

# HOLOMORPHIC VECTOR FIELDS WITH A BARYCENTRIC CONDITION

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## HOLOMORPHIC VECTOR FIELDS WITH A BARYCENTRIC CONDITION

DOMINIQUE CERVEAU, JULIE DÉSERTI, AND ALCIDES LINS NETO

ABSTRACT. We study the *p*-tuples of holomorphic vector fields  $(X_1, X_2, ..., X_p)$  satisfying the barycentric property  $\sum_{k} \exp tX_k = p \cdot id$ , where  $\exp tX$  denotes the flow of *X*.

## 1. INTRODUCTION

Let  $\mathcal{U}$  be a connected open subset of  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ). Let  $X_1, X_2, ..., X_p$  be *p* analytic (resp. holomorphic) distinct vector fields on  $\mathcal{U}$ . Denote by  $\varphi_t^k = \exp(tX_k)$  the local one-parameter subgroup of  $X_k$ ; it is the solution of the following ordinary differential equation

$$\frac{d\varphi_t^k(x)}{dt} = X_k(\varphi_t^k(x))$$

with initial data  $\varphi_0^k(x) = x$ .

For any point  $x \in \mathcal{U}$ ,  $\varphi_t^k(x)$  is well-defined for *t* sufficiently small and we assume that

$$\sum_{k=1}^{p} X_k(\varphi_t^k(x)) = 0 \qquad \forall x \in \mathcal{U} \text{ and } t \text{ small.}$$
(1.1)

In particular  $\sum_{k=1}^{p} \exp(tX_k) = p$ id and by doing t = 0 in (1.1) we get

$$\sum_{k=1}^p X_k = 0.$$

Let us give an interpretation of (1.1): at any point *x* there are *p* identical particles transported by the vector fields  $X_k$  while preserving their barycenter at the initial position *x*. The condition (1.1) is called barycentric property. A set of *p* vector fields  $X_1, X_2, ..., X_p$  satisfying the barycentric property is called a *p*-chambar and is denoted Ch $(X_1, X_2, ..., X_p)$ . In §2 we give a long list of detailed examples.

**Remark 1.1.** The barycentric property is invariant by affine transformations. Let  $Ch(X_1, X_2, ..., X_p)$  be a *p*-chambar in some open subset  $\mathcal{U} \subset \mathbb{C}^n$  and let *T* be an affine transformation of  $\mathbb{C}^n$ . Then the vector fields  $T_*X_1, T_*X_2, ..., T_*X_p$  satisfy the barycentric condition.

In fact, if a biholomorphism  $f: \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{C}^n$  sends any set of vector fields on  $\mathcal{U}$  with the barycentric property into another set with the barycentric property, then f is an affine transformation. However, in some particular cases of p-chambars there are other types of biholomorphisms with this property (see for instance Theorem 2.13).

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If X is a vector field on  $\mathcal{U}$ , then  $\mathcal{F}_X$  denotes the foliation (maybe singular) whose leaves are the integral curves of X. Hence  $\mathcal{F}_X$  is a foliation by (real or complex) curves. From now on all the vector fields  $X_k$  are not identically zero.

In the case of a 2-chambar  $Ch(X_1, X_2)$  condition (1.1) implies that  $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$ . We have also (Theorem 3.2):

**Theorem A.** Let  $\mathcal{U}$  be an open subset of  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ). Let  $X_1, X_2$  be two analytic (resp. holomorphic) vector fields on  $\mathcal{U}$ . Assume that  $X_1$  and  $X_2$  satisfy the barycentric property.

Then  $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$ , and it is a foliation by straight lines:

- $\diamond$  the closure of the generic leaves are intersection of lines with the open subset  $\mathcal{U}$ ;
- $\diamond$  on each line the flow  $\varphi_t^k = \exp(tX_k)$ , k = 1, 2, coincides with the flow of a constant vector field.

In §2 we will construct explicit examples satisfying Theorem A. It is sufficient to consider any foliation by straight lines  $\mathcal{F}$  (maybe singular) and to take a vector field *X* whose restriction to each leaf is "constant".

In the algebraic case the foliations by straight lines are classified on  $\mathbb{P}^2_{\mathbb{C}}$  and  $\mathbb{P}^3_{\mathbb{C}}$ . We will see that in this case the flows associated to a global algebraic 2-chambar are some special birational flows (§3).

We will consider the case of colinear vector fields (a condition satisfied by the 2-chambars), *i.e.* the case where  $X_i = a_i X$  with  $a_i$  constant for any  $1 \le i \le p$ ; such chambars are called rigid chambars. The barycentric property implies that  $\mathcal{F}_X$  is a foliation by straight lines in the real case (Theorem 4.6) but not in the complex case. We will see the two following results (§4, Theorem 4.8 and Corollary 4.11):

**Theorem B.** If  $Ch(a_1X, a_2X, ..., a_pX)$ ,  $a_k \in \mathbb{C}^*$ , is a rigid *p*-chambar on the connected open set  $\mathcal{U} \subset \mathbb{C}^n$ , then the flow  $\exp tX$  of *X* is polynomial of degree at most p-1 as a function of the time *t*. In particular, the orbits of *X* are contained in some rational curves.

**Theorem C.** Let  $Ch(a_1X, a_2X, ..., a_pX)$  be a rigid *p*-chambar on an open set  $U \subset \mathbb{C}^n$ . If *X* has a singular point, then the set Sing(X) of *X* has dimension  $\geq 1$ .

We will see also examples where the  $X_i$ 's are polynomial vector fields, and more generally rational vector fields. In particular, in the linear case we get (§6, Theorem 6.1):

**Theorem D.** Let  $X_1, X_2, ..., X_p$  be some linear vector fields on  $\mathcal{U} \subset \mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ).

If they satisfy the barycentric property ,then they are nilpotent. In particular, the flows  $\exp(tX_k)$  are polynomials in *t*.

In the case of 3-chambars one gets (Theorem 6.10):

#### **Theorem E.** Let $X_1, X_2, X_3$ be some linear vector fields on $\mathbb{C}^n$ .

If they satisfy the barycentric property, then, up to conjugacy, they are contained in the Heisenberg Lie algebra  $\mathfrak{h}_n$  (we identify  $X_i$  with its matrix).

We then give the classification of the 3-chambars in dimension 1, all chambars appearing in this classification are rigid (§5, Theorem 5.1):

#### **Theorem F.** Let $Ch(X_1, X_2, X_3)$ be a 3-chambar in one variable.

In the real case  $Ch(X_1, X_2, X_3)$  is constant (*i.e.* the  $X_i$ 's are distinct constant vector fields). In the complex case

- $\diamond$  either Ch( $X_1, X_2, X_3$ ) is constant,
- ♦ or Ch(*X*<sub>1</sub>, *X*<sub>2</sub>, *X*<sub>3</sub>) = Ch( $a(x)\frac{\partial}{\partial x}$ ,  $ja(x)\frac{\partial}{\partial x}$ ,  $j^2a(x)\frac{\partial}{\partial x}$ ), where  $j^3 = 1$ , and  $a(x) = \sqrt{\lambda x + \mu}$  with  $\lambda \in \mathbb{C}^*$ ,  $\mu \in \mathbb{C}$ .

Note that the classification implies that the global 3-chambars in one variable have no singularities where they are defined; this is not the case in higher dimensions (consider the nilpotent linear cases). Whereas 2-chambars and 3-chambars on an open subset of  $\mathbb{C}$  are rigid the 4-chambars are not. The classification of *p*-chambars on  $\mathbb{C}$  for  $p \ge 4$  is a difficult problem in particular because of irreducibility problems. Nevertheless we obtain interesting properties of such chambars.

In §7 we deal with chambars generated by homogeneous vector fields (homogeneous chambars). Among other results we will see the classification of homogeneous chambars of degree 2 (Theorem 7.6):

**Theorem G.** Let  $Ch(X_1, X_2, X_3)$  be an homogeneous 3-chambar of  $\mathbb{C}^2$  of degree 2. Then, after a change of variables,  $X_i$  can be written as  $a_i y^2 \frac{\partial}{\partial x}$ , and the  $a_i$ 's satisfy:  $a_1 + a_2 + a_3 = 0$ . In particular, any homogeneous 3-chambar of  $\mathbb{C}^2$  of degree 2 is rigid.

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#### 2. REMARKS AND EXAMPLES

Let  $\mathcal{U}$  be a connected open subset of  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ). Denote by  $O(\mathcal{U})$  the ring of analytic (resp. holomorphic) functions and by  $\chi(\mathcal{U})$  the  $O(\mathcal{U})$ -module of vector fields on  $\mathcal{U}$ . We denote also by  $O(\mathbb{C}^n, a)$  and by  $\chi(\mathbb{C}^n, a)$  the germs of the previous spaces at  $a \in \mathcal{U}$ . Let  $X_1, X_2, \ldots, X_p, Y_1, Y_2, \ldots, Y_q$  be some analytic or holomorphic vector fields on  $\mathcal{U}$ . If the *p*-tuple  $(X_1, X_2, \ldots, X_p)$  and the *q*-tuple  $(Y_1, Y_2, \ldots, Y_q)$  satisfy the barycentric property, then the (p+q)-tuple  $(X_1, X_2, \ldots, X_p, Y_1, Y_2, \ldots, Y_q)$  satisfy the barycentric property. This type of example is called a *reducible chambar*. A chambar is *irreducible* if it is not reducible. vector fields. Let  $v_1, v_2, ..., v_p$  be p distinct constant vector fields on  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ) such that

$$v_1 + v_2 + \ldots + v_p = 0$$

The translation flows  $T_t^{v_k}(x) = x + tv_k$  satisfy the barycentric property

$$\sum_{k=1}^{p} T_t^{\nu_k}(x) = \sum_{k=1}^{p} (x + t\nu_k) = \sum_{k=1}^{p} x + \sum_{k=1}^{p} t\nu_k = px + t \times 0 = px$$

and the vector fields  $v_1, v_2, ..., v_p$  define a *p*-chambar. Such a chambar is called a *constant p*-chambar. The trajectories of the  $v_k$  are straight lines. The constant chambar  $(v_1, v_2, ..., v_p)$  is reducible if and only if there is a subfamily  $(v_{j_1}, v_{j_2}, ..., v_{j_\ell})$  such that  $\sum_{k=1}^{\ell} v_{j_k} = 0$ .

Let us give a simple variant of this example. Fix some coordinates

$$(x,y) = (x_1, x_2, \dots, x_q, y_1, y_2, \dots, y_{n-q});$$

take p vector fields

$$X_k = f_1^k(x)\frac{\partial}{\partial y_1} + f_2^k(x)\frac{\partial}{\partial y_2} + \ldots + f_{n-q}^k(x)\frac{\partial}{\partial y_{n-q}}$$

where the  $f_i^k$ 's denote some analytic functions. Assume that

$$X_1 + X_2 + \ldots + X_p = 0.$$

The  $X_k$ 's satisfy the barycentric property since for any value of the parameter x the  $X_k$  are constant vector fields in the linear subspaces x = constant.

We can enrich this family of examples as follows. On the open subset  $\mathcal{U}$  consider a regular foliation  $\mathcal{F}$  of codimension q whose leaves are of the form  $A \cap \mathcal{U}$  where the A's are affine subspaces of codimension q. Take now analytic vector fields  $X_k$  constant on any leaf of  $\mathcal{F}$  and such that  $X_1 + X_2 + \ldots + X_p = 0$ . Then  $(X_1, X_2, \ldots, X_p)$  is a p-chambar.

These examples play an important role in the article.

Another kind of construction that will be used is the formula expressing the flow of a vector field. Let  $X = \sum_{k=1}^{n} A_k(x) \frac{\partial}{\partial x_k}$  be an analytic vector field on an open subset  $\mathcal{U}$  of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , considered as a derivation on  $O(\mathcal{U})$ : if  $f \in O(\mathcal{U})$ , then

$$X(f) = \sum_{k=1}^{n} A_k \frac{\partial f}{\partial x_k}.$$

Let  $(t,x) \mapsto \varphi_t(x)$  be the flow of X. For  $x \in \mathcal{U}$  fixed set  $h(t) = f(\varphi_t(x))$ . The Taylor series of h at t = 0 is of the form  $h(t) = h(0) + \sum_{k=1}^{\infty} \frac{h^{(k)}(0)}{k!} t^k$ .

On the other hand, h(0) = x and  $h^{(k)}(0) = X^k(f)$ . In particular we get

$$f(\mathbf{\phi}_t(x)) = x + \sum_{k \ge 1} \frac{1}{k!} X^k(f)(x) t^k.$$

If we specialize the above formula doing  $f(x) = x_j$ , the *j*-th coordinate of  $x = (x_1, x_2, ..., x_n)$ , then  $\varphi_t(x) = (\varphi_t^1(x), \varphi_t^2(x), ..., \varphi_t^n(x))$  where

$$\varphi_t^j(x) = x_j + \sum_{k \ge 1} \frac{1}{k!} X^k(x_j) t^k$$
(2.1)

Formula (2.1) will appear in some examples. Let us now give a consequence of (2.1):

**Proposition 2.1.** Let  $\mathcal{U} \subset \mathbb{C}^n$  be an open subset. Let  $X_1, X_2, ..., X_p$  be some distinct elements of  $\chi(\mathcal{U})$ . Then  $X_1, X_2, ..., X_p$  define a *p*-chambar if and only if for any  $1 \leq j \leq n$ 

$$\sum_{k=1}^{p} X_k^{\ell}(x_j) = 0 \qquad \forall \ell \ge 1$$

where  $x_j$  denotes the *j*-th coordinate of  $x = (x_1, x_2, ..., x_n)$ .

2.2. Barycentric property and integrability. Let  $Ch(X_1, X_2, ..., X_p)$  be a *p*-chambar. Examples seen in §2.1 and 2-chambars may suggest that the Pfaff system generated by  $X_1, X_2, ..., X_p$  is an integrable system, *i.e.* tangent to a foliation. The following example of 3-chambar in dimension 3 shows that this is not the case. Let us consider

$$X_1 = -2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \qquad X_2 = \frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \qquad X_3 = \frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial x_3}.$$

The flows of the  $X_i$  are

$$\exp tX_1 = (x_1 - 2t, x_2, x_3 + t),$$
  

$$\exp tX_2 = \left(x_1 + t, x_2 + tx_1 + \frac{t^2}{2}, x_3 + t\right),$$
  

$$\exp tX_3 = (x_1 + t, x_2 - x_1t - \frac{t^2}{2}, x_3 - 2t).$$

The barycentric property is satisfied; the leaves of  $X_1$  are lines and the generic leaves of  $X_2$  and  $X_3$  are parabolas. Let  $\omega = -x_1 dx_1 + dx_2 - 2x_1 dx_3$ . Then  $\omega(X_i) = 0$ , so  $\omega$  defines the Pfaffian system associated to the  $X_i$ . A direct computation yields to

$$\boldsymbol{\omega} \wedge \mathbf{d}\boldsymbol{\omega} = 2\mathbf{d}x_1 \wedge \mathbf{d}x_2 \wedge \mathbf{d}x_3,$$

*i.e.* the 2-plane field associated to  $\omega$  is a contact structure hence is not integrable.

2.3. Fundamental example in dimension 1 and generalization. Let us consider the translation flow  $\psi_t(x) = x + t$  on  $\mathbb{C}$ . Let v be an integer  $\geq 2$ . Denote by  $x^{\frac{1}{v}}$  the principal branch of the v-th root. Then

$$\varphi_{\mathbf{v},t}(x) = \left(\Psi_t(x^{\frac{1}{\mathbf{v}}})\right)^{\mathbf{v}} = \left(x^{\frac{1}{\mathbf{v}}} + t\right)^{\mathbf{v}}$$

defines a flow, at least in a neighborhood of 1 since it is a conjugate of the translation flow. This flow is polynomial in the time t and corresponds to the vector field

$$Z_{\mathsf{v}} = \mathsf{v} x^{\frac{\mathsf{v}-1}{\mathsf{v}}} \frac{\partial}{\partial x} = \mathsf{v} \frac{x}{x^{\frac{1}{\mathsf{v}}}} \frac{\partial}{\partial x}$$

well defined at least in a neighborhood of 1.

Let  $\sigma$  be a primitive (v + 1)-th root of unity. Then

$$\varphi_{\mathbf{v},\mathbf{\sigma}t}(x) = \left(x^{\frac{1}{\mathbf{v}}} + \mathbf{\sigma}t\right)^{\mathbf{v}}$$

is the flow of the vector field

$$\sigma Z_{\mathsf{v}} = \mathsf{v} \sigma \frac{x}{x^{\frac{1}{\mathsf{v}}}} \frac{\partial}{\partial x}.$$

Of course 
$$\sum_{p=0}^{\nu} \sigma^p \cdot Z_{\nu} = 0$$
 and  

$$\sum_{p=0}^{\nu} \left( x^{\frac{1}{\nu}} + \sigma^p t \right)^{\nu} = \sum_{p=0}^{\nu} \sum_{k=0}^{\nu} {\nu \choose k} x^{\frac{\nu-k}{\nu}} \sigma^{pk} t^k = \sum_{k=0}^{\nu} \left( \sum_{p=0}^{\nu} \sigma^{pk} \right) t^k {\nu \choose k} x^{\frac{\nu-k}{\nu}} = (\nu+1)x$$

We can thus state

**Proposition 2.2.** Let  $Z_v$  be the vector field defined in a neighborhood of 1 by

$$Z_{\mathbf{v}} = \mathbf{v} x^{\frac{\mathbf{v}-1}{\mathbf{v}}} \frac{\partial}{\partial x} = \mathbf{v} \frac{x}{x^{\frac{1}{\mathbf{v}}}} \frac{\partial}{\partial x}.$$

The (v+1)-tuple  $(Z_v, \sigma Z_v, \dots, \sigma^v Z_v)$  is an irreducible (v+1)-chambar in a neighborhood of 1.

One can conjugate a chambar by an affine map; hence

$$\left((\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x},\sigma(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x},\sigma^{2}(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x},\ldots,\sigma^{\nu}(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x}\right)$$

produces a (v + 1)-chambar where it makes sense.

For v = 2 the previous construction gives the flow  $\varphi_{2,t}(x) = x + 2t\sqrt{x} + t^2$  associated to the vector field  $Z_2 = 2\sqrt{x}\frac{\partial}{\partial x}$  and the 3-chambar Ch $(Z_2, \mathbf{j}Z_2, \mathbf{j}^2Z_2)$ ,  $\mathbf{j}^3 = 1$ , but also its affine conjugates.

An immediate generalization in any dimension is the following. Consider  $P(x) = (P_1(x), P_2(x), \dots, P_n(x))$  such that

- $◊ P_j ∈ ℂ[x_1, x_2, ..., x_n], \deg P_1 = ν ≥ 2 \text{ and } \deg P_j ≤ ν,$ ◊ P(0) = 0,
- ♦ and  $DP(0) = \rho \cdot id$  where id is the identity of  $\mathbb{C}^n$  and  $|\rho| > 1$ .

There exists a neighborhood U of  $0 \in \mathbb{C}^n$  such that  $V = P(U) \supset U$  and  $P_{|U}$  has an inverse  $\phi \colon V \to U$ . To any  $a = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$  we can associate a flow defined in a neighborhood of  $(0,0) \in \mathbb{C} \times \mathbb{C}^n$  by

$$\varphi_t(x) = P(\varphi(x) + ta).$$

The vector field associated to this flow is

$$X(x) = DP(\phi(x)) \cdot a \tag{2.2}$$

**Proposition 2.3.** Let *X* be as in (2.2) and let  $\sigma$  be a primitive (v+1)-th root of unity. Then the (v+1)-tuple  $(X, \sigma X, \dots, \sigma^{v}X)$  is an irreducible (v+1)-chambar in a neighborhood of  $0 \in \mathbb{C}^{n}$ .

