} \frac{\partial}{\partial x}\right) = \exp\left(\left[\exp x^{k+1} \frac{\partial}{\partial x}, \exp x^{\ell+1} \frac{\partial}{\partial x}\right] + \chi\right)$$

where χ is a vector field of the form

$$\chi = x^{\ell+k+1+\inf(k,\ell)} a(x) \frac{\partial}{\partial x};$$

in other words

$$\begin{aligned} \left(\exp x^{k+1} \frac{\partial}{\partial x} \right) \left(\exp x^{\ell+1} \frac{\partial}{\partial x} \right) \left(\exp -x^{k+1} \frac{\partial}{\partial x} \right) \left(\exp -x^{\ell+1} \frac{\partial}{\partial x} \right) &= \exp \left((\ell - k) x^{k+\ell+1} \frac{\partial}{\partial x} + \chi \right) \\ &= \exp \left((\ell - k) x^{k+\ell+1} \frac{\partial}{\partial x} \right) h \end{aligned}$$

where h denotes an element of $\widehat{G}_{\ell+k+\inf(k,\ell)}^1$ (still by Baker-Campbell-Hausdorff formula). Applying σ we get

$$\begin{aligned} \left(\exp \varepsilon_k x^{k+1} \frac{\partial}{\partial x} \right) \left(\exp \varepsilon_\ell x^{\ell+1} \frac{\partial}{\partial x} \right) \left(\exp -\varepsilon_k x^{k+1} \frac{\partial}{\partial x} \right) \left(\exp -\varepsilon_\ell x^{\ell+1} \frac{\partial}{\partial x} \right) \\ = \exp \left(\varepsilon_k \varepsilon_\ell \left[x^{k+1} \frac{\partial}{\partial x}, x^{\ell+1} \frac{\partial}{\partial x} \right] + \tilde{\chi} \right) \\ = \exp \left(\varepsilon_{k+\ell} x^{k+\ell+1} \frac{\partial}{\partial x} \right) \sigma(h) \end{aligned}$$

where $\tilde{\chi}$ is given by (6.1) and $\sigma(h)$ is controlled by Lemma 6.4. By truncating to a suitable order we see that $\varepsilon_k \varepsilon_\ell = \varepsilon_{k+\ell}$. Up to the action by an homothety we can assume that $\varepsilon_1 = 1$; as a result $\varepsilon_\ell = \varepsilon_{\ell+1}$. Hence by induction $\varepsilon_k = 1$ for any k . The automorphism σ thus fixes the homotheties and the $\exp t x^{k+1} \frac{\partial}{\partial x}$. The quotient groups $\widehat{G}_0^1 / \widehat{G}_k^1$ are generated by the projections of the homotheties and by the projections of the $\exp t x^{k+1} \frac{\partial}{\partial x}$, $t \in \mathbb{C}$, $\ell \leq k-1$. According to Lemma 6.4 the automorphism σ induces an automorphism of the quotient groups $\widehat{G}_0^1 / \widehat{G}_k^1$ that coincides with the identity on any $\widehat{G}_0^1 / \widehat{G}_k^1$. Therefore, σ coincides with the identity:

Theorem 6.6. *The group $\text{Aut}(\widehat{G}_0^1)$ is generated by the inner automorphisms and the automorphisms of the field \mathbb{C} . In other words*

$$\text{Out}(\widehat{G}_0^1) \simeq \text{Aut}(\mathbb{C}, +, \cdot)$$

where $\text{Out}(\widehat{G}_0^1)$ denotes the non-inner automorphisms of \widehat{G}_0^1 .

6.2. Automorphisms groups of G_0^1 . We are going to adapt the above approach to the holomorphic case, *i.e.* to the description of $\text{Out}(G_0^1)$. Theorem 6.1 has no analogue but [É75] and [CM88] imply:

Theorem 6.7. *The uncountable maximal abelian subgroups A of G_0^1 such that $A \cap G_1^1$ is uncountable are the conjugate of the groups A_0 and $A_{k,\lambda}$ where $k \geq 1$ denotes an integer, and λ a complex number.*

The group A_0 is a maximal abelian subgroup of G_0^1 .

