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Orientation mixing in active suspensions

Michele Coti Zelati, Helge Dietert, and David Gérard-Varet

ABSTRACT. We study a popular kinetic model introduced by Saintillan and Shelley for the dynamics of suspensions of active elongated particles where the particles are described by a distribution in space and orientation. The uniform distribution of particles is the stationary state of incoherence which is known to exhibit a phase transition. We perform an extensive study of the linearised evolution around the incoherent state. We show (i) in the non-diffusive regime corresponding to spectral (neutral) stability that the suspensions experience a mixing phenomenon similar to Landau damping and we provide optimal pointwise in time decay rates in weak topology. Further, we show (ii) in the case of small rotational diffusion ν that the mixing estimates persist up to time scale $\nu^{-1/2}$ until the exponential decay at enhanced dissipation rate $\nu^{1/2}$ takes over.

The interesting feature is that the usual velocity variable of kinetic models is replaced by an orientation variable on the sphere. The associated *orientation mixing* leads to limited algebraic decay for macroscopic quantities. For the proof, we start with a general pointwise decay result for Volterra equations that may be of independent interest. While, in the non-diffusive case, explicit formulas on the sphere allow to conclude the desired decay, much more work is required in the diffusive case: here we prove mixing estimates for the advection-diffusion equation on the sphere by combining an optimized hypocoercive approach with the vector field method. One main point in this context is to identify good commuting vector fields for the advection-diffusion operator on the sphere. Our results in this direction may be useful to other models in collective dynamics, where an orientation variable is involved.

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1. Active suspensions

Over the last decade, a significant mathematical effort has been put into the understanding of mixing mechanisms in kinetic equations. The easiest example is free transport

$$\partial_t f + v \cdot \nabla_x f = 0, \quad f(0,x,v) = f^{in}(x,v), \quad x \in \mathbb{T}^d, v \in \mathbb{R}^d. \tag{1.1}$$

The explicit representation $f(t,x,v) = f^{in}(x - vt, v)$ shows a filamentation of the support of the solution through time, leading to convergence of the solution as time goes to infinity, in weak topology, despite the absence of diffusive mechanisms. The rate of convergence depends on the regularity of the data, from

exponential convergence for analytic data, to polynomial convergence for Sobolev data. When diffusion is added, leading to

$$\partial_t f + v \cdot \nabla_x f - \nu \Delta_v f = 0,$$

the small scales created by the filamentation allow for an acceleration of the diffusive process; this leads to an enhanced dissipation on time scales $\nu^{-1/3}$ shorter than the usual ν^{-1}.

Identifying these phenomena in more complex kinetic equations, either of transport or weakly dissipative type, linearly and nonlinearly, has attracted a lot of attention. A main example is the analysis of Landau damping in the Vlasov-Poisson equation of plasma physics, with pioneering works [26, 30]. More recent works were dedicated to weakly dissipative cases, trying to exhibit both Landau damping and enhanced diffusion at the same time [2, 5].

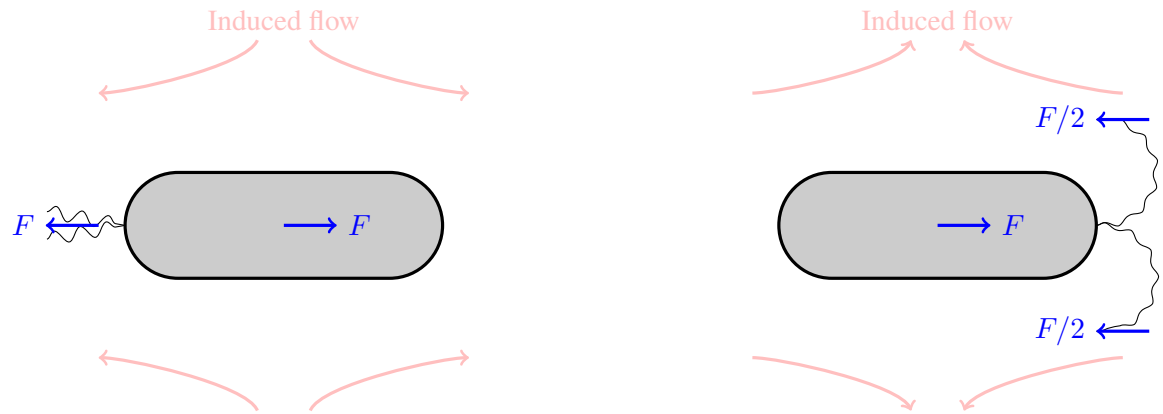
Another field where similar mixing phenomena are identified is the one of collective dynamics. A famous example in this direction is the Kuramoto model, a system of ODEs describing the dynamics of coupled oscillators, known to exhibit phase transition, from incoherence to synchronization. At the mathematical level, the mean-field limit of this system of ODEs leads to a kinetic equation, with unknown $f(t, \theta, \omega)$ describing the fraction of oscillators with phase $\theta \in \mathbb{T}$ and natural frequency $\omega \in \mathbb{R}$. Depending on a bifurcation parameter quantifying the intensity of the coupling, solutions converge weakly either to the uniform distribution (incoherence), or to the so-called partially locked states modeling synchronization (and containing Dirac masses in θ). It turns out that this convergence is again due to a mixing process, as noticed for the first time in [37]. Such phenomenon is now fully confirmed mathematically, both linearly and nonlinearly [7, 15, 20, 4, 17, 16].

In this paper, we will study mixing properties of a popular model describing a dilute suspension of elongated active particles in a Stokes flow.

1.1. The model. In [36], see also [35] for a review, D. Saintillan and M. Shelley have introduced a system of PDEs to describe the dynamics of a dilute suspension of elongated active particles in a viscous Stokes flow. The word "active" refers to the fact that they convert chemical energy into mechanical work. A typical example are bacteria, which are able to swim and create stress through the use of their flagellas. To write down the model, the first step is to consider a collection of N elongated particles (ellipsoids) immersed in a Stokes flow. They are described by the position of their center of mass $x_i \in \mathbb{T}^3_L = (\mathbb{R}/(L\mathbb{Z}))^3$ and their director $p_i \in \mathbb{S}^2$, $1 \leq i \leq N$. The choice of a torus of size L for the spacial domain instead of a real container is for mathematical convenience: it introduces a typical length scale L without problems related to the boundary of the domain. Neglecting interaction between the particles, one can use the work of Jeffery about a single passive ellipsoidal particle in a Stokes flow, see [24, 38]. If the typical size of the particle is very small compared to L, the torque experienced by the particle i is approximated by $\mathbb{P}_{p_i^\perp}\left(\gamma E(u) + W(u)\right)|_{x_i} p_i$, where $\mathbb{P}_{p_i^\perp} = \mathrm{I} - p_i \otimes p_i$ is the projection orthogonal to p_i, while $E(u)$ and $W(u)$ are the symmetric and skew-symmetric parts of ∇u. The parameter γ is related to the aspect ratio of the ellipsoid, with $\gamma = 1$ in the limit case of a rod. Note that in the work of Jeffery, u refers to the unperturbed velocity field, that is the one forgetting the particles. It is thus defined everywhere on $\mathbb{T}^3_L$, especially in x_i. The dynamics is then given by

$$\dot{x}_i = u(x_i) + U_0 p_i, \quad \dot{p}_i = \mathbb{P}_{p_i^\perp}\big(\gamma E(u) + W(u)\big)|_{x_i} p_i + \text{possible Brownian noise.}$$

The first equation describes the velocity of the particles as a sum of two contributions: the first one corresponds to advection by the velocity field u of the flow. The second one corresponds to swimming along the director p_i, at constant speed $U_0 > 0$. The second equation describes the rotation of the particle, which is known in the absence of noise to experience periodic trajectories (Jeffery's orbits). As far as we understand, these orbits are not really observed in experiments, and it is more accurate to perturb them by adding additional rotational Brownian motion, of small amplitude $\nu \ll 1$. On the other hand, we neglect Brownian motion in x, that is translational diffusion. As pointed out in [36, Page 7], it is not expected to play a big role in the linear stability properties of the model. We stress though that it could be included in the analysis below, and all the results would then hold independently of the strength of this diffusion.



(A) Pusher: bacteria with flagella at the back, e.g. *E. coli*.

(B) Puller: bacteria with flagella at the side at the front, e.g. *C. reinhardtii*.

FIGURE 1. Illustration of pusher and puller active particles moving to the right. Depending on configuration, the force for propulsion is differently applied on the fluid, which yields a different induced flow around the active particle [35].

Besides the swimming velocity $U_0 p_i$, another feature of active suspensions that needs to be retained is the stress that they create on the fluid. Saintillan and Shelley depart here from the work of Jeffery by including this effect at the level of the Stokes equation on u. The extra stress due to particle i can be thought as a dipole: the sum of two opposite and close point forces, along the direction of p_i: typically, for bacteria, one point force is centered at the body of the particles, while the other is centered at the flagella, see Fig. 1 distinguishing between two kinds of bacteria: pushers and pullers. One ends up with

$$-\Delta u + \nabla q = \alpha_N \sum_{i=1}^{N} \nabla_x \cdot \left((p_i \otimes p_i) \delta_{x_i} \right), \quad \nabla_x \cdot u = 0,$$

where this appearance of the divergence operator in front of a Dirac mass is typical of a dipole. Depending on the orientation of the dipole, the sign of α_N can be positive (pullers) or negative (pushers). We refer to [6] for a more rigorous derivation.

Eventually, with the right scaling of α_N, and performing a (formal) mean-field limit, one obtains the Saintillan-Shelley model, with two unknowns:

- $\Psi = \Psi(t, x, p)$, the distribution of particles in space and orientation,
- $u = u(t, x)$, the velocity field of the fluid.

It reads

$$\begin{aligned} &\partial_t \Psi + (U_0 p + u) \cdot \nabla_x \Psi + \nabla_p \cdot \left(\mathbb{P}_{p^\perp} \left[(\gamma E(u) + W(u)) p \right] \Psi \right) = \nu \Delta_p \Psi, \\ &-\Delta_x u + \nabla_x q = \alpha \nabla_x \cdot \int_{\mathbb{S}^2} \Psi(t, x, p)\, p \otimes p \, \mathrm{d}p, \\ &\nabla_x \cdot u = 0, \end{aligned} \tag{1.2}$$

where we remind that

$$E(u) = \frac{1}{2} \left[\nabla u + (\nabla u)^T \right], \qquad W(u) = \frac{1}{2} \left[\nabla u - (\nabla u)^T \right]. \tag{1.3}$$

The first equation on Ψ reflects transport along the stream lines associated to

$$\dot{x} = u(x) + U_0 p, \quad \dot{p} = \mathbb{P}_{p^\perp} \left[(\gamma E(u) + W(u))(x) p \right]$$

inspired by the dynamics of the particles mentioned above. It also contains some diffusion term $\nu \Delta_p \Psi$ ($\nu \ll 1$), where Δ_p refers to the Laplace-Beltrami operator associated to possible Brownian rotational

noise. The other equations describe the Stokes flow, which is forced by the extra stress $\operatorname{div}\sigma$, with σ given by the marginal in p of $\alpha\Psi p\otimes p$.

Let us notice that this model is very close to the Doi model [18] which describes passive suspensions of rodlike polymers. In this setting, $U_0 = 0$ (passive particles), and $\alpha > 0$ is proportional to the Boltzmann constant and the temperature: the tensor σ models viscoelastic stress. While local existence and uniqueness of smooth solutions of both the Doi and the Saintillan-Shelley models can be obtained by standard methods, the question of global in time well-posedness is harder. In the case of the Doi model, global well-posedness was proved in [32] for $\nu > 0$, and in [9] for $\nu \geq 0$. Actually a look at the proof of [9] shows that it is still valid for α of any sign. However, for non-zero U_0, as far as we know, global well-posedness is unknown (except for the addition of diffusion in x, see [6]). It seems to be an interesting open problem, especially when $\nu = 0$: indeed, one can check that the force field in the equation for $\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]$ has the same regularity in x as σ. This is to be compared with Vlasov-Poisson equation, where the force field has one degree of regularity more than the density ρ. In the present paper, we will not contribute to this well-posedness theory, but will rather investigate qualitative properties at the linear level.

1.2. Asymptotic stability of incoherence: main results. A class of equilibria of interest is given by

$$\Psi^S = \Psi^S(p), \qquad u^S = 0 \tag{1.4}$$

of which the isotropic suspension $\Psi^{iso} = 1/4\pi$ is a particular case. By analogy with the Kuramoto model, we will call Ψ^{iso} the incoherent state, as it reflects zero alignment between the orientation of the particles. The linearized system around $(\Psi^{iso}, u^S = 0)$ reads

$$\partial_t\psi + U_0 p\cdot\nabla_x\psi + \frac{1}{4\pi}\nabla_p\cdot\Big(\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]\Big) = \nu\Delta_p\psi, \tag{1.5}$$

$$-\Delta_x u + \nabla_x q = \nabla_x\cdot\alpha\int_{\mathbb{S}^2}\psi(t,x,p)\,p\otimes p\,\mathrm{d}p, \tag{1.6}$$

$$\nabla_x\cdot u = 0. \tag{1.7}$$

Since $E(u)$ is symmetric, $W(u)$ is skew-symmetric and u is divergence-free, we have

$$\nabla_p\cdot((\mathrm{I} - p\otimes p)E(u)p) = -3p\otimes p : E(u), \qquad \nabla_p\cdot((\mathrm{I} - p\otimes p)W(u)p) = 0. \tag{1.8}$$

Hence the equations become

$$\begin{aligned}
&\partial_t\psi + U_0 p\cdot\nabla_x\psi - \frac{3\gamma}{4\pi}p\otimes p : E(u) = \nu\Delta_p\psi,\\
&-\Delta_x u + \nabla_x q = \nabla_x\cdot\alpha\int_{\mathbb{S}^2}\psi(t,x,p)\,p\otimes p\,\mathrm{d}p,\\
&\nabla_x\cdot u = 0.
\end{aligned} \tag{1.9}$$

A partial analysis of system (1.9) is performed in [36, 23]

- in the case $\nu = 0$, looking for unstable eigenmodes, the authors manage to calculate an explicit dispersion relation. They show that in the case of suspensions of pullers ($\alpha > 0$), no unstable eigenvalue exists. As pointed out by the authors, this is consistent with the fact that for the full nonlinear system (1.9), for any $\alpha > 0$, $\nu \geq 0$, the relative entropy of any solution ψ with respect to ψ^{iso} decays. However, the situation changes completely for pushers ($\alpha < 0$). In this case, unstable eigenvalues exist at low enough x frequencies. In other words, the length L of the torus is a bifurcation parameter: there is a critical value L_c, computed numerically in [36], such that for $L < L_c$, there exists no unstable eigenvalue, while for $L > L_c$, there exists at least one. Above this threshold, the incoherent state loses its stability, some partial alignment of the ellipsoidal particles is observed numerically, while the corresponding velocity field develops patterns that are favourable to mixing of passive scalars advected by the flow. See the nice recent work [31] for more on the nature of the bifurcation process.

- in the case of small $\nu > 0$, no explicit dispersion relation is available. But numerical computations in [36], confirm the picture given at $\nu = 0$, with some threshold close to L_c. This numerical observation will be confirmed rigorously here.

Our focus in the present paper is on what we call the incoherent regime, that is $L < L_c$, both in the case $\nu = 0$ and $\nu > 0$. Again, some theoretical observations are already contained in [36], see also [23]. For $\nu = 0$, the absence of unstable eigenvalue for the linearized operator in (1.9) does not imply automatically linear stability, due to the possible unstable continuous spectrum. Numerical simulations in [23] show high frequency oscillations but no instability. Moreover, some decay at rate $(kt)^{-2}$, where k is the x-frequency of the perturbation, is seen for some moment of the solution ψ with respect to p. Stability of the incoherent state is conjectured on the basis of these simulations. Regarding $\nu > 0$, stability is confirmed by simulations, but no clear rate of convergence with respect to ν is given.

Our aim is to clarify most of these aspects, through a detailed mathematical study of the linearized equation (1.9). Since $x \in \mathbb{T}^3_L$, we take the Fourier transform of ψ

$$\psi_k(t,p) = \int_{\mathbb{T}^3_L} \mathrm{e}^{-ik\cdot x}\psi(t,x,p)\,\mathrm{d}x, \qquad k \in \frac{2\pi}{L}\mathbb{Z}^3, \tag{1.10}$$

$$\psi(t,x,p) = \frac{1}{L^3}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^3} \psi_k(t,p)\,\mathrm{e}^{ik\cdot x}, \tag{1.11}$$

and similarly for u_k, the Fourier transform of u. The Fourier transform of the equation is then

$$\begin{aligned}
&\partial_t\psi_k + U_0 ik\cdot p\,\psi_k = \frac{3\gamma}{4\pi}p\otimes p : E_k(u_k) + \nu\Delta_p\psi_k,\\
&u_k := \frac{i}{k^2}\mathbb{P}_{k^\perp}\sigma_k k, \quad \mathbb{P}_{k^\perp} := \left(\mathrm{I} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right), \quad E_k(u) := \frac{i}{2}(k\otimes u + u\otimes k),\\
&\sigma_k := \alpha\int_{\mathbb{S}^2}\psi_k(t,p)\,p\otimes p\,\mathrm{d}p.
\end{aligned}$$

Note that for $k = 0$, the equation reduces to the simple heat equation

$$\partial_t\psi_0 - \nu\Delta_p\psi_0 = 0$$

so that we restrict to $k \neq 0$. Moreover, through the change of variables

$$t := \frac{t}{U_0|k|}, \quad u := \frac{u}{|\alpha k|}, \quad \Gamma := \frac{\gamma|\alpha|}{U_0|k|}, \quad \nu := \frac{\nu}{U_0|k|}, \quad k := \frac{k}{|k|}, \quad \varepsilon := \frac{\alpha}{|\alpha|}, \tag{1.12}$$

the system becomes

$$\begin{aligned}
&\partial_t\psi_k + ik\cdot p\,\psi_k = \frac{3\Gamma}{4\pi}p\otimes p : E(u_k) + \nu\Delta_p\psi_k,\\
&u_k = i\mathbb{P}_{k^\perp}\sigma_k k, \quad \mathbb{P}_{k^\perp} = (I - k\otimes k)\,,\\
&\sigma_k = \varepsilon\int_{\mathbb{S}^2}\psi_k(t,p)\,p\otimes p\,\mathrm{d}p,
\end{aligned} \tag{1.13}$$

where

$$k \in \mathbb{S}^2, \quad \varepsilon = \pm 1 \quad \text{with } \varepsilon = 1 \text{ for pullers, } \varepsilon = -1 \text{ for pushers.} \tag{1.14}$$

In this rescaled setting, our main results are the following:

THEOREM 1 (**Inviscid case, pointwise decay through mixing**). *Let $\nu = 0$, $\psi_k^{in} = \psi_k^{in}(p) \in H^{3+\delta}(\mathbb{S}^2)$ for some $\delta > 0$, $k \in \mathbb{S}^2$, $\varepsilon = \pm 1$ and $\Gamma \in \mathbb{R}_+$. Let $\psi_k = \psi_k(t,p)$ be the solution of (1.13) such that $\psi_k|_{t=0} = \psi_k^{in}$. There is an absolute constant $\Gamma_c \in (0,+\infty]$, with $\Gamma_c = +\infty$ for $\varepsilon = 1$, and $C = C\big(\delta,\Gamma\big)$,*

such that if $\Gamma < \Gamma_c$*, then for all* $t \geq 0$

$$|u_k(t)| \leq \frac{C}{(1+t)^2} \|\psi_k^{in}\|_{H^{3+\delta}(\mathbb{S}^2)}, \tag{1.15}$$

$$\|\psi_k(t,\cdot)\|_{H^{-1-\delta}(\mathbb{S}^2)} \leq \frac{C}{(1+t)} \|\psi_k^{in}\|_{H^{3+\delta}(\mathbb{S}^2)}. \tag{1.16}$$

THEOREM 2 (**Diffusive case, mixing persists for small** ν **up to time** $\nu^{-1/2}$). *Let* $\nu \in (0,1)$*,* $\psi_k^{in} = \psi_k^{in}(p) \in H^{2+\delta}(\mathbb{S}^2)$ *for some* $\delta > 0$*,* $k \in \mathbb{S}^2$*,* $\varepsilon = \pm 1$ *and* $\Gamma \in [0, \Gamma_c)$*, with* Γ_c *as in Theorem 1. Let* $\psi_k = \psi_k(t,p)$ *be the solution of* (1.13) *such that* $\psi_k|_{t=0} = \psi_k^{in}$*. There exist* $\nu_0 = \nu_0(\Gamma) > 0$*,* $C = C(\delta, \Gamma) > 0$*, such that for* $0 < \nu < \nu_0$ *and* $t \in [0, \nu^{-1/2}]$ *it holds that*

$$|u_k(t)| \leq \frac{C \ln(2+t)}{(1+t)^2} \|\psi_k^{in}\|_{H^{2+\delta}(\mathbb{S}^2)}, \tag{1.17}$$

$$\|\psi_k(t,\cdot)\|_{H^{-1-\delta}(\mathbb{S}^2)} \leq \frac{C \ln(2+t)}{(1+t)} \|\psi_k^{in}\|_{H^{2+\delta}(\mathbb{S}^2)}. \tag{1.18}$$

After the time $\nu^{-1/2}$, we have the mixing estimates with an additional log factor until the enhanced dissipation takes over.

