A gradient system for a higher-gradient generalization of Fourier's law of heat conduction

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Abstract

We derive a generalized heat conduction problem for a rarefied gas at slip regime from a gradient system where the driving functional is the entropy. Specifically, we construct an Onsager system $(\boldsymbol{X}, \mathscr{S}, \mathbb{K}_{\text{heat}})$ such that the associated evolution of the system is given by $\partial_t u = +\mathbb{K}_{\text{heat}}(u)\mathrm{D}\mathscr{S}(u)$, where the Onsager operator, $\mathbb{K}_{\text{heat}}(u)$, contains higher-gradients of the absolute temperature u. Moreover, through Legendre-Fenchel theory we write the Onsager system as a classical gradient system $(\boldsymbol{X}, \mathscr{S}, \mathscr{G})$ with an induced gradient flow equation, $\partial_t u = \nabla_{\mathscr{G}}\mathrm{D}\mathscr{S}(u)$. We demonstrate the usefulness of the approach by modeling scale-size thermal effects in periodic media that have been recently observed experimentally.

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1 Introduction

Fourier's law of heat conduction, in its simplest form, relates the heat flux and the temperature gradient, that is:

$$\boldsymbol{q} = -\kappa \nabla u, \tag{1.1}$$

where u is the absolute temperature and $\kappa > 0$ the thermal conductivity that depends on properties of the material. In general κ may depend on the temperature, space, or

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time but often varies so little in cases of interest that it is reasonable to neglect this variation. By now, Fourier's law of heat conduction is widely regarded as a limiting approximation of some general (potentially nonlinear) constitutive law for the heat flux (see e.g. [2], [14], [1], [6]). For instance, the temperature of a rarefied gas at the slip regime, namely $0.001 < \mathrm{K}_n < 0.3$ where K_n is the non-dimensional Knudsen number, deviates from Fourier's law of heat conduction [14]. Moreover, in the same article a generalized heat conduction model, from a phenomenological point of view, was postulated under the assumption that the gas is isotropic. The authors' speculation was that the heat flux in a rarefied gas in the slip regime, linearly depends on the temperature gradient but also on higher-order temperature derivatives,

$$\boldsymbol{q} = \mathcal{R}_1 \nabla u + \mathcal{R}_2 \!:\! \nabla\nabla u + \mathcal{R}_3 \vdots \nabla\nabla\nabla u, \tag{1.2}$$

where $\mathcal{R}_1$ is a second order tensor, $\mathcal{R}_2$ is a third order tensor, and $\mathcal{R}_3$ is a fourth order tensor with $:$ and $\vdots$ denoting second and third order contraction, respectively. Since the gas is usually assumed to be isotropic (and once a non-dimensionalization was performed), the above flux reduces to the following,

$$\boldsymbol{q} = -\eta_0(u)\nabla u + \alpha\eta_0(u)\mathrm{K}_n^2 \,\mathrm{div}(\nabla\nabla u), \tag{1.3}$$

where η_0 is the heat conductivity, and α is a parameter that depends on the gaseous type only. The postulated, higher-gradient model recovers Fourier's law of heat conduction when the Knudsen number vanishes ($\mathrm{K}_n = 0$). As a matter of fact, the authors in [14] showed numerically that when $\mathrm{K}_n = 0.001$ the temperature distribution profile was (almost) linear, indicating that the higher-gradient model recovers Fourier's classical law of heat conduction for $\mathrm{K}_n < 0.001$.

In addition to the above mentioned works, emerging primarily from the physics community, non-classical laws of Fourier's heat conduction have, for many years now, attracted considerable attention from the theoretical mechanics community as well (see e.g. [2], [6]). In recent years, the motivation for deriving non-classical heat conduction models in the mechanics field stemmed from trying to understand the presence of thermal fluctuation fields in heterogeneous materials with a microstructure. Specifically, the authors in [6] postulate the existence of a free energy function that has an added dependence, in a non-trivial way, on the gradient of the entropy density variable. Based on this enhanced free energy, an enhanced heat equation was derived that has the structure of the Cahn–Hilliard equation in mass transport theory and contains a term with a characteristic length related to material parameters that *can account for scale-size effects in microheterogeneous bodies*. Upon setting the aforementioned characteristic length equal to zero the classical law of heat conduction is recovered.