Remarks 6.8. (1) The group G_1^1 contains many countable abelian maximal subgroups; in fact generically the group generated by a generic element of G_1^1 is maximal ([É75]).
 (2) There are diffeomorphisms of type $\lambda z + \text{h.o.t.}$, $\lambda = \exp(2i\pi\gamma)$, $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, which are not holomorphically linearizable f . The maximal abelian group that contains f is not necessarily of the previous type. It can be uncountable (see [PM95]).

Let σ be an element of $\text{Aut}(G_0^1)$. We can, as in the formal case, characterize the groups A_0 and $A_{k,\lambda}$ by their torsion elements. Recall that if f belongs to G_1^1 , then f is a commutator, and $\sigma(f)$ also. As a consequence, $\sigma(G_1^1) = G_1^1$, and if H is a maximal abelian subgroup of G_1^1 , then $\sigma(H)$ also. Hence, as in the formal context, $\sigma(A_{k,0})$ is holomorphically conjugate to one of the $A_{k,\lambda}$. We want to prove that $\sigma(A_0)$ is conjugated to A_0 .

Lemma 6.9. *Let f be an element of G_0^1 . Assume that f acts by conjugation on $A_{k,\lambda}$, i.e. $fA_{k,\lambda}f^{-1} = A_{k,\lambda}$. Then*

- if $\lambda \neq 0$, then $f = \xi \exp \varepsilon \frac{x^{p+1}}{1+\lambda x^p} \frac{\partial}{\partial x}$ for some ε, ξ in \mathbb{C} such that $\xi^p = 1$;
- if $\lambda = 0$, then $f = \mu \exp \varepsilon x^{p+1} \frac{\partial}{\partial x}$ for some μ in \mathbb{C}^* , and ε in \mathbb{C} .

Proof. Left to the reader. □

Now, let us fixed k ; we can assume, up to conjugacy, that $\sigma(A_{k,0}) = A_{k,\lambda}$ for some λ . But $\sigma(A_0)$ contains periodic elements of all periods so $\lambda = 0$ and $\sigma(A_0)$ is an abelian subgroup of the "affine" group $\{\mu \exp \varepsilon x^{p+1} \frac{\partial}{\partial x} \mid \mu \in \mathbb{C}^*, \varepsilon \in \mathbb{C}\}$ (Lemma 6.9). Such a group is either conjugate to a subgroup of A_0 , or conjugate to a subgroup of

$$\left\{ \xi \exp \varepsilon \frac{x^{p+1}}{1+\lambda x^p} \frac{\partial}{\partial x} \mid \varepsilon, \xi \in \mathbb{C}, \xi^p = 1 \right\}.$$

The fact that $\sigma(A_0)$ contains an infinite number of periodic elements implies that $\sigma(A_0)$ is a subgroup of A_0 . As $\sigma(A_0)$ is maximal, one gets: $\sigma(A_0) = A_0$.

Recall that $\sigma(A_0) = A_0$ acts on all the $\sigma(A_{\ell,0})$; an argument similar to that used in the formal case (§6.1) implies that $\sigma(A_{\ell,0}) = A_{\ell,0}$ for all $\ell \in \mathbb{N}$.

As before $\sigma: A_0 \rightarrow A_0$ corresponds to an automorphism of the field \mathbb{C} .

Lemma 6.10. *Any $\sigma \in \text{Aut}(G_0^1)$ extends into an automorphism $\widehat{\sigma}: \widehat{G}_0^1 \rightarrow \widehat{G}_0^1$. Moreover, $\sigma(G_k^1) = G_k^1$.*

Proof. The projections of the homotheties and the $\exp x^\ell \frac{\partial}{\partial x}$'s, $\ell \leq k-1$, generate $G_0^1/G_0^k \simeq \widehat{G}_0^1/\widehat{G}_0^k$. Since σ preserves A_0 and the $A_{\ell,0}$'s, it induces an automorphism

$$\sigma_k: \widehat{G}_0^1/\widehat{G}_0^k \rightarrow \widehat{G}_0^1/\widehat{G}_0^k$$

for any k . By construction these automorphisms are compatible with the filtration induced by the G_0^k 's

$$\begin{array}{ccc} \widehat{G}_0^1/\widehat{G}_0^k & \xrightarrow{\sigma_k} & \widehat{G}_0^1/\widehat{G}_0^k \\ \uparrow & & \uparrow \\ \widehat{G}_0^1/\widehat{G}_0^{k+\ell} & \xrightarrow{\sigma_{k+\ell}} & \widehat{G}_0^1/\widehat{G}_0^{k+\ell} \end{array}$$