THEOREM 3 (**Diffusive case, enhanced dissipation at timescale** $\nu^{-1/2}$). *Under the assumptions of Theorem 2, there exist* $M, \eta, \nu_0 > 0$ *depending on* Γ*, and* $C = C(\delta, \Gamma)$ *such that for* $0 < \nu < \nu_0$ *and* $t \geq \nu^{-1/2}$ *it holds that*

$$|u_k(t)| \leq C \min\left(|\ln(\nu)|^M \nu, \mathrm{e}^{-\eta \nu^{1/2} t} \right) \|\psi_k^{in}\|_{H^{2+\delta}(\mathbb{S}^2)}, \tag{1.19}$$

$$\|\psi_k(t)\|_{L^2(\mathbb{S}^2)} \leq C(1+t) \mathrm{e}^{-\eta \nu^{1/2} t} \|\psi_k^{in}\|_{H^{2+\delta}(\mathbb{S}^2)}. \tag{1.20}$$

A few remarks are in order.

Remark 1.1. On the torus the largest length scale is L, so that in the unscaled equation the original wavenumber $|k|$ is at least $2\pi/L$. This leads to the upper bound

$$\Gamma \leq \frac{\gamma |\alpha| L}{2\pi U_0}. \tag{1.21}$$

Hence the stability condition on Γ can be translated into a maximal size of the torus.

Remark 1.2. Theorem 1 is the expression of an inviscid damping: it leads to a $O(t^{-2})$ decay for the velocity field u_k. Notice that u_k involves ψ_k through an average in p: this average allows to take advantage of the mixing phenomenon. As a byproduct, one has a $O(t^{-1})$ decay of ψ_k itself in weak topology (negative Sobolev space). The difference in the rate of decay is very much related to the special structure of u_k in terms of ψ_k. We stress that these polynomial rates cannot be improved, even taking analytic initial data. This is a major difference with Landau damping, and is related to the fact that the orientation variable $p \in \mathbb{S}^2$ replaces the velocity variable $v \in \mathbb{R}^3$.

Remark 1.3. In our proof of Theorem 1, the stability constant Γ_c that we exhibit when $\varepsilon = -1$ is sharp: it means that for $\Gamma > \Gamma_c$ there exist eigenmodes with u_k growing. Concretely, condition $\Gamma < \Gamma_c$ is equivalent to the fact that some dispersion relation has no root in the unstable half-plane:

$$F_\gamma(z) \neq 0, \qquad \forall \operatorname{Re}(z) \geq 0. \tag{1.22}$$

Our rigorous analysis of this dispersion relation, including the identification of Γ_c, is inspired by the work of Penrose [33] on the stability of plasmas. It substitutes for the numerical analysis carried in [36].

Remark 1.4. Theorem 2, dealing with the weakly diffusive setting, shows that for $\nu > 0$ small enough, the mixing phenomenon persists up to time $\nu^{-1/2}$, more generally up to time $C\nu^{-1/2}$ for arbitrary $C > 0$. The fact that $\nu^{-1/2}$ is a natural time threshold for our problem is confirmed by Theorem 3: solutions of

the equation experience an exponential decay after this typical threshold. This exponential decay before the natural diffusive timescale ν^{-1} reflects the well-known phenomenon of enhanced dissipation [3, 10, 12, 13], evoked in the introduction, although one can notice again a strong qualitative difference between variable $p \in \mathbb{S}^2$ and variable $v \in \mathbb{R}^3$. In the latter case, the enhanced dissipation would hold with typical time scale $\nu^{-1/3}$. Here, the typical time scale is much longer, which creates strong mathematical difficulties in showing mixing up to this time scale, and enhanced diffusion afterwards.

For the transition time until the enhanced dissipation takes over, we show the persistence of the mixing estimates with additional log factors. In advection-diffusion problems, a diffusion-limited mixing behavior is often observed [29]. That is, inviscid mixing does not persist as $t \to \infty$, rather the ratio of the $\dot{H}^{-1}$ to the L^2 norm converges to a positive constant. This corresponds to the existence of a characteristic filamentation length scale, often referred to as the Batchelor scale in the physics literature.

Remark 1.5. While completing our manuscript, we noticed the release of the interesting preprint [1], about the same Saintillan-Shelley model (except for the introduction of an additional diffusion in variable x). Although paper [1] and ours share a few common features, we believe that they are different enough to provide distinct and valuable insights into the stability of active suspensions.

Regarding mixing, which is the focus of the present paper, [1] only contains a weaker version of our Theorem 1, showing that under condition (1.22), a L^2 in time stability estimate

$$\int_{\mathbb{R}_+} |u_k(t)|^2 (1+t)^{3-\epsilon} \mathrm{d}t < \infty$$

holds. Our extensive discussion of (1.22), see Remark 1.3, as well as our optimal pointwise in time decay estimates are not covered. We stress that these pointwise estimates are based on the general Theorem 5 on Volterra equations, which is of independent interest.

More importantly, [1] does not contain any analogue of our Theorem 2, which is the heart of our paper, and requires completely different arguments from the inviscid case.

On the other hand, [1] contains the derivation of Taylor dispersion estimates when $x \in \mathbb{R}^d$, and two nonlinear stability results (diffusion in x is crucial there). First, in the case of pullers ($\varepsilon = 1$), using an approach à la Guo [22], the authors prove nonlinear stability of the incoherent state Ψ^{iso}, both for $x \in \mathbb{T}^d$ and $x \in \mathbb{R}^d$ (but without enhanced dissipation). Second, in the case of pushers, they prove nonlinear stability of the incoherent state with enhanced diffusion, under the stringent assumption $\Gamma = o(\nu^{1/2})$, which allows to treat the model as a perturbation of the advection-diffusion equation. With regards to these last two results, and although we restrict here to a linear analysis, our Theorem 3 is a significant progress.

1.3. Key ideas. The evolution (1.13) can be split as

$$\partial_t \psi_k = L_{1,k}\psi_k + \bar{L}_{2,k} \cdot u_k[\psi_k] \tag{1.23}$$

where

$$L_{1,k} := -ik \cdot p + \nu \Delta_p \tag{1.24}$$

is the advection-diffusion operator, and u_k is the low-dimensional linear map from the kinetic distribution to the macroscopic velocity field given by

$$u_k[\psi] := i\mathbb{P}_{k^\perp}\sigma k, \quad \mathbb{P}_{k^\perp} = (I - k \otimes k), \quad \sigma := \varepsilon \int_{\mathbb{S}^2} \psi(p)\, p \otimes p \,\mathrm{d}p. \tag{1.25}$$

Finally, $\bar{L}_{2,k}$ is the vector field independent of time defined by

$$\bar{L}_{2,k}(p) := \frac{3i\Gamma}{4\pi}(p \cdot k)\mathbb{P}_{k^\perp} p. \tag{1.26}$$

Here, we have used that u_k satisfies $k \cdot u = 0$ so that

$$\frac{3\Gamma}{4\pi} p \otimes p : E_k(u_k) = \bar{L}_{2,k} \cdot u_k.$$

By Duhamel's formula

$$\psi_k(t,\cdot) = \mathrm{e}^{tL_{1,k}}\psi_k^{in} + \int_0^t \mathrm{e}^{(t-s)L_{1,k}}\left(\bar{L}_{2,k}\cdot u_k[\psi_k(s,\cdot)]\right)\,\mathrm{d}s. \tag{1.27}$$

Applying the linear map u_k then yields the Volterra equation

$$u_k(t) + \int_0^t K_k(t-s)\,u_k(s)\,\mathrm{d}s = U_k(t) \tag{1.28}$$

with the source

$$U_k(t) = u_k[\mathrm{e}^{tL_{1,k}}\psi_k^{in}] \tag{1.29}$$

and the matrix kernel K_k defined by

$$K_k(t)v = -u_k[\mathrm{e}^{tL_{1,k}}(\bar{L}_{2,k}\cdot v)], \quad v\in\mathbb{C}^3. \tag{1.30}$$

In the proof of Theorems 1, 2 and 3 the key point is to work on the Volterra equation (1.28) and to show that u_k decays like t^{-2} at infinity. Once this is obtained, to go back to the evolution of ψ_k and estimate its decay in negative Sobolev norms is direct. Regarding the decay of u_k, there are two main steps:

i) To show that the source term U_k and the kernel K_k decay like in the corresponding theorem.
ii) To show that under appropriate conditions on Γ, this decay passes to the solution u_k of the Volterra equation.

Let us stress that the mathematical questions raised by the second step are not specific to our model, and can be asked for any Volterra equation. They will be examined in Section 2. Obviously, before relating the decay rate of the solution to the decay rates of the source and the kernel, a prerequisite is the stability of the solution. Using the classical theory of Volterra equation, one knows that it is equivalent to the spectral condition

$$\det(I+\mathcal{L}K_k(z)) \neq 0, \quad \forall\,\mathrm{Re}(z)\geq 0, \tag{1.31}$$

where $\mathcal{L}$ is the Laplace transform. Under this condition, we achieve Step ii) by proving Theorem 5 in Section 2. We provide a short proof, that uses the notion of Volterra kernel of type L^∞ and the underlying structure of Banach algebra of this class of kernels

As regards Step i), there is a main difference between $\nu=0$ and $\nu>0$. In the inviscid case $\nu=0$, considered in Section 3, the evolution $\mathrm{e}^{tL_{1,k}}$ can be solved explicitly: the kernel and sources are given by Fourier transforms on the sphere, whose decay properties are well-known, and analyzed through stationary phase arguments. This implies that U_k and K_k are $O(t^{-2})$. Moreover, the spectral stability condition can be fully understood through an analysis à la Penrose, see Section 3, completing the proof of Theorem 1.

For the diffusive case $\nu>0$, we cannot solve the evolution $\mathrm{e}^{tL_{1,k}}$ explicitly and the main effort in proving Theorem 2 and 3 is to obtain optimal mixing estimates on the advection-diffusion equation. This is performed in Section 4. In terms of the advection-diffusion evolution e^{tL_1}, the first result is an enhanced dissipation result, which can be proved by adapting hypocoercive methods:

PROPOSITION 1.6. *There exist constants $C_0,\varepsilon_0,\nu_0>0$ with the following property: for every $\nu_0>\nu>0$ and ψ^{in} the advection-diffusion evolution is bounded for all $t\geq 0$ as*

$$\|\mathrm{e}^{tL_{1,k}}\psi^{in}\|_{L^2(\mathbb{S}^2)} \leq C_0\mathrm{e}^{-\varepsilon_0\nu^{1/2}t}\|\psi^{in}\|_{L^2(\mathbb{S}^2)}. \tag{1.32}$$

The main novelty is that the inviscid mixing estimates persists until the enhanced dissipation takes over:

PROPOSITION 1.7. *There exist $\nu_0>0$ and $M\geq 2$ such that for $c,\delta>0$ there exists a constant C with the following bounds: For any $\nu\leq\nu_0$, any scalar function F and $t\leq\nu^{-1/2}$ it holds that*

$$\left|\int_{\mathbb{S}^2}(\mathrm{e}^{tL_{1,k}}\psi^{in})F\,\mathrm{d}p\right| \leq C\frac{\sqrt{\ln(2+t)}}{(1+t)}\|F\|_{H^{1+\delta}}\|\psi^{in}\|_{H^{1+\delta}} \tag{1.33}$$

and

$$\left|\int_{\mathbb{S}^2}(\mathrm{e}^{tL_{1,k}}\psi^{in})F\,\nabla(p\cdot k)\,\mathrm{d}p\right| \leq C\frac{\sqrt{\ln(2+t)}}{(1+t)^2}\|F\|_{H^{2+\delta}}\|\psi^{in}\|_{H^{2+\delta}}. \tag{1.34}$$

Further, for $t \in [\nu^{-1/2}, c\nu^{-1/2}|\ln \nu|]$ *it holds that*

$$\left| \int_{\mathbb{S}^2} (\mathrm{e}^{tL_{1,k}} \psi^{in}) F \, \mathrm{d}p \right| \leq C\nu^{1/2} |\ln \nu|^M \|F\|_{H^{1+\delta}} \|\psi^{in}\|_{H^{1+\delta}} \tag{1.35}$$

and

$$\left| \int_{\mathbb{S}^2} (\mathrm{e}^{tL_{1,k}} \psi^{in}) F \, \nabla(p \cdot k) \, \mathrm{d}p \right| \leq C\nu |\ln \nu|^M \|F\|_{H^{2+\delta}} \|\psi^{in}\|_{H^{2+\delta}}. \tag{1.36}$$

The key idea is the use of the vector field method. This method, introduced for the analysis of decay properties of wave equations [25], was used recently in the context of Vlasov type equations on $\mathbb{R}^d$ [5]. It allowed to exhibit mixing when $0 < \nu \ll 1$, despite the loss of explicit representation formula. Let us just mention at this stage that the key point is to construct good vector fields, meaning that they commute to the advection-diffusion operator $ik \cdot p - \nu \Delta_p$. In the Euclidean setting (1.1), the natural choice $J = \nabla_v + itk$ works well over the time scale $\nu^{-1/3}$, after which enhanced dissipation dominates. On the sphere this has no good analogue as the obvious generalization $J = \nabla_p + i\nabla(p \cdot k)t$, when applied to a solution ψ of the advection-diffusion equation, creates commutators of the form $\nu t \nabla \psi$, with a time integral that can not be controlled over the large time scale $\nu^{-1/2}$. We overcome this difficulty by combining two ideas. We first introduce a better vector field,

$$J_\nu = \alpha(t)\nabla + i\beta(t)\nabla(p \cdot k), \quad \text{where} \quad \alpha(t) := \cosh(\sqrt{-2i\nu}\, t), \quad \beta(t) := \frac{1}{\sqrt{-2i\nu}} \sinh(\sqrt{-2i\nu}\, t).$$

Roughly, it allows to replace the bad commutator by one that behaves like $\nu t(1 - k \cdot p)\nabla\psi$, hence vanishing near $\pm k$. Then, we adapt Villani's hypocoercivity method [39], using additional weights in time. The key point of this adaptation is that, besides proving enhanced diffusion at time scale $\nu^{-1/2}$, see Remark 1.4, it provides extra decay information for some quantities vanishing near the poles $\pm k$ of the sphere. This allows to control the commutator and apply the vector field method. All details will be found in Section 4.

Using the decay estimate, the proof of Theorem 2 is achieved in Section 5.1. Regarding the enhanced dissipation estimates of Theorem 3, a key point is to understand the behaviour of the diffusive Volterra kernel past time $\nu^{-1/2}$, and notably to check the spectral condition (1.31) for small ν. In the Euclidean case, see for instance [5], this can be proved perturbatively from the inviscid case, using the decay of the diffusive kernel uniformly in ν. In our spherical setting, this is where we need the second part of Proposition 1.7. This analysis and the proof of Theorem 3 are performed in Section 5.2.

2. Volterra equations

In this section we study a general Volterra equation:

$$u(t) + K \star u(t) = v(t), \quad t \geq 0, \tag{2.1}$$

with unknown $u : \mathbb{R}_+ \to \mathbb{C}^n$, and data $K : \mathbb{R}_+ \to M_n(\mathbb{C})$ (the kernel), $v : \mathbb{R}_+ \to \mathbb{C}^n$ (the source). The convolution is here on $\mathbb{R}_+$, defined by $F \star g(t) = \int_0^t F(s)\, g(t-s)\, \mathrm{d}s$. We are interested in the global in time solvability of this equation, and in accurate polynomial decay estimates for solution u under assumptions of polynomial decay on K and v. The classical path to study this equation, detailed in [21], is to construct the so-called resolvent of the equation, that is a matrix-valued $R : \mathbb{R}_+ \to M_n(\mathbb{C})$ satisfying

$$R(t) + K \star R(t) = R(t) + R \star K(t) = K(t). \tag{2.2}$$

Note that the convolution product does not commute when $n > 1$, so that in this case, the resolvent satisfies two distinct equations. It is then straightforward to check that if K and v are integrable, and if there exists an integrable solution R of (2.2), then an integrable solution u of (2.1) is given by $u = v - R \star v$ and is unique.

For the construction of the resolvent, a vague idea of proof is the following. Assuming there is a solution R to (2.2), and extending both R and K by zero on $\mathbb{R}_-$, the relation (2.2) holds now for all t on $\mathbb{R}$, replacing

the convolution on $\mathbb{R}_+$ by the usual convolution on $\mathbb{R}$. Taking the Fourier transform yields in particular

$$(I + \hat{K}(\xi))\hat{R}(\xi) = \hat{K}(\xi), \quad \forall \xi \in \mathbb{R}. \tag{2.3}$$

This suggests to impose the condition

$$\det(I + \hat{K}) \neq 0 \quad \text{on } \mathbb{R} \tag{2.4}$$

and to define

$$R = \mathcal{F}^{-1}(I + \hat{K})^{-1}\hat{K}. \tag{2.5}$$

This is however too quick. We remind that, in order to obtain (2.3), we have extended our hypothetical solution R by zero on $\mathbb{R}_-$. Hence, we must not only verify that the formula (2.5) makes sense, but also that it gives a function that vanishes on $\mathbb{R}_-$. Condition (2.3) is not sufficient for that. Still, under the stronger condition

$$\det(I + \mathcal{L}K(z)) \neq 0 \quad \text{for all } \operatorname{Re} z \geq 0, \tag{2.6}$$

where $\mathcal{L}K(z) = \int_0^{+\infty} \mathrm{e}^{-zt} K(t)\,\mathrm{d}t$ is the Laplace transform of the kernel, everything goes nicely. This is the content of following theorem.