Proof. Since P has degree v

$$\varphi_t(x) = P(\varphi(x) + ta)$$
  
=  $P(\varphi(x)) + tDP(\varphi(x)) \cdot a + \sum_{j=2}^{\nu} \frac{t^j}{j!} D^{(j)} P(\varphi(x)) \cdot a$   
=  $x + tH_1(x, a) + \sum_{j=2}^{\nu} t^j H_j(x, a)$ 

where  $H_i(x, a)$  is homogeneous of degree j with respect to  $a \in \mathbb{C}^n$ . Hence the flow of  $\sigma^k X$  is

$$\varphi_{\sigma^k \cdot t}(x) = x + \sigma^k t H_1(x, a) + \sum_{j=2}^{\nu} \sigma^{jk} t^j H_j(x, a)$$

and so

and so  

$$\sum_{k=0}^{\nu} \varphi_{\sigma^{k} \cdot t}(x) = \sum_{k=0}^{\nu} \left( x + \sigma^{k} t H_{1}(x, a) + \sum_{j=2}^{\nu} \sigma^{jk} t^{j} H_{j}(x, a) \right) = (\nu + 1) x$$
because  $\sum_{k=0}^{\nu} \sigma^{jk} = 0$  if  $1 \le j \le \nu$ .

**Remark 2.4.** The construction produces vector fields X whose flow exptX is polynomial in the variable time t.

Example 2.5. A global example of this kind (Proposition 2.3) can be given by a polynomial diffeomorphism  $P: \mathbb{C}^n \to \mathbb{C}^n$ . For instance

$$P(x_1, x_2, \dots, x_n) = (x_1, x_2 + q_2(x_1), x_3 + q_3(x_1, x_2), \dots, x_n + q_n(x_1, x_2, \dots, x_{n-1}))$$

where  $q_i \in \mathbb{C}[x_1, x_2, ..., x_{i-1}], 2 \le j \le n$ .

As a particular example, consider the polynomial diffeomorphism of  $\mathbb{C}^2$ 

$$\phi(x,y) = (x+y^2,y).$$

Conjugating the flow

$$(x+a_kt,y+b_kt)$$
  $a_k,b_k\in\mathbb{C}$ 

with  $\varphi$  we get the flow

$$\phi_k^t = (x + a_k t + 2b_k t y + b_k^2 t^2, y + b_k t);$$

one can check that it is the flow of the affine vector field

$$X_k = \left(a_k + 2b_k y\right) \frac{\partial}{\partial x} + b_k \frac{\partial}{\partial y}.$$

Remark that this flow is polynomial in the time *t*.

As soon as  $b_k \neq 0$  the trajectories are the parabola

$$f_k = a_k y + b_k y^2 - b_k x = \text{constant}$$

For  $p \ge 3$  if we choose  $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$  such that

$$a_1 + a_2 + \ldots + a_p = b_1 + b_2 + \ldots + b_p = b_1^2 + b_2^2 + \ldots + b_p^2 = 0$$
 (2.3)

then the  $X_k$  satisfy the barycentric property and produce a *p*-chambar. For a generic choice of the parameters  $a_k$  and  $b_k$  the  $X_k$  are not  $\mathbb{C}$ -colinear. Note that for p = 3 if (2.3) holds, then the web W( $X_1, X_2, X_3$ ) is an hexagonal web (see for instance [5]) since  $f_1 + f_2 + f_3 = 0$ .

## 2.4. Polynomial vector fields that satisfy the barycentric property.

**Proposition 2.6.** In dimension 1 the polynomial vector fields that satisfy the barycentric property are the constant vector fields

$$a_k \frac{\partial}{\partial x}$$

with  $a_k \in \mathbb{C}^*$  and  $\sum_{k=1}^p a_k = 0$ .

*Proof.* The proof is based on Proposition 2.1. Let  $X = P(x)\frac{\partial}{\partial x}$  where  $P \in O(\mathbb{C})$  is viewed as a derivation on  $O(\mathbb{C})$ . According to (2.1) the flow  $\varphi_t$  of *X* is

$$\varphi_t(x) = x + \sum_{k \ge 1} \frac{1}{k!} X^k(x) t^k.$$

If  $P \in \mathbb{C}[x]$  is a polynomial of degree  $d \ge 1$ , then  $X^k(x)$  is also a polynomial for any  $k \ge 1$ . Let us write  $X^k(x)$  as  $X^k(x) = \sum_{j=0}^{d(k)} a_j^k x^j$ . If we set  $d(\ell) := \deg(X^\ell(x))$ , then

- (1) since deg(X) = d, then  $a_d^1 \neq 0$ ;
- (2)  $d(\ell) = (d-1)\ell + 1$  because  $d(\ell+1) = \deg(X(x)) + d(\ell) 1 = d + d(\ell) 1;$
- (3) the equality  $a_{d(\ell+1)}^{\ell+1} = d(\ell)a_d^1 a_{d(\ell)}^\ell$  holds.

By recurrence we get from (3) that  $a_{d(\ell)}^{\ell} = A(\ell)(a_d^1)^{\ell}$  where

(4) A(1) = 1 and  $A(\ell + 1) = d(\ell)A(\ell)$  for  $\ell \ge 1$ .

On the one hand if d = 0, then  $X(x) \neq 0$  and  $X^{\ell}(x) = 0$  for all  $\ell \geq 2$ . On the other hand it follows from (2), (3) and (4) that if  $d \geq 1$ , then  $d(\ell) \geq 1$  and  $A(\ell) \geq 1$  for all  $\ell \geq 1$ .

Now assume that  $(X_1, X_2, ..., X_p)$  is a polynomial *p*-chambar on  $\mathbb{C}$ . Let  $d = \max_{1 \le j \le p} \deg(X_j)$ . Suppose by contradiction that  $d \ge 1$ . Without lost of generality we can assume that

$$\{j \mid \deg(X_j) = d\} = \{1, 2, \dots, q\} \subset \{1, 2, \dots, p\}.$$
  
Set  $X_k = P_k(x) \frac{\partial}{\partial x}$  where  $P_k(x) = \sum_{j=0}^d a_{kj} x^j, 1 \le k \le p$ , where  
 $a_{jd} \ne 0$  if  $1 \le j \le d$  and  $a_{jd} = 0$  if  $q < j \le p$ .

**Claim 1.** *For any*  $\ell \geq 1$  *we have* 

$$(a_{1d})^{\ell} + (a_{2d})^{\ell} + \ldots + (a_{qd})^{\ell} = 0$$

The statement follows from the Claim (indeed if  $(a_{1d})^{\ell} + (a_{2d})^{\ell} + \ldots + (a_{qd})^{\ell} = 0$  for any  $\ell \ge 1$ , then  $a_{1d} = a_{2d} = \ldots = a_{qd} = 0$ ). Let us now justify it:

Proof of the Claim. Set  $d(k, \ell) = \deg(X_k^{\ell}(x)), 1 \le k \le p$ . Note that:

- ◇ if d = 1, then  $d(k, \ell) = 1$  for all  $1 \le k \le q$  and all  $\ell \ge 1$ ; furthermore if  $q < k \le p$ , then  $X_k^{\ell}(x) = 0$  for all  $\ell \ge 2$ .
- ◇ id d > 1 and 1 ≤ k ≤ q, then d ≤ d(k, ℓ) = (d − 1)k + 1 and so d(k, ℓ) < d(k, ℓ + 1) for all ℓ ≥ 1.</li>
   Moreover if q < k ≤ p, then either d(k, ℓ) < (d − 1)k + 1 or X<sub>k</sub><sup>ℓ</sup>(x) = 0 for all ≥ 2.

Given  $1 \le k \le p$  let  $a(k, \ell)$  be the coefficient of  $x^{d(k,\ell)}$  in the polynomial  $X_k^{\ell}(x)$ . If follows from the above computations that

- ♦ if  $1 \le k \le q$ , then  $a(k, \ell) = A(\ell)(a_{kd})^{\ell}$  where  $A(\ell) \ne 0$ ,
- ♦ if  $q < k \le p$ , then  $a(k, \ell) = 0$ .

According to Proposition 2.1 we get that

$$X_1^{\ell}(x) + X_2^{\ell}(x) + \ldots + X_p^{\ell}(x) = 0$$

implies

$$A(\ell)\Big((a_{1d})^{\ell} + (a_{2d})^{\ell} + \ldots + (a_{qd})^{\ell} = 0\Big) = 0;$$

as  $A(\ell) \neq 0$  we finally obtain that  $(a_{1d})^{\ell} + (a_{2d})^{\ell} + \ldots + (a_{qd})^{\ell} = 0$ .

**Remark 2.7.** If p = 3, then Proposition 2.6 is a consequence of Theorem 5.1.

**Remark 2.8.** If *X* is a holomorphic vector field on the Riemann sphere  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , then in the affine chart  $\mathbb{C}$  there exists a polynomial function *a* of degree  $\leq 2$  such that  $X = a(x)\frac{\partial}{\partial x}$ . The only *p*-tuple of global vector fields that satisfy the barycentric property in this chart are the constant vector fields.

## 2.5. Examples produced by those of dimension 1. We need a definition:

**Definition 2.9.** A *p*-chambar of the form  $Ch(a_1X, a_2X, \ldots, a_pX)$ , with  $a_i$  constant, is called *rigid*.

Propositions 2.2 and 2.3 give examples of rigid *p*-chambars.

Let us give a construction presented in dimension 2 for simplicity but that can be generalized in any dimension n and for any p.

Consider the vector field  $X(x) = 2\sqrt{x}\frac{\partial}{\partial x}$  that induces the flow  $\varphi_t(x) = x + 2t\sqrt{x} + t^2$ , a special case of §2.3. A first 3-chambar in dimension 2 is

$$Ch(X(x) + X(y), \mathbf{j}(X(x) + X(y)), \mathbf{j}^{2}(X(x) + X(y)))$$

which is rigid. Similarly one can consider

$$\operatorname{Ch}(X(x) + X(y), \mathbf{j}X(x) + \mathbf{j}^{2}X(y), \mathbf{j}^{2}X(x) + \mathbf{j}X(y))$$

which is non-rigid. These examples are well defined on any simply connected open subset that do not intersect the axis x = 0 and y = 0.

Let us now give an example of a non-rigid irreducible 4-chambar still in dimension 2

$$\operatorname{Ch}\left(X(x),\mathbf{j}X(x)+X(y),\mathbf{j}^{2}X(x)+\mathbf{j}X(y),\mathbf{j}^{2}X(y)\right)$$

that can be generalized to a 5-chambar as follows

$$\operatorname{Ch}\left(X(x),\mathbf{j}X(x),\mathbf{j}^{2}X(x)+X(y),\mathbf{j}X(y),\mathbf{j}^{2}X(y)\right).$$

**Example 2.10.** Another way to obtain examples is by taking the real part of a complex *p*-chambar on  $\mathbb{C}^n$ . For instance, if we set  $z = x + \mathbf{i}y$ , then  $\frac{d}{dz} = \frac{1}{2} \left( \frac{d}{dx} - \mathbf{i} \frac{d}{dy} \right)$ ,

$$\sqrt{z} = \underbrace{\sqrt{\sqrt{x^2 + y^2} + x}}_{A(x,y)} + \mathbf{i} \underbrace{\sqrt{\sqrt{x^2 + y^2} - x}}_{B(x,y)}$$

and

$$\operatorname{Re}\left(2\sqrt{z}\frac{d}{dz}\right) = 2A(x,y)\frac{d}{dx} + 2B(x,y)\frac{d}{dy}.$$

The three vector fields Re  $(2\sqrt{z}\frac{d}{dz})$ , Re  $(\mathbf{j}2\sqrt{z}\frac{d}{dz})$ , Re  $(\mathbf{j}^2 2\sqrt{z}\frac{d}{dz})$  give a real 3-chambar but if we consider *x*, *y* as complex variables we get a 3-chambar on a suitable open set of  $\mathbb{C}^2$ .

Let us remark that we can iterate this process: take a chambar on  $\mathbb{C}^n$ , its real part gives a chambar on  $\mathbb{R}^{2n}$  whose complexification is a chambar on  $\mathbb{C}^{2n}$  and so on...

## 2.6. Examples associated to some polynomial flows in t.

2.6.1. *Polynomial examples*. Let  $P = p_0 + p_1 x + ... + p_N x^{\nu}$  be a polynomial of degree  $\nu$ . Consider the vector field

$$X = a\frac{\partial}{\partial x} + P(x)\frac{\partial}{\partial y}$$

where  $a \in \mathbb{C}^*$ . Its flow is polynomial in *t*:

$$\varphi_t(x,y) = \left(x + at, y + \sum_{k=0}^{\nu} p_k \left(\frac{(x+at)^{k+1}}{a(k+1)} - \frac{x^{k+1}}{a(k+1)}\right)\right)$$

that can we rewritten

$$\varphi_t(x,y) = \left(x + at, y + \widetilde{P}_a(x + at) - \widetilde{P}_a(x)\right)$$

where  $\widetilde{P}_a(y) = \sum_{k=0}^{\nu} p_k \frac{y^{k+1}}{a(k+1)}$ .

Let us consider p vector fields  $X_1, X_2, ..., X_p$  of the following form

$$X_k = a_k \frac{\partial}{\partial x} + P_k(x) \frac{\partial}{\partial y}.$$

The barycentric property is equivalent to

$$\sum_{k=1}^{p} a_k = 0 \tag{2.4}$$

and

$$\sum_{k=1}^{p} \widetilde{P}_{k,a_k}(x+a_k t) - \widetilde{P}_{k,a_k}(x) = 0$$
(2.5)

Note that (2.5) holds if and only if

$$\frac{\partial}{\partial t} \left( \sum_{k=1}^{p} \widetilde{P}_{k,a_{k}}(x+a_{k}t) \right) = 0$$

$$\sum_{k=1}^{\nu} P_{k}(x+a_{k}t) = 0$$
(2.6)

if and only if

As soon as we have fixed the constants  $a_1, a_2, ..., a_p$  the equality (2.6) is a linear system in the coefficients of the polynomials  $P_k$ , system that sometimes has non-trivial solutions.

Consider for instance the case p = 3 and v = 2. Set

$$P_1 = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$
,  $P_2 = \beta_0 + \beta_1 x + \beta_2 x^2$ ,  $P_3 = \gamma_0 + \gamma_1 x + \gamma_2 x^2$ .

Conditions (2.4) and (2.6) are equivalent to

$$(I) \begin{cases} a_1 + a_2 + a_3 = 0\\ \alpha_0 + \beta_0 + \gamma_0 = 0\\ \alpha_1 + \beta_1 + \gamma_1 = 0\\ \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 = 0 \end{cases} (II) \begin{cases} \alpha_2 + \beta_2 + \gamma_2 = 0\\ \alpha_2 a_1 + \beta_2 a_2 + \gamma_2 a_3 = 0\\ \alpha_2 a_1^2 + \beta_2 a_2^2 + \gamma_2 a_3^2 = 0 \end{cases}$$

In other words (2.4) and (2.6) give seven equations in the parameters space  $\alpha$ ,  $\beta$ ,  $\gamma$ , *a* of dimension 12. The set of solutions is not irreducible. Assume that the parameters  $a = \underline{a}$  satisfies  $\underline{a_1} \neq \underline{a_2} \neq \underline{a_3}$ . Then in a neighborhood of  $a = \underline{a}$  the system (*II*) is a Vandermonde one so has for solution  $\alpha_2 = \overline{\beta_2} = \gamma_2 = 0$ . Then (*I*) and (*II*) are equivalent to

$$\begin{cases} a_1 + a_2 + a_3 = 0\\ \alpha_0 + \beta_0 + \gamma_0 = 0\\ \alpha_1 + \beta_1 + \gamma_1 = 0\\ \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 = 0\\ \alpha_2 = \beta_2 = \gamma_2 = 0 \end{cases}$$

that defines a quadric of dimension 12 - 7 = 5. But there are solutions such that two of the  $\underline{a_i}$  are equal. For instance if  $\underline{a_1} = \underline{a_2} = \underline{a_3} = 0$ , then (I) and (II) are equivalent to

$$\underline{a_1} = \underline{a_2} = \underline{a_3} = \alpha_0 + \beta_0 + \gamma_0 = \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 0$$

which is a linear space of dimension 12 - 6 = 6.

Hence the set  $\Sigma$  of vector fields of this type satisfying the barycentric property is not irreducible. In fact  $\Sigma$  consists of three vector spaces of dimension 6, one vector space of dimension 6 and one quadric of dimension 5.

2.6.2. Birational examples. Take 
$$(a_1, a_2, \dots, a_p)$$
 a p-tuple of  $\mathbb{C}^n$  and set for  $1 \le k \le p$ 

$$a_k = (a_{k,1}, a_{k,2}, \dots, a_{k,n}).$$

Consider the translation flow

$$T_t^{a_k}(x_1, x_2, \dots, x_n) = (x_1 + a_{k,1}t, x_2 + a_{k,2}t, \dots, x_n + a_{k,n}t).$$

Denote by  $\psi$  the blow-up

$$\Psi: (x_1, x_2, \dots, x_n) \dashrightarrow (x_1, x_1 x_2, \dots, x_1 x_n).$$

The lift  $F_t^k$  of  $T_t^{a_k}$  by  $\psi$  can be written

$$F_t^k(x) = \Psi \circ T_t^{a_k} \circ \Psi^{-1}(x) \\ = \left( x_1 + a_{k,1}t, \left( x_1 + a_{k,1}t \right) \left( \frac{x_2}{x_1} + a_{k,2}t \right), \dots, \left( x_1 + a_{k,1}t \right) \left( \frac{x_n}{x_1} + a_{k,n}t \right) \right)$$

The condition  $\sum_{k=1}^{p} F_t^k(x) = px$  is satisfied if  $\diamond$  for any  $1 \le \ell \le n$ 

$$\sum_{k=1}^p a_{k,\ell} = 0$$

♦ and for any  $2 \le \ell \le n$ 

$$\sum_{k=1}^p a_{k,1}a_{k,\ell}=0.$$

**Remark 2.11.** In the previous examples we assume that the  $a_k$ 's are not all zero. Up to a linear conjugation (such a conjugation preserves a barycentric property) we can assume that  $a_1 = (1, 0, 0, ..., 0)$ . The previous conditions can be rewritten

$$\begin{cases} \sum_{k=1}^{p} a_{k,\ell} = 0 \quad 1 \le \ell \le n\\ a_{1,\ell} = 0 \quad 2 \le \ell \le n \end{cases}$$

that thus form a linear subspace of the space of coefficients  $a_{j,i}$ . These examples of *p*-chambars are given by birational flows quadratic in the time *t* (see [3] for other examples).

2.7. Examples of chambars whose flows are non-algebraic/non-polynomial in *t*. Let *k* be an integer; consider  $q_k$  vector fields of the following form

$$X_k^j = a_k \frac{\partial}{\partial x} + b_{k,j} e^{\lambda_k x} \frac{\partial}{\partial y} \qquad 1 \le j \le q_k$$

where  $a_k$ ,  $b_{k,j}$  and  $\lambda_k$  belong to  $\mathbb{C}^*$ . The flows of  $X_k^j$  is

$$(\exp tX_k^j)(x,y) = \left(x + a_k t, y + \frac{b_{k,j}}{\lambda_k a_k} e^{\lambda_k x} (e^{\lambda_k a_k t} - 1)\right)$$

Set  $\ell = \sum_{k=1}^{\nu} q_k$ . The  $\ell$  vector fields  $X_k^j$  form a  $\ell$ -chambar if and only if for any  $1 \le k \le p$  the following equalities hold

$$\sum_{k=1}^{p} q_k a_k = 0,$$
  $\sum_{j=1}^{q_k} b_{k,j} = 0.$ 

Contrary to the previous example the flows  $\exp tX_k^j$  are non-polynomial: their orbits are the levels of the functions

$$\lambda_k a_k y - b_{k,i} e^{\lambda_k x}$$

This construction starts with  $\ell = 4$  and produces global chambars on  $\mathbb{C}^2$ . It can be generalized to higher dimensions.

2.8. Compatible diffeomorphisms. The concept of *p*-chambar is an affine one, that is the barycentric property is invariant under the action of the group of affine transformations; if C is a local p-chambar and  $\phi$ a diffeomorphism, then, in general,  $\phi_* C$  is not a chambar.

**Problem 2.12.** Let  $Ch_c$  be a constant chambar; what are the diffeomorphisms  $\phi$  such that  $\phi_*Ch_c$  is a pchambar ? What is the structure of such a set of diffeomorphisms ?