Our aim in this communication is to show that we can derive the evolution of such a system from an Onsager system where the driving functional is the entropy. Consequently, one can construct a (classical) gradient system due to the one-to-one correspondence between Onsager systems and classical gradient systems and derive a gradient flow equation for such a system. The Onsager system approach achieves, primarily, two objectives: The

first, it demonstrates that the higher-gradient heat evolution problem is of a geometric nature and can be completely described by a functional and a geometric structure. The second, modelling natural phenomena using Onsager systems permits us to exploit the property of the associated Onsager operator to add different dissipative mechanisms so long they are driven by the same functional. Furthermore, an Onsager system can easily be incorporated into an even more general framework that combines reversible dynamics, often expressed through a Poisson structure and an energy functional, as long as we impose non-interaction conditions (see e.g. [9] for some examples of classical equations).

Finally, we demonstrate the applicability and the effectiveness of the above model in modeling scale-size thermal effects in heterogeneous media (see e.g. [4], [3]) by introducing an additional length scale parameter based on the absolute size of the constituents of the heterogeneous medium. Furthermore, from a mathematical point of view this model enjoys considerably better regularity properties than its classical counterpart.

2 Onsager system for Fourier's law with higher-gradients

Definition of an Onsager system: An Onsager system is a triple $(\boldsymbol{X}, \Phi, \mathbb{K})$ where $\boldsymbol{X}$ is the state space containing the states $u \in \boldsymbol{X}$, $\Phi : \boldsymbol{X} \mapsto \mathbb{R} \cup \{\infty\}$ is a driving functional, and $\mathbb{K}(u) : \boldsymbol{X}^* \mapsto \boldsymbol{X}$ is a linear, symmetric, and positive semi-definite operator often referred to as the Onsager operator. The state space $\boldsymbol{X}$ is usually assumed to be a reflexive Banach space with dual $\boldsymbol{X}^*$ and the driving functional Φ is assumed to be differentiable. The evolution of the state u is given by,

$$\partial_t u = -\mathbb{K}(u)\mathrm{D}\Phi(u). \tag{2.1}$$

The rate equation (2.1) is also called *Onsager equation*. Here onwards, we proceed formally without specifying the function space $\boldsymbol{X}$ (this is standard procedure in thermo-mechanics modelling as is pointed out in [9]). Moreover, functional derivatives can be interpreted as variational derivatives by assuming that all functions are sufficiently smooth.

An Onsager system for a rarefied gas at slip regime: We consider a rarefied gas that is diffusing inside a domain $\Omega \subset \mathbb{R}^3$. Define the triple $(\boldsymbol{X}, \mathscr{S}, \mathbb{K}_{\text{heat}})$ where we choose the entropy,

$$\mathscr{S}(u) = \int_\Omega \log(u) d\boldsymbol{x}, \tag{2.2}$$

as the driving entropy functional and the following Onsager operator,

$$\mathbb{K}_{\text{heat}}(u)\square = -\mathrm{div}\left(\eta_0(u)u^2\nabla\square - \alpha\eta_0(u)\mathrm{K}_n^2\nabla\mathrm{div}(u^2\nabla\square)\right). \tag{2.3}$$

Then the evolution of the temperature u is given by the rate equation[1],

[1] As it is pointed out in [10], the driving potential for the gradient system is $-\mathscr{S}$. Throughout the letter, we are careful in using the correct sign in the formulas. Hence, the current Onsager system gives rise to $\partial_t u = +\mathbb{K}_{\text{heat}}(u)\mathrm{D}\mathscr{S}(u)$. The reader is encouraged to consult reference [10] for more details regarding correct corresponding signs in different type of gradient systems

$$\partial_t u = +\mathbb{K}_{\text{heat}}(u)\mathrm{D}\mathscr{S}(u). \tag{2.4}$$

We remark that, since u is the absolute temperature and the $s \mapsto \log(s)$ is non-decreasing the entropy is non-decreasing in accordance with the second law of thermodynamics.