THEOREM 4 ([**21**, Chapter 2]). *Assume that $K \in L^1(\mathbb{R}_+, M_n(\mathbb{C}))$, and that the spectral condition (2.6) is satisfied. Then, there exists a unique solution $R \in L^1(\mathbb{R}_+, M_n(\mathbb{C}))$ of (2.2). As a consequence, for any $v \in L^1(\mathbb{R}_+, \mathbb{C}^n)$, (2.1) has a unique solution $u \in L^1(\mathbb{R}_+, \mathbb{C}^n)$.*

Once global existence of an L^1 solution has been obtained, a natural question is its decay, depending on the decay of the kernel and data. While many works, for instance on Landau damping, treat this kind of question, most results fall into one of the following two types. Either they use weighted spaces and use the stronger assumption $\int_0^\infty |K(t)|\,(1+t)^\alpha\,\mathrm{d}t$ to conclude from v in $O((1+t)^{-\alpha})$ that u is $O((1+t)^{-\alpha})$, e.g. [**15, 20**]. Or they establish L^p in time estimates without loss for $p = 1, 2$. Indeed, while weighted L^1 estimates behave well with respect to convolution formulas such as (2.2), L^2 estimates may be established using (2.3) and Plancherel formula, see [**1**].

Note that this is a significant loss of information, and a look at (2.1) shows that we may expect much more: if K, v are $O((1+t)^{-\alpha})$ for $\alpha > 1$, one may hope that u is $O((1+t)^{-\alpha})$ as well because this property is stable by convolution. Our main result exactly shows this:

THEOREM 5. *Assume that (2.6) holds. Let $\alpha > 1$, and assume that K satisfies*

$$|K(t)| \leq C_K(1+t)^{-\alpha}, \quad t \geq 0, \tag{2.7}$$

for $\alpha > 1$ and a constant C_K. Then, for any v, the solution u of (2.1) satisfies

$$\sup_{t\in\mathbb{R}_+} (1+t)^\alpha |u(t)| \lesssim \sup_{t\in\mathbb{R}_+} (1+t)^\alpha |v(t)|.$$

Remark 2.1. Faou, Horsin and Rousset [**19**, Corollary 3.3] have a related result for specific weights. Our proof is different and easily applies for general weights, notably weight $\ln(2+t)(1+t)^{-\alpha}$, that will be used later.

Our proof relies on the analysis of Volterra equations of non-convolution type carried in [**21**, Chapter 9]. These equations take the form

$$\tilde{u}(t) + \int_J k(t,s)\,\tilde{u}(s)\,\mathrm{d}s = \tilde{v}(t), \quad \text{for almost every } t \text{ in } J, \tag{2.8}$$

where J is a subinterval of $\mathbb{R}_+$. An important notion developed there is the one of *Volterra kernel of type L^p*, that we introduce here only for $p = \infty$.

Definition 2.2. A Volterra kernel on J is a measurable mapping $k : J \times J \to M_n(\mathbb{C})$, such that $k(t,s) = 0$ for all $s, t \in J$ with $s > t$.

Definition 2.3. A Volterra kernel is said to be of type L^∞ on J if it satisfies

$$|||k|||_{\infty,J} < \infty, \quad \text{where} \quad |||k|||_{\infty,J} := \sup_{t\in J} \int_J |k(t,s)|\,\mathrm{d}s.$$

On J we can define a generalized convolution product

$$(k_1 \star k_2)(t,s) := \int_J k_1(t,u)\, k_2(u,s)\,\mathrm{d}u$$

and one can directly verify that the set of Volterra kernels of type L^∞ on J, equipped with $(+,\star)$, is a Banach algebra for the norm $|||\cdot|||_{\infty,J}$. Moreover, one can show that the space $L^\infty(J,\mathbb{C}^n)$ is a left Banach module over it, through $(kv)(t) = \int_J k(t,u)\, v(u)\,\mathrm{d}u$.

As for classical Volterra equations, one has a notion of resolvent:

Definition 2.4. Given a Volterra kernel k on J, a resolvent of k on J is another Volterra kernel satisfying

$$r + k\star r = r + r\star k = k.$$

As in the convolution case, the resolvent determines the solution.

Lemma 2.5 ([21, Chapter 9, Lemma 3.4])**.** *If k is a Volterra kernel of type L^∞ on J, which has a resolvent r of type L^∞ on J, then, for any $\tilde{v} \in L^\infty(J,\mathbb{C}^n)$, equation (2.8) has a unique solution $\tilde{u} \in L^\infty(J,\mathbb{C}^n)$, given by*

$$\tilde{u}(t) = \tilde{v}(t) - \int_J r(t,u)\,\tilde{v}(u)\,\mathrm{d}u.$$

In particular, $\|\tilde{u}\|_{L^\infty(J)} \lesssim \|\tilde{v}\|_{L^\infty(J)}$.

Using the standard von Neumann series for perturbations of the resolvent map in the Banach algebra, we can control the resolvent around a known resolvent.

PROPOSITION 2.6 ([21, Chapter 9, Theorem 3.9]). *If $k = k_1 + k_2$ is the sum of two Volterra kernels of type L^∞ on J, if k_1 has a resolvent r_1 of type L^∞ on J, and if*

$$|||k_2|||_{\infty,J} < \frac{1}{1 + |||r_1|||_{\infty,J}}$$

then k has a resolvent of type L^∞ on J.

PROOF OF THEOREM 5. For $\epsilon > 0$ (to be chosen later sufficiently small), we consider

$$\tilde{u}(t) := (1+\epsilon t)^\alpha u(t), \quad \tilde{v}(t) := (1+\epsilon t)^\alpha v(t).$$

By (2.1), they satisfy

$$\tilde{u}(t) + \int_{\mathbb{R}_+} k(t,s)\,\tilde{u}(s)\,\mathrm{d}s = \tilde{v}(t), \quad \forall t \in \mathbb{R}_+ \tag{2.9}$$

with the Volterra kernel

$$k(t,s) := \frac{(1+\epsilon t)^\alpha}{(1+\epsilon s)^\alpha} K(t-s) \quad \text{if } t > s, \quad k(t,s) := 0 \quad \text{otherwise.}$$

By Lemma 2.5 it suffices to show that k is of type L^∞ on $\mathbb{R}_+$ and has a resolvent of type L^∞ on $\mathbb{R}_+$. We decompose k as

$$k(t,s) = K(t-s)1_{s<t} + \left(\frac{(1+\epsilon t)^\alpha}{(1+\epsilon s)^\alpha} - 1\right) K(t-s)1_{s<t} =: k_1(t,s) + k_2(t,s).$$

Clearly, $|||k_1|||_{\infty,\mathbb{R}_+} \le \|K\|_{L^1}$ so that k_1 is of type L^∞. By Theorem 4, under (2.6), the convolution Volterra kernel K has a resolvent R, solution of (2.2). Setting

$$r_1(t,s) := R(t-s) \quad \text{if } t > s, \quad r_1(t,s) := 0 \quad \text{otherwise,}$$

yields the resolvent r_1 of k_1 on $\mathbb{R}_+$. Moreover, $|||r_1|||_{\infty,\mathbb{R}_+} \leq \|R\|_{L^1}$ so that r_1 is of type L^∞. By Proposition 2.6, it then suffices to show that we can make $|||k_2|||_{\infty,\mathbb{R}_+}$ arbitrarily small by choosing ϵ small enough. For this last claim, first consider $t \leq \lambda/\epsilon$ for a parameter $\lambda > 0$. Then

$$\int_0^t |k_2(t,s)|\,\mathrm{d}s \lesssim \int_0^t ((1+\lambda)^\alpha - 1)\,\frac{\mathrm{d}s}{(1+t-s)^\alpha} \leq ((1+\lambda)^\alpha - 1)\int_{\mathbb{R}_+} \frac{\mathrm{d}s'}{(1+s')^\alpha},$$

which can be made arbitrary small by choosing λ small enough. This λ being fixed, we consider now $t > \lambda/\epsilon$ and split the integral at δt for $0 < \delta < 1$. First,

$$\begin{aligned}\int_{\delta t}^t |k_2(t,s)|\,\mathrm{d}s &\lesssim \int_{\delta t}^t \left(\left(\frac{1+\epsilon t}{1+\epsilon\delta t}\right)^\alpha - 1\right)\frac{\mathrm{d}s}{(1+t-s)^\alpha}\\ &\lesssim \left(\left(\frac{1+\epsilon t}{1+\epsilon\delta t}\right)^\alpha - 1\right)\int_{\mathbb{R}_+}\frac{\mathrm{d}s'}{(1+s')^\alpha} \lesssim \sup_{t'\in\mathbb{R}_+}\left(\frac{1+t'}{1+\delta t'}\right)^\alpha - 1,\end{aligned}$$

which can be made arbitrary small by choosing some $\delta < 1$ close to 1. Then, for fixed λ, δ we find

$$\begin{aligned}\int_0^{\delta t} |k_2(t,s)|\,\mathrm{d}s &\leq \sup_{s\in(0,\delta t)} |K(t-s)| \int_0^{\delta t} \frac{(1+\epsilon t)^\alpha}{(1+\epsilon s)^\alpha}\mathrm{d}s \lesssim \frac{(1+\epsilon t)^\alpha}{(1+t)^\alpha}\int_{\mathbb{R}_+}\frac{\mathrm{d}s}{(1+\varepsilon s)^\alpha}\\ &\lesssim \frac{(1+\epsilon t)^\alpha}{\varepsilon(1+t)^\alpha} \lesssim \sup_{\tau>\frac{\lambda}{\varepsilon}} \frac{(1+\epsilon\tau)^\alpha}{\varepsilon\tau^\alpha} \lesssim \left(\lambda^{-1}+1\right)^\alpha \varepsilon^{\alpha-1},\end{aligned}$$

which again can be made arbitrary small by choosing ϵ sufficiently small. □

3. Isotropic suspensions in the inviscid case

As the incoherent state is rotational invariant, we can always choose a coordinate system so that $k \in \mathbb{S}^2$ equals the coordinate vector $e := (0,0,1)^t$. More precisely, if $\psi_k[\psi^{in}]$ is the solution of (1.13) with initial data ψ^{in}, one has for any rotation matrix R:

$$\psi_{Rk}[\psi^{in}(R^{-1}\cdot)](t,p) = \psi_k[\psi^{in}](t, R^{-1}p),$$

so that it is enough to assume $k = e$ to prove Theorem 1 or Theorem 2. For easier readability we then also drop the explicit dependence in the index k. The system (1.13) reduces to

$$\partial_t\psi = L_1\psi + \bar{L}_2\cdot u[\psi] \tag{3.1}$$

where $L_1 := L_{1,e}$, $u[\psi] = u_e[\psi]$, $\bar{L}_2 = \bar{L}_{2,e}$, see definitions (1.24)-(1.25)-(1.26). By standard methods, for any $\psi^{in} \in H^s(\mathbb{S}^2)$, $s \geq 0$, there exists a unique solution $\psi \in C(\mathbb{R}_+, H^s(\mathbb{S}^2)) \cap C^1(\mathbb{R}_+, H^{s-1}(\mathbb{S}^2))$ if $\nu = 0$, resp. $\psi \in C(\mathbb{R}_+, H^s(\mathbb{S}^2)) \cap L^2_{loc}(\mathbb{R}_+, H^{s+1}(\mathbb{S}^2))$ if $\nu > 0$, of (3.1). The point is to obtain the decay estimates, first for u, then for ψ itself. We consider in this section the case $\nu = 0$.

3.1. Volterra equation on u. As indicated in Section 1.3, we can rewrite the evolution for u as a Volterra equation

$$u(t) + K\star u(t) = U(t)$$

with

$$U(t) := u[\mathrm{e}^{tL_1}\psi_e^{in}], \quad K(t)v := -u[\mathrm{e}^{tL_1}(\bar{L}_2\cdot v)], \quad v \in \mathbb{C}^3. \tag{3.2}$$

As $\nu = 0$, by the definition of $u[\cdot]$, we find explicitly for a test function ϕ that

$$u[\mathrm{e}^{tL_1}\phi] = i\varepsilon\int_{\mathbb{S}^2} \mathbb{P}_{e^\perp}p\,(e\cdot p)(\mathrm{e}^{tL_1}\phi)(p)\,\mathrm{d}p = i\varepsilon\int_{\mathbb{S}^2}\mathbb{P}_{e^\perp}p\,(e\cdot p)\phi(p)\,\mathrm{e}^{-ie\cdot pt}\mathrm{d}p. \tag{3.3}$$

We recognize a Fourier transform over the sphere, with well-known decay properties, quantified in

Lemma 3.1. *Let $M \in \mathbb{N}$, $\delta > 0$ and $F \in H^{2M+1+\delta}(\mathbb{S}^2)$ be a function over $\mathbb{S}^2$. For a unit vector $e \in \mathbb{S}^2$ define the integral*

$$I(t) = \int_{\mathbb{S}^2} \mathrm{e}^{i\, e\cdot p\, t} F(p)\, \mathrm{d}p, \quad t \geq 0.$$

Then there exist complex numbers $c_{m,\pm}$, $1 \leq m \leq M$, and $I_M = I_M(t)$ such that

$$c_{m,\pm} \in \operatorname{span}\Big(\big\{\partial^\beta F(\pm e), \quad |\beta| \leq 2m-2\big\}\Big) \tag{3.4}$$

$$\forall t \geq 0, \quad |I_M(t)| \leq \frac{C}{(1+t)^{M+1}} \|F\|_{H^{2M+1+\delta}(\mathbb{S}^2)} \tag{3.5}$$

and

$$I(t) = \sum_{m=1}^{M} \big(c_{m,+}\mathrm{e}^{it} + c_{m,-}\mathrm{e}^{-it}\big)(1+t)^{-m} + I_M(t). \tag{3.6}$$

For completeness, we shall provide the proof of this lemma, directly inspired from the lecture notes [8], at the end of this subsection. In the special case of our source U and kernel K, the integrand F of the lemma contains the projection $\mathbb{P}_{k^\perp}$, that is vanishing at $p = \pm k$ so that Lemma 3.1 with $M = 1$ implies the bounds

$$|U(t)| \lesssim (1+t)^{-2} \|\psi^{in}\|_{H^{3+\delta}},$$
$$|K(t)| \lesssim (1+t)^{-2}.$$

Hence we find by Theorem 5 that under the spectral condition

$$\det(I + \mathcal{L}K(\lambda)) \neq 0 \quad \text{for all} \;\; \operatorname{Re}\lambda \geq 0, \tag{3.7}$$

the first bound (1.15) on the decay of u holds. To complete the proof of Theorem 1, it remains to check when assumption (3.7) is satisfied, and finally to analyse the decay of ψ itself. This will be done in the following Subsections 3.2 and 3.3.

PROOF OF LEMMA 3.1. We introduce a smooth partition of unity χ_0, $\chi_\pm$, with $\chi_0 + \chi_+ + \chi_- = 1$ over $\mathbb{S}^2$, where χ_0 is supported away from $p = \pm e$ and $\chi_\pm$ is supported in a neighborhood of $\pm e$. We decompose

$$I(t) = I_0(t) + I_+(t) + I_-(t), \quad I_l(t) = \int_{\mathbb{S}^2} F(p)\chi_l(p)\, \mathrm{d}p, \quad l \in \{0, +, -\}.$$

By the rotational symmetry, one can introduce coordinates such that e is along the z-axis. Then any point $p \in \mathbb{S}^2$ can be parametrized as

$$p = p(\gamma, z) = (\sqrt{1-z^2}\, \sin\gamma, \sqrt{1-z^2}\, \cos\gamma, z)^T \tag{3.8}$$

for $z \in [-1, 1]$ and $\gamma \in [0, 2\pi)$. The surface measure on the sphere is $\mathrm{d}\sigma = \mathrm{d}\gamma\, \mathrm{d}z$ so that

$$I_0(t) = \int_{-1}^{1} \mathrm{e}^{-izt} F_0(z)\, \mathrm{d}z, \quad F_0(z) = \int_0^{2\pi} (F\chi_0)(p(\gamma, z))\, \mathrm{d}\gamma. \tag{3.9}$$

The key point is that F_0 is compactly supported in $(-1, 1)$ so that its extension by zero to $\mathbb{R}$, still denoted F_0, is in $H_c^{2M+2}(\mathbb{R})$. We find in particular that

$$\forall\, t \geq 1, \quad |t^{M+1} I_0(t)| = |t^{M+1} \hat{F}_0(t)| \leq \|F_0\|_{W^{M+1,1}(\mathbb{R})} \leq \|F_0\|_{W^{M+1,1}(\mathbb{S}^2)} \lesssim \|F\|_{H^{2M+1+\delta}(\mathbb{S}^2)}. \tag{3.10}$$

It remains to treat I_+ (I_- can be handled in the same way). Let

$$\varphi : B(0,1) \subset \mathbb{R}^2 \to \mathbb{S}^2, \quad x \to (x, \sqrt{1-|x|^2}).$$

We find

$$I_+(t) = \int_{B(0,1)} \mathrm{e}^{i\sqrt{1-|x|^2}t} \left(F_0\, \chi_+\, j\right)(\varphi(x))\, \mathrm{d}x$$

where j is the Jacobian from the change of variable.

The key point is that the phase $\sqrt{1-|x|^2}$ has a non-degenerate critical point at $x=0$, with Hessian matrix at zero being $-2I$. By Morse lemma, there exists a smooth diffeomorphism ψ from a neighborhood U of 0 in $\mathbb{R}^2$ to $B(0,\eta)$, for η small enough, so that $\sqrt{1-|\psi(y)|^2}=1-|y|^2$. By taking the support of χ_+ sufficiently small, we can perform another change of variables and arrive at

$$I_+(t)=\mathrm{e}^{it}\int_U \mathrm{e}^{-i|y|^2t}F_+(y)\,\mathrm{d}y$$

where F_+ is the product of $F_0\circ\varphi\circ\psi$ with a smooth function, compactly supported in U coming from the Jacobian. Extending F_+ by zero outside U, we end up with

$$I_+(t)=\mathrm{e}^{it}\int_{\mathbb{R}^2}\mathrm{e}^{-i|y|^2t}F_+(y)\,\mathrm{d}y=\frac{\mathrm{e}^{it-i\frac{\pi}{4}}}{t}\int_{\mathbb{R}^2}\mathrm{e}^{-i\frac{|\xi|^2}{t}}\hat{F}_+(\xi)\,\mathrm{d}\xi$$

with the Fourier transform $\hat{F}_+$ of F_+ and where the last line comes from Plancherel identity.

We can then perform a Taylor expansion

$$\mathrm{e}^{-i\frac{|\xi|^2}{t}}=\sum_{m=1}^{M}\frac{1}{(m-1)!}\left(-i\frac{|\xi|^2}{t}\right)^{m-1}+O\Big(\frac{|\xi|^{2M}}{t^M}\Big).$$

Setting

$$c_{m,+}=\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{(m-1)!}(-i)^{m-1}\int_{\mathbb{R}^2}|\xi|^{2m-2}\hat{F}_+(\xi)\,\mathrm{d}\xi=2\pi\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{(m-1)!}(-i)^{m-1}[(-\Delta)^{m-1}F_+](0)$$

we obtain that

$$I_+(t)=\sum_{m=1}^{M}\frac{c_{m,+}\mathrm{e}^{it}}{t^m}+I_{M,+}(t),\quad |I_{M,+}(t)|\le\frac{C}{t^{M+1}}\||\xi|^{2M}\hat{F}_+\|_{L^1}.$$

One can then notice that

$$\begin{aligned}\||\xi|^{2M}\hat{F}_+\|_{L^1(\mathbb{R}^2)}&\le\|(1+|\xi|)^{-1-\delta}\|_{L^2(\mathbb{R}^2)}\|(1+|\xi|)^{2M+1+\delta}\hat{F}_+\|_{L^2(\mathbb{R}^2)}\\&\lesssim\|F_+\|_{H^{2M+1+\delta}(\mathbb{R}^2)}\lesssim\|F\|_{H^{2M+1+\delta}(\mathbb{S}^2)}.\end{aligned}$$

The result of the lemma follows directly. □

3.2. Stability condition. We now study the stability condition (3.7). This condition was already studied through explicit numerical computations in [36], but here we provide another angle through the argument principle which allows a complete solution.