Let us give an answer to this problem in the special case p = 3, n = 2. Let  $Ch(X_1, X_2, X_3)$  a constant 3-chambar in  $\mathbb{C}^2$ . We say that  $Ch(X_1, X_2, X_3)$  is generic if the  $X_i$ 's are linearly independent. We immediately notice that a generic constant 3-chambar is linearly conjugate to the "standard" 3-chambar

$$\operatorname{Ch}_{0} = \operatorname{Ch}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right).$$

Let  $\phi$  be a local diffeomorphism; we say that  $\phi$  is compatible with Ch<sub>0</sub> if  $\phi_*$ Ch<sub>0</sub> is a 3-chambar. We have the following statement (recall that **j**,  $\mathbf{j}^2$  are the roots of  $t^2 + t + 1$ ):

**Theorem 2.13.** A local diffeomorphism of  $\mathbb{C}^2$  is compatible with  $Ch_0$  if and only if it can be written L+Fwhere *L* denotes an affine inversible transformation and F = (f,g) with

$$f, g \in \langle (y + \mathbf{j}x)^2, (y + \mathbf{j}^2x)^2, xy(y - x) \rangle_{\mathbb{C}}.$$

**Remark 2.14.** A local compatible diffeomorphism is in fact a global application, but not in general a global diffeomorphism.

Let us first state and prove the following result we use in the proof of Theorem 2.13:

Lemma 2.15. If h is a holomorphic function satisfying the P.D.E's

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial x \partial y} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad \qquad \frac{\partial^3 h}{\partial x^2 \partial y} + \frac{\partial^3 h}{\partial x \partial y^2} = 0$$

then h is a polynomial of degree 3 of the form

$$h(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 (x + \mathbf{j}y)^2 + \alpha_4 (x + \mathbf{j}^2 y)^2 + \alpha_5 x y (y - x)$$

with  $\alpha_0, \alpha_1, \ldots, \alpha_5 \in \mathbb{C}$ .

*Proof.* To simplify the notations let us consider the differential operators

$$S = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \qquad \qquad T = \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}$$

The inclusion  $\langle 1, x, y, (y + \mathbf{j}x)^2, (y + \mathbf{j}^2x)^2, xy(y - x) \rangle_{\mathbb{C}} \subset \ker(S) \cap \ker(T)$  is straightforward.

Note that

$$\frac{\partial}{\partial x} \cdot S = \frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \frac{\partial^2}{\partial y^2} = \frac{\partial^3}{\partial x^3} + T$$

so ker(S)  $\cap$  ker(T)  $\subset$  ker $\left(\frac{\partial^3}{\partial x^3}\right)$ . Similarly  $\frac{\partial}{\partial y} \cdot S = \frac{\partial^3}{\partial y^3} + T$  and thus  $\ker(S) \cap \ker(T) \subset \ker\left(\frac{\partial^3}{\partial y^3}\right)$ . As a result  $\ker(S) \cap \ker(T) \subset \ker\left(\frac{\partial^3}{\partial x^3}\right) \cap \ker\left(\frac{\partial^3}{\partial y^3}\right)$ . In particular if *h* belongs to  $\ker(S) \cap \ker(T)$ , then  $\frac{\partial^3 h}{\partial x^3} = \frac{\partial^3 h}{\partial y^3} = 0.$ 

Let  $h = \sum_{k,\ell} h_{k,\ell} x^k y^\ell$  be the Taylor series of h at (0,0). If  $\frac{\partial^3 h}{\partial x^3} = \frac{\partial^3 h}{\partial y^3} = 0$ , then  $h_{k,\ell} \neq 0$  if and only if  $k, \ell \leq 2$ .

However if  $k = \ell = 2$ , then we have  $S(x^2y^2) = 2y^2 + 2x^2 + 4xy \neq 0$  and so

$$\ker(S) \cap \ker(T) = \langle 1, x, y, (y+\mathbf{j}x)^2, (y+\mathbf{j}^2x)^2, xy(y-x) \rangle_{\mathbb{C}}.$$

Proof of Theorem 2.13. If  $\phi$  is a local diffeomorphism of  $\mathbb{C}^2$  compatible with Ch<sub>0</sub>, then the barycentric condition asserts that

$$\phi(x+t,y) + \phi(x,y+t) + \phi(x-t,y-t) = 3\phi(x,y).$$
(2.7)

We can assume that  $\phi$  is defined in a neighborhood of (0,0). Let us write  $\phi$  as L + (f,g) where L is affine and  $f, g \in O(\mathbb{C}^2, 0)$  satisfy (f,g)(0,0) = D(f,g)(0,0) = (0,0). By derivating (2.7) twice with respect to t, we get that both components f and g satisfy the P.D.E.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial x \partial y} + \frac{\partial^2 h}{\partial y^2} = 0.$$

The solutions of such P.D.E. are of the following type

$$h = \mathbf{\varphi}_{+}(\mathbf{y} + \mathbf{j}\mathbf{x}) + \mathbf{\varphi}_{-}(\mathbf{y} + \mathbf{j}^{2}\mathbf{x})$$
(2.8)

with **j**, **j**<sup>2</sup> the roots of  $t^2 + t + 1$  and  $\varphi_+$ ,  $\varphi_-$  holomorphic in one variable defined on suitable domains.

A third derivation with respect to t shows that f and g also satisfy the P.D.E.

$$0 = \frac{\partial^3 h}{\partial x^2 \partial y} + \frac{\partial^3 h}{\partial x \partial y^2} = \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \right).$$
(2.9)

Lemma 2.15 allows to conclude (note that, with the notations of Lemma 2.15 an element of ker  $S \cap \ker T$  satisfies relation (2.7)).

More generally, one can state:

**Theorem 2.16.** Let  $f: \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{C}^n$  be a biholomorphism from the open set  $\mathcal{U} \subset \mathbb{C}^n$  to  $f(\mathcal{U}), n \ge 2$ . Assume that the vector fields

$$f_*\frac{\partial}{\partial x_1}, \qquad f_*\frac{\partial}{\partial x_2}, \qquad \dots, \qquad f_*\frac{\partial}{\partial x_n}, \qquad f_*\left(-\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}-\dots-\frac{\partial}{\partial x_n}\right)$$

satisfy the barycentric property. Then all the components  $f_j$  of f are polynomial.

**Lemma 2.17.** Let  $h \in O(\mathcal{U})$  be a holomorphic function with the property that

$$\sum_{j=1}^{n} h(x_1, x_2, \dots, x_{j-1}x_j + t, x_{j+1}, x_{j+2}, \dots, x_n) + h(x_1 - t, x_2 - t, \dots, x_n - t) = (n+1)h(x_1, x_2, \dots, x_n)$$
(2.10)

for all  $x \in U$  and  $t \in \mathbb{C}$  with |t| small enough. Then *h* satisfies the system of *P.D.Es* 

$$\begin{cases} T_2(h) = 0 \\ T_3(h) = 0 \\ \dots \end{cases}$$

where  $T_k$  is the differential operator

$$T_k = \frac{\partial^k}{\partial x_1^k} + \frac{\partial^k}{\partial x_2^k} + \ldots + \frac{\partial^k}{\partial x_n^k} + (-1)^k \Big( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \ldots + \frac{\partial}{\partial x_n} \Big)^k.$$

*Proof.* Let  $e_1 = (1, 0, 0, ..., 0)$ ,  $e_2 = (0, 1, 0, 0, ..., 0)$ , ...,  $e_n = (0, 0, ..., 0, 1)$  and  $v = -\sum_{j=1}^n e_j$ . The idea is to prove by induction on  $k \ge 1$  that for any  $t \in (\mathbb{C}, 0)$ 

$$\sum_{j=1}^{n} \frac{\partial^{k}}{\partial x_{j}} h(x+te_{j}) + (-1)^{k} \left(\frac{\partial}{\partial x_{1}} + \frac{\partial}{\partial x_{2}} + \dots + \frac{\partial}{\partial x_{n}}\right)^{k} h(x+tv) = 0;$$
(2.11)

indeed if t = 0 in (2.11), then we get (2.10).

Let  $\varphi(t,x) = \sum_{j=1}^{n} h(x+te_j) + h(x+tv)$ . According to (2.10) the function  $\varphi(t,x)$  depends only of x. In

particular differentiating k times with respect to t we get

$$\frac{\partial^k \varphi(t,x)}{\partial t^k} = \sum_{j=1}^n \frac{\partial^k}{\partial x_j} h(x+te_j) + (-1)^k \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \dots + \frac{\partial}{\partial x_n}\right)^k h(x+tv) = 0$$

Furthermore doing t = 0 we get  $T_k(h) = 0$ .

Proof of Theorem 2.16. Now suppose that  $f: \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{C}^n$  is a biholomorphism such that the vector fields  $f_*\frac{\partial}{\partial x_1}, f_*\frac{\partial}{\partial x_2}, \dots, f_*\frac{\partial}{\partial x_n}, f_*\left(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \dots - \frac{\partial}{\partial x_n}\right)$  satisfy the barycentric property. Setting  $f = (f_1, f_2, \dots, f_n)$  we see that is equivalent to

$$\sum_{j=1}^{n} f_{\ell}(x+te_{j}) + f_{\ell}(x+tv) = (n+1)f_{\ell}(x) \qquad \forall 1 \le \ell \le n$$

Therefore each component  $f_{\ell}$  of f satisfies (2.10) so that  $f_{\ell}$  belongs to  $\bigcap_{k \ge 2} \ker(T_k)$  for any  $1 \le \ell \le n$  (Lemme

2.17). The idea is to prove that  $\bigcap_{k\geq 2} \ker(T_k) \subset \mathbb{C}[x_1, x_2, \dots, x_n]$ : if  $h \in \bigcap_{k\geq 2}^{k\geq 2} \ker(T_k)$ , then *h* is a polynomial.

Let  $\mathcal{P}$  be the Noetherian ring of linear differential operators on  $\mathcal{O}(\mathcal{U})$  with constant coefficients

$$\mathcal{P} = \left\{ P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) \mid P \in \mathbb{C}[z_1, z_2, \dots, z_n] \right\}$$

and let  $I = \langle T_k | k \ge 2 \rangle$  be the ideal of  $\mathcal{P}$  generated by all the operators  $T_k$ ,  $k \ge 2$ . Note that if *S* belongs to *I*, then  $\bigcap_{k\ge 2} \ker(T_k)$  is contained in ker(*S*).

**Claim 2.** There exists  $p \in \mathbb{N}$  such that  $\frac{\partial^p}{\partial x_j^p}$  belongs to *I* for all  $1 \le j \le n$ .

Claim 2 implies that if *h* belongs to  $\bigcap_{k\geq 2} \ker(T_k)$ , then *h* is a polynomial of degree at most n(p-1).

*Proof of Claim 2.* Let  $\Phi: \mathcal{P} \to \mathcal{O}_n$  be the unique ring homomorphism satisfying

$$\Phi\left(\frac{\partial}{\partial x_j}\right) = z_j \qquad \forall 1 \le j \le n.$$

Note that  $\Phi(T_k) = z_1^k + z_2^k + \ldots + z_n^k + (-1)^k (z_1 + z_2 + \ldots + z_n)^k$ . Let us set

$$P_{k}(z) = z_{1}^{k} + z_{2}^{k} + \dots + z_{n}^{k} + (-1)^{k} (z_{1} + z_{2} + \dots + z_{n})^{k}, \qquad \widetilde{I} = \langle P_{k} | k \ge 2 \rangle, \qquad \Phi(I) = \widetilde{I}$$

Claim 3. One has

$$Z(\widetilde{I}) = \left\{ z \in \mathbb{C}^n \, | \, P_k(z) = 0 \quad \forall k \ge 2 \right\} = \left\{ 0 \right\}.$$

From  $Z(\tilde{I}) = \{0\} = Z(\mathfrak{m}_n)$  one gets (using the definition of  $\sqrt{\tilde{I}}$ ) that  $\sqrt{\tilde{I}} = \mathfrak{m}_n$ . According to Hilbert's theorem (Nullstellensatz) one obtains that  $\tilde{I} \supset \mathfrak{m}_n^p$  for some p. As a result  $z_j^p$  belongs to  $\tilde{I}$  for all  $1 \le j \le n$  and so  $\frac{\partial^p}{\partial z_i^p}$  belongs to I for all  $1 \le j \le n$ .

Proof of Claim 3. Define  $S := -(z_1 + z_2 + \ldots + z_n)$  so that  $P_k = z_1^k + z_2^k + \ldots + z_n^p + S^k$ . Therefore if z belongs  $Z(\widetilde{I})$ , then

$$(**) \begin{cases} z_1 + z_2 + \ldots + z_n + S = 0\\ z_1^2 + z_2^2 + \ldots + z_n^2 + S^2 = 0\\ \ldots\\ z_1^n + z_2^n + \ldots + z_n^n + S^n = 0\\ z_1^{n+1} + z_2^{n+1} + \ldots + z_n^{n+1} + S^{n+1} = 0 \end{cases}$$

Doing  $S = z_{n+1}$  system (\*\*) is equivalent to  $Q_{n+1}v^t = 0$  where  $Q_{n+1}$  is the matrix

$$Q_{n+1}(z) = \begin{pmatrix} z_1 & z_2 & \dots & z_{n+1} \\ z_1^2 & z_2^2 & \dots & z_{n+1}^2 \\ \vdots & & \vdots \\ z_1^{n+1} & z_2^{n+1} & \dots & z_{n+1}^{n+1} \end{pmatrix}$$

and v = (1, 1, ..., 1). Finally it can be checked by induction on  $n \ge 0$  that if  $Q_{n+1}(z)v^t = 0$  for some  $u = (u_1, u_2, ..., u_{n+1})$ , where  $u_j > 0$  for all  $1 \le j \le n+1$ , then z = 0.

#### 3. Description of the 2-chambars

3.1. Examples coming from foliations by straight lines. In order to precise the previous statements we recall the classification of foliations by straight lines on  $\mathbb{P}^3_{\mathbb{C}}$  that can be found in [2] (according to Jorge Pereira this classification was already known to Kummer). We do not know if such a classification exists on  $\mathbb{P}^3_{\mathbb{R}}$ .

Let  $\mathcal{F}$  be a holomorphic foliation on  $\mathbb{P}^n_{\mathbb{C}}$ . Chow theorem asserts that  $\mathcal{F}$  is algebraic; such a foliation  $\mathcal{F}$  has singularities. We say that  $\mathcal{F}$  is a *foliation by straight lines* if the generic leaf is contained in a line (in fact a line without a few points). Let us mention the difference between the real case: foliations by straight lines of  $\mathbb{P}^3_{\mathbb{R}}$  without singularities exist. The typical example is produced by Hopf fibration: the real projectivization of complex vector lines of  $\mathbb{C}^2 \simeq \mathbb{R}^4$  gives such a foliation  $\mathcal{H}$ . Setting  $z = x_1 + \mathbf{i}x_2$  and  $w = x_3 + \mathbf{i}x_4$  these foliations have the first integral

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{x_1x_3 - x_2x_4 + \mathbf{i}(x_1x_4 + x_2x_3)}{x_3^2 + x_4^2}$$

In particular  $\frac{x_1x_3-x_2x_4}{x_3^2+x_4^2}$  and  $\frac{x_1x_4+x_2x_3}{x_3^2+x_4^2}$  are real first integrals of  $\mathcal{H}$ .

Let us recall the classification of foliations by straight lines of  $\mathbb{P}^3_{\mathbb{C}}$ :

**Theorem 3.1** ([2]). Every holomorphic foliation by straight lines in  $\mathbb{P}^3_{\mathbb{C}}$  is, up to linear equivalence, of one of the following types

- 1. a radial foliation at a point,
- 2. a radial foliation "in the pages of an open book", *i.e.* a family of radial foliations of dimension 2 each contained in a plane of the family of planes containing a fixed line;
- 3. a foliation associated with the twisted cubic  $t \mapsto (t, t^2, t^3)$ ; here the (closure of the) leaves of the foliation are the chords and the lines tangent to the twisted cubic.

Foliations of the first type correspond to foliations by parallel lines in a well-chosen affine chart (singular point at infinity).

To construct a foliation of the second type we consider an open book, *i.e.* a pencil of hyperplanes, for instance  $\frac{x_1}{x_2}$  = constant; in any page  $\frac{x_1}{x_2} = c$  we fix a point  $(\underline{x_1}, c\underline{x_2}, \underline{x_3})$  and ask that any leaf of  $\mathcal{F}$  is a line contained in a page  $\frac{x_1}{x_2} = c$  and passes through the prescribed point  $(\underline{x_1}, c\underline{x_2}, \underline{x_3})$  (see [2] for further details).



Remark that Theorem 3.1 gives the description of algebraic foliations by straight lines in the affine space  $\mathbb{C}^3$ .

Let us now explain how we can construct a 2-chambar from a foliation  $\mathcal{F}$  by lines defined on an open subset  $\mathcal{U}$  of  $\mathbb{C}^n$ . For a good choice of the affine coordinates  $x_i$  the foliation  $\mathcal{F}$  is defined by a vector field

$$X = \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} + \ldots + \alpha_n \frac{\partial}{\partial x_n}$$

on  $\mathcal{U}$ . Of course, in general, the  $\alpha_i$ 's are meromorphic and we consider  $\mathcal{U}^* = \mathcal{U} \setminus \bigcup_{i=2}^n (\text{poles of } \alpha_i)$ . Then if *m* belongs to  $\mathcal{U}^*$  the trajectory of *X* passing through *m* is a line  $D_m$  and  $\exp(tX)|_{D_m}$  is a translation flow on  $D_m$ . The pair (X, -X) thus defines a 2-chambar.

One can next consider  $f \cdot X$ , where f is any meromorphic first integral of X, instead of X. Since f is constant on the trajectories of X,  $f \cdot X$  still defines a translation flow on any trajectory of X, and  $(f \cdot X, -f \cdot X)$  is also a 2-chambar.

3.2. Some properties. The barycentric property for a 2-chambar  $Ch(X_1, X_2)$  implies that  $X_1 + X_2 = 0$  and can be rewritten as

$$\varphi_t(x) + \varphi_{-t}(x) = 2x \qquad \forall x \in \mathcal{U}$$

where  $\varphi_t$  denotes the flow of  $X = X_1$ .

Differentiating the previous equality with respect to time t, we get

$$\overset{\bullet}{\boldsymbol{\phi}_{t}}(x) - \overset{\bullet}{\boldsymbol{\phi}_{-t}}(x) = X\left(\boldsymbol{\phi}_{t}(x)\right) - X\left(\boldsymbol{\phi}_{-t}(x)\right) = 0;$$

differentiating a second time with respect to t, we obtain

$$DX(\varphi_t(x))\phi_t(x) + DX(\varphi_{-t}(x))\phi_{-t}(x) = 0$$

where  $DX: \mathcal{U} \to \mathbb{R}^n$  (or  $DX: \mathcal{U} \to \mathbb{C}^n$ ) denotes the differential of *X*. If  $X = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$ , the above relation is equivalent to

$$DX(X) = \sum_{i=1}^{n} X(\alpha_i) \frac{\partial}{\partial x_i} = 0$$

In particular the coefficients  $\alpha_k$  are first integrals of X,  $2 \le k \le n$ . As a result the  $\alpha_k$  are constant along the trajectories of X; these trajectories are thus (contained in) lines.

Note that in dimension 1 we can write  $X = \alpha \frac{\partial}{\partial x}$  and the above relation is equivalent to  $\alpha \frac{\partial \alpha}{\partial x} = 0$ ; hence  $\alpha$  is constant. On any of its trajectories the flow of X thus coincides with the flow of a constant vector field. As a result one can state:

**Theorem 3.2.** Let  $\mathcal{U}$  be an open subset of  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ). Let  $X_1, X_2$  be two analytic (resp. holomorphic) vector fields on  $\mathcal{U}$ . Assume that  $X_1$  and  $X_2$  satisfy the barycentric property.

Then the leaves of  $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$  are contained in lines; on each of these lines the flows  $\exp(tX_k)|_D$  are translation flows.

In particular in dimension 1 any 2-chambar (X, -X) is produced by a constant vector field. Remark also that any local 2-chambar in one variable can be globalized.

**Corollary 3.3.** Let *X* be a rational vector field on  $\mathbb{C}^n$ . Assume that (X, -X) defines a 2-chambar. Then  $\exp(tX) = id + tX^0$  defines a flow of birational maps of  $\mathbb{C}^n$ .

Note that in  $\exp(tX) = id + tX^0$  the letter  $X^0$  denotes the map whose components are the components of the vector field *X*, a system of coordinates having been chosen.