As a result, the σ_k 's determine an automorphism $\widehat{\sigma}: \widehat{G}_0^1 \rightarrow \widehat{G}_0^1$. This automorphism extends σ in the following sense: if we fix a coordinate z , we get an embedding $G_0^1 \hookrightarrow \widehat{G}_0^1$ that associates to the convergent element $f = \sum a_n z^n$ the element $f = \sum a_n z^n$ seen as a formal serie. By construction $\widehat{\sigma}(f) = \sigma(f)$, i.e. $\widehat{\sigma}$ sends a holomorphic diffeomorphism onto a holomorphic diffeomorphism. \square

According to what we have done before we can assume that $\widehat{\sigma}$ is associated to an automorphism τ of the field \mathbb{C} : if $f = \sum a_n z^n$, then

$$\widehat{\sigma}(f) = \sum_{n \geq 1} \tau(a_n) z^n.$$

The automorphism τ satisfies the following property: if $\sum_{n \geq 1} a_n z^n$ converges, then $\sum_{n \geq 1} \tau(a_n) z^n$ also converges.

Lemma 6.11. *Either τ is the identity $z \mapsto z$, or τ is the complex conjugation $z \mapsto \bar{z}$.*

Proof. If τ preserves \mathbb{R} , that is if $\tau(\mathbb{R}) = \mathbb{R}$, then τ is either the identity, or the complex conjugation.

If $\tau(\mathbb{R}) \neq \mathbb{R}$, then the image of the unit disk $\mathbb{D}(0,1)$ is dense in \mathbb{C} (see [Kes51]). Suppose that τ is neither the identity, nor the complex conjugation. By density there exists $a_n \in \mathbb{D}(0,1)$, $n \geq 2$, such that $|\tau(a_n) - n!| < 1$. If $f = z + \sum a_n z^n$, then on the one hand f belongs to G_0^1 , and on the other hand $z + \sum a_n z^n$ diverges: contradiction. \square

We finally can state:

Theorem 6.12. *The group $\text{Out}(G_0^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

7. AUTOMORPHISMS GROUPS OF \widehat{G}_0^n , AND G_0^n

We have in mind the following two conjectures:

Conjecture 7.1. *The automorphisms group of \widehat{G}_0^n is generated by the inner automorphisms, and the automorphisms of the field \mathbb{C} .*

Conjecture 7.2. *The group $\text{Out}(G_0^n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Let σ be an element of $\text{Aut}(\widehat{G}_0^n)$. Denote by

$$j^1: \widehat{G}_0^n \rightarrow \text{GL}(\mathbb{C}^n), \quad f \mapsto Df_{(0)}$$

the map that associates to f its linear part at 0. Set $\widetilde{\varphi} = j^1 \circ \sigma$, i.e.

$$\begin{array}{ccccc} \widetilde{\varphi}: \widehat{G}_0^n & \xrightarrow{\sigma} & \widehat{G}_0^n & \xrightarrow{j^1} & \text{GL}(\mathbb{C}^n) \\ f & \mapsto & \sigma(f) & \mapsto & D(\sigma(f))_{(0)} \end{array}$$

and consider $\varphi = \widetilde{\varphi}|_{\text{GL}(\mathbb{C}^n)}$ the restriction of $\widetilde{\varphi}$ to $\text{GL}(\mathbb{C}^n)$.

Lemma 7.3. *The map*

$$\varphi: \mathrm{GL}(\mathbb{C}^n) \rightarrow \mathrm{GL}(\mathbb{C}^n), \quad A \mapsto D(\sigma(A))_{(0)}$$

is an injective homomorphism.

Proof. Denote by \mathcal{H} the subgroup of homotheties $\mathbb{C}^* \mathrm{id}$ of $\mathrm{GL}(\mathbb{C}^n)$.