We first compute the determinant.

Lemma 3.2. *For $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\ge 0$ we have*

$$\det(1+\mathcal{L}K(\lambda))=\left(1+\frac{3\Gamma\varepsilon}{4}\int_{-1}^{+1}\frac{z^2(1-z^2)}{\lambda+iz}\,\mathrm{d}z\right)^2\qquad \text{if } \lambda\notin i(-1,1). \tag{3.11}$$

$$\det(1+\mathcal{L}K(\lambda))=\left(1+\frac{3\Gamma\varepsilon\pi}{4}b^2(1-b^2)+\mathrm{PV}\,\frac{3\Gamma\varepsilon}{4i}\int_{-1}^{+1}\frac{z^2(1-z^2)}{b+z}\,\mathrm{d}z\right)^2\quad \text{if } \lambda=ib \text{ for } b\in(-1,1). \tag{3.12}$$

PROOF. We remind that

$$\bar{L}_2=\frac{3\Gamma i}{4\pi}(p\cdot e)p$$

We again take the parametrization (3.8). We then deduce from (3.2) and (3.3) for $\operatorname{Re}\lambda>0$ that

$$\mathcal{L}K(\lambda)=\frac{3\Gamma\varepsilon}{4\pi}\int_{z=-1}^{1}\int_{\gamma=0}^{2\pi}\frac{z^2(1-z^2)}{\lambda+iz}\begin{pmatrix}\sin^2\gamma & \sin\gamma\cos\gamma & 0\\ \sin\gamma\cos\gamma & \cos^2\gamma & 0\\ 0 & 0 & 0\end{pmatrix}\mathrm{d}\gamma\,\mathrm{d}z.$$

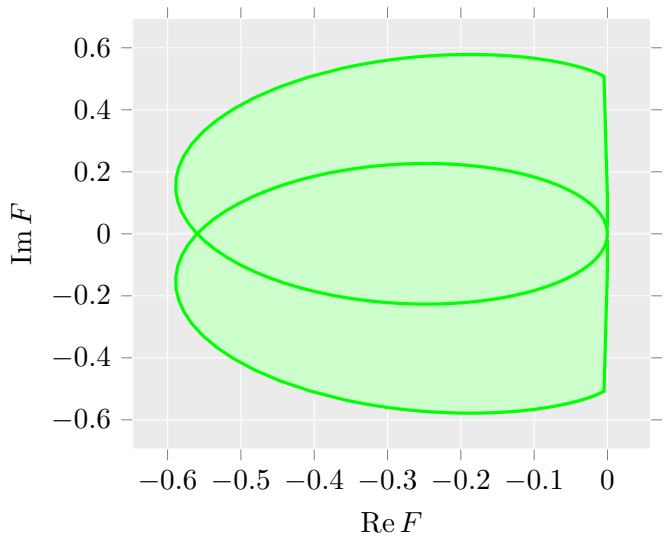


FIGURE 2. Encircled area of the curve $b \mapsto F(b)$ for the spectral condition in Proposition 3.4.

This directly yields the expression for $\operatorname{Re}\lambda > 0$. For $\operatorname{Re}\lambda = 0$ the result follows from continuity and Plemelj formula. □

Remark 3.3. The principal value can be computed explicitly: for all $b \in (-1, 1)$,

$$\mathrm{PV}\int_{-1}^{1} \frac{z^2(1-z^2)}{b+z}\mathrm{d}z = 2b^3 - \frac{4}{3}b + (b^4 - b^2)\ln\frac{1-b}{1+b}. \tag{3.13}$$

This is an odd function, that vanishes at 0 and at $\pm b_c$ where $b_c > 0$ can be evaluated numerically to

$$b_c \approx 0.62375.$$

Moreover, the function is negative on $(0, b_c)$, positive on $(b_c, 1)$.

PROPOSITION 3.4. *For the inviscid system, one has depending on ε*

(1) If $\varepsilon = 1$, then the spectral condition (3.7) is satisfied for all $\Gamma \in \mathbb{R}_+$.

(2) If $\varepsilon = -1$, the spectral condition is satisfied if and only if $\Gamma < \Gamma_c$, where

$$\frac{1}{\Gamma_c} = \frac{3\pi}{4} b_c^2 (1 - b_c^2).$$

PROOF. For $\operatorname{Re}\lambda \geq 0$ define the analytic function $F(\lambda)$ by

$$F(a+ib) = \frac{3i}{4}\int_{z=-1}^{+1} \frac{z^2(1-z^2)}{b+z}\,\mathrm{d}z \qquad \text{if } (a,b) \notin \{0\} \times (-1,1),$$

$$F(ib) = -\frac{3\pi}{4}b^2(1-b^2) + \frac{3i}{4}\,\mathrm{PV}\int_{z=-1}^{+1} \frac{z^2(1-z^2)}{b+z}\,\mathrm{d}z \quad \text{if } b \in (-1,1).$$

By Lemma 3.2, there exists then an eigenmode if and only if F attains in the right half plane the value ε/Γ.

By the explicit expression, we also see that $F(\lambda) \to 0$ as $|\lambda| \to \infty$. Hence by the argument principle the attained values are exactly those values encircled by the curve $b \mapsto F(ib)$. This curve is plotted in Fig. 2 but can also be understood analytically.

For $|b| \geq 1$, the expression directly shows that $F(b)$ is purely imaginary, while for $|b| \leq 1$ we find that $\operatorname{Re} F \leq 0$. Hence it cannot encircle positive real numbers, which proves the first statement of the proposition.

In $\{|b|<1\}$ the curve crosses the real axis at $b=0$ and $b=\pm b_c$. By the expression of F, we see that it indeed crosses the real axis at $-3\pi b_c^2(1-b_c^2)/4$. Hence we then have an eigenmode if and only if

$$\frac{1}{\Gamma} \leq \frac{3\pi}{4} b_c^2(1-b_c^2),$$

which shows the claimed stability. □

3.3. Control on ψ. With the control of u we go back to prove (1.16). Again, it is enough to treat the case $k=e$. Let $\varphi \in H^{1+\delta}(\mathbb{S}^2)$. From (1.27), we deduce

$$\int_{\mathbb{S}^2} \psi(t,p)\varphi(p)\,\mathrm{d}p = \int_{\mathbb{S}^2} \mathrm{e}^{ite\cdot p}\psi^{in}(p)\varphi(p)\,\mathrm{d}p + \int_0^t \int_{\mathbb{S}^2} \mathrm{e}^{i(t-s)e\cdot p}\varphi(p)\bar{L}_2(p)\cdot u(s)\,\mathrm{d}p\,\mathrm{d}s =: I_1(t)+I_2(t).$$

We apply Lemma 3.1 with $M=0$, to obtain

$$|I_1(t)| \leq \frac{C}{(1+t)}\|\psi^{in}\varphi\|_{H^{1+\delta}} \lesssim \frac{1}{1+t}\|\psi^{in}\|_{H^{1+\delta}}\|\varphi\|_{H^{1+\delta}},$$

as $H^{1+\delta}(\mathbb{S}^2)$ is an algebra. As regards I_2, we write

$$\begin{aligned} I_2(t) &= \int_0^{t/2}\Big(\int_{\mathbb{S}^2} \mathrm{e}^{i(t-s)e\cdot p}\varphi(p)\bar{L}_2(p)\mathrm{d}p\Big)\cdot u(s)\,\mathrm{d}s + \int_{t/2}^t\int_{\mathbb{S}^2} \mathrm{e}^{i(t-s)e\cdot p}\varphi(p)\bar{L}_2(p)\cdot u(s)\,\mathrm{d}p\,\mathrm{d}s \\ &=: I_{2,1}(t)+I_{2,2}(t). \end{aligned}$$

We bound the parenthesis in the first term using again Lemma 3.1 with $M=0$: it follows

$$\begin{aligned} |I_{2,1}(t)| &\lesssim \int_0^{t/2} \frac{1}{1+(t-s)}\|\varphi\bar{L}_2\|_{H^{1+\delta}}|u(s)|\,\mathrm{d}s \\ &\lesssim \frac{1}{1+t}\|\varphi\|_{H^{1+\delta}}\int_0^{t/2}\frac{1}{(1+s)^2}\,\mathrm{d}s\,\|\psi^{in}\|_{H^{3+\delta}} \\ &\lesssim \frac{1}{1+t}\|\varphi\|_{H^{1+\delta}}\|\psi^{in}\|_{H^{3+\delta}}, \end{aligned}$$

where we have used (1.15) for the second inequality. Eventually, we use again (1.15) to get

$$\begin{aligned} |I_{2,2}(t)| &\lesssim \int_{t/2}^t \frac{1}{(1+s)^2}\,\mathrm{d}s\,\|\psi^{in}\|_{H^{3+\delta}}\|\varphi\bar{L}_2\|_{L^\infty} \\ &\lesssim \frac{1}{1+t}\|\varphi\|_{H^{1+\delta}}\|\psi^{in}\|_{H^{3+\delta}}, \end{aligned}$$

where we applied the Sobolev imbedding $H^{1+\delta}(\mathbb{S}^2)\hookrightarrow L^\infty(\mathbb{S}^2)$. This concludes the proof of (1.16) and of Theorem 1.

4. Decay estimates for advection-diffusion on the sphere

In this central section of our paper we prepare for the results under small angular diffusion ν by proving Proposition 1.6 and Proposition 1.7.

Here we study mixing estimates for the semigroup $\mathrm{e}^{tL_{1,k}}$ where $L_{1,k}=-ik\cdot p+\nu\Delta$, $k\in\mathbb{S}^2$. As explained at the beginning of Section 3, there is no loss of generality in assuming $k=e=(0,0,1)$. We shall later use spherical coordinates (θ,φ), with colatitude $\theta\in(0,\pi)$ and longitude $\varphi\in(0,2\pi)$, so that $p=\sin\theta\cos\varphi\, e_x+\sin\theta\sin\varphi\, e_y+\cos\theta\, e$.

Setting $\psi(t)=\mathrm{e}^{tL_{1,k}}\psi^{in}$, we thus study decay estimates for solutions to

$$\partial_t\psi + ip\cdot e\,\psi = \nu\Delta\psi, \qquad p\in\mathbb{S}^2, \tag{4.1}$$

where, from now on, we suppress the subscript on all differential operators, as it is understood that they are all with respect to the variable p on the sphere $\mathbb{S}^2$.

A first feature of (4.1) is that the interaction of transport and diffusion leads to an enhanced diffusion time scale $O(1/\sqrt{\nu})$ that is much shorter than the heat equation one $O(1/\nu)$. This phenomenon is analysed in details in Section 4.1. Furthermore, we exhibit an auxiliary time $\nu^{-1/3}$ for the decay of $\|\nabla(p\cdot e)\psi\|_{L^2}$, a quantity that vanishes near the pole. This additional time scale is coherent with classical results for enhanced dissipation in the Euclidean setting, where the absence of critical points also leads to typical time $\nu^{-1/3}$.

In the following Section 4.2, we build a vector field J_ν under the form

$$J_\nu = \alpha_\nu(t)\nabla + i\beta_\nu(t)\nabla(p\cdot e)$$

such that, roughly:

(i) $\alpha_\nu \sim 1$, $\beta_\nu \sim t$ over times $\nu^{-1/2}$;
(ii) $J_\nu\psi$ is well-controlled over times $\nu^{-1/2}$.

In the usual Euclidean setting, where $e\cdot p$ is replaced by $e\cdot v$, $v\in\mathbb{R}^3$, the vector field $J=\nabla_v + ite$ is an easy and convenient choice: it commutes with the advection-diffusion operator, so that $J\psi$ can be controlled easily for all times. In the case of the sphere, we do not know how to construct such a commuting vector field. Designing a J_ν for which we can show properties (i) and (ii) is difficult, and relies on the refined estimates of Section 4.1.

By these vector fields we can obtain the first part of the mixing estimates of Proposition 1.7 up to time $\nu^{-1/2}$ in Section 4.3. For the estimates on the longer time scale, we adapt the estimates in Section 4.4 which then concludes the proof of Proposition 1.7.

4.1. Hypocoercive estimate for advection-diffusion. In this subsection we will introduce the hypocoercive estimates which will yield the enhanced dissipation of Proposition 1.6.

Using the covariant derivatives ∇ as discussed in Appendix A, we find in our case of $\mathbb{S}^2$ that

$$\nabla\Delta\psi = \Delta\nabla\psi - \nabla\psi$$

where the Laplacian $\Delta = \mathrm{tr}(\nabla^2)$ is the connection Laplacian (the correction comes from the Ricci curvature tensor which equals the metric on the sphere). Taking the covariant derivative of (4.1), we therefore get

$$\partial_t\nabla\psi + ip\cdot e\nabla\psi + i\nabla(p\cdot e)\psi = \nu\Delta\nabla\psi - \nu\nabla\psi. \tag{4.2}$$

We explicitly compute $\Delta(\nabla(p\cdot e)\psi)$ in Lemma A.1 and this concrete expression yields

$$\partial_t(\nabla(p\cdot e)\psi) + ip\cdot e(\nabla(p\cdot e)\psi) = \nu\Delta(\nabla(p\cdot e)\psi) + \nu\nabla(p\cdot e)\psi + 2\nu(p\cdot e)\nabla\psi. \tag{4.3}$$

We further note that $|p\cdot e|\le 1$ and $|\nabla(p\cdot e)|\le 1$. In spherical coordinates, we find explicitly that $|\nabla(p\cdot e)| = \sin\theta$ which is uniformly lower-bounded away from the poles where it vanishes linearly.

We now derive several energy identities for the solution to (4.1). In all what follows, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ stand for the complex scalar product (with the conjugate on the second variable) and norm on $L^2(\mathbb{S}^2)$ respectively. For simplicity, we will use the same notations for the scalar product and norm of L^2 vector fields, or more generally for L^2 sections of any tensor bundle over $\mathbb{S}^2$. A direct computation using the antisymmetry of the operator $ip\cdot e$ allows us to derive the L^2 balance

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2 + \nu\|\nabla\psi\|^2 = 0. \tag{4.4}$$

By testing (4.2) with $\nabla\psi$, we obtain the H^1 balance

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi\|^2 + \nu\|\nabla\nabla\psi\|^2 + \nu\|\nabla\psi\|^2 = -\,\mathrm{Re}\langle i\nabla(p\cdot e)\psi, \nabla\psi\rangle \le \|\nabla(p\cdot e)\psi\|\,\|\nabla\psi\|. \tag{4.5}$$

Next we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\langle i\nabla(p\cdot e)\psi,\nabla\psi\rangle+\|\nabla(p\cdot e)\psi\|^2 &= \nu\operatorname{Re}\langle i\Delta(\nabla(p\cdot e)\psi),\nabla\psi\rangle+\nu\langle i\nabla(p\cdot e)\psi,\nabla\psi\rangle\\ &\quad+2\nu\operatorname{Re}\langle i(p\cdot e)\nabla\psi,\nabla\psi\rangle+\nu\operatorname{Re}\langle i\nabla(p\cdot e)\psi,\nabla\Delta\psi\rangle\\ &= -\nu\operatorname{Re}\langle i\nabla(\nabla(p\cdot e)\psi),\nabla^2\psi\rangle-\nu\operatorname{Re}\langle i(p\cdot e)\psi,\Delta\psi\rangle\\ &\quad-\nu\operatorname{Re}\langle i\nabla\cdot\big(\nabla(p\cdot e)\psi\big),\Delta\psi\rangle\\ &\le 4\nu\|\nabla^2\psi\|\big(\|\psi\|+\|\nabla(\nabla(p\cdot e)\psi)\|\big).\end{aligned}\tag{4.6}$$

From (4.3) we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(p\cdot e)\psi\|^2+\nu\|\nabla(\nabla(p\cdot e)\psi)\|^2+\nu\|\nabla(p\cdot e)\psi\|^2 &\\ &= 2\nu\operatorname{Re}\langle\nabla(p\cdot e)\psi+(p\cdot e)\nabla\psi,\nabla(p\cdot e)\psi\rangle\\ &= 2\nu\|\nabla(p\cdot e)\psi\|^2+\nu\int(p\cdot e)\nabla(p\cdot e)\cdot\nabla(|\psi|^2)\\ &\le 2\nu\|\nabla(p\cdot e)\psi\|^2+2\nu\|\psi\|^2.\end{aligned}\tag{4.7}$$

For positive constants a,b,c to be chosen later *independently* of ν, define the energy functional

$$E[\psi]=\frac{1}{2}\Big[\|\psi\|^2+a\nu t\|\nabla\psi\|^2+2b\nu t^2\operatorname{Re}\langle i\nabla(p\cdot e)\psi,\nabla\psi\rangle+c\nu t^3\|\nabla(p\cdot e)\psi\|^2\Big].\tag{4.8}$$

Such a form of time-dependent functional takes inspiration from [**28, 14, 40**]. Assuming that $2b^2\le ac$, the mixed term can be controlled by the squares so that

$$E[\psi]\ge\frac{1}{2}\left[\|\psi\|^2+\frac{a\nu t}{2}\|\nabla\psi\|^2+\frac{c\nu t^3}{2}\|\nabla(p\cdot e)\psi\|^2\right].\tag{4.9}$$

Moreover, from (4.4)–(4.7), we deduce that $E[\psi]$ satisfies the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\psi]\le&-\left(\nu-\frac{a\nu}{2}\right)\|\nabla\psi\|^2-a\nu^2t\|\nabla^2\psi\|^2-\left(b\nu t^2-\frac{3c\nu t^2}{2}\right)\|\nabla(p\cdot e)\psi\|^2\\ &-c\nu^2t^3\|\nabla(\nabla(p\cdot e)\psi)\|^2+2b\nu t\|\nabla\psi\|\,\|\nabla(p\cdot e)\psi\|+a\nu t\|\nabla(p\cdot e)\psi\|\,\|\nabla\psi\|\\ &+4b\nu^2t^2\|\nabla^2\psi\|\big(\|\psi\|+\|\nabla(\nabla(p\cdot e)\psi)\|\big)+2c\nu^2t^3\big(\|\nabla(p\cdot e)\psi\|^2+\|\psi\|^2\big).\end{aligned}\tag{4.10}$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\psi]\le&-\left(\frac{3\nu}{4}-\frac{a\nu}{2}\right)\|\nabla\psi\|^2-\frac{a\nu^2t}{2}\|\nabla\nabla\psi\|^2\\ &-\left(b\nu t^2-\frac{3c\nu t^2}{2}-2c\nu^2t^3-2a^2\nu t^2-8b^2\nu t^2\right)\|\nabla(p\cdot e)\psi\|^2\\ &-\left(c\nu^2t^3-\frac{8b^2\nu^2t^3}{a}\right)\|\nabla(\nabla(p\cdot e)\psi)\|^2+\left(\frac{16b^2}{a}+2c\right)\nu^2t^3\|\psi\|^2.\end{aligned}\tag{4.11}$$

We will choose the constants according to the following lemma, which is crucial for the subsequent analysis and holds for the longer time-scale ν^{-1}.