**Remark 3.4.** In the real case there is an other proof of Theorem 3.2 which is geometric.

Let  $\Gamma$  be a generic leaf of  $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$ . Assume that  $\Gamma$  is not (contained in) a line. If  $x \in \Gamma$  is a generic point, then there exists an hyperplane  $\Sigma$  tangent to  $\Gamma$  at x such that

♦ the germ Γ<sub>,x</sub> is contained in one of the half spaces delimited by Σ,
 ♦ Γ<sub>,x</sub> ∩ Σ = {x}



If we set  $\varphi_t = \exp t X_1$ , then  $\varphi_t(x) - x + \varphi_{-t}(x) - x \neq 0$ : contradiction.

Let  $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$  be a germ of vector fields at the origin of  $\mathbb{C}^n$ . Denote by  $\operatorname{Sing}(X) = \{\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0\}$  the singular set of *X*.

The following statement is a special case of Theorem 4.10; its proof is algebraic in contrast with the geometric proof of Theorem 4.10.

**Theorem 3.5.** Let Ch(X, -X) be a 2-chambar at  $0 \in \mathbb{C}^n$ . Assume that X is singular at 0, that is  $\{0\} \subset Sing(X)$ .

Then dim  $Sing(X) \ge 1$ .

*Proof.* The condition  $X(\alpha_k) = 0, 1 \le k \le n$ , is equivalent to

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \alpha_k}{\partial x_i} = 0 \qquad 1 \le k \le n.$$

Hence the partial derivatives  $\left(\frac{\partial \alpha_k}{\partial x_1}, \frac{\partial \alpha_k}{\partial x_2}, \dots, \frac{\partial \alpha_k}{\partial x_n}\right)$  are relations of the ideal  $(\alpha_1, \alpha_2, \dots, \alpha_n)$ .

Assume by contradiction that  $\dim \text{Sing}(X) = 0$ . Then according to [6] the relations are generated by the trivial relations

$$(0,0,\ldots,0,\underbrace{\alpha_j}_{i\text{th coordinate}},0,\ldots,0,\underbrace{-\alpha_i}_{j\text{th coordinate}},0,0,\ldots,0);$$

this gives a contradiction with the following fact: the algebraic multiplicity at 0 of one of the  $\frac{\partial \alpha_k}{\partial x_i}$  is less than the algebraic multiplicity at 0 of  $\alpha_k$ .

**Remark 3.6.** Let  $u \in O^*(\mathbb{C}^n, 0)$  be a unit. Then the vector field  $u \cdot \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$  which has linear trajectories can not belong to a 2-chambar; but the rational field  $\frac{1}{x_1} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$  can.

## 4. RIGID CHAMBARS

### 4.1. Flows which are polynomial in the time t.

**Definition 4.1.** Let X be an holomorphic vector field on the open set  $\mathcal{U} \subset \mathbb{C}^n$ . We say that X is a *t*-polynomial vector field if  $t \mapsto \exp tX$  is polynomial. The *t*-degree of X is the usual degree in the variable t and is denoted by  $t.d(X) \in \mathbb{N} \cup \{\infty\}$ .

We have seen a lot of examples of *t*-polynomial vector fields: constant vector fields, nilpotent vector fields, the vector field  $2\sqrt{x}\frac{\partial}{\partial x}$ , ...

If  $\mathcal{U} = \mathbb{C}^n$ , then the trajectories of a *t*-polynomial vector field are points or rational curves.

**Proposition 4.2.** Let *X* be a *t*-polynomial vector field of *t*-degree v on the open set  $\mathcal{U} \subset \mathbb{C}^n$ . Write exp*tX* as  $\mathrm{Id} + tF_1 + t^2F_2 + \ldots + t^vF_v$ , with  $F_k \in O(\mathcal{U})$ . Then the components  $F_{v,1}, F_{v,2}, \ldots, F_{v,n}$  of  $F_v$  are first integrals of *X*.

In particular in the 1-dimensional case,  $F_v$  is a non-zero constant.

*Proof.* It is a direct consequence of the identity  $\exp tX \circ \exp sX = \exp(s+t)X$ : the coefficient of  $t^{\vee}$  in that identity is exactly

$$F_{\mathsf{v}}(\exp sX)=F_{\mathsf{v}}.$$

This implies the statement.

Remark the  $F_{v,k}$  may be constant; this is the case for the flow of  $X = 2\sqrt{x}\frac{\partial}{\partial x}$ . A contrario if a *t*-polynomial vector field *X* of degree v is singular at a point, say 0 (*i.e.* X(0)0), then obviously some of the  $F_{v,k} = \frac{X^v(x_k)}{v!}$  are non identically 0. In particular in dimension 2, a *t*-polynomial vector field *X* singular at the origin  $0 \in \mathbb{C}^2$ , X(0) = 0, has a non-constant holomorphic first integral *f*. The generic leaves of *X* are the levels of *f*; note that since the flow is polynomial one has the following important property:  $X_{|f^{-1}(0)} \equiv 0$ .

The *t*-polynomial vector fields produce examples of *p*-chambars as we have seen previously. Typically if  $\sigma$  is a primitive v-th root of unity and  $t \cdot d(X) = v$ , then  $X, \sigma X, \dots, \sigma^{v-1}X$  defines a (rigid) v-chambar.

If  $t \cdot d(X) = 1$ , then  $\exp tX = \operatorname{Id} + tF_1$  and the foliation associated to X is a foliation by straight lines. Conversely to a foliation by straight lines we can associate a (meromorphic) *t*-polynomial vector field X such that  $t \cdot d(X) = 1$ .

In dimension 2, consider a foliation given by the vector field  $X = f \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ . Then X is a *t*-polynomial vector field of degree 1 if and only if the foliation  $\mathcal{F}_X$  is a foliation by straight lines; this means that *f* satisfies the non-linear PDE

$$0 = X(f) = f\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y};$$

note that this PDE is the famous inviscid Burgers' equation, a well-known PDE in fluid mechanic. Similarly *t*-polynomial vector fields of degree 2 on open set of  $\mathbb{C}^2$  correspond to foliations in parabolas etc. In that case appear generalizations of Burgers' equation as the reader can see.

The following result gives the classification of the *t*-polynomial vector field on the complex line.

**Theorem 4.3.** Let  $X(x) = a(x)\frac{\partial}{\partial x}$  be a germ at  $0 \in \mathbb{C}$  of a holomorphic vector field. Assume that the flow of *X* is polynomial in *t* of *t*-degree  $\ell$ . Then  $a = f' \circ \phi$  where

- ♦ *f* is a polynomial of degree  $\ell$  with f(0) = 0 and  $f'(0) = a(0) \neq 0$ ;
- $\diamond \phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$  is a local inverse of  $f : f \circ \phi(x) = x$ .

In other words X is conjugate to the constant vector field  $\frac{\partial}{\partial x}$  via a polynomial (local) diffeomorphism.

*Proof.* Suppose that  $a(0) \neq 0$ . In this case the vector field X is conjugated to a constant vector field, say  $Y = \frac{\partial}{\partial x}$ . Let f be an element of Diff( $\mathbb{C}, 0$ ) such that  $f_*Y = X$ . The flow  $\varphi_t$  of X can be written as

$$\varphi_t(x) = f(f^{-1}(x) + t),$$

where  $f^{-1} \in \text{Diff}(\mathbb{C},0)$  is the local inverse of f. We thus have  $a(x) = f' \circ f^{-1}(x)$ . As we have seen in §2 (2.1)

$$\varphi_t(x) = x + \sum_{k \ge 1} \frac{1}{k!} X^k(x) t^k;$$

since  $t \cdot d(X) = d$  we must have  $X^k(x) = 0$  for all  $k \ge d + 1$ . Note that the functions  $f_k(x) = X^k(x), k \ge 1$ , satisfy the recurrence rule:

- (i)  $f_1 = a$ ,
- (ii)  $f_{k+1} = af'_k, \forall k \ge 1$ .

Let us define another sequence of germs at  $0 \in \mathbb{C}$  as  $g_k = f_k \circ f$ ,  $k \ge 1$ . This new sequence satisfies the recurrence rule:

(i')  $g_1 = f_1 \circ f = a \circ f = f'$ ,

(ii')  $g_{k+1} = f_{k+1} \circ f = a \circ f \cdot f'_k \circ f = f'_k \circ f \cdot f' = (f_k \circ f)' = g'_k, \forall k \ge 1.$ 

Therefore from (i') and (ii') we get for all  $k \ge 1$ 

$$g_k = \frac{\partial^k f}{\partial x^k}.$$

Now, as  $f_{\ell+1} \equiv 0$  we have  $g_{\ell+1} \equiv 0$  and so f is a polynomial of degree at most  $\ell$ . But since the flow  $\varphi_t$  has degree  $\ell$ , f must be of degree exactly  $\ell$ .

Suppose by contradiction that a(0) = 0. In this case we can write  $a(x) = x^{\ell} h(x)$  where  $\ell \ge 1$  and  $h(0) \ne 0$ . But using the recurence rule (ii) it is possible to prove that  $f_k(x) = x^{\ell k - k + 1} h_k(0)$  where  $h_k(0) \neq 0$  for all  $k \ge 1$ . As a consequence the flow can not be polynomial in t.  $\square$ 

**Remark 4.4.** Fixing x = 0 in the third line of the proof we immediately get that f is polynomial; we followed a longer process because it is essential in the study of the case a(0) = 0.

Theorem 4.3 implies that a germ of holomorphic *t*-polynomial vector field in one variable has no singularities. This is not the case in  $n \ge 2$  variables (consider for instance  $x_2 \frac{\partial}{\partial x_1}$ ). Nevertheless Theorem 4.3 has a natural generalization in  $n \ge 2$  variables, but with an additional assumption of "non-singularities":

**Theorem 4.5.** Let  $X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$  a germ at 0 of a non-singular *t*-polynomial vector field,  $a_1(0) \neq 0$  for fixing ideas.

There exists  $f \in \text{Diff}(\mathbb{C}^n, 0)$  a germ of diffeomorphism which is polynomial in the variable  $x_1$  such that  $X = f_* \frac{\partial}{\partial x_1}$ , *i.e.*  $\varphi_t(x) = f(f^{-1} + te_1)$  where  $\varphi_t$  is the flow of X and  $f^{-1}$  the local inverse of f at 0.

*Proof.* Let f be a local conjugacy between X and  $\frac{\partial}{\partial x_1}$  satisfying  $f(0, x_2, x_3, \dots, x_n) = (0, x_2, x_3, \dots, x_n)$  (it is well-known that such a conjugacy exists). In particular  $\varphi_t(x) = f(f^{-1}(x) + te_1)$  and

$$\varphi_t(0, x_2, x_3, \dots, x_n) = f(t, x_2, x_3, \dots, x_n);$$

in particular f is thus polynomial in the variable  $x_1$ .

4.2. Rigid chambars on  $\mathbb{R}^n$  and foliations by straight lines. The following statement generalizes to the real case the property satisfied by the 2-chambars:

**Theorem 4.6.** If  $Ch(a_1X, a_2X, ..., a_pX)$  is a rigid *p*-chambar on an open subset of  $\mathbb{R}^n$ , then the foliation  $\mathcal{F}_X$  associated to *X* is a foliation by straight lines.

Proof. As in the proof of Theorem 3.2 we get by successive derivations the equalities

$$\begin{cases} \sum_{k=1}^{p} a_k = 0\\ \left(\sum_{k=1}^{p} a_k^2\right) DX \cdot X = 0 \end{cases}$$

Since  $a_k \neq 0$  for any  $1 \le k \le p$  one has  $DX \cdot X = 0$ . As a result all the non-singular trajectories of X are straight lines.

Theorem 4.6 can not be generalized to the complex case. Let us give a counter example of Theorem 4.6 in the complex case in dimension 2. Consider on  $\mathbb{C}^2$  the linear vector field

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}.$$

The closure of its trajectories are the parabola  $y = cx^2$  with  $c \in \mathbb{P}^1_{\mathbb{C}}$  (if  $c \in \{0, \infty\}$ , then the trajectory is a line). Let us consider the vector field

$$Y = \frac{1}{x}X = \frac{\partial}{\partial x} + \frac{2y}{x}\frac{\partial}{\partial y}$$

which is holomorphic outside x = 0. Its 1-parameter group is the group of birational maps

$$(\exp tY)(x,y) = \left(x+t, \left(\frac{x+t}{x}\right)^2 y\right).$$

Hence if  $a_k$  belongs to  $\mathbb{C}^*$ , then one has

$$(\exp ta_k Y)(x,y) = \left(x + a_k t, \left(\frac{x + a_k t}{x}\right)^2 y\right).$$

Take some non zero constants  $a_1, a_2, ..., a_p, p \ge 3$ , such that

$$\sum_{k=1}^{p} a_k = \sum_{k=1}^{p} a_k^2 = 0.$$

Then the vector fields  $Y_k = a_k Y$ ,  $1 \le k \le p$ , form a *p*-chambar on the open set  $\mathcal{U} = \mathbb{C}^2 \setminus \{x = 0\}$ . But the trajectories of *Y*, that are almost the trajectories of *X*, are not straight lines.

**Remark 4.7.** Let *X* be a germ at  $0 \in \mathbb{C}^n$  of holomorphic vector field. Suppose that there exist some constants  $a_1, a_2, \ldots, a_p$  such that the  $X_k = a_k X$  generate a *p*-chambar. If *X* is not singular at 0,  $X(0) \neq 0$ , then  $Ch(a_1X, a_2X, \ldots, a_pX)$  is locally conjugate to the constant *p*-chambar  $Ch\left(a_1\frac{\partial}{\partial x}, a_2\frac{\partial}{\partial x}, \ldots, a_p\frac{\partial}{\partial x}\right)$ . Indeed if  $\phi$  is a local diffeomorphism that conjugates *X* to  $\frac{\partial}{\partial x}$  and if *a* belongs to  $\mathbb{C}$ , then  $\phi$  conjugates *aX* to  $a\frac{\partial}{\partial x}$ . Be

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careful it does not mean that the image of a constant *p*-chambar via a diffeomorphism is a *p*-chambar (see Theorem 2.13).

#### 4.3. Rigid and semi-rigid chambars on $\mathbb{C}^n$ .

## 4.3.1. Rigid chambars on $\mathbb{C}^n$ and t-polynomial vector fields.

**Theorem 4.8.** Let  $Ch(X, a_1X, a_2X, \dots, a_{p-1}X)$  be a germ at  $0 \in \mathbb{C}^n$  of rigid *p*-chambar.

Then the flow  $\varphi_t$  of X is polynomial of degree at most p - 1, as a function of the time t.

If t.d(X) = d, then  $a_1, a_2, \ldots, a_p$  satisfy

$$a_1^\ell + a_2^\ell + \ldots + a_p^\ell = 0 \quad \forall 1 \le \ell \le d$$

In particular if d = p - 1, then  $a_1^p = a_2^p = \ldots = a_p^p$ .

Moreover if the *p*-chambar  $(a_1X, a_2X, ..., a_pX)$  is irreducible, then  $\frac{a_k}{a_1}$  is a primitive *p*-th root of unity for some  $1 \le k \le p$ .

*Proof.* Write X as  $\sum_{k=1}^{n} X_k \frac{\partial}{\partial x_k}$ ; the barycentric condition is the following

$$px_{j} = px_{j} + t(a_{1} + a_{2} + \dots + a_{p})X_{j} + \frac{t^{2}}{2}(a_{1}^{2} + a_{2}^{2} + \dots + a_{p}^{2})X(X_{j})$$
$$+ \dots + \frac{t^{k}}{k!}(a_{1}^{k} + a_{2}^{k} + \dots + a_{p}^{k})X^{k-1}(X_{j}) + \dots$$

for j = 1, 2, ..., n.

The fact that the coefficients  $a_k$  are different from zero implies that a Newton formula

 $a_1^\ell + a_2^\ell + \ldots + a_p^\ell$ 

is non zero for an  $\ell \leq p$ . As a consequence  $X^m(X_j) \equiv 0$  for all  $m \geq \ell - 1$  and  $1 \leq j \leq n$ . This implies that the flow of *X*, and the flows of the  $a_k X_k$ , are polynomial in *t*.

The other facts can be checked by the reader.

4.3.2. A property of the singular set. Let X be a holomorphic vector field defined on an open subset  $\mathcal{U}$  of  $\mathbb{C}^n$ . Denote by  $\mathcal{F}_X$  the singular one dimensional foliation defined by X on  $\mathcal{U}$ . A separatrix  $\gamma$  of X through  $x_0 \in \text{Sing}(X)$  is a germ of analytic curve at  $x_0$  such that  $x_0$  belongs to  $\gamma$  and  $\gamma \setminus \{x_0\}$  is a leaf of the germ of  $\mathcal{F}_X$  at  $x_0$ . This means that  $x_0$  belongs to  $\gamma$  and if x belongs to  $\gamma \setminus \{x_0\}$ , then  $X(x) \neq 0$  and  $T_x \gamma = \mathbb{C} \cdot X(x)$ .

Let *X* be an holomorphic vector field defined on a closed ball  $B = \overline{B(0,r)}$  with X(0) = 0. We suppose that *X* is a *t*-polynomial vector field, that is  $t \mapsto \varphi_t(x)$  is polynomial in  $t, x \in B$ ,  $\varphi_t = \exp tX$ . Note that for any  $x \in B$ ,  $t \mapsto \varphi_t(x)$  can be extended on all the line  $\mathbb{C}$ . As a consequence if  $x \in B$ , the leaf  $\mathcal{L}_x$  of  $\mathcal{F}_X$  in *B* is

- either the point x (case  $x \in \text{Sing}(X)$ ),
- or the connected component of  $\mathcal{L}'_x \cap B$  containing x where  $\mathcal{L}'_x$  is the rational curve image of  $t \mapsto \varphi_t(x)$ .

**Lemma 4.9.** Suppose that x does not belong to Sing(X); then 0 does not belong to the closure  $\overline{\mathcal{L}_x}$  of  $\mathcal{L}_x$  in B.

*Proof.* Assume by contradiction that 0 belongs to  $\overline{\mathcal{L}_x}$ . Then there is a sequence  $(t_n)_n$  of complex numbers such that  $\lim_{n \to +\infty} \varphi_{t_n}(x) = 0$ . Since  $0 \in \text{Sing}(X)$  one has  $\lim_{n \to +\infty} |t_n| = +\infty$ , and as  $t \mapsto \varphi_t(x)$  is polynomial (non constant)  $\lim_{n \to +\infty} |\varphi_{t_n}(x)| = +\infty$ : contradiction.

**Theorem 4.10.** Let  $X \in \chi(\mathbb{C}^n, 0)$  be a germ of a *t*-polynomial vector field at the origin of  $\mathbb{C}^n$ .

Assume that  $\operatorname{Sing}(X) \neq \emptyset$ . Then  $\operatorname{dim} \operatorname{Sing}(X) \geq 1$ .

Moreover X has no separatrices through a singularity.

Proof. Assume that X is defined on the ball  $B = \overline{B(0,r)}$  and that 0 is an isolated singularity of X. Let  $(x_n)_n$  be a sequence of points of B such that  $\lim_{n \to +\infty} x_n = 0$ . The leaf  $\mathcal{L}_{x_n}$  is closed in B and cuts the sphere  $S(0,r) = B \setminus B(0,r)$ . Let  $y_n$  be a point in  $\mathcal{L}_{x_n} \cap S(0,r)$  and  $y_0$  a limit point of  $y_n$ , up to extraction  $y_0 = \lim_{n \to +\infty} y_n$ . According to Lemma 4.9 the point 0 does not belong to  $\overline{\mathcal{L}_{y_0}}$  and  $\mathcal{L}_{y_0}$  can be seen as the leaf of the restriction of  $\mathcal{F}_{X|B \setminus B(0,r')}$  for r' sufficiently small. The fact that  $y_0 = \lim_{n \to +\infty} y_n$  implies that  $\mathcal{L}_{y_n}$  is contained in  $B \setminus B(0,r')$  for n sufficiently large: contradiction with  $\lim_{n \to +\infty} x_n = 0$ .

**Corollary 4.11.** Let  $Ch(a_1X, a_2X, \ldots, a_pX)$  be a rigid *p*-chambar on an open set  $\mathcal{U}$  of  $\mathbb{C}^n$ . Then

- $\diamond$  either Sing(X) =  $\emptyset$ , that is X is regular;
- ◇ or dim Sing(X) ≥ 1.