Lemma 7.4. *The periodic elements of $\mathrm{GL}(\mathbb{C}^n) \setminus \{\mathrm{id}\}$ do not belong to $\ker \varphi$.*

Proof. Let $A \in \mathrm{GL}(\mathbb{C}^n)$ be a non-trivial periodic element of period p , i.e. $A^p = \mathrm{id}$, and $A^k \neq \mathrm{id}$ for any $1 \leq k \leq p-1$. Then $\sigma(A)$ is also a non-trivial periodic element of $\mathrm{GL}(\mathbb{C}^n)$, and its period is p . As a consequence $\sigma(A)$ is conjugate to its linear part $\varphi(A)$, and $\varphi(A)$ is a periodic element of $\mathrm{GL}(\mathbb{C}^n)$ of period p . Therefore A does not belong to $\ker \varphi$. \square

The normal subgroups of $\mathrm{GL}(\mathbb{C}^n)$ are either subgroups of \mathcal{H} , or overgroups of $\mathrm{SL}(\mathbb{C}^n)$. But $\mathrm{SL}(\mathbb{C}^n)$ contains periodic elements, so Lemma 7.4 implies that the normal subgroup $\ker \varphi$ of $\mathrm{GL}(\mathbb{C}^n)$ is a subgroup of \mathcal{H} .

Let us consider an element B of $\ker \varphi$. Then $\sigma(B) = \mathrm{id} + \text{h.o.t.}$, and $\sigma(B) = \exp X$ where X denotes a formal vector field of order $v \geq 2$. Let us remark that

$$\exp tX = \mathrm{id} + \sum P_l(t)x^l$$

where the $P_l(t)$'s are polynomial transformations. Hence if f commutes to $\exp X$, then f commutes to all $\exp tX$, and $f_*X = X$. Since B belongs to $\ker \varphi \subset \mathcal{H}$, B commutes to $\mathrm{GL}(\mathbb{C}^n)$, and $\sigma(B)$ commutes to $\sigma(\mathrm{GL}(\mathbb{C}^n))$: if ψ belongs to $\sigma(\mathrm{GL}(\mathbb{C}^n))$, then $\psi_*X = X$. As a result, if X_v is the homogeneous part of lower degree of X , then for any $\phi_* \in \varphi(\mathrm{GL}(\mathbb{C}^n))$ we have $\phi_*X_v = X_v$.

Let $C = \mu \mathrm{id}$ be an homothety such that μ is a p -th root of unity with the following property: $\mu^{v-1} \neq 1$. As we have seen $\sigma(C)$ is periodic, non-trivial, of period p ; hence $\sigma(C)$ is conjugate to $\varphi(C)$. As a consequence $\varphi(C)$ is a periodic element of $\mathrm{GL}(\mathbb{C}^n)$.

- i) Assume that $\varphi(C)$ is in \mathcal{H} , i.e. $\varphi(C) = \alpha \mathrm{id}$ with $\alpha \in \mathbb{C}^*$ such that $\alpha^p = 1$ and $\alpha^{v-1} \neq 1$. On the one hand $\phi_*X_v = X_v$, and on the other hand

$$\phi_*X_v = (\alpha \mathrm{id})_*X_v = \alpha^{v-1}X_v.$$

As a consequence $\alpha^{v-1} = 1$: contradiction.

ii) Suppose that $\varphi(C)$ is not in \mathcal{H} . Since $\varphi(C)$ is periodic, $\varphi(C)$ is semi-simple. We can thus write $\varphi(C)$ as follows

$$\left(\begin{array}{ccc} \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{array} \right] & & \\ & \left[\begin{array}{ccc} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{array} \right] & & \\ & & & & \ddots \end{array} \right)$$

with $\lambda_i \neq \lambda_j$ as soon as $i \neq j$. Note that we have the following diagram:

$$\begin{array}{ccc} \mathrm{GL}(\mathbb{C}^n) & \xrightarrow{\varphi} & \varphi(\mathrm{GL}(\mathbb{C}^n)) \\ \downarrow & \nearrow \simeq & \\ \mathrm{GL}(\mathbb{C}^n) / \ker \varphi & & \end{array}$$