Lemma 4.1. *For any $\delta>0$, there exists b_0 such that for the constants $a=b^{2/3}$, $0<b<b_0$ and $c=32b^2/a$ and for all times $t\le\delta\nu^{-1}$ it holds for $0<\nu\le1$ that*

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\psi]+\frac{\nu}{2}\|\nabla\psi\|^2+\frac{a\nu^2t}{2}\|\nabla\nabla\psi\|^2+\frac{b\nu t^2}{2}\|\nabla(p\cdot e)\psi\|^2+\frac{c\nu^2t^3}{2}\|\nabla(\nabla(p\cdot e)\psi)\|^2\le\left(\frac{16b^2}{a}+2c\right)\nu^2t^3\|\psi\|^2.$$

PROOF. By the choices $a = b^{2/3}$, $c = 32b^2/a$ and the constraint $t \leq \delta\nu^{-1/2}$ the result follows from (4.11) as soon as

$$\frac{3c}{2} + 2c\delta + 2a^2 + 8b^2 \leq \frac{b}{2} \quad \Leftrightarrow \quad \left(\frac{3}{2} + 2\delta\right) 16b^{4/3} + 2b^{4/3} + 8b^2 \leq \frac{b}{2}.$$

Hence we find the result for a small enough b_0. □

To conclude the enhanced dissipation from the previous lemma, we need to control $\|\psi\|$. This is achieved by an interpolation inequality involving $\|\nabla(p \cdot e)\psi\|$, which gives a good control apart from the poles $\pm e$, and $\|\nabla\psi\|$, see also [1, Lemma 4.3].

Lemma 4.2. *For all $\sigma \in (0, 1]$, all vectors $e \in \mathbb{S}^2$ and all complex-valued $g \in H^1(\mathbb{S}^2)$, the following inequality holds*

$$\sigma^{1/2}\|g\|^2 \leq \frac{\sigma}{2}\|\nabla g\|^2 + 2\|\nabla(p \cdot e)\, g\|^2. \tag{4.12}$$

PROOF. Introducing the spherical coordinates (θ, φ), with latitude $\theta \in (0, \pi)$ and longitude $\varphi \in (0, 2\pi)$, the inequality (4.12) becomes

$$\sigma^{1/2}\|g\|^2 \leq \frac{\sigma}{2}\int_{\mathbb{S}^2}\left(|\partial_\theta g|^2 + \frac{1}{\sin^2\theta}|\partial_\varphi g|^2\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi + 2\|g\sin\theta\|^2. \tag{4.13}$$

Now, for $\sigma \leq 1$, we have

$$\sigma^{1/2}\|g\|^2 = \sigma^{1/2}\|g\sin\theta\|^2 + \sigma^{1/2}\|g\cos\theta\|^2 \leq \|g\sin\theta\|^2 + \sigma^{1/2}\|g\cos\theta\|^2. \tag{4.14}$$

Moreover, an integration by parts entails

$$\begin{aligned}\|g\cos\theta\|^2 &= \int_{\mathbb{S}^2}\cos\theta\partial_\theta(\sin\theta)|g|^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \|g\sin\theta\|^2 - \|g\cos\theta\|^2 - 2\,\mathrm{Re}\langle\partial_\theta g\cos\theta, g\sin\theta\rangle \\ &\leq \sigma^{-1/2}\|g\sin\theta\|^2 - \|g\cos\theta\|^2 + \sigma^{1/2}\|\partial_\theta g\|^2,\end{aligned}$$

implying

$$\sigma^{1/2}\|g\cos\theta\|^2 \leq \|g\sin\theta\|^2 + \frac{\sigma}{2}\|\partial_\theta g\|^2. \tag{4.15}$$

Combining the above estimate with (4.14), we obtain the desired estimate (4.13). □

We can now conclude the enhanced dissipation.

PROOF OF PROPOSITION 1.6. Fix the constants a, b, c according to Lemma 4.1. Let $\lambda > 0$ to be fixed later independently of ν. Let ν_0 such that $\frac{4a\nu_0}{c\lambda^2} \leq 1$. From the definition of $E[\psi]$, we have, for all $\nu \leq \nu_0$, at time $t = \lambda\nu^{-1/2}$

$$\begin{aligned}E[\psi] &\geq \frac{1}{2}\|\psi\|^2 + \frac{a\nu^{1/2}\lambda}{2}\|\nabla\psi\|^2 + \frac{c\lambda^3\nu^{-1/2}}{2}\|\nabla(p \cdot e)\psi\|^2 \\ &\geq \frac{1}{2}\|\psi\|^2 + \frac{c\lambda^3\nu^{-1/2}}{4}\left(2\|\nabla(p \cdot e)\psi\|^2 + \frac{2a\nu}{c\lambda^2}\|\nabla\psi\|^2\right) \\ &\geq \frac{1}{2}\left(1 + (ac)^{1/2}\lambda^2\right)\|\psi\|^2,\end{aligned}$$

where the last line comes from Lemma 4.2, with $\sigma := \frac{4a\nu}{c\lambda^2} \leq 1$. By the evolution estimate of Lemma 4.1 we also find (as $\|\psi\|$ is non-increasing), that at time $t = \lambda\nu^{-1/2}$,

$$E[\psi] \leq \frac{1}{2}\|\psi^{in}\|^2 + c\lambda^4\|\psi^{in}\|^2.$$

Hence we find that at time $t = \lambda\nu^{-1/2}$,

$$\|\psi\|^2 \leq \frac{1 + 2c\lambda^4}{1 + (ac)^{1/2}\lambda^2}\|\psi^{in}\|^2.$$

Taking λ small enough (depending on a, b, c but independently of ν), the factor at the right-hand side is less than 1, which implies exponential decay with a rate proportional to $\nu^{1/2}$. □

4.2. Hypocoercive estimate for vector fields. In this subsection we introduce the vector fields to show the mixing estimate Proposition 1.7 for $\nu > 0$ when the semigroup e^{tL_1} cannot solved explicitly. This has already been used in a few instances for mixing estimates [**5, 11**]. In terms of L_1, the evolution of ψ of (4.1) can be written as

$$(\partial_t - L_1)\psi = 0, \quad L_1 = -ie \cdot p + \nu\Delta.$$

and the idea is to find a vector field J for which we control $J\psi$. A natural candidate for a vector field is $J = \nabla + it\nabla(p \cdot e)$, which commutes with the inviscid part of the equation. However, it does not commute well with the diffusion operator. Namely, we find that

$$(\partial_t - L_1)J\psi + \nu J\psi = 2i\nu t\big(\nabla(p \cdot e)\psi + (p \cdot e)\nabla\psi\big).$$

By Duhamel's formula,

$$J\psi(t) = \mathrm{e}^{(L_1-\nu)t}J\psi(0) + 2i\nu \int_0^t \mathrm{e}^{(L_1-\nu)(t-s)}\big(\nabla(p \cdot e)\psi(s) + (p \cdot e)\nabla\psi(s)\big)\mathrm{d}s.$$

If we were to rely only on the straightforward (yet optimal on time scales $O(1)$) bounds

$$\|\psi(s)\|_{L^2} \le C, \quad \|\nabla\psi(s)\|_{L^2} \le C(1+s),$$

we would get

$$\|J\psi(t)\|_{L^2} \le C\Big(1 + \nu\int_0^t s(1+s)\,\mathrm{d}s\Big), \qquad \forall t \le \nu^{-1/2}.$$

The second term behaves like $C\nu t^3$, and therefore diverges for $t \gg \nu^{-1/3}$, which is a faster time scale than the enhanced dissipation one for this problem.

To overcome this issue, we introduce the *viscosity-adapted* vector field J_ν of the form

$$J_\nu\psi = \alpha(t)\nabla\psi + i\beta(t)\nabla(p \cdot e)\psi. \tag{4.16}$$

for scalar functions $\alpha = \alpha_\nu$, $\beta = \beta_\nu$. For the evolution, we then find

$$\Big(\partial_t + i(p \cdot e) - \nu\Delta\Big)(\alpha\nabla\psi) = \alpha'\nabla\psi - i\alpha\nabla(p \cdot e)\psi - \nu\alpha\nabla\psi,$$
$$\Big(\partial_t + i(p \cdot e) - \nu\Delta\Big)(i\beta\nabla(p \cdot e)\psi) = i\beta'\nabla(p \cdot e)\psi - i\beta\nu\nabla(p \cdot e)\psi + 2i\beta\nu\nabla\big((p \cdot e)\psi\big).$$

In order to control the commutator error terms, we set

$$\beta' = \alpha \quad \text{and} \quad \alpha' = -2i\nu\beta$$

and thus take

$$\alpha(t) = \cosh(\sqrt{-2i\nu}\,t) \quad \text{and} \quad \beta(t) = \frac{1}{\sqrt{-2i\nu}}\sinh(\sqrt{-2i\nu}\,t).$$

For the time frame $t \le \nu^{-1/2}$ we see that $\alpha \sim 1$ and $\beta \sim t$. By this choice, we find that now $X = J_\nu\psi$ solves

$$\Big(\partial_t + i(p \cdot e) - \nu\Delta\Big)X + \nu X = 2i\beta\nu\nabla([p \cdot e - 1]\psi) \tag{4.17}$$

or

$$\Big(\partial_t + i(p \cdot e) - \frac{2i\beta\nu}{\alpha}[p \cdot e - 1] - \nu\Delta\Big)X + \nu X = -\frac{2\beta^2}{\alpha}\nu[p \cdot e - 1]\nabla(p \cdot e)\psi + 2i\beta\nu\nabla(p \cdot e)\psi. \tag{4.18}$$

The gain from the adapted vector field J_ν is that the right-hand side now vanishes at the north pole $p = e$. As the enhanced dissipation estimate from Section 4.1 provides better decay properties for quantities that vanish at the poles $\pm e$, this will provide a better control of the source term, and in turn a better control of X. Obviously, the right-hand side still does not vanish at the south pole $p = -e$, so that we need to localize

the estimates away from this pole. Symmetrically, one could construct another vector field $\tilde{X}$ for which the roles of the north and south poles would be reversed.

To obtain good control of X (away from the south pole), our starting point is Lemma 4.1 with $\delta = 1$. After integration in time, using that $\|\psi(t)\| \leq \|\psi^{in}\|$ for all t, we get for all $t_* \leq \nu^{-1}$ that

$$\sup_{t\in(0,t_*)} \Big(\|\psi\|^2 + \nu t\|\nabla\psi\|^2 + \nu t^3\|\nabla(p\cdot e)\psi\|^2\Big) + \int_0^{t_*} \Big(\nu\|\nabla\psi\|^2 + \nu t^2\|\nabla(p\cdot e)\psi\|^2 + \nu^2 t\|\nabla^2\psi\|^2 + \nu^2 t^3\|\nabla(\nabla(p\cdot e)\psi)\|^2\Big) \lesssim (1+\nu^2 t_*^4)\|\psi^{in}\|^2. \tag{4.19}$$

From there, as a preliminary step, we deduce a few easy bounds on X and R:

Lemma 4.3. *There exists $\nu_0 > 0$, such that for all $0 < \nu \leq \nu_0$ and all $t_* \leq \nu^{-1}$ it holds that*

$$\int_0^{t_*} \nu\|X\|^2\,\mathrm{d}t + \int_0^{t_*} \nu^2 t\|\nabla X\|^2\,\mathrm{d}t \lesssim \sup_{[0,t_*]} \Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)(1+\nu^2 t_*^4)\|\psi^{in}\|^2.$$

Furthermore, for

$$R = 2i\beta\nu\nabla\big([p\cdot e - 1]\psi\big) \tag{4.20}$$

and any cutoff χ excluding the south pole $-e$ we have

$$\int_0^{t_*} t\|R\chi\|^2\,\mathrm{d}t \lesssim \sup_{[0,t^*]} \Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)(1+\nu^2 t_*^4)\|\psi^{in}\|^2.$$

In particular,

$$\int_0^{\nu^{-1/2}} \nu\|X\|^2\,\mathrm{d}t + \int_0^{\nu^{-1/2}} \nu^2 t\|\nabla X\|^2\,\mathrm{d}t + \int_0^{\nu^{-1/2}} t\|R\chi\|^2\,\mathrm{d}t \lesssim \|\psi^{in}\|^2.$$

PROOF. The bounds relative to X follow from the definition $X = \alpha\nabla\psi + i\beta\nabla(p\cdot e)\psi$ and from the hypocoercive estimate (4.19). For the control of R, we decompose

$$t\|R\chi\|^2 \leq 2\nu^2|\beta|^2 t\left(\|[p\cdot e-1]\nabla\psi\chi\|^2 + \|\psi\chi\|^2\right) \leq C\left(\frac{|\beta|^2}{t^2}\right)\nu^2 t^3\left(\|\nabla(\nabla(p\cdot e)\psi)\chi\|^2 + \|\psi\chi\|^2\right)$$

and the estimate also follows from (4.19). □

We now state the key estimates of this paragraph, where we first focus on times up to $\nu^{-1/2}$. In the first iteration, we use the nested cutoffs χ and χ' excluding the south pole $-e$.

Lemma 4.4. *There exists $\nu_0 > 0$, such that for all $0 < \nu \leq \nu_0$ and cutoffs all $\chi, \chi' : \mathbb{S}^2 \to [0,1]$ such that $\chi' = 1$ on the support of χ and such that $-e$ does not belong to the support of χ', one has for some $C > 0$ the estimate*

$$\sup_{t_*\in[0,\nu^{-1/2}]} \Big(\|X\chi\|^2 + \nu t\|\nabla X\chi\|^2 + \nu t^3\|\nabla(p\cdot e)\otimes X\chi\|^2\Big) + \int_0^{\nu^{-1/2}} \Big(\nu\|\nabla X\chi\| + \nu^2\|\nabla^2 X\chi\|^2 + \nu t^2\|\nabla(p\cdot e)\otimes X\chi\|^2 + \nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\Big)\,\mathrm{d}t \leq C\left(\|\psi^{in}\|_{H^1}^2 + \int_0^{\nu^{-1/2}} \nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2\mathrm{d}t\right).$$

Before the proof, we show that iterating this estimates shows the estimate with the sharp rates.

Corollary 4.5. *There exists $\nu_0 > 0$, such that for all $0 < \nu \le \nu_0$ and cutoff $\chi : \mathbb{S}^2 \to [0,1]$ such that $-e$ does not belong to the support of χ, one has for some $C > 0$ the estimate*

$$\sup_{t\in[0,\nu^{-1/2}]} \Big(\|X\chi\|^2 + \nu t\|\nabla X\chi\|^2 + \nu t^3\|\nabla(p\cdot e)\otimes X\chi\|^2\Big)$$
$$+ \int_0^{\nu^{-1/2}} \Big(\nu\|\nabla X\chi\| + \nu^2\|\nabla^2 X\chi\|^2 + \nu t^2\|\nabla(p\cdot e)\otimes X\chi\|^2 + \nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\Big)\,\mathrm{d}t$$
$$\le C\|\psi^{in}\|_{H^1}^2.$$

PROOF. Consider nested cutoffs $\chi, \chi', \chi'' : \mathbb{S}^2 \to [0,1]$ such that $\chi' = 1$ on $\operatorname{supp}\chi$ and $\chi'' = 1$ on $\operatorname{supp}\chi'$ and χ'' excludes $-e$.

In a first step, applying Lemma 4.4 with χ' and χ'' yields in particular with the factor $\nu^{1/2}$ the bound

$$\int_0^{\nu^{-1/2}} \nu^{3/2}t^2\|\nabla(p\cdot e)\otimes X\chi'\|^2 \le C\nu^{1/2}\|\psi^{in}\|_{H^1}^2 + C\nu^{1/2}\int_0^{\nu^{-1/2}} \nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi''\|^2$$
$$\le C\nu^{1/2}\|\psi^{in}\|_{H^1}^2 + C\nu\int_0^{\nu^{1/2}} \|X\|^2 \le C'\|\psi^{in}\|_{H^1}^2$$

where the last inequality comes from Lemma 4.3. Hence we gain the control

$$\int_0^{\nu^{-1/2}} \nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2 \le \int_0^{\nu^{-1/2}} \nu^{3/2}t^2\|\nabla(p\cdot e)\otimes X\chi'\|^2 \le C'\|\psi^{in}\|_{H^1}^2,$$

so that the corollary now follows by another application of Lemma 4.4. □

We now come to the proof of Lemma 4.4.