**Example 4.12.** Let X be a linear nilpotent vector field on  $\mathbb{C}^n$ . Then the flow  $\exp tX$  is polynomial of degree  $d = \operatorname{rk} X$ . Moreover dim  $\operatorname{Sing}(X) = n - d$ . For instance if  $X^{n-1} \neq 0$ , then dim  $\operatorname{Sing}(X) = 1$ .

Problem 4.13. Does there exist a vector field with an isolated singularity belonging to a p-chambar?

**Remark 4.14.** Recall that the Camacho-Sad theorem ([1]) says that a holomorphic foliation G by curves at the origin 0 of  $\mathbb{C}^2$  has an invariant curve passing through 0. As a consequence if X is a *t*-polynomial vector field at the origin 0 of  $\mathbb{C}^2$ , with X(0) = 0, then the invariant curves of the foliation associated to X are contained in the singular set Sing(X).

The previous considerations suggest in dimension  $\geq 3$  the following question:

**Question 4.1.** Let *X* be a germ at  $0 \in \mathbb{C}^n$  of holomorphic vector field. Assume that the closure of the integral curves are analytic. Does *X* preserve an invariant curve passing through 0 ?

4.3.3. Semi-rigid chambars on  $\mathbb{C}^n$ .

**Definition 4.15.** A *p*-chambar  $Ch(X_1, X_2, ..., X_p)$  on an open subset of  $\mathbb{C}^n$  is *semi-rigid* if the  $X_k$  are colinears, that is if  $X_1 \wedge X_k = 0$  for any  $2 \le k \le p$ .

In dimension 1 all chambars are semi-rigid.

**Example 4.16.** The 3-chambar  $\operatorname{Ch}\left(\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -(y+1)\frac{\partial}{\partial x}\right)$  on  $\mathbb{C}^2$  is semi-rigid but not rigid.

**Example 4.17.** The 4-chambar  $\operatorname{Ch}\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -y\frac{\partial}{\partial x}\right)$  on  $\mathbb{C}^2$  is semi-rigid but not rigid. Note that it is a non-irreducible chambar.

**Proposition 4.18.** Let  $Ch(X_1, X_2, X_3)$  be a semi-rigid 3-chambar on an open subset of  $\mathbb{C}^n$ . Then one of the following holds<sup>1</sup>:

 $\diamond \mathcal{F}_{X_1} = \mathcal{F}_{X_2} = \mathcal{F}_{X_3}$  and  $\mathcal{F}_{X_i}$  is a foliation by straight lines;

 $\diamond$  Ch( $X_1, X_2, X_3$ ) is a rigid chambar.

*Proof.* Let  $\mathcal{U}$  be an open subset of  $\mathbb{C}^n$  where the  $X_i$ 's are defined. Set  $X_1 = X$ ; then  $X_2 = fX$  where f denotes a meromorphic function defined on  $\mathcal{U}$ . The barycentric condition implies that  $X_3 = -(1+f)X$ . The equality

$$\sum_{k=1}^{3} DX_k \cdot X_k = 0$$

obtained by derivation from the barycentric property can be rewritten as follows

$$2(1+f+f^2)DX \cdot X + (1+2f)X(f) \cdot X = 0.$$
(4.1)

that implies that

$$(1+f+f^2)X \wedge DX \cdot X = 0.$$

If  $1 + f + f^2 = 0$ , then *f* is constant and  $Ch(X_1, X_2, X_3)$  is rigid. Otherwise, we have  $X \wedge DX \cdot X = 0$  and so  $\mathcal{F}_X$  is a foliation by lines.

**Question 4.2.** Does there exist a generalization of Proposition 4.18 for *p*-chambars,  $p \ge 3$ ? The answer is positive in the real case:

**Proposition 4.19.** Let  $Ch(X_1, X_2, ..., X_p)$  be a semi-rigid *p*-chambar on an open subset  $U \subset \mathbb{R}^n$ ,  $n \ge 2$ . Then  $\mathcal{F}_{X_1} = \mathcal{F}_{X_2} = ... = \mathcal{F}_{X_p}$  is a foliation by straight lines.

*Proof.* Since the chambar is semi-rigid we can write  $X_j = f_j \cdot X$  where X is a vector field on  $\mathcal{U}$  and  $f_j \colon \mathcal{U} \to \mathbb{R}$ ,  $1 \leq j \leq p$ . Note that

$$DX_j \cdot X_j = D(f_j \cdot X) \cdot (f_j X) = f_j \cdot X(f_j) \cdot X + f_j^2 \cdot DX \cdot X.$$

In particular we get

$$0 = \sum_{k=1}^{p} DX_k \cdot X_k = \left(\sum_{k=1}^{p} f_k \cdot X(f_k)\right) \cdot X + \left(\sum_{k=1}^{p} f_k^2\right) \cdot DX \cdot X.$$

Taking the wedge product with X in the above relation, we get

$$\left(\sum_{k=1}^p f_k^2\right) X \wedge DX \cdot X = 0.$$

Since the  $f_k$ 's are non identically zero, we get  $X \wedge DX \cdot X \equiv 0$ . Therefore,  $\mathcal{F}_X$  is a foliation by straight lines.

<sup>&</sup>lt;sup>1</sup>Note that the two properties are not mutually exclusive.

#### 5.1. Description of 3-chambars in one variable.

**Theorem 5.1.** Let  $\mathcal{B}$  be a holomorphic 3-chambar on some connected open subset of  $\mathbb{C}$ . Then

 $\diamond$  either  $\mathcal{B}$  is a constant 3-chambar;

$$\diamond \text{ or } \mathcal{B} = \operatorname{Ch}\left(a(x)\frac{\partial}{\partial x}, \mathbf{j}a(x)\frac{\partial}{\partial x}, \mathbf{j}^2a(x)\frac{\partial}{\partial x}\right), \text{ where } a(x) = \sqrt{\lambda x + \mu} \text{ with } \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}.$$

In particular,  $\mathcal{B}$  is a rigid chambar.

**Remark 5.2.** In a certain sense Theorem 5.1 shows that the set of 3-chambars on a connected set of  $\mathbb{C}$  has two "irreducible components".

Proof of Theorem 5.1. Set  $\mathcal{B} = Ch(X_1, X_2, X_3)$ . We can write  $X_k = a_k(x) \frac{\partial}{\partial x}$ , where  $a_k \in O_1$ ,  $1 \le k \le 3$ .

Lemma 5.3. The 3-chambar  $\mathcal{B}$  is rigid. Moreover:

 $\diamond$  either it is a constant chambar, *i.e.* the *a<sub>k</sub>*'s are constant and *a*<sub>1</sub>+*a*<sub>2</sub>+*a*<sub>3</sub> = 0;  $\diamond$  or *a*<sub>2</sub>(*x*) = α*a*<sub>1</sub>(*x*) and *a*<sub>3</sub>(*x*) = α<sup>2</sup>*a*<sub>1</sub>(*x*) where 1 + α + α<sup>2</sup> = 0.

Proof of Lemma 5.3. If  $X = a(x)\frac{\partial}{\partial x}$ , then by formula (2.1):

$$(\exp tX)(x) = x + ta(x) + \frac{t^2}{2}a(x)a'(x) + \frac{t^3}{3!}\left(a(x)a'(x)^2 + a^2(x)a''(x)\right) + \frac{t^4}{4!}a(x)\left(a(x)a'(x)^2 + a^2(x)a''(x)\right)' + \dots$$
(5.1)

The barycentric property implies the following equalities:

$$a_1 + a_2 + a_3 = 0 \tag{5.2}$$

$$a_1' + a_2' + a_3' = 0 \tag{5.3}$$

$$a_1a_1' + a_2a_2' + a_3a_3' = 0 \tag{5.4}$$

$$a_1''a_1 + a_1'^2 + a_2''a_2 + a_2'^2 + a_3''a_3 + a_3'^2 = 0$$
(5.5)

$$a_1(a_1''a_1 + a_1'^2) + a_2(a_2''a_2 + a_2'^2) + a_3(a_3''a_3 + a_3'^2) = 0$$
(5.6)

Note that (5.3) (resp. (5.5)) is obtained by derivating (5.2) (resp. (5.4)).

According to (5.2) the 3-tuple  $(a_1, a_2, a_3)$  is not a holomorphic multiple of (1, 1, 1). As a consequence all non trivial solutions of the linear system (seen on the field  $\mathcal{M}$ ) of meromorphic functions

$$\begin{cases} y_1 + y_2 + y_3 = 0\\ a_1y_1 + a_2y_2 + a_3y_3 = 0 \end{cases}$$
(5.7)

are thus  $\mathcal{M}$ -colinear. Note that one can assume that all the  $a'_k$  are non zero; indeed if all the  $a_k$  are constant, then the statement holds. There thus exists a meromorphic function f such that

$$\left(a_1''a_1 + a_1'^2, a_3''a_3 + a_3'^2, a_3''a_3 + a_3'^2\right) = f(a_1', a_2', a_3')$$
(5.8)

Remark that the 3-tuple  $\left((a_1''a_1 + a_1'^2)', (a_2''a_2 + a_2'^2)', (a_3''a_3 + a_3'^2)'\right)$  is also solution of (5.7) (one can see it by looking at both the derivation of (5.5) and the coefficient of  $t^4$  in  $\sum_{k=1}^{3} \exp tX_k(x)$ ). There thus exists a meromorphic function g such that

$$\left((a_1''a_1+a_1'^2)',(a_2''a_2+a_2'^2)',(a_3''a_3+a_3'^2)'\right)=g(a_1',a_2',a_3').$$

By derivating (5.8) we get

$$f(a_1'', a_2'', a_3'') + f'(a_1', a_2', a_3') = g(a_1', a_2', a_3')$$
(5.9)

♦ Note that if f = 0, then the  $a_k a'_k$  are constant

$$a_k a'_k = \frac{c_k}{2}, \qquad \qquad c_k \in \mathbb{C}^*$$

and the  $a_k^2 = c_k x + d_k$  are affine and  $a_k = (c_k x + d_k)^{\frac{1}{2}}$ . The equality (5.2) that can be rewritten

$$\sum_{k=1}^{3} (c_k x + d_k)^{\frac{1}{2}} = 0$$

implies that  $\frac{c_1}{d_1} = \frac{c_2}{d_2} = \frac{c_3}{d_3}$ , *i.e.* the  $X_k$ 's are colinear. Let us remark that the previous case coincides with the example described in §2.5.

♦ Assume now that  $f \neq 0$ . The equality (5.9) implies that the  $a'_k$  satisfy a linear differential equation

$$a_k'' = ha_k', \qquad \qquad h = \frac{g - f'}{f}.$$

We integrate and get

$$a_k = \alpha_k H + \beta_k$$

where the  $\alpha_k$ ,  $\beta_k$  are some constants and *H* is holomorphic. As the 3-tuple  $(a_1, a_2, a_3)$  is supposed to be non-constant, *H* is non-constant. The equality (5.2) becomes

$$\sum_{k=1}^3 \alpha_k = \sum_{k=1}^3 \beta_k = 0$$

and (5.4) becomes

$$\left(\sum_{k=1}^{3} \alpha_k^2\right) HH' + \sum_{k=1}^{3} \beta_k \alpha_k H' = 0.$$

Since H is non constant, H' is non zero, and

$$\sum_{k=1}^3 \alpha_k^2 = \sum_{k=1}^3 \alpha_k \beta_k = 0.$$

We obtain the following alternative:

- either  $(\alpha_1, \alpha_2, \alpha_3) = \gamma(1, \mathbf{j}, \mathbf{j}^2)$  and  $(\beta_1, \beta_2, \beta_3) = \varepsilon(1, \mathbf{j}, \mathbf{j}^2)$  with  $\gamma \in \mathbb{C}^*$  and  $\varepsilon \in \mathbb{C}$ ;
- or  $(\alpha_1, \alpha_2, \alpha_3) = \gamma(1, \mathbf{j}^2, \mathbf{j})$  and  $(\beta_1, \beta_2, \beta_3) = \varepsilon(1, \mathbf{j}^2, \mathbf{j})$  with  $\gamma \in \mathbb{C}^*$  and  $\varepsilon \in \mathbb{C}$ .

In the two cases the  $a_k = \alpha_k H + \beta_k$  are  $\mathbb{C}$ -colinear.

According to Lemma 5.3 one can write  $X_k$  as follows:

$$X_k = c_k a(x)$$

where *a* is a holomorphic function and  $c_i$  are non-zero complex numbers such that  $c_1 + c_2 + c_3 = 0$ .

The barycentric property (see equation (5.4) of Proof of Lemma 5.3) implies

$$(c_1^2 + c_2^2 + c_3^2)aa' = 0.$$

◇ If c<sub>1</sub><sup>2</sup> + c<sub>2</sub><sup>2</sup> + c<sub>3</sub><sup>2</sup> ≠ 0, then *a* is constant (and Corollary 5.4 is proved).
◇ If c<sub>1</sub><sup>2</sup> + c<sub>2</sub><sup>2</sup> + c<sub>3</sub><sup>2</sup> = 0, then up to multiplication by a constant either (c<sub>1</sub>, c<sub>2</sub>, c<sub>3</sub>) = (1, **j**, **j**<sup>2</sup>), or (c<sub>1</sub>, c<sub>2</sub>, c<sub>3</sub>) = (1, **j**, **j**<sup>2</sup>).

By (5.1) we have 
$$\left(\sum_{k=1}^{3} c_{k}^{3}\right) \cdot \left(a''a^{2} + a'^{2}a = 0\right)$$
 and  $a''a^{2} + a'^{2}a = 0$ 

since 
$$c_1^3 + c_2^3 + c_3^3 = 3$$
. Therefore  $0 = a''a^2 + a'^2a = a(a''a + a'^2) = a(aa')'$  and  $aa' = \frac{\lambda}{2}$  for some  $\lambda$  in  $\mathbb{C}$ . As a result  $a^2 = \lambda x + \mu$  for some  $\mu \in \mathbb{C}$ .

**Corollary 5.4.** Let  $\mathcal{B} = Ch(X_1, X_2, X_3)$  be a local 3-chambar on  $\mathbb{R}$ . Then  $\mathcal{B}$  is a constant 3-chambar  $Ch\left(c_1\frac{\partial}{\partial x}, c_2\frac{\partial}{\partial x}, c_3\frac{\partial}{\partial x}\right)$  with  $c_i$  non-zero real numbers such that  $c_1 + c_2 + c_3 = 0$ .

## 5.2. *p*-chambar with weights.

**Definition 5.5.** Let us consider *p* analytic vector fields  $X_1, X_2, ..., X_p$ , defined on some open subset  $\mathcal{U}$  of  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ), with flows  $t \mapsto \varphi_t^{\ell}$ ,  $1 \le \ell \le p$ . Consider also non-zero real (resp. complex) numbers  $\alpha_1, \alpha_2, ..., \alpha_p$  and  $\alpha = \sum \alpha_{\ell}$ .

We say that  $X_1^{\iota}, X_2, ..., X_p$  define a holomorphic *p*-chambar with weights  $\alpha_1, \alpha_2, ..., \alpha_p$  if

$$\alpha_1 \varphi_t^1(x) + \alpha_2 \varphi_t^2(x) + \ldots + \alpha_p \varphi_t^p(x) = \alpha x, \qquad (5.10)$$

for all (t,x) where the above formula makes sense.

**Remark 5.6.** This definition is equivalent to

$$\alpha_1 X_1^k(x_\ell) + \alpha_2 X_2^k(x_\ell) + \ldots + \alpha_p X_p^k(x_\ell) = 0 \quad \forall k \ge 1, \forall 1 \le \ell \le n.$$

We remark that the condition is not equivalent to consider the flows of the vector fields  $\alpha_{\ell} X_{\ell}$ ,  $1 \le \ell \le n$ .

The classification of 3-chambars (Theorem 5.1) can be extended to this type of chambars with an adaptation in the second case:

**Theorem 5.7.** Assume that  $X_1$ ,  $X_2$  and  $X_3$  define a holomorphic 3-chambar  $\mathcal{B}$  with weights  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$  on some connected open subset of  $\mathbb{C}$ . Then

 $\diamond$  either  $\mathcal{B}$  is a constant 3-chambar,

$$\Rightarrow \text{ or } \mathcal{B} = \operatorname{Ch}\left(\beta_1 a(x) \frac{\partial}{\partial x}, \beta_2 a(x) \frac{\partial}{\partial x}, \beta_3 a(x) \frac{\partial}{\partial x}\right) \text{ where } a(x) = \sqrt{\lambda x + \mu} \text{ with } \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \text{ and} \\ \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 = \alpha_1 \beta_1^2 + \alpha_2 \beta_2^2 + \alpha_3 \beta_3^2 = 0.$$

In particular,  $\mathcal{B}$  is a rigid chambar.

#### 5.3. Almost *p*-chambar.

**Definition 5.8.** Let X be a vector field. We say that X is almost a p-chambar if there exist non-zero vector fields  $X_2, X_3, \ldots, X_p$  such that  $(X, X_2, X_3, \ldots, X_p)$  is a *p*-chambar.

We say that X is almost a chambar if there exists an integer p such that X is almost a p-chambar.

**Remark 5.9.** If *X* is almost a *p*-chambar, then *X* is almost a (p+q)-chambar for any  $q \ge 2$ .

**Example 5.10.** The constant vector fields are almost *p*-chambars for any  $p \ge 2$ .

**Example 5.11.** Let X be a nilpotent linear vector field, and let p be its index of nilpotence. Then X is almost a *p*-chambar.

We suspect that most vector fields are not almost chambars. Let us give an explicit example in (real or complex) dimension 1:

**Proposition 5.12.** If  $\lambda$  is a non-zero constant, then the vector field  $\lambda x \frac{\partial}{\partial x}$  is

- ◊ not almost a 2-chambar in a neighborhood of 0;
- $\diamond$  not almost a 3-chambar in a neighborhood of 0.

Remark 5.13. The first assertion of the statement is clear.

The second one is a direct consequence of the classification of the 3-chambars (Theorem 5.1). Note that the argument does not use the property of nilpotency of linear chambar; indeed if  $(X_1, X_2, \dots, X_p)$  is a pchambar containing  $X = \lambda x \frac{\partial}{\partial x}$  then it is possible that one of the  $X_k(0)$  is non zero. We conjecture that any semi-simple linear vector field  $\sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$ ,  $\lambda_i \neq 0$ , is not almost a *p*-chambar.

5.4. Some remarks on 4-chambars in one variable. The 2-chambars and 3-chambars on an open subset of  $\mathbb{C}$  are rigid. This property is not satisfied by all the 4-chambars. Consider the vector fields  $X = 2\sqrt{x} \frac{d}{dx}$ and  $Y = 2\sqrt{x+\epsilon} \frac{\partial}{\partial x}$ ,  $\epsilon \neq 0$ , on a suitable domain of  $\mathbb{C}$ . As we know the flows of X and Y are

$$\exp tX = x + 2t\sqrt{x} + t^2$$
  $\exp tY = x + 2t\sqrt{x+\varepsilon} + t^2$ 

and it is easy to see that the 4-chambar Ch(X, -X, iY, -iY) is irreducible and non rigid. Such a 4-chambar is said to be special.

**Conjecture 5.14.** Up to affine conjugacy a 4-chambar on an open subset of  $\mathbb{C}$  is of one of the following type:

- ◊ constant Ch (a<sub>1</sub> ∂/∂x, a<sub>2</sub> ∂/∂x, a<sub>3</sub> ∂/∂x, a<sub>4</sub> ∂/∂x), a<sub>k</sub> ∈ C\*;
  ◊ rigid of *t*-degree 2: Ch(a<sub>1</sub>X, a<sub>2</sub>X, a<sub>3</sub>X, a<sub>4</sub>X) with X = 2√x ∂/∂x and a<sub>k</sub> constants satisfying a<sub>1</sub> + a<sub>2</sub> +  $a_3 + a_4 = a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0;$
- $\diamond$  rigid of *t*-degree 3: Ch(X,  $\sigma X, \sigma^2 X, \sigma^3 X$ ) with X of *t*-degree 3 and  $\sigma$  a root of unity of order 4;
- $\diamond$  special Ch(X, -X, Y, -Y) with X and Y of t-degree 2.