As $\ker \varphi$ is a group of homotheties we have a surjection

$$\mathrm{GL}(\mathbb{C}^n) / \ker \varphi \twoheadrightarrow \mathrm{PGL}(\mathbb{C}^n)$$

and so a surjection

$$\varphi(\mathrm{GL}(\mathbb{C}^n)) \twoheadrightarrow \mathrm{PGL}(\mathbb{C}^n).$$

Since C commutes with $\mathrm{GL}(\mathbb{C}^n)$, $\varphi(C)$ commutes with $\varphi(\mathrm{GL}(\mathbb{C}^n))$. As a result $\varphi(\mathrm{GL}(\mathbb{C}^n))$ is a group of block matrices. But there is no surjection from a group of block matrices into $\mathrm{PGL}(\mathbb{C}^n)$; indeed, such a group contains non-trivial proper normal subgroups whereas $\mathrm{PGL}(\mathbb{C}^n)$ is simple. \square

According to Lemma 7.3, $\ker \varphi = \{\mathrm{id}\}$. Consider a non-periodic homothety $\lambda \mathrm{id}$. Then, $\varphi(\lambda \mathrm{id})$ is non-periodic, and as $\lambda \mathrm{id}$ commutes to $\mathrm{GL}(\mathbb{C}^n)$, $\varphi(\lambda \mathrm{id})$ commutes to $\varphi(\mathrm{GL}(\mathbb{C}^n)) \simeq \mathrm{GL}(\mathbb{C}^n)$.

Claim 7.5. *The image $\varphi(\lambda \mathrm{id})$ of $\lambda \mathrm{id}$ by φ is an homothety.*

Proof of Claim 7.5. Assume by contradiction that $\varphi(\lambda \mathrm{id})$ is not an homothety. Then, $\mathrm{Cent}(\varphi(\lambda \mathrm{id}), \mathrm{GL}(\mathbb{C}^n))$ is isomorphic to a block matrices group H . In particular, $\varphi(\mathrm{GL}(\mathbb{C}^n))$ is a subgroup of H , and thus can not be isomorphic to $\mathrm{GL}(\mathbb{C}^n)$: contradiction. \square

Since $\varphi(\lambda \mathrm{id})$ is a non-periodic homothety, $\varphi(\lambda \mathrm{id})$ can be written as $\mu \mathrm{id}$ for some $\mu \in \mathbb{C}^*$ that is not a root of unity, and

$$\sigma(\lambda \mathrm{id}) = \mu \mathrm{id} + \text{h.o.t.}$$

is (maybe only formally if $|\mu| = 1$) conjugate to μid . After conjugacy let us assume that $\sigma(\lambda \text{id}) = \varphi(\lambda \text{id})$. As $\sigma(\text{GL}(\mathbb{C}^n))$ commutes to μid , the inclusion $\sigma(\text{GL}(\mathbb{C}^n)) \subset \text{GL}(\mathbb{C}^n)$ holds after conjugacy. Let us now prove the reciprocal inclusion, that is $\text{GL}(\mathbb{C}^n) \subset \sigma(\text{GL}(\mathbb{C}^n))$. Let B be an element of $\text{GL}(\mathbb{C}^n) \setminus \sigma(\text{GL}(\mathbb{C}^n))$. But B commutes to μid , so $\sigma^{-1}(B)$ commutes to $\sigma^{-1}(\mu \text{id}) = \lambda \text{id}$. Since λ is not a root of unity, $\sigma^{-1}(B)$ is linear, *i.e.* belongs to $\text{GL}(\mathbb{C}^n)$: contradiction with $\text{GL}(\mathbb{C}^n) \setminus \sigma(\text{GL}(\mathbb{C}^n))$. Finally $\text{GL}(\mathbb{C}^n) = \sigma(\text{GL}(\mathbb{C}^n))$, and one can state:

Lemma 7.6. *If σ belongs to $\text{Aut}(\widehat{\mathbb{G}}_0^n)$, then up to conjugacy $\sigma|_{\text{GL}(\mathbb{C}^n)}$ is an automorphism of $\text{GL}(\mathbb{C}^n)$.*

Proposition 7.7. *Up to conjugacy and up to the action of an element of $\text{Aut}(\mathbb{C}, +, \cdot)$, the restriction of an element of $\text{Aut}(\widehat{\mathbb{G}}_0^n)$ to $\text{GL}(\mathbb{C}^n)$ is the identity map.*

Proof. Let σ be an element of $\text{Aut}(\widehat{\mathbb{G}}_0^n)$. According to Lemma 7.6, up to the action of an automorphism of the field \mathbb{C} , either $\sigma|_{\text{GL}(\mathbb{C}^n)}$ is the identity map, or $\sigma|_{\text{GL}(\mathbb{C}^n)}$ is the involution $A \mapsto {}^tA^{-1}$.