PROOF OF LEMMA 4.4. The basic idea is to reproduce the hypocoercive estimates of Section 4.1, where we have additional contributions from the cut-off χ and the remainder R. We rewrite (4.17) as

$$(\partial_t + ip\cdot e - \nu\Delta)X + \nu X = R. \tag{4.21}$$

We find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X\chi\|^2 + \nu\|X\chi\|^2 = \nu\,\mathrm{Re}\langle X\chi, \Delta X\chi\rangle + \mathrm{Re}\langle X\chi, R\chi\rangle$$
$$\le -\nu\|\nabla X\chi\|^2 + 2\nu\|\nabla X\chi\|\,\|\nabla\chi\otimes X\| + \mathrm{Re}\langle X\chi, R\chi\rangle.$$

For the last term, we use the explicit expression (4.20) of R and integrate by parts to find

$$\mathrm{Re}\langle X\chi, R\chi\rangle \le 2|\beta|\nu\big(2\|\nabla\chi X\| + \|\chi\nabla X\|\big)\|(1-p\cdot e)\psi\chi\| \le C\nu|\beta|\big(\|\nabla\chi X\| + \|\chi\nabla X\|\big)\|\nabla(p\cdot e)\psi\|.$$

It follows from Young's inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X\chi\|^2 + \frac{\nu}{2}\|\nabla X\chi\|^2 \le E_1(t), \tag{4.22}$$

where, for some absolute constant C,

$$E_1(t) := C\big(\nu\|X\|^2 + \nu|\beta|^2\|\nabla(p\cdot e)\psi\|^2\big). \tag{4.23}$$

For the gradient, we find, using the commutation $\nabla\Delta X = \Delta\nabla X + O(|\nabla X|)$, see Appendix A:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla X\chi\|^2 + \nu\|\nabla X\chi\|^2 = \nu\,\mathrm{Re}\langle\nabla X\chi, \nabla\Delta X\chi\rangle - \mathrm{Re}\langle\nabla X\chi, i\nabla(p\cdot e)\otimes X\chi\rangle + \mathrm{Re}\langle\nabla X\chi, \nabla R\chi\rangle$$
$$\le -\nu\|\nabla\nabla X\chi\|^2 + 2\nu\|\nabla\chi\otimes\nabla X\|\|\nabla\nabla X\chi\| + C\nu\|\nabla X\chi\|^2$$
$$+ \|\nabla X\chi\|\|\nabla(p\cdot e)\otimes X\chi\| + \|\Delta X\chi\|\,\|R\chi\| + 2\|\nabla X\chi\|\,\|R\otimes\nabla\chi\|.$$

Hence, by Young's inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla X\chi\|^2 + \frac{\nu}{2}\|\nabla\nabla X\chi\|^2 \le \|\nabla X\chi\|\|\nabla(p\cdot e)\otimes X\chi\| + E_2(t), \tag{4.24}$$

where, for some absolute constant C,

$$E_2(t) := C\left(\nu\|\nabla X\|^2 + \frac{1}{\nu}\|R\chi'\|^2\right). \tag{4.25}$$

For the mixed term we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla X\chi\rangle + 2\nu\operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla X\chi\rangle + \|\nabla(p\cdot e)\otimes X\chi\|^2\\
&= \nu\operatorname{Re}\langle i\nabla(p\cdot e)\otimes \Delta X\chi, \nabla X\chi\rangle + \nu\operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla\Delta X\chi\rangle\\
&\quad + \operatorname{Re}\langle i\nabla(p\cdot e)\otimes R\chi, \nabla X\chi\rangle + \operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla R\chi\rangle\\
&\le C_0\nu\|\Delta X\chi\|\left(\|\nabla(p\cdot e)\otimes\nabla X\chi\| + \|X\chi\| + \|\nabla\chi\cdot\nabla(p\cdot e)X\|\right)\\
&\quad + C_0\left(\|\nabla(\nabla(p\cdot e)\otimes X)\chi\| + \|X\chi\|\right)\|R\chi\| + C_0\|\nabla(p\cdot e)\otimes X\chi\|\,\|\nabla\chi\otimes R\|.
\end{aligned}$$

It follows by Young's inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla X\chi\rangle + \|\nabla(p\cdot e)\otimes X\chi\|^2\\
&\le C_0\nu\|\Delta X\chi\|\left(\|\nabla\big(\nabla(p\cdot e)\otimes X\big)\chi\| + \|X\chi\|\right) + E_3(t),
\end{aligned}$$

where

$$E_3(t) := C_0\nu\|\Delta X\chi\|\|\nabla\chi\cdot\nabla(p\cdot e)X\| + C_0\left(\|\nabla\cdot(\nabla(p\cdot e)\otimes X)\chi\| + \|X\chi\|\right)\|R\chi'\|. \tag{4.26}$$

Finally, we find using the explicit calculation of Lemma A.1 that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(p\cdot e)\otimes X\chi\|^2 + \nu\|\nabla(p\cdot e)\otimes X\chi\|^2\\
&= \nu\operatorname{Re}\langle\nabla(p\cdot e)\otimes X\chi, \Delta(\nabla(p\cdot e)\otimes X)\chi\rangle + \nu\operatorname{Re}\langle\nabla(p\cdot e)\otimes X\chi, \nabla(p\cdot e)\otimes X\chi\rangle\\
&\quad + 2\nu\operatorname{Re}\langle\nabla(p\cdot e)\otimes X\chi, (p\cdot e)\nabla X\chi\rangle + \operatorname{Re}\langle\nabla(p\cdot e)\otimes X\chi, \nabla(p\cdot e)\otimes R\chi\rangle.
\end{aligned}$$

For the contribution of R, use the explicit expression of R and the definition of X to find that

$$\begin{aligned}
\nabla(p\cdot e)\otimes R &= \nabla(p\cdot e)\otimes 2i\beta\nu\nabla([p\cdot e-1]\psi)\\
&= 2i\nu\nabla([p\cdot e-1](X-\alpha\nabla\psi)) - 2i\beta\nu\nabla^2(p\cdot e)[p\cdot e-1]\psi,
\end{aligned}$$

so that

$$\begin{aligned}
&\operatorname{Re}\langle\nabla(p\cdot e)\otimes X\chi, \nabla(p\cdot e)\otimes R\chi\rangle\\
&\le C\nu\Big(\|\nabla(\nabla(p\cdot e)\otimes X)\chi\| + \|\nabla(p\cdot e)\otimes X\chi\|\Big)\Big(\|\nabla(p\cdot e)\otimes X\chi'\| + |\alpha|\|\nabla(\nabla(p\cdot e)\psi)\| + |\alpha|\|\psi\|\Big)\\
&\quad + C\nu|\beta|\|\nabla(p\cdot e)\otimes X\chi\|\|\nabla(p\cdot e)\psi\|.
\end{aligned}$$

Hence we find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(p\cdot e)\otimes X\chi\|^2 + \nu\|\nabla(p\cdot e)\otimes X\chi\|^2\\
&\qquad\le -\nu\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2 + 2\nu\|\nabla(p\cdot e)\otimes X\chi\|^2 + C_1\nu\|X\chi\|^2 + E_4(t),
\end{aligned}$$

where

$$\begin{aligned}
E_4(t) := {}& C_1\nu\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|\Big(\|\nabla(p\cdot e)\otimes X\chi'\| + |\alpha|\|\nabla(\nabla(p\cdot e)\psi)\| + |\alpha|\|\psi\|\Big)\\
&+ C_1\nu\|\nabla(p\cdot e)\otimes X\chi\|\Big(\|X\| + \|\nabla(p\cdot e)\otimes X\chi'\| + |\alpha|\|\nabla(\nabla(p\cdot e)\psi)\| + |\alpha|\|\psi\|\Big)\\
&+ C_1\nu|\beta|\|\nabla(p\cdot e)\otimes X\chi\|\|\nabla(p\cdot e)\psi\|.
\end{aligned}$$

From here, we can proceed exactly as in the proof of Lemma 4.1: for suitable positive constants a, b, c, ν_0, for all $\nu \le \nu_0$, an energy of the form

$$E[X] := E[\psi] = \frac{1}{2}\Big[\|X\chi\|^2 + a\nu t\|\nabla X\chi\|^2 + 2b\nu t^2\operatorname{Re}\langle i\nabla(p\cdot e)\otimes X\chi, \nabla X\chi\rangle + c\nu t^3\|\nabla(p\cdot e)X\chi\|^2\Big]$$

is both coercive

$$E[X] \geq \frac{1}{2}\Big[\|X\|^2 + \frac{a}{2}\nu t\|\nabla X\chi\|^2 + \frac{c}{2}\nu t^3\|\nabla(p\cdot e)X\chi\|^2\Big]$$

and satisfies for times $t \leq \nu^{-1}$ the inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E[X] + \frac{\nu}{2}\|\nabla X\chi\|^2 + \frac{a\nu^2 t}{2}\|\nabla\nabla X\chi\|^2 + \frac{b\nu t^2}{2}\|\nabla(p\cdot e)\otimes X\chi\|^2 + \frac{c\nu^2 t^3}{2}\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\\ &\leq C\nu^2 t^3\|X\chi\|^2 + E_1(t) + a\nu t E_2(t) + b\nu t^2 E_3(t) + c\nu t^3 E_4(t).\end{aligned} \tag{4.27}$$

Thanks to (4.19) and Lemma 4.3, we have for all $t_* \leq \nu^{-1}$ that

$$\int_0^{t_*} E_1 + \int_0^{t_*} \nu t E_2 \lesssim \sup_{[0,t_*]}\Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)(1+\nu^2 t_*^4)\|\psi^{in}\|^2. \tag{4.28}$$

In particular,

$$\int_0^{\nu^{-1/2}} E_1 + \int_0^{\nu^{-1/2}} \nu t E_2 \lesssim 1.$$

Regarding E_3, we have by Young's inequality that for all $\kappa > 0$:

$$\begin{aligned}\nu t^2 E_3 &\leq \kappa\nu^2 t\|\nabla\nabla X\|^2 + C_\kappa \nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2\\ &\quad + \kappa\nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2 + \kappa\nu^2 t^3\|X\chi\|^2 + C_\kappa t\|R\chi'\|^2\\ &\leq \kappa\big(\nu^2 t\|\nabla\nabla X\|^2 + \nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\big)\\ &\quad + C_\kappa\nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2 + \nu^2 t^3\|X\chi\|^2 + \tilde{E}_3,\end{aligned}$$

where still using Lemma 4.3, we have for all $t_* \leq \nu^{-1}$,

$$\int_0^t \tilde{E}_3 \lesssim \int_0^{t_*} t\|R\chi'\|^2\mathrm{d}t \lesssim \sup_{[0,t_*]}\Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)(1+\nu^2 t_*^4)\|\psi^{in}\|^2. \tag{4.29}$$

In particular,

$$\int_0^{\nu^{-1/2}} \tilde{E}_3 \lesssim \int_0^{\nu^{-1/2}} t\|R\chi'\|^2 \lesssim 1.$$

Regarding E_4, we estimate for all $t \leq \nu^{-1}$

$$\begin{aligned}\nu t^3 E_4 &\leq \kappa\nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\\ &\quad + C_\kappa\nu^2 t^3\Big(\|\nabla(p\cdot e)\otimes X\chi'\|^2 + |\alpha|^2\|\nabla(\nabla(p\cdot e)\psi)\|^2 + |\alpha|^2\|\psi\|^2\Big)\\ &\quad + \kappa\nu t^2\|\nabla(p\cdot e)\otimes X\chi\|^2 + C_\kappa\nu^3 t^4\|X\|^2 + C_\kappa\nu^3 t^4|\beta|^2\|\nabla(p\cdot e)\psi\|^2\\ &\leq \kappa\big(\nu^2 t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2 + \nu t^2\|\nabla(p\cdot e)\otimes X\chi\|^2\big) + C_\kappa\nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2 + \tilde{E}_4,\end{aligned}$$

where, for all $t_* \leq \nu^{-1}$

$$\begin{aligned}\int_0^{t_*}\tilde{E}_4 &\lesssim \nu^2 t_*^4\int_0^{t_*}\nu\|X\|^2\\ &\quad + \int_0^{t_*}\Big(\nu^2 t^3|\alpha|^2\|\nabla(\nabla(p\cdot e)\psi)\chi'\|^2 + \nu^2 t^3|\alpha|^2\|\psi\chi'\|^2 + \nu^3 t^4|\beta|^2\|\nabla(p\cdot e)\psi\|^2\Big)\\ &\lesssim \sup_{[0,t_*]}\Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)(1+\nu^4 t_*^8)\|\psi^{in}\|^2.\end{aligned} \tag{4.30}$$

Here we used in the last step (4.19) and Lemma 4.3. In particular, this shows

$$\int_0^{\nu^{-1/2}}\tilde{E}_4 \lesssim 1.$$

For κ small enough independently of ν, and for all $t \leq \nu^{-1}$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E[X] + \frac{\nu}{4}\|\nabla X\chi\|^2 + \frac{a\nu^2 t}{4}\|\nabla\nabla X\chi\|^2 + \frac{b\nu t^2}{4}\|\nabla(p\cdot e)\otimes X\chi\|^2 + \frac{c\nu^2 t^3}{4}\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\\ &\leq C'\nu^2 t^3\|X\chi\|^2 + C'\nu^2 t^3\|\nabla(p\cdot e)\otimes X\chi'\|^2 + E_1(t) + a\nu t E_2(t) + b\tilde{E}_3(t) + c\tilde{E}_4(t).\end{aligned} \tag{4.31}$$

The desired inequality follows by time integration from 0 to $\nu^{-1/2}$, using the bounds on $E_1, E_2, \tilde{E}_3, \tilde{E}_4$. □

To obtain a good decay rate for the velocity field of our linearized model, we will need to control also the L^∞ norm of ψ near the pole. This is the purpose of the next lemma.

Lemma 4.6. *There exist absolute constants $C > 0$ and $\nu_0 > 0$ such that for all $\nu \leq \nu_0$ and $t \leq \nu^{-1/2}$ it holds that*

$$|X(t,p)|^2 \leq \|\nabla\psi^{in}\|_{L^\infty}^2 + C\left([1-p\cdot e]^3\nu t^4 + [1-p\cdot e]^2\nu^2 t^5 + [1-p\cdot e]\nu t^2 + 1\right)\|\psi^{in}\|_{L^\infty}^2.$$

PROOF. Using the Leibniz rule for the covariant derivative, we have

$$\frac{1}{2}\Delta|X|^2 = g(\Delta X, X) + |\nabla X|^2,$$

where ΔX still refers to the connection Laplacian of the vector field X and where $|X| = \sqrt{g(X,X)}$ is the usual norm induced by the metric on tensors. Back to (4.18), it follows that

$$(\partial_t - \nu\Delta)\frac{|X|^2}{2} + \nu|\nabla X|^2 + \nu|X|^2 = g(-\frac{2\beta^2}{\alpha}\nu[p\cdot e - 1]\nabla(p\cdot e)\psi + 2i\beta\nu\nabla(p\cdot e)\psi, X) \tag{4.32}$$

so that

$$\begin{aligned}(\partial_t - \nu\Delta)\frac{|X|^2}{2} &\leq \nu t|X|^2 + C\nu\frac{|\beta|^4}{|\alpha|^2 t^4}t^3[1-p\cdot e]^3|\psi|^2 + C\frac{|\beta|^2}{t^2}\nu t[1-p\cdot e]|\psi|^2\\ &\leq \nu t|X|^2 + C'\Big(\nu t^3[1-p\cdot e]^3 + \nu t[1-p\cdot e]\Big)\|\psi^{in}\|_{L^\infty}^2,\end{aligned}$$

where the latter inequality holds for all $t \leq \nu^{-1/2}$. It is then easy to find absolute constants $c_0 > 0$, $\nu_0 > 0$, $a_0, a_1, a_2, a_3 > 0$, such that for all $\nu \leq \nu_0$, the function

$$f = \Big(a_0\nu t^4[1-p\cdot e]^3 + a_1\nu^2 t^5[1-p\cdot e]^2 + a_2\nu t^2[1-p\cdot e] + a_3\Big)\|\psi^{in}\|_{L^\infty}^2$$

satisfies

$$(\partial_t - \nu\Delta)\Big(\mathrm{e}^{-c_0\nu t^2}|X|^2 - f\Big) \leq 0 \qquad \forall t \leq \nu^{-1/2}.$$

The lemma then follows from the maximum principle for the scalar heat flow on the sphere. □

The estimates on $X = J_\nu\psi$ can roughly yield a decay like t^{-1}. For the sharp decay of t^{-2} for the velocity field, we need to iterate and thus also need a control of $J_\nu X = \alpha\nabla X + i\beta\nabla(p\cdot e)\otimes X$. From (4.17), we get

$$\Big(\partial_t + i(p\cdot e) - \nu\Delta\Big)J_\nu X + \nu J_\nu X = -\nu\alpha[\Delta,\nabla]X - i\beta\nu\nabla(p\cdot e)\otimes X + 2i\beta\nu\nabla\big([p\cdot e - 1]X\big) + J_\nu R,$$

where we recall $R = 2i\beta\nu\nabla([p\cdot e - 1]\psi)$. By Ricci's formula, see Appendix A, one has $[\Delta,\nabla]X = \mathcal{R}\nabla X$ for some tensor $\mathcal{R}$ so that

$$-\nu\alpha[\Delta,\nabla]X = -\nu\mathcal{R}J_\nu X + \nu i\beta\mathcal{R}\nabla(p\cdot e)\otimes X$$

and eventually

$$\Big(\partial_t + i(p\cdot e) - \nu\Delta\Big)J_\nu X = -\nu(\mathcal{R}+1)J_\nu X + \nu i\beta(\mathcal{R}-1)\nabla(p\cdot e)\otimes X + 2i\beta\nu\nabla\big((p\cdot e - 1)X\big) + J_\nu R. \tag{4.33}$$

Using the previous estimates, we shall prove:

Lemma 4.7. *There exists $\nu_0 > 0$, such that for all $0 < \nu \leq \nu_0$ and cutoffs $\chi : \mathbb{S}^2 \to [0,1]$ such that $-e$ does not belong to the support of χ, one can find $C > 0$ independent of ν satisfying for all $t \leq \nu^{-1/2}$*

$$\|J_\nu X(t)\chi\|^2 \leq C\|\psi^{in}\|_{H^2}^2.$$

PROOF. Let χ, χ' such that $\chi' = 1$ on the support of χ and such that $-e$ does not belong to the support of χ'. Performing an L^2 estimate on (4.33), we find after standard integrations by parts that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_\nu X\chi\|^2 + \nu\|\nabla J_\nu X\chi\|^2 \leq{}& 2\nu\|\nabla J_\nu X\chi\|\,\|J_\nu X \otimes \nabla\chi\| + C\nu\|J_\nu X\chi\|^2\\ &+ C\nu|\beta|\|\nabla(p\cdot e)\otimes X\chi\|\|J_\nu X\chi\|\\ &+ 2\nu|\beta|\|(p\cdot e-1)X\chi\|\Big(\|\nabla J_\nu X\chi\| + 2\|J_\nu X\otimes\nabla\chi\|\Big)\\ &+ \mathrm{Re}\langle J_\nu R\chi, J_\nu X\chi\rangle.\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_\nu X\chi\|^2 + \frac{\nu}{2}\|\nabla J_\nu X\chi\|^2 \leq C'\nu\|J_\nu X\chi'\|^2 + C'\nu|\beta|^2\|\nabla(p\cdot e)\otimes X\chi\|^2 + \langle J_\nu R\chi, J_\nu X\chi\rangle.$$

We first notice that

$$\nu\|JX\chi'\|^2 + \nu\beta^2\|\nabla(p\cdot e)\otimes X\chi\|^2 \lesssim \nu|\alpha|^2\|\nabla X\chi'\|^2 + \nu|\beta|^2\|\nabla(p\cdot e)\otimes X\chi'\|^2.$$

For the last term, we use that

$$\begin{aligned}J_\nu R &= 2i\nu\beta J_\nu\nabla([p\cdot e-1]\psi) = 2i\nu\beta\nabla([p\cdot e-1]J_\nu\psi) + 2i\nu\beta[J_\nu, \nabla([p\cdot e-1]\cdot)]\psi\\ &= 2i\nu\beta\nabla([p\cdot e-1]X) + 2i\nu\beta[J_\nu, \nabla([p\cdot e-1]\cdot)]\psi =: R_1 + R_2.\end{aligned}$$

For the first part, we find by integration by parts that

$$\mathrm{Re}\langle R_1\chi, J_\nu X\chi\rangle \leq 2\nu|\beta|\,\|(p\cdot e-1)X\chi\|\Big(\|\nabla J_\nu X\chi\| + 2\|J_\nu X\otimes\nabla\chi\|\Big).$$

This right-hand side was already encountered above and we bound it as

$$\mathrm{Re}\langle R_1\chi, J_\nu X\chi\rangle \leq \frac{\nu}{8}\|\nabla J_\nu X\chi\|^2 + \nu\|J_\nu X\chi'\|^2 + C\nu|\beta|^2\|\nabla(p\cdot e)\otimes X\chi\|^2.$$

For the other component, we further compute that

$$R_2 = 2i\nu\beta\alpha\nabla\big(\nabla(p\cdot e)\psi\big) + 2\nu\beta^2\nabla(p\cdot e)(p\cdot e-1)\psi := R_{2,1} + R_{2,2}.$$

One has

$$\mathrm{Re}\langle R_{2,1}\chi, J_\nu X\chi\rangle \leq C\nu|\alpha\beta|\|\nabla(p\cdot e)\psi\chi\|\Big(\|\nabla J_\nu X\chi\| + \|J_\nu X\otimes\nabla\chi\|\Big),$$

so that one ends up with

$$\mathrm{Re}\langle R_{2,1}\chi, J_\nu X\chi\rangle \leq \frac{\nu}{8}\|\nabla J_\nu X\chi\|^2 + C\nu\|J_\nu X\chi'\|^2 + C\nu|\alpha\beta|^2\|\nabla(p\cdot e)\psi\chi\|^2.$$

Eventually, we notice that

$$R_{2,2} = \frac{2}{i}\nu\beta(p\cdot e-1)(X - \alpha\nabla\psi),$$

resulting after integrations by part in

$$\begin{aligned}\mathrm{Re}\langle R_{2,2}\chi, J_\nu X\chi\rangle &\leq C\nu|\beta|\|(p\cdot e-1)X\chi\|\|J_\nu X\chi\| + C\nu|\alpha\beta|\|\nabla(p\cdot e)\psi\|\big(\|\nabla J_\nu X\chi\| + \|J_\nu X\chi'\|\big)\\ &\leq \frac{\nu}{8}\|\nabla J_\nu X\chi\|^2 + C\nu\|J_\nu X\chi'\|^2 + C\nu|\beta|^2\big(\|\nabla(p\cdot e)\otimes X\chi\|^2 + |\alpha|^2\|\nabla(p\cdot e)\psi\|^2\big).\end{aligned} \tag{4.34}$$

Hence, collecting all these bounds, we find for all $t \leq \nu^{-1}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_\nu X\chi\|^2 + \frac{\nu}{8}\|\nabla J_\nu X\chi\|^2 \lesssim E(t) \tag{4.35}$$

where

$$E(t) := \nu|\alpha|^2\|\nabla X\chi'\|^2 + \nu|\beta|^2\|\nabla(p\cdot e)\otimes X\chi'\|^2 + \nu|\alpha\beta|^2\|\nabla(p\cdot e)\psi\|^2. \tag{4.36}$$

For $t \le \nu^{-1/2}$,

$$E(t) \lesssim \nu\|\nabla X\chi'\|^2 + \nu t^2\|\nabla(p\cdot e)\otimes X\chi'\|^2 + \nu t^2\|\nabla(p\cdot e)\psi\|^2$$

and thanks to Lemma 4.4, one has $\int_0^{\nu^{-1/2}} E(t)\mathrm{d}t \le C$, independent of ν. This yields the claimed estimate for $J_\nu X$. □

4.3. Mixing estimates. We now show how the obtained estimates can be used to obtain the mixing estimates up to time $\nu^{-1/2}$ of Proposition 1.7.