**Remark 5.15.** The classification of *p*-chambars on  $\mathbb{C}$  for  $p \ge 4$  is a difficult problem in particular because of irreducibility problems. Indeed if p = 6 for instance one can consider the vector field  $Z_5 = 5x^{\frac{4}{5}} \frac{\partial}{\partial x}$  to which one can associate the 6-chambar

$$Ch(Z_5, \sigma Z_5, \sigma^2 Z_5, \sigma^3 Z_5, \sigma^4 Z_5, \sigma^5 Z_5)$$

which is irreducible. But one can also consider the non-irreducible 6-chambar obtained as follows

$$\operatorname{Ch}\left(X_1,\mathbf{j}X_1,\mathbf{j}^2X_1,X_2,\mathbf{j}X_2,\mathbf{j}^2X_2\right)$$

where  $X_k = \sqrt{\lambda_k x + \mu_k} \frac{\partial}{\partial x}$  and  $\lambda_k$ ,  $\mu_k$  are complex numbers such that  $\lambda_1 \mu_2 - \lambda_2 \mu_1 \neq 0$ .

**Problem 5.16.** *Classify irreducible p-chambars in dimension* 1, *for*  $p \ge 4$ .

**Theorem 5.17.** Let  $Ch(X_1, X_2, X_3, X_4)$  be a holomorphic 4-chambar on some open set  $\mathcal{U} \subset \mathbb{C}$ . Set  $X_k = y_k(x) \frac{\partial}{\partial x}$  with  $y_k \in O(\mathcal{U})$  for  $1 \le k \le 4$ .

Then there exists a polynomial  $P: \mathbb{C}^3 \to \mathbb{C}^4$  independent of the  $y_k$ 's such that the vector  $y = (y_1, y_2, y_3, y_4)$  satisfies a differential equation of the form

$$\Delta(y) \cdot y''' = P(y, y', y'')$$
(5.11)

where  $\Delta(y) = \prod_{i < j} (y_j - y_i)$ .

Furthermore the polynomial P is homogeneous of degree 7.

*Proof.* Let us recall some basic facts. The operator  $X_k$  on  $O(\mathcal{U})$  acts as  $X_k(f) = y_k \cdot f'$ . In particular

$$X_k(x) = y_k, \qquad X_k^2(x) = y_k y'_k, \qquad X_k^3(x) = p(y_k, y'_k) + y_k^2 y''_k, \qquad X_k^4(x) = q(y_k, y'_k, y''_k) + y_k^3 y''_k$$

where  $p(y,z) = yz^2$  and  $q(y,z,w) = yz^3 + 4y^2zw$ . More generally we have

$$X_{k}^{\ell}(x) = P_{\ell}(y_{k}, y_{k}', \dots, y_{k}^{(\ell-2)}) + y_{k}^{\ell-1} \cdot y_{k}^{(\ell-1)}$$
(5.12)

where  $P_{\ell}$  denotes a homogeneous polynomial of degree  $\ell$ .

Using (5.12) we get by an induction argument

$$\frac{\partial^n X_k^{\ell}(x)}{\partial x^n} = P_{\ell,n}(y_k, y'_k, \dots, y_k^{(\ell+n-2)}) + y_k^{\ell-1} \cdot y_k^{(\ell+n-1)}$$
(5.13)

where  $P_{\ell,n}$  is homogeneous of degree  $\ell$  and  $P_{\ell,0} = P_{\ell}$ . Note that  $P_{\ell,n}$  is independent of the open set  $\mathcal{U}$  and of the function  $y: \mathcal{U} \to \mathbb{C}^4$ .

Since the  $X_k$ 's satisfy the barycentric condition, we have  $\sum_{k=1}^{4} X_k^{\ell} = 0, 1 \le k \le 3$  and so

$$\sum_{k=1}^{4} \frac{\partial^n X_k^{\ell}}{\partial x^n} = 0 \quad \forall 1 \le \ell \le 4, \quad \forall n \ge 0.$$

From the above relations and (5.12) we get the following system of equations

$$\begin{cases} y_1''' + y_2''' + y_3''' + y_4''' = 0\\ y_1y_1''' + y_2y_2''' + y_3y_3''' + y_4y_4''' = Q_2(y, y', y'')\\ y_1^2y_1''' + y_2^2y_2''' + y_3^2y_3''' + y_4^2y_4''' = Q_3(y, y', y'')\\ y_1^3y_1''' + y_2^3y_2''' + y_3^3y_3''' + y_4^3y_4''' = Q_4(y, y', y'') \end{cases}$$

with

$$\begin{aligned}
Q_2(y,y',y'') &= -3\sum_{i=1}^4 y'_i y''_i \\
Q_3(y,y',y'') &= -\sum_{i=1}^4 \left( (y'_i)^3 + 4y_i y'_i y''_i \right) \\
Q_4(y,y',y'') &= -\sum_{i=1}^4 \left( y_i (y'_i)^3 + 4y_i^2 y'_i y''_i \right)
\end{aligned}$$

Writing the above system in the matrix form we get  $W(y) \cdot {}^{t}(y'') = {}^{t}Q(y, y', y'')$  where  ${}^{t}v$  denotes the transpose of *v* and *W* the Wronskian

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 \end{pmatrix}$$

Solving (5.13) we get that the vector function y satisfies the ODE

$$\Delta^{\mathsf{t}}(y''') = \operatorname{adj}(W)(y) \cdot {}^{\mathsf{t}}Q(y, y', y'')$$
(5.14)

where adj(W) is the adjoint of the matrix W,  $\Delta = det(W) = \prod_{i < j} (y_j - y_i)$  and  $Q = (0, Q_2, Q_3, Q_4)$ . Set  $P(y, y', y'') = adj(W)(y) \cdot {}^tQ(y, y', y'')$ . By looking carefully at the right hand side of the above relation, we see that *P* is homogeneous of degree 7.

**Remark 5.18.** According to Theorem 5.17 if  $y_1 \frac{\partial}{\partial x_1}$ ,  $y_2 \frac{\partial}{\partial x_2}$ , ...,  $y_4 \frac{\partial}{\partial x_4}$  are holomorphic vector fields that define a 4-chambar on an open set  $\mathcal{U} \subset \mathbb{C}$ , then the vector function  $x \in \mathcal{U} \mapsto y(x) = (y_1(x), y_2(x), \dots, y_4(x))$  satisfies an ODE of the form

$$\Delta y''' = P(y, y', y''), \tag{5.15}$$

where  $\Delta = \prod_{i < j} (y_j - y_i)$ . In particular if we fix an initial condition  $y(x_0)$ ,  $y'(x_0)$ ,  $y''(x_0)$  where  $y_i(x_0) \neq y_j(x_0)$  for any i < j, then (5.15) has an unique solution  $x \in \mathcal{U} \mapsto y(x)$  such that  $y_i(x) \neq y_j(x)$  for any i < j and any  $x \in \mathcal{U}$ .

However, if  $y_1(x_0) = y_2(x_0)$  for instance, then  $y_1(x) = y_2(x)$  for any  $x \in \mathcal{U}$ . The condition on the flows is now

$$2\varphi_t^1(x) + \varphi_t^3(x) + \varphi_t^4(x) = 4x,$$

which is a particular case of (5.10).

**Remarks 5.19.** Let us fix three (constant) vectors  $\alpha_0$ ,  $\alpha_1$  and  $\alpha_2$  in  $\mathbb{C}^4$  and assume that the components of  $\alpha_0$  are two by two different. Then there exists an unique germ  $y = (y_1, y_2, y_3, y_4) \in O(\mathbb{C}^4, 0)$  satisfying (5.11) with initial conditions  $y(0) = \alpha_0$ ,  $y'(0) = \alpha_1$  and  $y''(0) = \alpha_2$ .

Since the differential equation (5.11) is meromorphic on  $\mathbb{C}^4$  the solution  $x \mapsto y(x)$  can be extended until it reaches the codimension one submanifold  $\bigcup_{i < j} (y_i = y_j)$  of  $\mathbb{C}^4$ .

For instance, the constant vectors  $y = (a_1, a_2, a_3, a_4)$  are solutions of the ODE (5.14). In fact, if y is a constant vector then y' = y'' = 0 and Q(y, y', y'') = 0.

Next we will study the solutions with initial condition of the form  $y_i(0) = y_j(0)$ ,  $i \neq j$ . The idea is to lift the ODE to a first order ODE on  $\mathbb{C}^{12}$ .

Consider the ODE (5.11) of order 3 on  $\mathcal{U} \subset \mathbb{C}^4$ . Introducing new variables z = y', and w = z' = y'', this ODE can be lifted to a system of meromorphic ODE's of order 1 on  $\mathcal{V} = \mathcal{U} \times \mathbb{C}^4 \times \mathbb{C}^4$  as

$$\begin{cases} y' = z \\ z' = w \\ w' = \Delta^{-1} \cdot P(y, z, w) \end{cases}$$
(5.16)

Multiplying (5.16) by  $\Delta$  we obtain a tangent holomorphic vector field on  $\mathcal{V}$ 

$$\chi(y,z,w) = \Delta \sum_{j=1}^{4} z_j \frac{\partial}{\partial y_j} + \Delta \sum_{j=1}^{4} w_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{4} P_j(y,z,w) \frac{\partial}{\partial w_j}.$$
(5.17)

**Theorem 5.20.** The following submanifolds of  $\mathbb{C}^{12}$  are  $\chi$ -invariant:

$$\begin{split} &\diamond \ \Sigma_{ij} := \mathcal{Z}\big(\langle y_j - y_i \,|\, 1 \leq i < j \leq 4 \rangle\big); \\ &\diamond \ \Sigma_1 := \mathcal{Z}\big(\langle \sum_j y_j, \sum_j z_j, \sum_j w_j \rangle\big); \\ &\diamond \ \Sigma_2 := \mathcal{Z}\big(\langle \sum_j y_j z_j, \sum_j (z_j^2 + y_j w_j) \rangle\big); \\ &\diamond \ \Sigma_3 := \mathcal{Z}\big(\langle \sum_j (y_j z_j^2 + y_j^2 w_j) \rangle\big). \end{split}$$

The notation  $\mathcal{Z}(\mathcal{I})$  stands for the zeroes of the ideal  $\mathcal{I}$ .

All these submanifolds are complete intersections and the codimensions coincide with the number of generators of the ideal. Furthermore, the submanifolds  $\Sigma_i$ ,  $1 \le i \le 3$ , coincide with the initial conditions corresponding to the barycentric conditions

$$\sum_{k=1}^{4} \frac{\partial^n X_k^{\ell}}{\partial x^n} = 0 \quad \forall 1 \le n + \ell \le 4, \quad \forall n \ge 0.$$

Let us now give a Lemma that will be useful for the proof of Theorem 5.20.

**Lemma 5.21.** The components  $P_1$ ,  $P_2$ ,  $P_3$ ,  $P_4$  of  $\chi$  satisfy the following relations:

Proof. Recall that on the one hand

$${}^{\mathsf{t}}P(y,y',y'') = \mathsf{ajd}(W)(y){}^{\mathsf{t}}Q(y,y',y'')$$

so

$${}^{\mathsf{t}}P(y,z,w) = \operatorname{ajd}(W)(y){}^{\mathsf{t}}Q(y,z,w).$$

On the other hand the four relations of the statement are equivalent to  $W(y)^{t}P(y,z,w) = \Delta^{t}Q(y,z,w)$ . Finally, we know from linear algebra that W(y) adj $(W)(y) = \Delta \cdot id$ , where id is the identity matrix. As a consequence

$$W(y)^{t}P(y,z,w) = W(y)\operatorname{adj}(W)(y)^{t}Q(y,z,w) = \Delta^{t}Q(y,z,w).$$

*Proof of Theorem 5.20.* Let  $\mathcal{J}$  be an ideal of  $\mathbb{C}[y, z, w]$ . Recall that the submanifold  $\mathcal{Z}(\mathcal{J})$ , defined by  $\mathcal{J}$ , is  $\chi$ -invariant if, and only if,  $\mathcal{Z}(\mathcal{J}) \subset \mathcal{J}$ . So, for instance

$$\boldsymbol{\chi}(y_k - y_\ell) = (z_k - z_\ell) \prod_{i < j} (y_j - y_i)$$

and  $\chi(y_k - y_\ell)$  belongs to  $\langle y_k - y_\ell \rangle$ ; in particular  $\Sigma_{k\ell}$  is  $\chi$ -invariant.

Consider the ideal 
$$\mathcal{J}_1 = \langle \sum_j y_j, \sum_j z_j, \sum_j w_j \rangle$$
. We have  
 $Z\left(\sum_i y_i\right) = \sum_i Z(y_i) = \Delta \sum_i z_i \in \mathcal{J}_1$   
 $Z\left(\sum_i z_i\right) = \sum_i Z(z_i) = \Delta \sum_i w_i \in \mathcal{J}_1$   
 $Z\left(\sum_i w_i\right) = \sum_i Z(w_i) = \sum_i P_i = 0 \in \mathcal{J}_1$  by the first assertion of Lemma 5.21

With a similar computation, using the other assertions of Lemma 5.21 it is possible to prove that  $\Sigma_2$ ,  $\Sigma_3$  and  $\Sigma_4$  are  $\chi$ -invariant.

**Corollary 5.22.** Let  $Ch(X_1, X_2, X_3, X_4)$  be a 4-chambar on an open set  $\mathcal{U} \subset \mathbb{C}$ , with  $X_j = y_j \frac{\partial}{\partial x}$ ,  $y_j \in O(\mathcal{U})$ ,  $1 \le j \le 4$ .

Suppose that  $y_k(x_0) = y_\ell(x_0)$  for some  $x_0 \in \mathcal{U}$  and  $k \neq \ell$ . Then  $y_k(x) = y_\ell(x)$  for all  $x \in \mathcal{U}$ . Moreover, if k = 1 and  $\ell = 2$ , for instance, then either the chambar is constant and  $2a_1 + a_3 + a_4 = 0$  or  $y_j(x) = a_j \sqrt{\lambda x + \mu}$  with  $\lambda \neq 0$ ,  $a_1 = a_2 = -\frac{1}{3}$  and  $a_3$  and  $a_4$  the roots of  $3z^2 + 2z + 3 = 0$ .

*Proof.* The first assertion is consequence of Theorem 5.20 and the other of Theorem 5.7.

Let us denote by Ch(4,1) the set of 4-tuples  $(X_1, X_2, X_3, X_4)$  of germs at  $0 \in \mathbb{C}$  of holomorphic vector fields whose flows satisfy the barycentric conditions.

**Corollary 5.23.** The set Ch(4, 1) is isomorphic to an algebraic submanifold of  $\mathbb{C}^{12}$  whose irreducible components have dimension at most six.

Proof. According to Theorems 5.17 and 5.20 any 4-chambar on  $\mathbb{C}$  gives origin to a trajectory (y, z, w):  $(\mathbb{C}, 0) \rightarrow \mathbb{C}^{12}$  tangent to the  $\chi$ -invariant submanifold  $\Sigma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$  of  $\mathbb{C}^{12}$ . This defines an embedding of Ch(4, 1) on  $\Sigma$ .

#### 6. LINEAR CHAMBARS

**Theorem 6.1.** Let  $X_1, X_2, ..., X_p$  be some linear vector fields on  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ).

If they satisfy the barycentric property, then they are nilpotent.

*Proof.* The flow  $\varphi_t^k$  of  $X_k$  can be written

$$\varphi_t^k(x) = (\exp tA_k)(x)$$

where the  $A_k$  belong to  $\text{End}(\mathbb{R}^n)$  or  $\text{End}(\mathbb{C}^n)$ . We identify the  $A_k$  to some matrices. The barycentric property is equivalent to

$$\sum_{k=1}^{p} \sum_{\ell=0}^{\infty} \frac{t^{\ell}}{\ell!} A_{k}^{\ell} = p \operatorname{Id}$$

that implies  $\sum_{k=1}^{p} A_k^{\ell} = 0$  for any  $\ell \ge 1$ . Let  $\lambda_{k,j}$  be the eigenvalues of  $A_k$ ,  $1 \le j \le n$ . We get for all  $\ell \ge 1$ 

$$0 = \operatorname{Tr}\left(\sum_{k=1}^{p} A_{k}^{\ell}\right) = \sum_{k=1}^{p} \sum_{j=1}^{n} \lambda_{k,j}^{n}.$$

As a result all the  $\lambda_{k,j}$  are equal to zero.

**Remark 6.2.** The  $\varphi_t^k$  are polynomial in *x* and *t*.

**Remark 6.3.** If p = 2, then the indices of nilpotence are 2 (*i.e.*  $A^2 = 0$ ) and we recover the fact that the trajectories are straight lines. Note also that if X is a nilpotent vector field of index 2, then the pair (X, -X) is a 2-chambar.

**Example 6.4.** Let *X* be a nilpotent linear vector field of order *p*. Let  $\sigma = \exp\left(\frac{2i\pi}{p}\right)$  be a primitive *p*-th root of unity. Then the vector fields *X*,  $\sigma X$ ,  $\sigma^2 X$ , ...,  $\sigma^{p-1} X$  satisfy the barycentric property.

**Remark 6.5.** Let  $Ch(X_1, X_2, ..., X_p)$  be a linear *p*-chambar. Denote by *k* the maximal order of nilpotence of the  $X_i$ 's. Take  $\ell < k$  an integer. Then  $Ch(X_1^{\ell}, X_2^{\ell}, ..., X_p^{\ell})$  is a *q*-chambar for some  $q \le p$ . The inequality comes from the fact that two  $X_k^{\ell}$  can be equal or  $X_k^{\ell}$  can be zero. The fact that q < p measures some degeneration and if q = p for any  $\ell < k$  it gives some condition of transversality.

**Remark 6.6.** Let  $Ch(X_1, X_2, ..., X_p)$  be a singular *p*-chambar such that  $X_k(0) = 0$ . Denote by  $A_i$  the linear part of  $X_i$  for  $1 \le i \le p$ .

Assume that the  $A_i$ 's generate a linear *p*-chambar  $Ch(A_1, A_2, ..., A_p)$ . Consider the homothety  $h_s: x \mapsto sx, s \in \mathbb{C}^*$  and

$$X_k^s = h_{s*}X_k = A_k + s(\ldots)$$

We construct in this way a family  $Ch^s = Ch(X_1^s, X_2^s, \dots, X_p^s)$  of *p*-chambars, all conjugate for  $s \neq 0$ , and that joins the initial chambar  $Ch^1 = Ch(X_1, X_2, \dots, X_p)$  to the linear chambar  $Ch^0 = Ch(A_1, A_2, \dots, A_p)$ .

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#### 6.1. Linear *p*-chambars in dimension 2.

**Lemma 6.7.** Let *B* be  $(2 \times 2)$ -matrix with complex coefficients. If Tr(B) = 0, then *B* is the sum of two nilpotent matrices.

First proof. If B = 0, then the result holds.

Let us now assume that  $B \neq 0$ . Let us write B as  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ . We are looking forward two nilpotent matrices

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \qquad \qquad A' = \begin{pmatrix} x' & y' \\ z' & -x' \end{pmatrix}$$

such that B = A + A'. We thus have to solve the following system

$$\begin{cases} x + x' = a \\ y + y' = b \\ z + z' = c \\ x^{2} + yz = 0 \\ x'^{2} + y'z' = 0 \end{cases}$$

(the last two conditions guaranteeing nilpotence). After elimination of x', y' and z' we get

$$\begin{cases} x^2 + yz = 0\\ (a-x)^2 + (b-y)(c-z) = 0 \end{cases}$$

that is

$$\begin{cases} x^2 + yz = 0\\ 2ax + bz + cy - a^2 - bc = 0 \end{cases}$$

which is the non-trivial intersection of a quadric and of a plane. These two sets intersect along a plane conic.  $\Box$ 

Second proof. Since Tr(B) = 0, then *B* is conjugate to  $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$  for some *x*, *y* in  $\mathbb{C}$ . We conclude using the fact that

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}}_{\text{nilpotent}} + \underbrace{\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}}_{\text{nilpotent}}$$

**Corollary 6.8.** Let  $A_3, A_4, \ldots, A_p$  be (p-2) nilpotent  $(2 \times 2)$ -matrices.

There exist two nilpotent  $(2 \times 2)$ -matrices  $A_1$ ,  $A_2$  such that the flows  $\varphi_t^k = \exp t A_k$ ,  $1 \le k \le p$ , satisfy the barycentric property.

*Proof.* Let  $A_1$  and  $A_2$  be two nilpotent matrices such that

$$A_1 + A_2 + A_3 + \ldots + A_p = 0$$

As  $\exp tA_k = \operatorname{Id} + tA_k$  in dimension 2, the *p*-tuple  $(A_1, A_2, \dots, A_p)$  suits.