Computing the variational derivative of the entropy we have,

$$\langle \mathrm{D}\mathscr{S}(u) \mid v\rangle_{\boldsymbol{X}^*,\boldsymbol{X}} = \frac{d}{d\varepsilon}\mathscr{S}(u+\varepsilon v)\big|_{\varepsilon=0} = \int_\Omega \frac{1}{u} v \, d\boldsymbol{x}. \tag{2.5}$$

Calculations in (2.3) then yield,

$$\partial_t u = \operatorname{div}\left(\eta_0(u)\nabla u - \alpha\eta_0(u)\mathrm{K}_n^2\nabla\operatorname{div}(\nabla u)\right). \tag{2.6}$$

Furthermore, when the coefficient $\eta_0(u)$ is independent of temperature u, the Onsager operator takes the following form,

$$\mathbb{K}_{\text{heat}}(u)\square = -\operatorname{div}\left(\eta_0 u^2\nabla\square - \operatorname{div}\left(\alpha\eta_0\mathrm{K}_n^2\nabla(u^2\nabla\square)\right)\right), \tag{2.7}$$

leading to the following, more convenient, Onsager equation,

$$\partial_t u = \operatorname{div}\left(\eta_0\nabla u - \operatorname{div}\left(\alpha\eta_0\mathrm{K}_n^2\nabla\nabla u\right)\right). \tag{2.8}$$

Equation (2.8) is precisely the higher-gradient generalization of Fourier's heat conduction law employed in [14, eq. (8), pg. 849] in the one dimensional setting with no temporal effects. Moreover, it is immediately clear that when the Knudsen number becomes negligible then we recover the Onsager system for the temperature of a gas that obeys Fourier's law of heat conduction (see [10]).

3 From an Onsager system to a gradient system

In this passage, we simply reproduce the train of thought in [9] that relates gradient systems and Onsager systems only for completion purposes.

Definition of a gradient system: A gradient system is a triple $(\boldsymbol{X}, \Phi, \mathscr{G})$ where $\boldsymbol{X}$ is, again, the state space, $\Phi : \boldsymbol{X} \mapsto \mathbb{R}\cup\{\infty\}$ is a (differentiable) driving functional, and a metric tensor $\mathscr{G} : \boldsymbol{X} \mapsto \boldsymbol{X}^*$ that is linear, symmetric, and positive semi-definite. The above gradient system induces a gradient flow equation which in the abstract force balance reads as follows:

$$\mathscr{G}(u)\partial_t u = -\mathrm{D}\Phi(u) \quad \Longleftrightarrow \quad \partial_t u = -\nabla_{\mathscr{G}}\Phi(u). \tag{3.1}$$

The term $\nabla_{\mathscr{G}}\Phi(u)$ is called the "gradient" of Φ and it is calculated by $\mathscr{G}(u)^{-1}\mathrm{D}\Phi(u)$. The name "gradient" is indeed appropriate since its computation depends, additionally, on the geometry of the ambient space on which the driving functional is defined through

the metric tensor $\mathscr{G}$. Many well know, classical equations can be framed in terms of gradient systems with an induced gradient flow equation that describes the evolution of that system (see [9], [10] for an eloquent presentation and taxonomy regarding gradient systems, Onsager systems, and gradient flow equations).

As it is pointed out in [9], there is a one-to-one correspondence between Onsager systems and gradient systems. Hence, if we define the dual dissipation potential,

$$\Psi^*(u,\xi) = \frac{1}{2}\left\langle \xi \mid \mathbb{K}_{\text{heat}}(u)\xi \right\rangle_{\boldsymbol{X},\boldsymbol{X}^*}. \tag{3.2}$$

then (2.4) becomes $\partial_t u = \mathrm{D}_\xi \Psi^*(u, +D\mathscr{S}(u))$. Since we the dual dissipation potential is convex (quadratic) we can apply the Legendre-Fenchel theory to compute the (direct) dissipation potential $\Psi(u,v)$ via,

$$\Psi(u,v) = \sup\left\{ \langle \xi \mid v\rangle_{\boldsymbol{X}^*,\boldsymbol{X}} - \Psi^*(u,\xi) \mid \xi \in \boldsymbol{X}^* \right\} = \frac{1}{2}\left\langle \mathbb{K}^{-1}_{\text{heat}}(u)v \mid v \right\rangle_{\boldsymbol{X}^*,\boldsymbol{X}}. \tag{3.3}$$