PROOF OF PROPOSITION 1.7 FOR TIMES UP TO $\nu^{-1/2}$. We start with the proof of the first inequality. We split the integrand

$$\int_{\mathbb{S}^2}\psi(t)F\,\mathrm{d}p = \int_{\mathbb{S}^2}\psi(t)F\chi\,\mathrm{d}p + \int_{\mathbb{S}^2}\psi(t)F(1-\chi)\,\mathrm{d}p,$$

where χ is a smooth function which is 1 near e and 0 near $-e$. By symmetry consideration, it is enough to show the decay for $\int_{\mathbb{S}^2}\psi(t)F\chi$. We introduce a cutoff χ_ϵ which is 1 in a ball of radius ϵ around the pole $p=e$, zero outside a ball of radius 2ε, and satisfies $|\nabla\chi_\epsilon| \lesssim \epsilon^{-1}$. We write

$$\int_{\mathbb{S}^2}\psi F\chi\,\mathrm{d}p = \int_{\mathbb{S}^2}\psi F\chi_\epsilon\chi\,\mathrm{d}p + \int_{\mathbb{S}^2}\psi F(1-\chi_\epsilon)\chi\,\mathrm{d}p =: I_1 + I_2.$$

As $\|\psi\|_{L^\infty} \le \|\psi^{in}\|_{L^\infty}$, we get

$$|I_1| \le C\epsilon^2\|F\|_{L^\infty}\|\psi^{in}\|_{L^\infty}.$$

As regards I_2, we write

$$\begin{aligned}\int\psi F(1-\chi_\epsilon)\chi &= \int\nabla(p\cdot e)\cdot\nabla(p\cdot e)\psi\frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi\\ &= \frac{1}{i\beta}\int\nabla(p\cdot e)\cdot(X-\alpha\nabla\psi)\frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi\\ &= \frac{1}{i\beta}\int\nabla(p\cdot e)\cdot X\frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi\\ &\quad+\frac{\alpha}{i\beta}\int\psi\nabla\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi\right) = I_{2,1}+I_{2,2}.\end{aligned} \tag{4.37}$$

We introduce again spherical coordinates (θ,φ), with colatitude $\theta\in(0,\pi)$ and longitude $\varphi\in(0,2\pi)$, so that $p = \sin\theta\cos\varphi\, e_x + \sin\theta\sin\varphi\, e_y + \cos\theta\, e$, and $\nabla(p\cdot e) = -\sin\theta e_\theta$, while the surface measure on the sphere is $\mathrm{d}s = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. For the first term, we find that

$$\begin{aligned}|I_{2,1}| &\le \frac{C}{|\beta|}\|F\|_{L^\infty}\int_{|\theta|\ge\epsilon}\frac{|X|}{|\sin\theta|}\chi\,\mathrm{d}s \le \frac{C}{|\beta|}\|F\|_{L^\infty}\left(\int_{|\theta|\ge\epsilon}\frac{1}{|\sin\theta|^2}\,\mathrm{d}s\right)^{\frac12}\|X\chi\|\\ &\le \frac{C'}{|\beta|}\|F\|_{L^\infty}\left(\int_{|\theta|\ge\epsilon}\frac{1}{|\sin\theta|}\,\mathrm{d}\theta\right)^{\frac12}\|X\chi\| \le \frac{C''}{|\beta|}\|F\|_{L^\infty}\sqrt{|\ln\epsilon|}\|X\chi\|.\end{aligned}$$

In particular with Corollary 4.5 this shows for times $t\le\nu^{-1/2}$ that

$$|I_{2,1}| \le \frac{C}{t}\|F\|_{L^\infty}\sqrt{|\ln\epsilon|}\,\|\psi^{in}\|_{H^1}.$$

For the second term, we get

$$
\begin{aligned}
|I_{2,2}| &\leq C\Big|\frac{\alpha}{\beta}\Big| \int_{|\theta|\geq\epsilon} |\psi| \left(|\nabla\chi_\epsilon| \frac{|F|}{|\sin\theta|} + \frac{1}{|\sin\theta|^2}|F| + \frac{|\nabla F|}{|\sin\theta|} \right) \mathrm{d}s \\
&\leq C'\Big|\frac{\alpha}{\beta}\Big| \|\psi^{in}\|_{L^\infty} \Bigg(\|F\|_{L^\infty} \int_{2\varepsilon\geq|\theta|\geq\epsilon} \frac{1}{\varepsilon|\sin\theta|}\mathrm{d}s + \|F\|_{L^\infty} \int_{|\theta|\geq\epsilon} \frac{1}{|\sin\theta|^2}\mathrm{d}s \\
&\qquad\qquad + \|F\|_{H^1} \Big(\int_{|\theta|\geq\epsilon} \frac{1}{|\sin\theta|^2}\mathrm{d}s \Big)^{1/2} \Bigg) \\
&\leq C''\Big|\frac{\alpha}{\beta}\Big| \|\psi^{in}\|_{L^\infty} |\ln\epsilon| \big(\|F\|_{L^\infty} + \|F\|_{H^1} \big).
\end{aligned}
$$

In particular for times $t \leq \nu^{-1/2}$ it shows that

$$
|I_{2,2}| \leq \frac{C''}{t} |\ln\epsilon| \, \|\psi^{in}\|_{L^\infty} (\|F\|_{L^\infty} + \|F\|_{H^1}).
$$

Taking $\epsilon = \frac{1}{\sqrt{t}}$ yields the first inequality.

As regards the second inequality, we write

$$
\int \psi F \nabla(p\cdot e)\chi = \frac{1}{i\beta}\int F(X - \alpha\nabla\psi)\chi = \frac{1}{i\beta}\int FX\chi + \frac{\alpha}{i\beta}\int \psi\nabla\cdot(F\chi) = J_1 + J_2.
$$

Applying the first inequality to the integral in J_2, we find for times $t \leq \nu^{-1/2}$ that

$$
|J_2| \leq C\frac{\sqrt{\ln(2+t)}}{t^2} \|\psi^{in}\|_{H^{1+\delta}} \|F\|_{H^{2+\delta}}.
$$

The term J_1 is similar, except that ψ is replaced by X, on which we have a weaker control. Using the same strategy as in the proof of the first inequality, we can write

$$
J_1 = \frac{1}{i\beta}\int XF\chi = \frac{1}{i\beta}\int XF\chi_\epsilon\chi + \frac{1}{i\beta}\int XF(1-\chi_\epsilon)\chi =: K_1 + K_2.
$$

We treat K_1 as I_1, except that we use Lemma 4.6 as a substitute to the L^∞ bound on ψ. We get

$$
|K_1| \leq \frac{C}{t}\big(\epsilon^2 + \sqrt{\nu}t^2\epsilon^5 + \nu t^{5/2}\epsilon^4 + \nu^{1/2}t\epsilon^3\big)\|\psi^{in}\|_{W^{1,\infty}}\|F\|_{L^\infty}.
$$

We then treat K_2 as I_2, resulting in

$$
\begin{aligned}
K_2 = &\frac{1}{-\beta^2}\int \nabla(p\cdot e)\cdot J_\nu X \frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi \\
&+ \frac{\alpha}{-\beta^2}\int X\nabla\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}(1-\chi_\epsilon)\chi\right) = K_{2,1} + K_{2,2}.
\end{aligned} \tag{4.38}
$$

As before we find

$$
|K_{2,1}| \leq \frac{C}{|\beta|^2}\|J_\nu X\|\sqrt{|\ln\epsilon|}\|F\|_{L^\infty},
$$

which yields in particular for $t \leq \nu^{-1/2}$ that

$$
|K_{2,1}| \leq \frac{C'}{t^2}\|\psi^{in}\|_{H^2}\|F\|_{L^\infty}\sqrt{|\ln\epsilon|}.
$$

For $K_{2,2}$, we further decompose it as

$$\begin{aligned}K_{2,2} &= \frac{\alpha}{\beta^2}\int X\nabla\chi_\epsilon\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}\chi\right) - \frac{\alpha}{\beta^2}\int X(1-\chi_\epsilon)\nabla\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}\chi\right)\\ &= \frac{\alpha}{\beta^2}\int X\nabla\chi_\epsilon\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}\chi\right)\\ &\quad - \frac{\alpha}{i\beta^3}\int J_\nu X(1-\chi_\epsilon)\frac{\nabla(p\cdot e)}{|\nabla(p\cdot e)|^2}\nabla\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}\chi\right)\\ &\quad - \frac{\alpha^2}{i\beta^3}\int X\nabla\cdot\left((1-\chi_\epsilon)\frac{\nabla(p\cdot e)}{|\nabla(p\cdot e)|^2}\nabla\cdot\left(\nabla(p\cdot e)\frac{F}{|\nabla(p\cdot e)|^2}\chi\right)\right) =: H_1+H_2+H_3.\end{aligned}$$

For H_1 we obtain the bound

$$|H_1| \le \frac{C}{\epsilon}\frac{|\alpha|}{|\beta|^2}\|F\|_{L^\infty}\int_{\epsilon\le|\theta|\le c\epsilon}|X|$$

and in particular for times $t\le\nu^{-1/2}$ we get by Lemma 4.6 that

$$|H_1| \le \frac{C}{t^2\epsilon}\big(\epsilon+\sqrt{\nu}t^2\epsilon^4+\nu t^{5/2}\epsilon^3+\nu^{1/2}t\epsilon^2\big)\|F\|_{L^\infty}\|\psi^{in}\|_{W^{1,\infty}}.$$

For H_2, we compute

$$|H_2| \le C\frac{|\alpha|}{|\beta|^3}\|F\|_{W^{1,\infty}}\int_{|\theta|\ge\epsilon}\frac{|J_\nu X|}{|\sin(\theta)|^2}\,\mathrm{d}\theta\,\mathrm{d}\varphi \le \frac{C}{\epsilon^2}\frac{|\alpha|}{|\beta|^3}\|F\|_{W^{1,\infty}}\|J_\nu X\|.$$

In particular for $t\le\nu^{-1/2}$ it shows by Lemma 4.7 that

$$|H_2| \le \frac{C'}{t^3\epsilon^2}\|F\|_{W^{1,\infty}}\|\psi^{in}\|_{H^2}.$$

Finally,

$$|H_3| \le C\frac{|\alpha|^2}{|\beta|^3}\int_{|\theta|\ge\epsilon}|X|\left(|\nabla\chi_\epsilon|\frac{(|F|+|\nabla F|)}{|\sin\theta|^2}+\frac{(|F|+|\nabla F|)}{|\sin\theta|^3}+\frac{|\nabla^2F|}{|\sin\theta|}\right).$$

In particular, for $t\le\nu^{-1/2}$ we use Lemma 4.6 to find

$$|H_3| \le \frac{C}{t^3}\Big(\epsilon^{-2}+\sqrt{\nu}t^2+\nu t^{5/2}(1+|\ln\epsilon|)+\nu^{1/2}t\epsilon^{-1}\Big)\|\psi^{in}\|_{W^{1,\infty}}\|F\|_{H^{2+\delta}}.$$

Taking $\epsilon=\frac{1}{\sqrt{t}}$ yields the second inequality. □

4.4. Mixing estimates after time $\nu^{-1/2}$. In this subsection, we prove the remaining estimates in Proposition 1.7, i.e. for times $t\in[\nu^{-1/2}, c\nu^{-1/2}|\ln\nu|]$. The general idea is the same and it has already been anticipated in the previous subsections. The main difference is to notice that we cannot approximate anymore α by 1 and β by t but that α and β are exponentially growing in this time scale.

PROOF OF PROPOSITION 1.7 FOR TIMES AFTER $\nu^{-1/2}$. Performing the proof of Lemma 4.4, we do not substitute α and β in the last step but rather take out a supremum. As an analogue of Corollary 4.5 this yields for all $t_*\le\nu^{-1}$ the bound

$$\begin{aligned}&\sup_{t\in[0,t_*]}\big(\|X\chi\|^2+\nu t\|\nabla X\chi\|^2+\nu t^3\|\nabla(p\cdot e)\otimes X\chi\|^2\big)\\ &+\int_0^{t_*}\big(\nu\|\nabla X\chi\|+\nu^2\|\nabla^2X\chi\|^2+\nu t^2\|\nabla(p\cdot e)\otimes X\chi\|^2+\nu^2t^3\|\nabla(\nabla(p\cdot e)\otimes X)\chi\|^2\big)\,\mathrm{d}t\\ &\le C\sup_{[0,t^*]}\Big(|\alpha|^2+\frac{|\beta|^2}{t^2}\Big)(1+\nu^4t_*^8)\|\psi^{in}\|_{H^1}^2.\end{aligned}\tag{4.39}$$

Using this inequality in (4.35)-(4.36), we find as an analogue of Lemma 4.7 that for all $t_* \leq \nu^{-1}$ the bound

$$\sup_{t\in[0,t_*]} \|J_\nu X(t)\|^2 \leq C \sup_{[0,t^*]} \Big(|\alpha|^2 + \frac{|\beta|^2}{t^2}\Big)^2 (1+\nu^4 t_*^8)\|\psi^{in}\|_{H^2}^2. \tag{4.40}$$

Regarding the L^∞ bound on X near the poles, we restart from (4.32) to deduce

$$\begin{aligned}(\partial_t - \nu\Delta)\frac{|X|^2}{2} &\leq \frac{\nu t}{|\ln\nu|}|X|^2 + C|\ln\nu|\nu\frac{|\beta|^4}{|\alpha|^2 t^4} t^3[1-p\cdot e]^3|\psi|^2 + C|\ln\nu|\frac{|\beta|^2}{t^2}\nu t[1-p\cdot e]|\psi|^2\\ &\leq \frac{\nu t}{|\ln\nu|}|X|^2 + C'|\ln\nu|\frac{|\beta|^2}{t^2}\Big(\nu t^3[1-p\cdot e]^3 + \nu t[1-p\cdot e]\Big)\|\psi^{in}\|_{L^\infty}^2,\end{aligned}$$

where we used that $|\beta|^2/(|\alpha|^2 t^2) \lesssim 1$ for the second inequality. Mimicking the end of the proof of Lemma 4.6, we find that for all $t_* \leq c\nu^{-1/2}|\ln\nu|$,

$$\begin{aligned}&\sup_{t\in[0,t_*]} |X(t,p)|^2 \leq \|\nabla\psi^{in}\|_{L^\infty}\\ &+ C|\ln\nu|\Big(\sup_{[0,t_*]}\frac{|\beta|^2}{t^2}\Big)\left([1-p\cdot e]^3\nu t_*^4 + [1-p\cdot e]^2\nu^2 t_*^5 + [1-p\cdot e](\nu^3 t_*^6 + \nu t_*^2) + 1\right)\|\psi^{in}\|_{L^\infty}^2.\end{aligned} \tag{4.41}$$

Taking into account that

$$\frac{|\alpha|^2}{|\beta|^2} \sim \nu, \quad t^2 \lesssim \nu|\ln\nu|^2 \quad \text{for } t\in[\nu^{-1/2}, c\nu^{-1/2}|\ln\nu|]$$

and using (4.39)-(4.40)-(4.41), we can adapt the proof of Section 4.3 to obtain the second part of Proposition 1.7. □

5. Proof of mixing and enhanced diffusion for the diffusive Saintillan-Shelley model

The good decay estimates for the semigroup e^{tL_1} of the previous subsection now allow us to conclude the results by studying the Volterra equation (1.28).