**Remark 6.9.** If  $A_1$ ,  $A_2$ ,  $A_3$  are nilpotent  $(2 \times 2)$ -matrices that satisfy the barycentric property, then the  $A_i$  are  $\mathbb{C}$ -colinear, *i.e.* Ch $(A_1, A_2, A_3)$  is rigid. Indeed the nilpotent  $(2 \times 2)$ -matrices form a quadratic cone.

6.2. Linear 3-chambars. The following example illustrates that we can find solutions to the barycentric property in some Lie algebras of vector fields. In the particular case n = 3 one can find p-chambars in the Heisenberg Lie algebra  $h_3$  formed by matrices

$$M(lpha,eta,\gamma)=\left(egin{array}{ccc} 0 & lpha & \gamma \ 0 & 0 & eta \ 0 & 0 & 0 \end{array}
ight).$$

One has  $M^2(\alpha, \beta, \gamma) = M(0, 0, \alpha\beta)$ . The barycentric property for the vector fields  $X_k$  corresponding to the matrices  $M(\alpha_k, \beta_k, \gamma_k), k = 1, ..., p$ , is equivalent to the equalities

$$\sum_{k=1}^p \alpha_k = \sum_{k=1}^p \beta_k = \sum_{k=1}^p \gamma_k = \sum_{k=1}^p \alpha_k \beta_k = 0.$$

In the coefficients space  $(\mathbb{C}^3)^p$  the barycentric property is the intersection of three hyperplanes and one quadric which has thus dimension 3p - 4.

**Theorem 6.10.** Let  $Ch(X_1, X_2, X_3)$  be a linear 3-chambar on  $\mathbb{C}^3$ . Then up to conjugacy, the  $X_i$ 's (identified to their matrices) are contained in the Heisenberg Lie algebra  $\mathfrak{h}_3 \subset \mathfrak{gl}(3,\mathbb{C})$ .

*Proof.* Let us identified  $X_i$  to its matrix.

We will distinguish two cases according to the rank of the  $X_i$ 's.

 $\diamond$  If one of the X<sub>i</sub>'s has rank 2, for instance X<sub>1</sub>, then up to conjugacy one can assume that X<sub>1</sub> =

 $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ . We are now looking for  $X_2$  and  $X_3$  such that  $X_2$  and  $X_3$  are nilpotent (in particular

their traces are zero) and  $X_1 + X_2 + X_3 = X_1^2 + X_2^2 + X_3^2 = 0$ . A straightforward computation implies that  $X_2$  and  $X_3$  belong to  $\mathfrak{h}_3$ .

 $\diamond$  It suffices now to deal with the case where the three nilpotent matrices  $X_1, X_2$  and  $X_3$  have rank 1. Up

to conjugacy one can suppose that  $X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . As  $X_2$  has rank 1 the three columns of  $X_2$  are

colinear, *i.e.* 
$$X_2 = (\lambda E, \mu E, \nu E)$$
 where  $E = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$ . Then  $X_3 = -X_1 - X_2 = \begin{pmatrix} -\lambda E, -\mu E, -\nu E - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$ .

Let us distinguish three cases:

• First assume that  $\lambda = \mu = 0$ . Changing the notations if needed let us take  $\nu = 1$ . Then

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \qquad X_3 = -\begin{pmatrix} 0 & 0 & a+1 \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}.$$

Since  $X_1$  and  $X_2$  are nilpotent, c has to be 0; but c = 0 leads to  $X_2^2 = X_3^2 = 0$ , and the  $X_i$  belong to  $\mathfrak{h}_3$ .

• Now suppose  $\lambda \neq 0$ , *i.e.*  $\lambda = 1$ . Then

$$X_{2} = \begin{pmatrix} a & \mu a & \nu a \\ b & \mu b & \nu b \\ c & \mu c & \nu c \end{pmatrix}, \qquad X_{3} = -\begin{pmatrix} a & \mu a & \nu a + 1 \\ b & \mu b & \nu b \\ c & \mu c & \nu c \end{pmatrix}.$$
  
As  $X_{3}$  has rank 1, the coefficients  $b$  and  $c$  are zero. Therefore  $X_{2} = \begin{pmatrix} a & \mu a & \nu a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ; since

 $X_2$  is nilpotent, *a* has to be 0. As a consequence  $X_2 = 0$  which is impossible (the matrices are implicitly assumed to be non-zero).

• Finally assume that  $\lambda = 0$  and  $\mu \neq 0$ , that is  $\lambda = 0$  and  $\mu = 1$  and

$$X_{2} = \begin{pmatrix} 0 & a & va \\ 0 & b & vb \\ 0 & c & vc \end{pmatrix}, \qquad X_{3} = -\begin{pmatrix} 0 & a & va+1 \\ 0 & b & vb \\ 0 & c & vc \end{pmatrix}$$

The fact that  $\operatorname{rk} X_3 = 1$  leads to b = c = 0 and

$$X_2 = \begin{pmatrix} 0 & a & va \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = -\begin{pmatrix} 0 & a & va+1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

belong to  $\mathfrak{h}_3$ .

In fact the statement holds in any dimension but we keep the previous result and its proof because this last one is much more easier. Let us start by some definitions, notations and intermediate results of non-commutative algebra.

A monomial of k-variables on  $\operatorname{End}(\mathbb{C}^n)$  is a map  $f: \operatorname{End}(\mathbb{C}^n)^k \to \operatorname{End}(\mathbb{C}^n)$  of the form

$$f(X_1, X_2, \dots, X_k) = X_{i_1}^{k_1} X_{i_2}^{k_2} \dots X_{i_r}^{k_r}$$

where  $r \ge 1$ ,  $i_j \in \{1, 2, ..., k\}$  and  $k_j \ge 0$  for any  $1 \le j \le r$ . By convention  $X_i^0 = 1$ .

We say that the monomial is reduced if

 $\diamond k_j \ge 1$  for any  $1 \le j \le r$ ;

$$\diamond i_j \neq i_{j+1}$$
 for any  $1 \leq j \leq r-1$ .

The degree of f is deg  $f = \sum_{i=1}^{r} k_i$ . A polynomial of k variables on  $\text{End}(\mathbb{C}^n)$  is a linear combination of monomials of k variables on  $\text{End}(\mathbb{C}^n)$ :

$$P(X_1, X_2, ..., X_k) = \sum_{j=1}^{s} a_j F_j(X_1, X_2, ..., X_k)$$

with  $a_1, a_2, ..., a_s$  in  $\mathbb{C}$ . The degree of P is deg  $P = \max\{\deg(F_j) | a_j \neq 0\}$ . If deg  $F_j \ge 1$  for any  $1 \le j \le s$ , then we say that P is without constant term.

If  $Ch(X_1, X_2, X_3)$  is a 3-linear chambar on  $\mathbb{C}^n$ , we denote by  $\mathcal{G} = \langle X_1, X_2, X_3 \rangle \subset End(\mathbb{C}^n) \simeq gl(n, \mathbb{C})$  the sub-algebra generated by  $X_1, X_2$  and  $X_3$ . As previously we identify the linear vector field  $X_j$  with elements of  $End(\mathbb{C}^n)$ .

We can now state the result:

**Theorem 6.11.** Let  $Ch(X_1, X_2, X_3)$  be a linear 3-chambar on  $\mathbb{C}^n$ . Let  $\mathcal{G} = \langle X_1, X_2, X_3 \rangle$  be the algebra of linear transformations generated by  $X_1$ ,  $X_2$  and  $X_3$ .

If  $Y_1, Y_2, \ldots, Y_n$  belong to G, then  $Y_1Y_2 \ldots Y_n = 0$ .

In particular, up to conjugacy, the  $X_i$ 's (identified to their matrices) are contained in the Heisenberg Lie algebra  $\mathfrak{h}_n \subset \mathfrak{gl}(n, \mathbb{C}).$ 

*Proof.* Let us start the proof with the following statement:

**Lemma 6.12.** Let  $Ch(X_1, X_2, X_3)$  be a linear 3-chambar on  $\mathbb{C}^n$ .

Let *f* be a monomial of two variables on  $\text{End}(\mathbb{C}^n)$ . There exists  $n(f) \in \mathbb{Z}$  such that

$$f(X_1, X_2) + f(X_2, X_1) = n(f) \cdot X_3^{\deg f}.$$

Proof. For instance from

$$X_1^k + X_2^k = -X_3^k \qquad \forall k \ge 1$$

we get

$$X_3^{k+j} = (X_1^k + X_2^k)(X_1^j + X_2^j) = X_1^{k+j} + X_2^{k+j} + X_1^k X_2^j + X_2^k X_1^j = -X_3^{k+j} + X_1^k X_2^j + X_2^k X_1^j$$

and so  $X_1^k X_2^j + X_2^k X_1^j = 2X_3^{k+j}$ .

A reduced monomial g of two variables on  $End(\mathbb{C}^n)$  can be written as

$$g(X,Y) = X^{k_1}Y^{j_1}X^{k_2}\dots Y^{j_k}$$

where  $k_1 \ge 0, j_r \ge 0, k_2, k_3, \dots, k_r \ge 1$  and  $j_1, j_2, \dots, j_{r-1} \ge 1$ . Note that deg  $g = \sum_{i=1}^r (k_i + j_i)$ . Let us introduce the following definitions:

- $\diamond$  the X-length of g is  $\ell_X(g) = \#\{i \mid k_i > 0\};$
- $\diamond$  the *Y*-length of *g* is  $\ell_Y(g) = \#\{i \mid j_i > 0\};$
- ♦ the length of *g* is  $\ell(g) = \ell_X(g) + \ell_Y(g)$ .

The proof is by induction on  $\ell(f)$ . Let us state the induction assumption: given  $m \in \mathbb{N}$  the assertion of the Lemma is true for any reduced monomial g with  $\ell(g) < m$ .

The induction assumption is true if  $m \le 2$ :

- ♦ for l(f) = 1 it is a consequence of the equality X<sub>1</sub><sup>k</sup> + X<sub>2</sub><sup>k</sup> = -X<sub>3</sub><sup>k</sup>;
  ♦ for l(f) = 2 it is a consequence of the equality X<sub>1</sub><sup>k</sup>X<sub>2</sub><sup>j</sup> + X<sub>2</sub><sup>k</sup>X<sub>1</sub><sup>j</sup> = 2X<sub>3</sub><sup>k+j</sup>.

Assume that the assertion of the Lemma is true for  $m \ge 2$  and let us prove that it is true for m+1. Let f be a monomial with length  $m + 1 \ge 3$ . Without loss of generality we can assume that  $f(X, Y) = X^k Y^j X^m g(X, Y)$ ;

note that 
$$\ell(g) = \ell(f) - 3 = m - 2$$
. Using that  $X_1^k X_2^j + X_2^k X_1^j = 2X_3^{k+j}$  we have  

$$\begin{aligned} f(X_1, X_2) + f(X_2, X_1) \\ &= X_1^k X_2^j X_1^m g(X_1, X_2) + X_2^k X_1^j X_2^m g(X_2, X_1) \\ &= (2X_3^{k+j} - X_2^k X_1^j) X_1^m g(X_1, X_2) + (2X_3^{k+j} - X_1^k X_2^j) X_2^m g(X_2, X_1) \\ &= 2X_3^{k+j} (X_1^m g(X_1, X_2) + X_2^m g(X_2, X_1)) - X_2^k X_1^{j+m} g(X_1, X_2) - X_1^k X_2^{j+m} g(X_2, X_1) \\ &= 2X_3^{k+j} \Big( g_1(X_1, X_2) + g_1(X_2, X_1) \Big) - \Big( g_2(X_1, X_2) + g_2(X_2, X_1) \Big) \end{aligned}$$

where  $g_1(X,Y) = X^m g(X,Y)$  and  $g_2(X,Y) = Y^k Y^{j+m} g(X,Y)$ . Note that

$$\ell(g_1) = 1 + \ell(g) = m - 1$$
 and  $\ell(g_2) = \ell(g) + 2 = m$ 

Therefore the induction assumption implies that for  $i \in \{1, 2\}$ 

$$g_i(X_1, X_2) + g_i(X_2, X_1) = n(g_i)X_3^{\deg f}.$$

Hence

$$f(X_1, X_2) + f(X_2, X_1) = n(f)C^{\deg f}$$

where  $n(f) = 2n(g_1) - n(g_2)$ .

**Lemma 6.13.** Let  $Ch(X_1, X_2, X_3)$  be a linear 3-chambar on  $\mathbb{C}^n$ .

Let P(X,Y) be a polynomial of two variables on  $End(\mathbb{C}^n)$ . Assume that *P* is without constant term. Then  $P(X_1,X_2)$  is nilpotent, that is  $P(X_1,X_2)^n = 0$ .

*Proof.* Assume first that *P* is a reduced monomial. Set  $d = \deg P$ . Denote by  $\lambda_1, \lambda_2, ..., \lambda_n$  (resp. by  $\mu_1, \mu_2, ..., \mu_n$ ) the eigenvalues of  $P(X_1, X_2)$  (resp.  $P(X_2, X_1)$ ). It follows from Lemma 6.12 that

$$\sum_{j} \lambda_{j} + \sum_{j} \mu_{j} = \operatorname{tr} \left( P(X_{1}, X_{2}) + P(X_{2}, X_{1}) \right) = \operatorname{tr} \left( n(P) X_{3}^{d} \right) = 0.$$

Given any  $m \in \mathbb{N}$ , since  $P(X, Y)^m$  is a monomial we have

$$\sum_{j} \lambda_{j}^{m} + \sum_{j} \mu_{j}^{m} = \operatorname{tr} \left( P(X_{1}, X_{2})^{m} + P(X_{2}, X_{1})^{m} \right) \qquad \forall m \in \mathbb{N}.$$

This implies that  $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$  and so  $P(X_1, X_2)$  is nilpotent. In particular we get tr $(P(X_1, X_2)) = 0$ .

Suppose now that *P* is a polynomial of two variables on  $\text{End}(\mathbb{C}^n)$  without constant term. Since *P* is a linear combination of non constant monomials we get  $\text{tr}(P(X_1, X_2)) = 0$ . Similarly, given  $m \in \mathbb{N}$  then  $P(X,Y)^m$  is also a polynomial without constant term and so  $\text{tr}(P(X_1, X_2)^m) = 0$ . Therefore  $P(X_1, X_2)$  is nilpotent and as  $P(X_1, X_2)$  belongs to  $\text{End}(\mathbb{C}^n)$  we get  $P(X_1, X_2)^n = 0$ .

Let  $\mathfrak{g}$  be any Lie algebra. Recall some classical well known facts. If x belongs to  $\mathfrak{g}$ ,  $y \mapsto [x, y]$  is an endomorphism of  $\mathfrak{g}$ , which we denote  $\operatorname{ad} x$ . We say that x is *ad-nilpotent* if  $\operatorname{ad} x$  is a nilpotent endomorphism. If  $\mathfrak{g}$  is nilpotent, then all elements of  $\mathfrak{g}$  are ad-nilpotent. The converse is also true, it is the Engel Theorem ([4]). In particular a Lie algebra  $\mathfrak{g}$  whose all elements are ad-nilpotent is triangularizable. If now  $\mathfrak{g}$  is a matrices algebra whose all elements are nilpotent (for the multiplication), then the algebra is up to conjugacy contained in the Heisenberg Lie algebra  $\mathfrak{h}_n$ . This ends the proof of the theorem.

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6.3. Some remarks on linear 4-chambars. As previously we will identify the vector field  $X_i$  to its matrix.

**Definition 6.14.** A *p*-chambar  $Ch(X_1, X_2, ..., X_p)$  has rank *r* if at least one  $X_i$  (identified with its matrix) has rank *r*.

Let us start with the following property:

**Proposition 6.15.** Let  $Ch(X_1, X_2, X_3, X_4)$  be a linear 4-chambar. If  $Ch(X_1, X_2, X_3, X_4)$  has rank 2, then  $Ch(X_1, X_2, X_3, X_4)$  is irreducible.

*Proof.* Suppose, by contradiction, that  $Ch(X_1, X_2, X_3, X_4)$  is reducible. Then  $Ch(X_1, X_2, X_3, X_4)$  consists of two pairs of 2-chambars: the trajectories are thus lines and the  $X_i$ 's (identified with their matrices) have rank 1.

6.3.1. A first family of examples. Consider the four following matrices

$$X_{1} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_{2} = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix},$$
$$X_{3} = \begin{pmatrix} 0 & a & -\frac{ab}{c} \\ 0 & b & -\frac{b^{2}}{c} \\ 0 & c & -b \end{pmatrix}, \qquad X_{4} = \begin{pmatrix} 0 & d & \frac{db}{e} \\ 0 & -b & -\frac{b^{2}}{e} \\ 0 & e & b \end{pmatrix}$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , *a*, *b*, *c*, *d*, *e* are complex numbers satisfying the following conditions

$$\gamma + a + d = 0,$$
  $\alpha - \frac{ab}{c} + \frac{db}{e} = 0,$   $\beta - \frac{b^2}{c} - \frac{b^2}{e^2} = 0,$   $\delta + c + e = 0.$ 

These matrices define a 4-chambar generically irreducible whose elements are not contained in a nilpotent algebra. Indeed

 on the one hand the nilpotent algebras of matrices are triangularizable; in particular the eigenvalues of a commutator are zero;

 $\diamond$  on the other hand the eigenvalues of the commutator  $[X_1, X_2] = \begin{pmatrix} 0 & \alpha \delta & -\beta \gamma \\ 0 & \beta \delta & 0 \\ 0 & 0 & -\beta \delta \end{pmatrix}$  are non-zero as soon as  $\beta \delta \neq 0$ .

Remark that the  $X_i$ 's have a common kernel for generic values of the parameters.

6.3.2. A second family of examples. Let us consider

$$X_{1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad X_{2} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & -c - 2 & 0 \end{pmatrix},$$
$$X_{3} = \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix}, \qquad X_{4} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2b & 1 & 0 \end{pmatrix}$$

Then  $(X_1, X_2, X_3, X_4)$  is a linear 4-chambar of rank 2 in  $\mathbb{C}^3$  and the  $X_i$ 's (identified to their matrices) are not contained in a nilpotent algebra of matrices.

More generally for  $1 \le j \le 4$  set

$$X_j = \left(\begin{array}{cc} A_j & 0\\ B_j & 0 \end{array}\right)$$

where  $A_j$  is a  $(2 \times 2)$ -matrix and  $B_j$  is a  $(1 \times 2)$ -matrix such that

$$\begin{cases} A_j^2 = 0\\ \sum_{j=1}^4 B_j A_j = 0 \end{cases}$$

Then  $(X_1, X_2, X_3, X_4)$  is a linear 4-chambar of rank 2 in  $\mathbb{C}^3$  and the  $X_i$ 's (identified to their matrices) are not contained in a nilpotent algebra of matrices.

6.3.3. A third family of examples. Consider

$$X_{1} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix}, \qquad X_{2} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ -c & \gamma & 0 \end{pmatrix},$$
$$X_{3} = \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & -b \\ c & 0 & 0 \end{pmatrix}, \qquad X_{4} = \begin{pmatrix} 0 & -\alpha & 0 \\ 0 & 0 & 0 \\ -c & -\beta & 0 \end{pmatrix}$$

where  $a, b, c, \alpha, \beta$  denote some complex numbers. Note that

$$X_1 + X_2 = \begin{pmatrix} 0 & \alpha & 0 \\ a & 0 & b \\ 0 & \beta & 0 \end{pmatrix} \qquad \qquad X_1 + tX_2 = \begin{pmatrix} 0 & t\alpha & 0 \\ a & 0 & b \\ (1-t)c & t\beta & 0 \end{pmatrix}$$

so that

 $\diamond X_1 + X_2$  has rank 2 generically on *a*, *b*,  $\alpha$  and  $\beta$ ,

♦  $X_1 + tX_2$  has rank 3 generically on *t*.

The eigenvalues of the commutator  $[X_1, X_2] = \begin{pmatrix} -a\alpha & 0 & -b\alpha \\ -bc & a\alpha + b\beta & 0 \\ -a\beta & \alpha c & -b\beta \end{pmatrix}$  are non-zero as soon as  $\alpha bc \neq -a\beta$ 

0. As a consequence  $Ch(X_1, X_2, X_3, X_4)$  is a 4-chambar generically irreducible and the matrices associated to the  $X_i$ 's are not contained in a nilportent algebra of matrices.