This means that the metric tensor is the inverse of the Onsager operator, $\mathscr{G}(u) = \mathbb{K}^{-1}_{\text{heat}}(u)$ since in a classical gradient system the dissipation potential is defined as,

$$\Psi(u,v) = \frac{1}{2}\langle \mathscr{G}(u)v \mid v\rangle_{\boldsymbol{X},\boldsymbol{X}^*}. \tag{3.4}$$

Hence, implicitly through Legendre-Fenchel theory, we can describe the metric tensor of the gradient system that induces a gradient flow.

4 Application to diffusion in a composite

In light of recent experimental work [3], [4] regarding scale-size thermal effects in heterogeneous media, we apply the above generalized Fourier's law to model scale-dependent thermal effects. To that end, consider a composite Ω with a spatially periodic microstructure, so that the composite can be divided into periodic cells of typical dimension ℓ. Let Y_ℓ denote a typical cell and Y_ℓ^α the part of Y_ℓ occupied by material α, $\alpha = 1,2$ so that $Y_\ell^1 \cup Y_\ell^2 = Y_\ell$. Moreover, we assume that each Y_ℓ^α, $\alpha = 1,2$ is simply connected (fee Fig. 1).

We start with the following Onsager system $(\mathrm{H}^2(\Omega), \mathscr{S}, \mathbb{K}_{\text{heat}})$ with,

$$\mathscr{S}(u) = \int_\Omega \log(u) d\boldsymbol{x}, \tag{4.1}$$

the driving entropy functional and the following Onsager operator,

$$\mathbb{K}_{\text{heat}}(u)\square = -\text{div}\left(\mathsf{K}u^2\nabla\square - \mathsf{M}\nabla\text{div}(u^2\nabla\square)\right). \tag{4.2}$$

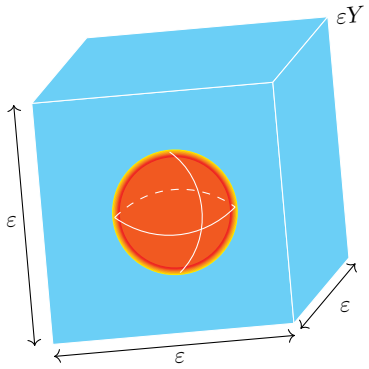

Figure 1: Schematic of the periodic microstructure of the composite.

The tensors K and M are Y_ℓ periodic and piece-wise constant in each phase. They stand for the thermal conductivity and spatial retardation, respectively. The above Onsager system leads to the following Onsager equation:

$$\partial_t u = \operatorname{div}\left(\mathsf{K}\nabla u - \operatorname{div}\left(\mathsf{M}{:}\nabla\nabla u\right)\right). \tag{4.3}$$

We normalize all variables as well as material properties the following way:

$$t^* = \frac{t}{\tau}, \quad \boldsymbol{x}^* = \frac{\boldsymbol{x}}{\mathrm{L}}, \quad u^*(\boldsymbol{x}^*) = \frac{u(\boldsymbol{x})}{\mathrm{L}}, \quad \mathcal{K}\mathsf{K}^* = \mathsf{K}, \quad \mathcal{M}\mathsf{M}^* = \mathsf{M}. \tag{4.4}$$