5.1. Proof of Theorem 2. We first prove Theorem 2. Coming back to the Volterra equation (1.28), we drop the subscript k and restore the superscript ν so that it now reads

$$u^\nu(t) + \int_0^t K^\nu(t-s)\, u^\nu(s)\,\mathrm{d}s = U^\nu(t), \tag{5.1}$$

with

$$U^\nu(t) = u[\mathrm{e}^{tL_1}\psi^{in}], \quad K^\nu(t)v = -u[\mathrm{e}^{tL_1}(\bar{L}_2\cdot v)], \quad v\in\mathbb{C}^3.$$

We remind that

$$L_1 = -i(p\cdot k) + \nu\Delta, \qquad \overline{L}_2 = \frac{3i\Gamma}{4\pi}(p\cdot k)\mathbb{P}_{k^\perp}p$$

and

$$u[\psi] := i\mathbb{P}_{k^\perp}\sigma k, \quad \mathbb{P}_{k^\perp} = (I - k\otimes k), \quad \sigma := \varepsilon\int_{\mathbb{S}^2}\psi(p)\, p\otimes p\,\mathrm{d}p.$$

Thanks to the identity

$$(p\cdot k)\mathbb{P}_{k^\perp}p = -\mathbb{P}_{k^\perp}\nabla(p\cdot k),$$

we find that

$$\begin{aligned}U^\nu(t) &= \int_{\mathbb{S}^2}(\mathrm{e}^{tL_1}\psi^{in})F\,\nabla(p\cdot k)\,\mathrm{d}p,\\ K^\nu(t)v &= \int_{\mathbb{S}^2}(\mathrm{e}^{tL_1}\bar{L}_2\cdot v)F\,\nabla(p\cdot k)\,\mathrm{d}p, \quad F := -i\varepsilon\mathbb{P}_{k^\perp},\end{aligned}$$

Then, Proposition 1.7 shows that there exists a constant C independent of ν such that

$$|U^\nu(t)| \leq \frac{C\ln(2+t)}{(1+t)^2}\|\psi^{in}\|_{H^{2+\delta}}, \qquad |K^\nu(t)| \leq \frac{C\ln(2+t)}{(1+t)^2}, \qquad \forall t \leq \nu^{-1/2}. \tag{5.2}$$

Introduce $\tilde{U}^\nu := U^\nu \mathbb{1}_{t\leq \nu^{-1/2}}$ and $\tilde{K}^\nu := K^\nu \mathbb{1}_{t\leq \nu^{-1/2}}$ that satisfy

$$|\tilde{U}^\nu(t)| \leq \frac{C\ln(2+t)}{(1+t)^2}\|\psi^{in}\|_{H^{2+\delta}}, \qquad |\tilde{K}^\nu(t)| \leq \frac{C\ln(2+t)}{(1+t)^2}, \qquad \forall t \geq 0, \tag{5.3}$$

and consider the corresponding modified Volterra equation

$$\tilde{u}^\nu(t) + \int_0^t \tilde{K}^\nu(t-s)\,\tilde{u}^\nu(s)\,\mathrm{d}s = \tilde{U}^\nu(t). \tag{5.4}$$

Let K^{inv} be the inviscid kernel considered in Section 3, that is the kernel for $\nu = 0$. We have seen that K^{inv} decays like t^{-2}. We now claim that

$$\lim_{\nu\to 0}\|\tilde{K}^\nu - K^{inv}\|_{L^1(\mathbb{R}_+)} = 0. \tag{5.5}$$

Indeed, let $\kappa > 0$ be arbitrarily small. We fix some large $T > 0$ independent of ν, so that

$$\int_T^{+\infty}|K^{inv}| + \int_T^{+\infty}|\tilde{K}^\nu| \leq \kappa.$$

Note that the second condition can be achieved because the constant C in (5.3) is independent of ν. Now, for all ν such that $\nu^{-1/2} > T$, we write

$$\begin{aligned}\|\tilde{K}^\nu - K^{inv}\|_{L^1(\mathbb{R}_+)} &\leq \|\tilde{K}^\nu - K^{inv}\|_{L^1(0,T)} + \int_T^{+\infty}\Big(|\tilde{K}^\nu| + |K^{inv}|\Big)\\ &\leq \|K^\nu - K^{inv}\|_{L^1(0,T)} + \kappa.\end{aligned}$$

Eventually, it is standard to show that, given any initial data in $L^2(\mathbb{S}^2)$, and any finite time interval $(0,T)$, the solution ψ^ν of the advection-diffusion equation

$$\partial_t\psi^\nu + i(p\cdot k)\psi^\nu - \nu\Delta\psi^\nu = 0$$

converges in $L^\infty(0,T,L^2(\mathbb{S}^2))$ to the solution ψ^{inv} of $\partial_t\psi^{inv} + i(p\cdot k)\psi^{inv} = 0$ (with the same initial data). With the choice of initial data $\psi_{in} = \bar{L}_2\cdot v$, it follows that $\lim_{\nu\to 0}\|K^\nu - K^{inv}\|_{L^1(0,T)} = 0$, and the claim (5.5) follows.

From there, we can conclude by the stability of solutions of Volterra equations for ν small enough equation (5.4) has a solution, that decays like $O(\ln(2+t)/t^2)$. More precisely, one can construct perturbatively the resolvent $\tilde{R}^\nu$, satisfying

$$\tilde{R}^\nu + \tilde{K}^\nu \star \tilde{R}^\nu = \tilde{K}^\nu,$$

see Proposition 2.6, which even applies to non-convolution kernels. In particular, the stability condition $\det(I + \mathcal{L}\tilde{K}^\nu(z)) \neq 0$ in $\{\mathrm{Re}\, z \geq 0\}$ is satisfied, and $\|\tilde{R}^\nu\|_{L^1} \leq C$. Using the estimates in (5.3) and Theorem 5 (see also Remark 2.1), we deduce that

$$|\tilde{u}^\nu(t)| \leq \frac{C\ln(2+t)}{(1+t)^2}\|\psi^{in}\|_{H^{2+\delta}}, \quad t \geq 0.$$

But, as $\tilde{K}^\nu = K^\nu$ on $[0,\nu^{-1/2})$, by uniqueness of the solutions of the Volterra equation on a finite time interval, one has $\tilde{u}^\nu = u^\nu$ on $[0,\nu^{-1/2})$, and the first inequality in Theorem 2 follows. From there, the second one can be obtained exactly as in Section 3.3, by relating ψ to u through Duhamel's formula. This concludes the proof of the theorem.

5.2. Proof of Theorem 3. Here we use the additional decay estimates from the enhanced dissipation of Proposition 1.6 and the extended time-frame of Proposition 1.7. Using the second inequality in Proposition 1.7, we have

$$|K^\nu(t)| \lesssim \nu|\ln\nu|^M, \quad |U^\nu(t)| \lesssim \nu|\ln\nu|^M\|\psi^{in}\|_{H^{2+\delta}}, \quad \forall t \in [\nu^{-1/2}, c\nu^{-1/2}|\ln\nu|]. \tag{5.6}$$

Moreover, by Proposition 1.6, we also have for ν small enough

$$|K^\nu(t)| \lesssim \mathrm{e}^{-\varepsilon_0\nu^{1/2}t}, \quad |U^\nu(t)| \lesssim \mathrm{e}^{-\varepsilon_0\nu^{1/2}t}\|\psi^{in}\|_{H^{2+\delta}}. \tag{5.7}$$

We first prove that $\|K^\nu - K\|_{L^1(\mathbb{R}_+)} \to 0$ as $\nu \to 0$. We fix c such that $c\varepsilon_0 > \frac{1}{2}$. We decompose

$$\|K^\nu - K\|_{L^1} \leq \int_0^T |K^\nu - K| + \int_T^\infty |K| + \int_T^{\nu^{-1/2}} |K^\nu| + \int_{\nu^{-1/2}}^{c\nu^{-1/2}|\ln\nu|} |K^\nu| + \int_{c\nu^{-1/2}|\ln\nu|}^\infty |K^\nu|.$$

The first three terms can be treated as in Section 5.1: for any $\kappa > 0$ there exists T large enough and ν small enough so that

$$\int_0^T |K^\nu - K| + \int_T^\infty |K| + \int_T^{\nu^{-1/2}} |K^\nu| \leq \kappa.$$

For the fourth term, we use (5.6), that yields

$$\int_{\nu^{-1/2}}^{c\nu^{-1/2}|\ln\nu|} |K^\nu| \lesssim \nu^{1/2}|\ln\nu|^{M+1} \leq \kappa,$$

for ν small enough. Finally,

$$\int_{c\nu^{-1/2}|\ln\nu|}^\infty |K^\nu| \lesssim \int_{c\nu^{-1/2}|\ln\nu|}^\infty \mathrm{e}^{-\varepsilon_0\nu^{1/2}t} \lesssim \nu^{c\varepsilon_0 - \frac{1}{2}} \leq \kappa,$$

for ν small enough. Hence, $\|K^\nu - K\|_{L^1(\mathbb{R}_+)} \to 0$ as $\nu \to 0$. Arguing as in Section 5.1, we deduce that the spectral condition $\det(I + \mathcal{L}K^\nu(z)) \neq 0$ in $\{\mathrm{Re}\, z \geq 0\}$ is satisfied, and that the resolvent R^ν associated to K^ν satisfies $\|R^\nu\|_{L^1} \leq C$.

We now would like to establish the bound

$$|u^\nu(t)| \lesssim \mathrm{e}^{-\eta_0\nu^{1/2}t}\|\psi^{in}\|_{H^{2+\delta}}, \qquad \eta_0 := \frac{\varepsilon_0}{4}. \tag{5.8}$$

We come back to the proof of Theorem 5, replacing the weight $(1+\varepsilon t)^\alpha$ by $\mathrm{e}^{-\eta_0\nu^{1/2}t}$. The non-convolution kernel $k(t,s) = \frac{(1+\varepsilon t^\alpha)}{(1+\varepsilon s^\alpha)}K(t-s)1_{t-s>0}$ is now replaced by

$$\begin{aligned} k^\nu(t,s) &:= \mathrm{e}^{\eta_0\nu^{1/2}(t-s)}K^\nu(t-s)1_{t-s>0} \\ &= K_\nu(t-s)1_{t-s>0} + \big(\mathrm{e}^{\eta_0\nu^{1/2}(t-s)} - 1\big)K^\nu(t-s)1_{t-s>0} =: k_1^\nu(t,s) + k_2^\nu(t,s). \end{aligned}$$

It is easily seen that k^ν is a Volterra kernel of type L^∞. Moreover, by the previous considerations, for ν small enough, k_1^ν has the resolvent $r_1(t,s) := R^\nu(t-s)1_{t-s>0}$, which satisfies $\|r_1^\nu\|_{\infty,\mathbb{R}_+} \leq \|R^\nu\|_{L^1(\mathbb{R}_+)} \leq C$ for a constant C independent of ν. By using Proposition 2.6, it is enough to show that $\|k_2^\nu\|_{\infty,\mathbb{R}_+}$ can be made arbitrarily small for ν small enough. We fix $c = \frac{4}{3\varepsilon_0} = \frac{1}{3\eta_0}$, and decompose for some $\delta > 0$ to be fixed

$$\begin{aligned} \|k_2^\nu\|_{\infty,\mathbb{R}_+} &\leq \int_0^\infty \Big|\big(\mathrm{e}^{\eta_0\nu^{1/2}s} - 1\big)K_1^\nu(s)\Big|\mathrm{d}s \\ &\leq \Big(\int_0^{\delta\nu^{-1/2}} + \int_{\delta\nu^{-1/2}}^{c|\ln\nu|\nu^{-1/2}} + \int_{c|\ln\nu|\nu^{-1/2}}^\infty\Big)\Big|\big(\mathrm{e}^{\eta_0\nu^{1/2}s} - 1\big)K_1^\nu(s)\Big|\,\mathrm{d}s =: I_1 + I_2 + I_3. \end{aligned}$$

Let $\kappa > 0$. Using the uniform bound $K^\nu = O(\ln(2+t)t^{-2})$ on $[0, \nu^{-1/2}]$, we find

$$I_1 \lesssim (\mathrm{e}^{\eta_0\delta} - 1) \leq \kappa \quad \text{for } \delta \text{ small enough.}$$

This δ being fixed, using the bound $|K^\nu| \lesssim \nu|\ln\nu|^M$ on $[\delta\nu^{-1/2}, c\nu^{-1/2}|\ln\nu|]$, which follows from (5.2) and (5.6), we find

$$I_2 \lesssim \nu^{-\eta_0 c}\nu^{1/2}|\ln\nu|^{M+1} \lesssim \nu^{1/2-1/3} \leq \kappa,$$

for ν small enough. Eventually, using (5.7), we get

$$I_3 \lesssim \nu^{(\varepsilon_0-\eta_0)c-1/2} \lesssim \nu^{1/2} \leq \kappa,$$

for ν small enough. Hence, for ν small enough, k^ν has a resolvent r^ν that satisfies $\|r^\nu\|_{\infty,\mathbb{R}_+} \leq C$. Also, by the bound (5.7), we know that $\tilde{U}^\nu = \mathrm{e}^{\eta_0\nu^{1/2}t}U^\nu$ is bounded by $C\|\psi^{in}\|_{H^{2+\delta}}$, and finally

$$\tilde{u}^\nu = \tilde{U}^\nu - \int_{\mathbb{R}_+} r^\nu(t,s)\tilde{U}^\nu(s)\mathrm{d}s,$$

is bounded as well, which yields that $u^\nu = \mathrm{e}^{-\eta_0\nu^{1/2}t}\tilde{u}^\nu$ satisfies (5.8).

For the intermediate time regime, we fix c such that $\eta_0 c = 2$. Combining (5.2) and (5.6), we find that

$$|K_\nu(t)| \lesssim \frac{\ln(2+t)^{M+2}}{(1+t)^2}, \quad \forall t \in [0, c\nu^{-1/2}|\ln\nu|].$$

Arguing exactly as in Section 5.1, we deduce:

$$|u^\nu(t)| \lesssim \frac{\ln(2+t)^{M+2}}{(1+t)^2}\|\psi^{in}\|_{H^{2+\delta}}, \quad \forall t \in [0, c\nu^{-1/2}|\ln\nu|],$$

which in particular gives for small enough ν:

$$|u^\nu(t)| \lesssim \nu|\ln(\nu)|^{M+2}\|\psi^{in}\|_{H^{2+\delta}}, \quad \forall t \in [\nu^{-1/2}, c\nu^{-1/2}|\ln\nu|], \tag{5.9}$$

Moreover, with our choice of c, for ν small enough and $t \geq c\nu^{-1/2}|\ln\nu|$,

$$\mathrm{e}^{-\eta_0\nu^{1/2}t} \lesssim \nu^2 \ll \nu|\ln\nu|^{M+2}$$

so that (5.8) and (5.9) imply the first inequality in Theorem 3 (replacing M by $M+2$).

To obtain the control of $\psi = \psi^\nu$, we use the expression

$$\psi^\nu(t) = \mathrm{e}^{tL_1}\psi^{in} + \int_0^t \left(\mathrm{e}^{(t-s)L_1}\bar{L}_2\right)\cdot u^\nu(s)\,\mathrm{d}s.$$

By Proposition 1.6 and (5.8), we get

$$\begin{aligned}\|\psi^\nu(t)\| &\lesssim \mathrm{e}^{-\varepsilon_0\nu^{1/2}t}\|\psi_{in}\| + \int_0^t \mathrm{e}^{-\varepsilon_0\nu^{1/2}(t-s)}\mathrm{e}^{-\eta_0\nu^{1/2}s}\,\mathrm{d}s\|\psi_{in}\|_{H^{2+\delta}} \\ &\lesssim \mathrm{e}^{-\varepsilon_0\nu^{1/2}t}\|\psi_{in}\| + \mathrm{e}^{-\min(\varepsilon_0,\eta_0)\nu^{1/2}t}\int_0^t \mathrm{d}s\|\psi_{in}\|_{H^{2+\delta}},\end{aligned}$$

which implies the last bound in Theorem 3.

Remark 5.1 (Stability of pullers)**.** The spectral stability condition

$$\det(I + \mathcal{L}\tilde{K}^\nu(z)) \neq 0, \qquad \text{in} \quad \{\mathrm{Re}\, z \geq 0\}$$

was shown above to be satisfied for small ν, through perturbation of the inviscid condition. Actually, in the case of pullers, there is a more straightforward way, valid for all ν and directly for the original kernel K^ν. Indeed, one can simply notice that

$$\mathcal{L}K^\nu(z)v\cdot\overline{v} = \frac{3\Gamma}{4\pi}\int (z-L_1)^{-1}\phi\,\overline{\phi},$$

where $\phi = (p\cdot k)\mathbb{P}_{k^\perp}p\cdot v$. In terms of $F = (z-L_1)^{-1}\phi$ we thus find

$$\mathcal{L}K^\nu(z)v\cdot\overline{v} = \frac{3\Gamma}{4\pi}\langle (z-L_1)F, F\rangle = \frac{3\Gamma}{4\pi}\left(\mathrm{Re}\, z\|F\|^2 + \nu\|\nabla F\|^2\right) \geq 0 \quad \text{for} \quad \mathrm{Re}\, z \geq 0.$$

This implies in particular that $\det(I + \mathcal{L}K^\nu)$ cannot vanish in the unstable half plane.

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Appendix A. Reminders on covariant derivatives

For the computations on the sphere, it is natural to rely on covariant derivatives from Riemannian geometry. Classical references are [34, 27], we just provide here a quick reminder. We start by introducing the natural metric g on the sphere induced by the Euclidean scalar product on $\mathbb{R}^3$. In spherical coordinates (θ, φ) it is explicitly given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad \Rightarrow \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix}.$$

Given this metric and the associated Levi-Civita connection, we introduce the covariant derivative of a tensor, denoted by ∇. We remind that the covariant derivative of a vector X, resp. of a covector ω, is the $(1,1)$-tensor, resp. the $(0,2)$ tensor, defined in a coordinate basis by

$$\nabla_i X^j = \partial_i X^j + \Gamma^j_{ik} X^k,$$

resp.

$$\nabla_i \omega_j = \partial_i \omega_j - \Gamma^k_{ij} \omega_k,$$

containing the Christoffel symbols Γ defined as

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}).$$

For general tensors, the covariant derivative is defined recursively through the formula

$$\nabla(T \otimes T') = \nabla T \otimes T' + T \otimes \nabla T'$$

The basic properties of the covariant derivatives are that they commute with the metric g and that they satisfy the Leibniz rule.

Thanks to this covariant derivative, we then define the connection Laplacian of a tensor as

$$\Delta = \mathrm{tr}(\nabla^2) = g^{ij} \nabla_i \nabla_j.$$

Higher-order derivatives do not commute due to the curvature. The Ricci identity captures the defect by the Riemann curvature tensor R. In a coordinate basis, it takes for a covector ω the form

$$[\nabla_a, \nabla_b]\omega_c = -R^d{}_{cab}\omega_d.$$

By the definition of the connection Laplacian, we thus find for a scalar function f that

$$[\Delta, \nabla] f = \mathrm{Ric}\, \nabla f$$

where Ric denotes the Ricci curvature, obtained by contracting the first and third argument in the Riemann curvature tensor. On a unit-sphere the Ricci curvature just equals the metric and thus we find

$$[\Delta, \nabla] f = \nabla f$$

which yields a good sign for our estimates in Section 4. The good sign simplifies the algebra even though the estimates would equally work for a bounded curvature.

Acting on a general tensor X, we find a similar expression with a successive application of the Riemann curvature tensor. As the Riemann curvature tensor is bounded on a sphere, we just note that

$$[\Delta, \nabla]X = \mathcal{R}\nabla X$$

for a bounded tensor $\mathcal{R}$.

In the enhanced dissipation estimates, we use the commutator between the Laplacian and the tensor $(p \cdot e)X$, for a fixed vector e. It is provided by

Lemma A.1. *Let X be a $(0, N)$-tensor. Then it holds that*

$$\Delta(\nabla(p \cdot e) \otimes X) = -\nabla(p \cdot e) \otimes X - 2(p \cdot e)\nabla X + \nabla(p \cdot e) \otimes \Delta X.$$

PROOF. We compute the expression in the spherical coordinates (θ, φ) where we take e to be the north pole. Using the Leibniz rule, we find that

$$\Delta(\nabla_{i_0}(p \cdot e)X_{i_1,\dots,i_n}) = (\Delta\nabla_{i_0}(p \cdot e))X_{i_1,\dots,i_n}) + 2g^{mn}(\nabla_m\nabla_{i_0}(p \cdot e))\nabla_n X_{i_1,\dots,i_n} + \nabla_{i_0}(p \cdot e)\Delta X_{i_1,\dots,i_n}.$$

We can explicitly compute that

$$\Delta(\nabla(p \cdot e)) = \nabla\Delta(p \cdot e) + \operatorname{Ric}\nabla(p \cdot e) = \nabla\Delta\cos\theta + \nabla\cos\theta = \sin\theta\,\mathrm{d}\theta = -\nabla(p \cdot e).$$

For the mixed term, we find

$$\nabla_m\nabla_{i_0}\cos\theta = \partial_m\partial_{i_0}\cos\theta - \Gamma^k_{mi_0}\partial_k\cos\theta.$$

In the spherical coordinates, the only non-zero Christoffel symbols are

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta \quad \text{and} \quad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}.$$

This then yields that

$$g^{mn}(\nabla_m\nabla_{i_0}(p \cdot e))\nabla_n X_{i_1,\dots,i_n} = (p \cdot e)\nabla_{i_0}X_{i_1,\dots,i_n}.$$

Hence we have arrived at the claimed expression. □

Conflict of interest. The authors declare that they have no conflict of interest.

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