Assume that $\sigma|_{\text{GL}(\mathbb{C}^n)}$ is the involution $A \mapsto {}^tA^{-1}$. Set $f = (x_1 + x_2^2, x_2, x_3, \dots, x_n)$,

$$g = (x_1 + x_2, x_2, x_3, \dots, x_n), \quad h_{a,b} = (a^2x_1, ax_2, b_3x_3, b_4x_4, \dots, b_nx_n)$$

with $a_i, b_i \in \mathbb{C}^*$. Note that

- g belongs to $\text{GL}(\mathbb{C}^n)$, so $\sigma(g) = (x_1, x_2 - x_1, x_3, x_4, \dots, x_n)$;
- $h_{a,b}$ belongs to $\text{GL}(\mathbb{C}^n)$, hence $\sigma(h) = \left(\frac{x_1}{a^2}, \frac{x_2}{a}, \frac{x_3}{b_3}, \frac{x_4}{b_4}, \dots, \frac{x_n}{b_n} \right)$;
- $fg = gf$;
- $fh_{a,b} = h_{a,b}f$.

As a consequence, $\sigma(f)$ commutes to both $(x_1, x_2 - x_1, x_3, x_4, \dots, x_n)$ and $\left(\frac{x_1}{a^2}, \frac{x_2}{a}, \frac{x_3}{b_3}, \frac{x_4}{b_4}, \dots, \frac{x_n}{b_n} \right)$. From $\sigma(f)\sigma(h_{a,b}) = \sigma(h_{a,b})\sigma(f)$, we get that there exist ε in \mathbb{C} and λ_i in \mathbb{C}^* such that

$$\sigma(f) = (\lambda_1x_1 + \varepsilon x_2^2, \lambda_2x_2, \lambda_3x_3, \dots, \lambda_nx_n);$$

furthermore, from $\sigma(f)\sigma(g) = \sigma(g)\sigma(f)$, we obtain $\varepsilon = 0$ and $\lambda_1 = \lambda_2$, *i.e.*

$$\sigma(f) = (\lambda_1x_1, \lambda_1x_2, \lambda_3x_3, \dots, \lambda_nx_n).$$

In particular $\sigma(f) = \sigma\left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_1}, \frac{x_3}{\lambda_3}, \frac{x_4}{\lambda_4}, \dots, \frac{x_n}{\lambda_n}\right)$: contradiction. □

Proposition 7.7 can be used as follows. Take an element A in $\text{GL}(n, \mathbb{C})$, typically $A_0 = (\lambda x_1, x_2, x_3, \dots, x_n)$, and consider the group $\text{Cent}(A, \widehat{\mathbb{G}}_0^n) = \{f \in \widehat{\mathbb{G}}_0^n \mid f \circ A = A \circ f\}$. In the case of the example of A_0 , for some generic λ , we have

$$\text{Cent}(A_0, \widehat{\mathbb{G}}_0^n) = \{(a(x_2, x_3, \dots, x_n)x_1, g(x_2, x_3, \dots, x_n)) \mid a \in \mathbb{C}[[x_2, x_3, \dots, x_n]]^*, g \in \widehat{\mathbb{G}}_0^{n-1}\}.$$

If $\sigma \in \text{Aut}(\widehat{\mathbb{G}}_0^n)$ is such that $\sigma|_{\text{GL}(n, \mathbb{C})} = \text{id}_{\text{GL}(n, \mathbb{C})}$, then $\sigma(\text{Cent}(A, \widehat{\mathbb{G}}_0^n)) = \text{Cent}(A, \widehat{\mathbb{G}}_0^n)$, and we can expect for instance with A of type A_0 to use an induction on the dimension to prove Conjecture 7.1.