We have denoted by L the characteristic length of the macroscale, by $\mathcal{K}{:=}\max_{\boldsymbol{z}\in Y_\ell}|\mathsf{K}(\boldsymbol{z})|$ the characteristic value for the heat conduction K and by $\mathcal{M}{:=}\max_{\boldsymbol{z}\in Y_\ell}|\mathsf{M}(\boldsymbol{z})|$ the characteristic value for the spatial retardation M. Moreover, two different time scales can be identified:

$$\tau{=}\mathrm{L}^2/\mathcal{K} \ \text{ and } \ \tau{=}\ell^2/\mathcal{K}, \tag{4.5}$$

which characterize the diffusion time over the macroscale and microscale, respectively. Since we are interested in deriving effective equations we concerns ourselves only with time behavior on the macroscale. Furthermore, unlike in classical homogenization problems, we can now introduce an additional length scale relation between $\mathcal{K}$ and $\mathcal{M}$ as follows:

$$\mathcal{M} = \ell_{\mathrm{TE}}^2\mathcal{K}. \tag{4.6}$$

Thus, the diffusion equation becomes,

$$\partial_{t^*}u^* = \operatorname{div}^*\left(\mathsf{K}^*\nabla^*u^* - \left(\frac{\ell_{\mathrm{TE}}}{\mathrm{L}}\right)^2\operatorname{div}^*\left(\mathsf{M}^*{:}\nabla^*\nabla^*u^*\right)\right). \tag{4.7}$$

In the above context, the spatial retardation tensor contains gaseous type parameters and the Knudsen number.

The approach here motivated from generalized continuum mechanic theories (see e.g. [5]). It is well understood that the effective properties of heterogeneous materials cannot depend only on the volume fraction of the phases or their geometrical distribution but also on the absolute size of the constituents (see e.g. [7], [13]). The intrinsic length ℓ_{TE} is directly related to scale-size thermal effects that have been observed experimentally (see e.g. [4], [3]) in heterogeneous materials with macroscale size and which cannot be modelled by the classical Fourier's law of heat conduction. Since we are interested only in the macroscopic behavior of the composite, we assume that the microscopic scale ℓ is much smaller than the macroscopic scale L, i.e. $\ell/\mathrm{L} = \varepsilon \ll 1$ and let ε tend to 0. For classical homogenization, these are the only two length scales one has to work with. However, in our case we must also consider how ℓ_{TE} compares relative to ℓ or L. Specifically, we consider the following scaling,

$$\ell_{\mathrm{TE}}/\ell \sim 1. \tag{4.8}$$

The above scaling asserts that the scale-size thermal effects are roughly of the same magnitude as the length of the heterogeneity. Alternatively, we could re-write the above equation as $\ell_{\mathrm{TE}}/\mathrm{L} \sim \varepsilon$ and hence, we have full scale separation. This is a distinction that we need to make since the proposed model allows for treatment where no scale separation is possible (i.e. $\ell_{\mathrm{TE}}/\mathrm{L} \sim 1$). We will not address such a case here. We refer the reader to [12] for different type of scalings in the context of non-simple materials. Therefore, dropping the star notation for expediency, and introducing the parameter ε on the temperature u to show the dependence of the the solution on ε, equation (4.3) becomes,

$$\partial_t u_\varepsilon = \mathrm{div}\left(\mathsf{K}(\frac{\boldsymbol{x}}{\varepsilon})\nabla u_\varepsilon - \varepsilon^2 \mathrm{div}\left(\mathsf{M}(\frac{\boldsymbol{x}}{\varepsilon}){:}\nabla\nabla u_\varepsilon\right)\right), \tag{4.9}$$

where the tensors K and M are assumed to be bounded and possess the property of strong ellipticity.

In the context of asymptotic analysis with two space variables characterizing slow and fast scales under the assumption of scale separation and enough smoothness of all functions involved, the temperature, regarded as functions of the slow, $\boldsymbol{x}$, and fast, $\boldsymbol{y}{=}\boldsymbol{x}/\varepsilon$, is expanded in a two-scale series as follows,

$$u_\varepsilon(t,\boldsymbol{x}){=}u^{(0)}(t,\boldsymbol{x},\boldsymbol{x}/\varepsilon){+}\varepsilon^1 u^{(1)}(t,\boldsymbol{x},\boldsymbol{x}/\varepsilon){+}\varepsilon^2 u^{(2)}(t,\boldsymbol{x},\boldsymbol{x}/\varepsilon){+}\cdots. \tag{4.10}$$