Note that for generic values of parameters the  $X_i$ 's do not all have the same kernel. As a consequence examples of §6.3.1 and §6.3.3 are not conjugated.

Finally one can state:

## Proposition 6.16. There exist linear, irreducible 4-chambars with the two following properties

- ♦ their flows are generically quadratic in t;
- ♦ the associated matrices are not contained in a nilpotent algebra of matrices.

#### 7. Homogeneous chambars

7.1. First properties. Let  $\mathcal{B} = Ch(X_1, X_2, ..., X_p)$  be a *p*-chambar at  $0 \in \mathbb{C}^n$ . We say that  $\mathcal{B}$  is homogeneous of degree v if any  $X_i$  is homogeneous of degree v.

**Remark 7.1.** Let Ch(X, -X) be a homogeneous 2-chambar on  $\mathbb{C}^2$ . Then up to linear conjugacy  $X = x^{\nu} \frac{\partial}{\partial y}$  (the proof is an exercise).

Given two holomorphic vector fields X and Y on  $\mathbb{C}^n$ , we define the set of colinearity between X and Y as

$$\operatorname{Col}(X,Y) := \{ m \in \mathbb{C}^n \, | \, X(m) \wedge Y(m) = 0 \}.$$

**Remarks 7.2.** We would like to remark the following facts:

- $\diamond$  Col(*X*,*Y*) is an analytic set;
- ◇ if  $\operatorname{Col}(X, Y) \neq \emptyset$ , then dim<sub>ℂ</sub>( $\operatorname{Col}(X, Y)$ ) ≥ 1;
- ♦ if *X* and *Y* are homogeneous vector fields, then dim<sub>ℂ</sub>(Col(*X*,*Y*)) ≥ 1;
- $\diamond$  il *X* is homogeneous and *Y* = *R* is the radial vector field of  $\mathbb{C}^n$ , then Col(*X*,*R*) is an union of straight lines through the origin 0 ∈  $\mathbb{C}^n$ . Il *X* ∧ *R* ≠ 0, then the vector fields *X* and *R* generate a singular foliation  $\mathcal{F}$  of dimension 2 of  $\mathbb{C}^n$ . There is a holomorphic foliation  $\widetilde{\mathcal{F}}$  on  $\mathbb{P}^{n-1}_{\mathbb{C}}$  such that  $\mathcal{F} = \pi^*(\widetilde{\mathcal{F}})$ . It is possible to prove that

$$\operatorname{Col}(X, R) = \pi^{-1} \left( \operatorname{Sing}(\mathcal{F}) \right) = \operatorname{Sing}(\mathcal{F}).$$

The various previous examples suggest the following conjecture:

**Conjecture 7.3.** Let  $Ch(X_1, X_2, ..., X_p)$  be a homogeneous *p*-chambar of degree  $v \ge 1$  on  $\mathbb{C}^n$ , where  $p \ge 2$ . Then  $Col(X_k, R) \subset Sing(X_k)^2$  for any  $k \ge 1$ . In particular dim  $Sing(X_k) \ge 1$ .

In the same spirit we have the following problem:

**Problem 7.4.** Let  $Ch(X_1, X_2, ..., X_p)$  be a (non-homogeneous) p-chambar such that  $X_k(0) = 0$ . Do the inequalities dim  $Sing(X_k) \ge 1$  hold ?

**Remark 7.5.** The problem is solved in the following cases:

- $\diamond v = 1$  (Theorem 6.1);
- $\diamond p = 2$  (Theorem 3.5);
- ♦ rigid-chambars (Corollary 4.11).

We proved the conjecture in the special case of homogeneous 3-chambar on  $\mathbb{C}^2$  of degree 2. In fact we will prove the following:

$$\operatorname{Sing}(X_k) = \left\{ m \in \mathbb{C}^n \, | \, X_k(m) = 0 \right\}.$$

<sup>&</sup>lt;sup>2</sup> Recall that  $Sing(X_k)$  is the singular set of  $X_k$ :

**Theorem 7.6.** Let  $Ch(X_1, X_2, X_3)$  be a homogeneous 3-chambar on  $\mathbb{C}^2$  of degree 2. Then, after a change of variables,  $X_j$  can be written as  $a_j y^2 \frac{\partial}{\partial x}$ , where  $a_1 + a_2 + a_3 = 0$ . In particular, any homogeneous 3-chambar on  $\mathbb{C}^2$  of degree 2 is rigid.

Let *X* be a homogeneous vector field of degree *d* on  $\mathbb{C}^2$ . Then *X* has *d* + 1 invariant straight lines through  $0 \in \mathbb{C}^2$ , counted with multiplicity. These lines are the solutions of f(x, y) = 0, where *f* is the homogeneous polynomial of degree *d* + 1 defined by

$$R \wedge X = f(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$
(7.1)

that is  $f(x,y) = \det \begin{pmatrix} x & y \\ X(x) & X(y) \end{pmatrix}$ . We will assume that  $f \neq 0$  (if not X is colinear to the radial vector field *R*).

Since f = 0 is X-invariant, then  $X(f) = h \cdot f$ , where h is a homogeneous polynomial of degree d - 1. Moreover, h = 0 if and only if f is a first integral of X. In this case, the foliations defined by X and by f must coincide: the relation X(f) = 0 gives  $X(x)\frac{\partial f}{\partial x} + X(y)\frac{\partial f}{\partial y}$ , and thus  $X(x)\frac{\partial f}{\partial x} = -X(y)\frac{\partial f}{\partial y}$ . Since the degrees of  $X(x), X(y), \frac{\partial f}{\partial x}$ , and  $\frac{\partial f}{\partial y}$  are equal, we obtain that

$$X = \alpha \left( \frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \right).$$

Using that R(f) = (d+1)f and (7.1) we get  $\alpha = \frac{1}{d+1}$  in the above relation.

In general, we have

$$(d+1)X - hR = H(f), (7.2)$$

where  $H(f) = \frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x}$ .

Another relation that we will use is

$$X(f) = X \left( \det \begin{pmatrix} x & y \\ X(x) & X(y) \end{pmatrix} \right) = \det \begin{pmatrix} x & y \\ X^2(x) & X^2(y) \end{pmatrix}.$$
(7.3)

**Lemma 7.7.** Let  $Ch(X_1, X_2, X_3)$  be a homogeneous 3-chambar of degree d on  $\mathbb{C}^2$ . For  $1 \le j \le 3$  define  $f_j$  by  $R \land X_j = f_j(x, y) \frac{\partial}{\partial x} \land \frac{\partial}{\partial y}$ . Suppose that the  $f_j$  are not identically 0. Then

- $\diamond$  either  $f_1$ ,  $f_2$  and  $f_3$  have two common linear factors,
- ◇ or  $f_j$  is a first integral of  $X_j$ ,  $1 \le j \le 3$ .

Proof. First of all, using relations (7.1), (7.3) and both

$$\sum_{j} X_{j}(x) = \sum_{j} X_{j}(y) = 0, \qquad \qquad \sum_{j} X_{j}^{2}(x) = \sum_{j} X_{j}^{2}(y) = 0$$

we obtain  $\sum_{j} f_j = 0$  and  $\sum_{j} X_j(f_j) = 0$ . If we set  $X_j(h_j) = h_j \cdot f_j$ ,  $1 \le j \le 3$ , then  $\sum_{j} h_j \cdot f_j = 0$ . On the other hand, since  $\sum_{j} X_j = 0$  and  $\sum_{j} f_j = 0$ , we get from (7.2) that  $0 = \sum_{j} ((d+1)X_j - h_jR - H(f_j)) = -\sum_{j} h_jR$ 

$$0 = \sum_{j} \left( (d+1)X_j - h_j R - H(f_j) \right) = -\sum_{j} h_j R$$

and so  $\sum_{j} h_j = 0$ .

Let us assume that  $h_j \neq 0$  for some  $1 \le j \le 3$ . In this case, from  $\sum_j h_j = 0$  there are  $i \ne j$  such that  $h_i \ne h_j$ . Suppose for instance that  $h_1 \ne h_2$ . Then the equalities

$$\begin{cases} f_1 + f_2 + f_3 = 0\\ h_1 f_1 + h_2 f_2 + h_3 f_3 = 0 \end{cases}$$

imply

$$(h_1 - h_3)f_1 = (h_3 - h_2)f_2. (7.4)$$

In particular both members of relation (7.4) are not identically zero. Since  $h_1 - h_3$  and  $h_3 - h_2$  have degree d - 1, and  $f_1$  and  $f_2$  degree d + 1,  $f_1$  and  $f_2$  must have two common factors. As  $f_3 = -f_1 - f_2$  these factors are also factors of  $f_3$ .

**Remark 7.8.** Lemma 7.7 implies that for a homogeneous 3-chambar on  $\mathbb{C}^2$  Problem 7.4 has a positive answer, maybe except when the  $f_i$  are first integral.

**Lemma 7.9.** Let  $Ch(X_1, X_2, X_3)$  be a homogeneous 3-chambar of degree 2 on  $\mathbb{C}^2$ , and let  $f_\ell$  be as in Lemma 7.7.

Then the  $f_{\ell}$ 's are not identically zero.

*Proof.* Suppose that  $f_1 \equiv 0$ ; up to a linear change of coordinates we can assume that  $X_1 = xR = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$ . Let  $\ell = 0$  be a  $X_2$ -invariant line; then  $\ell = 0$  is  $X_1$ -invariant, and also  $X_3$ -invariant since  $X_1 + X_2 + X_3 = 0$ . These facts imply that the restriction of  $X_1$ ,  $X_2$ ,  $X_3$  to  $\ell = 0$  define a 3-chambar on the line  $\ell = 0$ . The classification of 3-chambars on  $\mathbb{C}$  (Theorem 5.1) implies that the  $X_i$  are 0 on  $\ell = 0$ . In particular  $\ell = 0 = (x = 0)$  and  $X_1 = xR$ ,  $X_2 = xL_2$ ,  $X_3 = xL_3$ , with  $L_i$  linear vector field, and  $R + L_2 + L_3 = 0$ . The same argument as before implies that the invariant lines of  $L_2$ ,  $L_3$  are necessarily x = 0, *i.e.*:

$$L_2 = a_2 x \frac{\partial}{\partial x} + (b_2 x + c_2 y) \frac{\partial}{\partial y}, \qquad \qquad L_3 = a_3 x \frac{\partial}{\partial x} + (b_3 x + c_3 y) \frac{\partial}{\partial y}.$$

The first components of the flows of  $X_1$ ,  $X_2$ ,  $X_3$  are respectively  $\frac{x}{1-tx}$ ,  $\frac{x}{1-ta_2x}$ ,  $\frac{x}{1-ta_3x}$ ; the sum of these three homographies can not be 3x: contradiction.

Problem 7.10. Is Lemma 7.9 true in any degree ?

Assume that  $Ch(X_1, X_2, X_3)$  is homogeneous of degree 2, and that the  $f'_j$ s have two common factors. Let  $\ell_1$  and  $\ell_2$  be the two linear common factors of the  $f'_j$ s. We have the following two possibilities:

- i)  $\ell_1 \neq \ell_2$ : we can thus assume that *xy* is a factor of the  $f'_i$ s;
- ii)  $\ell_1 = \ell_2$ : we can thus suppose that  $y^2$  is a factor of the  $f'_i$ s.

Another fact is that a polynomial *p*-chambar in dimension 1 is constant (Proposition 2.6). Therefore, if a straight line  $\ell = 0$  is invariant for all vector fields of the chambar, then  $X_{j|\ell} = 0$ , and  $\ell$  is a factor of  $X_j$ . In dimension 2 this implies that  $X_j = \ell \cdot L_j$ , where  $L_j$  is a linear vector field,  $1 \le j \le 3$ .

In particular, i) and ii) imply the following possibilities:

i') if  $\ell_1 = x$  and  $\ell_2 = y$ , then we must have  $X_i = xyV_i$  where  $V_i$  is a constant vector field;

ii') if 
$$\ell_1 = \ell_2 = y$$
, then  $X_j = yL_j$  where  $R \wedge L_j = ym_j \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ ,  $m_j = a_j x + b_j y$ . In particular, we must have  
 $L_j = (\alpha_j x + \beta_j y) \frac{\partial}{\partial x} + \gamma_j y \frac{\partial}{\partial y}$ 
(7.5)

where  $a_j = \gamma_j - \alpha_j$  and  $b_j = -\beta_j$ .

Let us check that i') can not happen. In fact, let X = xyV, where  $V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ . By a direct computation we find

$$\begin{cases} \frac{X(x)}{xy} = a, & \frac{X^2(x)}{xy} = a^2y + abx, & \frac{X^3(x)}{xy} = a^3y^2 + \alpha xy + \beta x^2\\ \frac{X(y)}{xy} = b, & \frac{X^2(x)}{xy} = aby + b^2x, & \frac{X^3(y)}{xy} = b^3x^2 + \gamma xy + \delta y^2 \end{cases}$$

This implies with obvious notations: for any  $1 \le k \le 3$ 

$$a_1^k + a_2^k + a_3^k = 0$$
 and  $b_1^k + b_2^k + b_3^k = 0$ 

so  $V_1 = V_2 = V_3 = 0$ .

In situation ii') the vector fields  $X_j = yL_j$  are of the form  $X = y\left((ax+by)\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}\right)$ , and a direct computation shows that  $X(y) = \ldots + cy^2$ ,  $X^2(y) = 2c^2y^3$ , and  $X^3(y) = 6c^3y^4$ . This implies  $\sum_j c_j^k = 0$  for  $1 \le k \le 3$ ,

so that  $c_1 = c_2 = c_3 = 0$ , and  $X_j = y \ell_j \frac{\partial}{\partial x}$ , where  $\ell_j = a_j x + b_j y$  is linear. In particular, we get

$$X_j(x) = y\ell_j, \qquad \qquad X_j^2(x) = y^2 \frac{\partial \ell_j}{\partial x} \ell_j = a_j y^2 \ell_j \qquad \qquad X_j^3(x) = a_j^2 y^3 \ell_j;$$

as a consequence,  $a_1^k + a_2^k + a_3^k = 0$  for any  $1 \le k \le 3$ . This yields to  $a_1 = a_2 = a_3 = 0$ , and to  $X_j = b_j y^2 \frac{\partial}{\partial x}$  for any  $1 \le j \le 3$ . Note that the  $f_j$  are first integral of  $X_j$ .

It remains to consider the case where  $h_1 = h_2 = h_3 = 0$  and  $f_j$  is a first integral of  $X_j$ ,  $1 \le j \le 3$ . Let us come back to the definition of  $f_j := xX_j(y) - yX_j(x)$ , so that  $X_j(f_j) = 0$ . Remark first that X is a constant multiple of the hamiltonian of f

$$H(f_j) = \frac{\partial f_j}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f_j}{\partial x} \frac{\partial}{\partial y}$$

it can be checked that this follows from  $X_j(f_j) = 0$ . From the definition of  $f_j$  and Euler's identity we get  $X_j = -\frac{1}{3}H(f_j)$ . Let  $\ell$  be a straight line invariant for  $X_1$  and passing through 0. Suppose by contradiction that it is not invariant by  $X_2$ . We assert that, either  $X_{1|\ell} = 0$ , or the trajectories of  $X_2$  and of  $X_3$  are parallel straight lines. Assume that  $X_{1|\ell} \neq 0$ ; we will see that  $f_2$  is a perfect cube, *i.e.*  $f_2 = h^3$ , where *h* is linear, so that the trajectories of  $X_2$  are the levels of *h*. Without lost of generality we can suppose that  $\ell = (y = 0)$ . We can write  $f_2(x,y) = ax^3 + yq(x,y)$ , where *q* is homogeneous of degree 2 and  $a \neq 0$  because y = 0 is not  $X_2$ -invariant. If  $c \neq 0$ , then the level  $f_2 = c$  cuts  $\ell$  in three points  $z_j := (x_j, 0), 1 \leq j \leq 3$ , where the  $x_j$ 's are the roots of  $x^3 = \frac{c}{a}$ . If  $f_2$  is not a perfect cube, then the level  $f_2 = c$  is irreducible, and so it is connected. Denote by  $\varphi_t^j$  the flow of  $X_j$ ,  $1 \leq j \leq 3$ . Let us remark the following facts:

- a)  $\varphi_t^1(x,0) + \varphi_t^2(x,0) + \varphi_t^3(x,0) = 3(x,0)$  for all  $x \in \mathbb{C}$ , for all t where the flows are defined (barycentric property);
- b)  $X_{1|_{y=0}} = \alpha x^2 \frac{\partial}{\partial x}$  so  $\varphi_t^1(x,0) = \frac{x}{1-\alpha tx}$  and since we are assuming  $X_{1|_{y=0}} \neq 0$ ,  $\alpha$  is non-zero;
- c) as  $(f_2 = c) \cap (y = 0) = \{(x_j, 0) | 1 \le j \le 3\}$  and  $f_2 = c$  is connected, there exists  $\tau \ne 0$  such that  $\varphi_{\tau}^2(x_1, 0) = (x_2, 0)$ .

It is possible to prove that  $\varphi_{3\tau}^2(x_1, 0) = (x_1, 0)$ , and more generally  $\varphi_{3k\tau}^2(x_i, 0) = (x_i, 0)$  for all  $k \in \mathbb{Z}$ , i = 1, 2, 3. 3. Since  $f_3$  is a first integral of  $X_3$ , the leaf of the foliation generated by  $X_3$  through  $(x_1, 0)$  must cut  $\ell$  in at most three points. However, a) and b) imply that

$$\varphi_{3k\tau}^3(x_1,0) = \left(2x_1 - \frac{x_1}{1 - 3k\tau x_1}, 0\right),$$

contradicting that the number is finite. As a result,

- I) either  $f_2$  and  $f_3$  are perfect cubes,
- II) or  $X_{1|_{y=0}} = 0$ .

Let us deal with these two possibilities.

- I) Assume that  $f_2 = \ell_2^3$  and  $f_3 = \ell_3^3$ , where  $\ell_2$  and  $\ell_3$  are linear. In this case, the trajectories of  $X_2$ , and also of  $X_3$ , are parallel lines. We have the alternative:
  - A) either  $d\ell_2 \wedge d\ell_3 = 0$ ,
  - B) or  $d\ell_2 \wedge d\ell_3 \neq 0$ .

In case A), we have  $\ell_3 = \alpha \ell_2$ ,  $\alpha \neq 0$ , and  $\ell_2$  is a line invariant for the chambar. After a linear change of variables we can suppose that  $X_j = a_j y^2 \frac{\partial}{\partial x}$ , and the statement is proved. Note that in this case  $X_{j|_{\ell}} = 0$  for  $1 \leq j \leq 3$ .

In case B), after a linear change of variables, we can suppose that  $f_2 = -x^3$  and  $f_3 = -y^3$ , which implies  $X_2 = -x^2 \frac{\partial}{\partial y}$ , and  $X_3 = -y^2 \frac{\partial}{\partial x}$ . However, in this case we would have  $X_1 = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$ . This is not a 3-chambar because

$$X_2^2(x) = X_3^2(x) = 0$$
 and  $X_1^2(x) \neq 0$ .

II) Suppose that  $X_{1|_{y=0}} = 0$ . From the above we have the following consequences: the hamiltonian  $H(f_j) = X_j$  is identically zero on the lines  $f_j = 0$ . In particular all the irreducible components of  $f_j$  have multiplicity. Since the  $f_j$ 's have degree 3, the  $f_j$ 's are perfect cubes and we conclude as previously.

This ends the proof of Theorem 7.6.

#### REFERENCES

- C. Camacho and P. Sad. Invariant varieties through singularities of holomorphic vector fields. Ann. of Math. (2), (115):579–585, 1982.
- [2] D. Cerveau. Feuilletages en droites, équations des eikonales et autres équations différentielles. Astérisque, (323):101–122, 2009.
- [3] D. Cerveau and J. Déserti. Transformations birationnelles de petit degré, volume 19 of Cours Spécialisés. Société Mathématique de France, Paris, 2013.
- [4] J. E. Humphreys. Introduction to Lie algebras and representation theory, volume 9 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.
- [5] J. V. Pereira and L. Pirio. An invitation to web geometry, volume 2 of IMPA Monographs. Springer, Cham, 2015.

[6] J.-C. Tougeron. *Idéaux de fonctions différentiables*. Springer-Verlag, Berlin-New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71.

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