REFERENCES

- [Ban86] A. Banyaga. On isomorphic classical diffeomorphism groups. I. *Proc. Amer. Math. Soc.*, 98(1):113–118, 1986.
- [Ban97] A. Banyaga. *The structure of classical diffeomorphism groups*, volume 400 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.
- [BCLN96] M. Berthier, D. Cerveau, and A. Lins Neto. Sur les feuilletages analytiques réels et le problème du centre. *J. Differential Equations*, 131(2):244–266, 1996.
- [BS62] G. Baumslag and D. Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962.
- [CCD13] F. Cano, D. Cerveau, and J. Déserti. *Théorie élémentaire des feuilletages holomorphes singuliers*. Echelle. Belin, 2013.
- [CCGS20] S. Cantat, D. Cerveau, V. Guirardel, and J. Souto. Surface groups in the group of germs of analytic diffeomorphisms in one variable. *Enseign. Math.*, 66(1-2):93–134, 2020.
- [CL98] D. Cerveau and F. Loray. Un théorème de Frobenius singulier via l’arithmétique élémentaire. *J. Number Theory*, 68(2):217–228, 1998.
- [CLPT19] B. Claudon, F. Loray, J. V. Pereira, and F. Touzet. Holonomy representation of quasi-projective leaves of codimension one foliations. *Publ. Mat.*, 63(1):295–305, 2019.
- [CM88] D. Cerveau and R. Moussu. Groupes d’automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $ydy + \dots = 0$. *Bull. Soc. Math. France*, 116(4):459–488 (1989), 1988.
- [D06] J. Déserti. Sur les automorphismes du groupe de Cremona. *Compos. Math.*, 142(6):1459–1478, 2006.
- [D07] J. Déserti. Le groupe de Cremona est hopfien. *C. R. Math. Acad. Sci. Paris*, 344(3):153–156, 2007.
- [Die55] J. Dieudonné. *La géométrie des groupes classiques*. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Heft 5. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.
- [DS05] C. Druţu and M. Sapir. Non-linear residually finite groups. *J. Algebra*, 284(1):174–178, 2005.
- [É75] J. Écalle. Théorie itérative: introduction à la théorie des invariants holomorphes. *J. Math. Pures Appl. (9)*, 54:183–258, 1975.
- [Fil82] R. P. Filipkiewicz. Isomorphisms between diffeomorphism groups. *Ergodic Theory Dynam. Systems*, 2(2):159–171 (1983), 1982.
- [Kes51] H. Kestelman. Automorphisms of the field of complex numbers. *Proc. London Math. Soc. (2)*, 53:1–12, 1951.
- [Mal40] A. Malcev. On isomorphic matrix representations of infinite groups. *Rec. Math. N.S.*, 8 (50):405–422, 1940.
- [Mal65] A. I. Mal’tsev. On the faithful representation of infinite groups by matrices. *Transl., Ser. 2, Am. Math. Soc.*, 45:1–18, 1965.
- [PM95] R. Pérez Marco. Nonlinearizable holomorphic dynamics having an uncountable number of symmetries. *Invent. Math.*, 119(1):67–127, 1995.
- [Sie42] C. L. Siegel. Iteration of analytic functions. *Ann. of Math. (2)*, 43:607–612, 1942.
- [Sie52] C. L. Siegel. Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt.*, 1952:21–30, 1952.
- [Tit72] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.
- [Tol93] D. Toledo. Projective varieties with non-residually finite fundamental group. *Inst. Hautes Études Sci. Publ. Math.*, (77):103–119, 1993.
- [Whi63] J. V. Whittaker. On isomorphic groups and homeomorphic spaces. *Ann. of Math. (2)*, 78:74–91, 1963.

UNIV. RENNES, CNRS, IRMAR-UMR 6625, F-35000 RENNES, FRANCE

Email address: dominique.cerveau@univ-rennes1.fr

UNIVERSITÉ D’ORLÉANS, INSTITUT DENIS POISSON, ROUTE DE CHARTRES, 45067 ORLÉANS CEDEX 2, FRANCE

Email address: deserti@math.cnrs.fr