Likewise, any derivatives are split in a way to represent the two scales as follows,

$$\begin{aligned} \nabla &\rightsquigarrow \nabla_x{+}\varepsilon^{-1}\nabla_y, \\ \nabla\nabla &\rightsquigarrow \nabla_x\nabla_x{+}2\varepsilon^{-1}\nabla_x\nabla_y{+}\varepsilon^{-2}\nabla_y\nabla_y. \end{aligned} \tag{4.11}$$

The above rule is very general and can be applied to different linear differential operators such as the divergence or curl operators. Hence, using the above two-scale expansions we have,

$$\nabla u^\varepsilon = \varepsilon^{-1}\nabla_y u^{(0)} + \varepsilon^0\Big(\nabla_x u^{(0)} + \nabla_y u^{(1)}\Big) + \varepsilon^1\Big(\nabla_x u^{(1)} + \nabla_y u^{(2)}\Big) + \dots, \tag{4.12}$$

$$\begin{aligned}\nabla\nabla u^\varepsilon = {}& \varepsilon^{-2}\nabla_y\nabla_y u^{(0)} + \varepsilon^{-1}\Big(2\nabla_x\nabla_y u^{(0)} + \nabla_y u^{(1)}\Big) \\ & + \varepsilon^0\Big(\nabla_x\nabla_x u^{(0)} + 2\nabla_x\nabla_y u^{(1)} + \nabla_y\nabla_y u^{(2)}\Big) + \dots,\end{aligned} \tag{4.13}$$

Therefore, the generalized flux expands as follows:

$$\begin{aligned}\operatorname{div} \boldsymbol{q}^\varepsilon = {}& \varepsilon^{-2}\operatorname{div}_y \boldsymbol{q}^{(0)} + \varepsilon^{-1}(\operatorname{div}_x \boldsymbol{q}^{(0)} + \operatorname{div}_y \boldsymbol{q}^{(1)}) \\ & + \varepsilon^0(\operatorname{div}_x \boldsymbol{q}^{(1)} + \operatorname{div}_y \boldsymbol{q}^{(2)}) + \cdots.\end{aligned} \tag{4.14}$$

Matching at order ε^{-2}. At order ε^{-2} we obtain,

$$\begin{aligned}&-\operatorname{div}_y\left(\overbrace{\mathsf{K}(\boldsymbol{y})\nabla_y u^{(0)} - \operatorname{div}_y(\mathsf{M}(\boldsymbol{y}){:}\nabla_y\nabla_y u^{(0)})}^{\boldsymbol{q}^{(0)}}\right) = 0 \text{ in } Y, \\ &u^{(0)}(t,\boldsymbol{x},\boldsymbol{y}) \text{ is } Y\text{-periodic.}\end{aligned} \tag{4.15}$$

The above problem is a classical elliptic partial differential equation in the variable $\boldsymbol{y}$ that possesses a unique solution. Since any function of $\boldsymbol{x}$ satisfies the above equation we obtain that $u^{(0)}(t,\boldsymbol{x},\boldsymbol{y}) = u^{(0)}(t,\boldsymbol{x})$.

Matching at order ε^{-1}. At order ε^{-1} we obtain,

$$\begin{aligned}&-\operatorname{div}_y\left(\overbrace{\mathsf{K}(\boldsymbol{y})(\nabla_x u^{(0)} + \nabla_y u^{(1)}) - \operatorname{div}_y(\mathsf{M}(\boldsymbol{y}){:}\nabla_y\nabla_y u^{(1)})}^{\boldsymbol{q}^{(1)}}\right) = 0 \text{ in } Y, \\ &u^{(1)}(t,\boldsymbol{x},\boldsymbol{y}) \text{ is } Y\text{-periodic.}\end{aligned} \tag{4.16}$$

The above problem has a unique solution that is linearly dependent on the gradient (with respect to $\boldsymbol{x}$) of $u^{(0)}$,

$$u^{(1)}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{3} w_i(\boldsymbol{y})\frac{\partial u^{(0)}}{\partial x_i}(t,\boldsymbol{x}) + c(\boldsymbol{x}), \tag{4.17}$$

where w_i (often referred to as the corrector) satisfies the following problem,

$$\begin{cases} -\mathrm{div}_{\mathrm{y}}\left(\mathsf{K}(\boldsymbol{y})\left(\boldsymbol{e}_i+\nabla_y w_i\right)\right)-\mathrm{div}_{\mathrm{y}}\left(\mathsf{M}(\boldsymbol{y}){:}\nabla_y\nabla_y w_i\right)\right)=\mathbf{0} \text{ in } Y, \\ w_i(\boldsymbol{y}) \text{ is } Y-\text{periodic}, \end{cases} \tag{4.18}$$

and c an arbitrary function of $\boldsymbol{x}$. Problem (4.18) has a unique solution (up to a constant in $\boldsymbol{x}$) as a consequence of the Lax-Milgram lemma and contains information about the morphology of the heterogeneous medium and the volume fraction. One will further note, that (4.18) is a fourth order partial differential equation and not the usual second order equation that one obtains using Fourier's classical heat conduction law.

Matching at order ε^0. At order ε^0 we obtain the macroscopic problem. Averaging over Y and using the Y-periodicity of the flux $\boldsymbol{q}^{(i)}$, $i=1,2,3,\ldots$ we obtain,

$$\partial_t u^{(0)} = \mathrm{div}_x \int_Y \left(\overbrace{\mathsf{K}(\boldsymbol{y})(\nabla_x u^{(0)}+\nabla_y u^{(1)})-\mathrm{div}_y(\mathsf{M}(\boldsymbol{y}){:}\nabla_y\nabla_y u^{(1)})}^{\boldsymbol{q}^{(1)}} \right) d^3y \text{ in } \Omega. \tag{4.19}$$

Inserting formula (4.17) into the above equation we can re-write (4.19) as,

$$\partial_t u^{(0)} = \mathrm{div}_x \left(\mathbb{K}^{\mathrm{eff}} \nabla_x u^{(0)} \right) \text{ in } \Omega, \tag{4.20}$$

where $\mathbb{K}^{\mathrm{eff}}$ is the effective conductivity given by the following formula,

$$\mathbb{K}^{\mathrm{eff}}_{ik} := \sum_{j=1}^{3} \int_Y \mathsf{K}_{ij}(\boldsymbol{y}) \left(\delta_{jk} + \frac{\partial w_k}{\partial y_j} \right) d^3y, \tag{4.21}$$

for $i,k=1,2,3$. It is immediately obvious that (4.20) is Fourier's classical law of heat conduction.

Note that the effective conductivity is not simply the volume average of the thermal conductivity of the two materials. Moreover, in the limit the higher-gradient effects are present only in the microscopic problem (4.18). This is in contrast to classical results where no higher-gradient effects appear on any scale. Furthermore, the appearance of higher-gradient effects in the microscale is due to the scaling choice we made in (4.8). A different scaling could in some way "transfer" the higher-gradient effects on different scales or completely nullify them (see e.g. [11] in the context of non-simple materias).

5 Concluding remarks

Finally, we would like to remark that Fourier's law of heat conduction has also been recovered using probabilistic techniques (see [8, pg. 184–187] for a probabilistic derivation

of the heat equation based on Brownian motion). The postulated theories put fourth in [14], [1] all recover Fourier's classical law of heat conduction when higher gradients are negligible. An exception to this, is the article of [2] where they derive a nonlinear, higher-gradient theory for the heat flux based on the principle of *equipresence*[2], thermodynamics, and symmetry considerations. In the approach of [2], when higher gradients are negligible, the heat flux reduces to the following,

$$\boldsymbol{q} = \kappa(u)\nabla u + \beta(u)|\nabla u|^2\nabla u, \tag{5.1}$$

where κ and β are material parameters. Hence, Fourier's law of heat conduction can be recovered, if in addition to higher-gradients being negligible, $\beta \equiv 0$. However, since β is a material parameter it is not clear whether such an assumption can or should be made. Thus, we interpret the derived heat flux law in [2] as a replacement of Fourier's heat conduction law rather than a generalization.

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