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Christophe Gomez. An effective fractional paraxial wave equation for wave-fronts in randomly layered media with long-range correlations. 2022. hal-03723309v1

**HAL Id: hal-03723309**

**<https://hal.science/hal-03723309v1>**

Preprint submitted on 14 Jul 2022 (v1), last revised 13 Jun 2023 (v2)

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# An effective fractional paraxial wave equation for wave-fronts in randomly layered media with long-range correlations

Christophe Gomez \*

July 14, 2022

## Abstract

This work concerns the asymptotic analysis of high-frequency wave propagation in randomly layered media with fast variations and long-range correlations. The analysis takes place in the 3D physical space and weak-coupling regime. The role played by the slow decay of the correlations on a propagating pulse is two fold. First we observe a random travel time characterized by a fractional Brownian motion that appears to have a standard deviation larger than the pulse width, which is in contrast with the standard O'Doherty-Anstey theory for random propagation media with mixing properties. Second, a deterministic pulse deformation is described as the solution of a paraxial wave equation involving a pseudo-differential operator. This operator is characterized by the autocorrelation function of the medium fluctuations. In case of fluctuations with long-range correlations this operator is close to a fractional Weyl derivative whose order, between 1 and 2, depends on the power decay of the autocorrelation function. In the frequency domain, the pseudo-differential operator exhibits a frequency-dependent power law attenuation with exponent corresponding to the order of the fractional derivative, and a frequency-dependent phase modulation, both ensuring the causality of the limiting paraxial wave equation as well as the Kramers-Kronig relations. The mathematical analysis is based on an approximation-diffusion theorem for random ordinary differential equations with long-range correlations.

**Key words.** wave propagation, paraxial approximation, lossy wave equation, fractional derivative, random media, long-range processes.

## 1 Introduction

Wave propagation in rough media is attracting more attention as they proved their ability to reproduced experimental observations. In context such as geophysics, laser beam propagation through the atmosphere, or medical imaging propagating waves can undergo attenuation at rate proportional to

$$|\omega|^\lambda \quad \lambda \in (0, 2), \quad (1)$$

for a given  $\omega$  representing the angular frequency [9, 16, 24, 26, 33, 37]. To reproduce such power law decay several models have been proposed involving fractional derivatives (see [5, 15, 23, 24, 34, 39, 40] for instance). Another approach consists in considering the propagation media as random and exhibiting fractal correlation structures or long-range correlations (see [12, 14, 36] and [24, Chapter 9]). In other words, the correlation function of the medium fluctuations decays slowly enough to not be integrable at infinity. Collects of data have confirmed the possibility to encounter this situation in practice, such as in geophysics [6, 8] or laser beam propagation through the atmosphere [11, 35]. Another aspect of wave propagation that we address in this paper concerns the paraxial approximation. This approximation consists in studying the wave propagation along a privileged axis and have been extensively studied and used in applications (see [1, 3, 7, 13, 19, 42] for instance). Under suitable assumptions on physical parameters this approximation can greatly simplify the description of propagation phenomena as well as data analysis through numerical simulations or inverse problems.

The aim of this paper is to derive from first principles a paraxial wave equation model exhibiting a frequency-dependent attenuation with a power law (1), for  $\lambda \in (1, 2]$ , in the context of random propagation media with long-range correlations. Our analysis considers both aspects, paraxial approximation and random medium fluctuations with long-range correlations, under the same limit. The case of mixing medium fluctuations is also treated for comparison. In homogeneous propagation media the derivation of the paraxial

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approximation is relatively straightforward, but it becomes much more complex when waves are propagating in heterogeneous media. In applications wave frequencies are generally sufficiently high so that the interaction with the fine structure of the medium fluctuations cannot be ignored. Rigorous derivations of the paraxial wave equation in random media with mixing properties can be found in [1, 13] for instance, and in case of long-range correlations in [19]. However, in this later work the scaling regime is not the same as the one proposed in this paper. As waves propagate over large distances it is natural to expect some *universal behavior* to describe the statistical property of the multiple scattering effects. In case of long-range correlations this refers to the non-central limit theorem [41], as opposed to the standard central limit theorem for mixing propagation media. The non-central theorem scaling has been used to study wave propagation in 1D propagation media with long-range correlations through the use of the rough-path theory [31, 32] under general assumptions on the random fluctuations, and random waveguides with a moment technique [20]. In this context, the effects on the propagating waves can be described as a random time-shift driven by a fractional Brownian motion leading to anomalous diffusion phenomena but no power law attenuation in frequency. Here, we rather consider a central limit theorem scaling in the context of long-range correlations leading to mathematical challenges. This scaling regime has been considered for 1D propagation media [12, 14], but their approach, allowing to exhibit a power law attenuation of the form (1) as well as a fractional derivative in the effective wave equation, does not seem to apply in a 3D setting. Despite our assumptions on the random fluctuations are more restrictive than in [12, 14, 31, 32], the approach we proposed in this paper can also be applied for more general 3D settings with non-layered fluctuations, but with additional technical difficulties, and will be the aim of future works. As for random media with mixing fluctuations, more general models for the fluctuations with long-range correlations should not change the overall results as the asymptotic equations and scattering coefficients depend only on the correlation functions of the fluctuation models, and not their precise definitions.

As already pointed out in the context of the random Schrödinger equation with long-range correlations [18], the central limit theorem scaling can be seen as propagating the non-central limit scaling over longer propagation distances. However, since this later scaling already produces an effective phase modulation [2] driven by a fractional Brownian motion, the waves start to oscillate very fast over larger propagation distances, and then have to be treated properly to still exhibit effective nontrivial effects. For the random Schrödinger equation the Wigner transform is used to study the energy propagation by looking at correlations of the wave function, which naturally cancels out the rapid phases and provides an effective description of the energy propagation through a radiative transfer equation [17]. For classical wave propagation problems an equivalent approach consists in looking at the wave-front along a proper random characteristic. As a result, the rapid phases still have some effects by averaging the stochasticity in order to obtain a deterministic spreading for the wave-front. It is worthnoticing that under the central limit theorem scaling, but with long-range correlations, the random travel time has a standard deviation very large compared to the pulse width [12, 14]. This is in contrast with the standard O’Doherty-Anstey (ODA) theory with mixing medium fluctuations for which the standard deviation of the random travel time and the pulse width are of the same order (see [10, Chapter 8] and references therein). This *unstable* behavior of the random travel time may have a dramatic effect for applications based on travel time estimations, and a more deeper understanding on the effects on the propagating wave is required. In our context, the pulse deformation can be approximately characterized by a (deterministic) paraxial wave equation of the form

$$\partial_{tz}^2 \psi + \frac{c_0}{2} \Delta_{\mathbf{x}} \psi - r_0 D_t^{2+\gamma} \psi = 0 \quad \gamma \in (0, 1),$$

where the  $z$ -variable corresponds to the main propagation axis, the  $\mathbf{x}$ -variable to the transverse section,  $t$  to the time variable (see Figure 1),  $c_0$  to the background wave speed, and  $r_0 > 0$  is a constant. Also,  $D_t^{2+\gamma}$  stands for the Weyl fractional derivative with respect to time and order between 2 and 3, depending on the power decay rate  $\gamma \in (0, 1)$  of the correlation function of the medium fluctuations (see (19)). This fractional derivative ensures the causality of the paraxial wave equation in the sense that for a given time  $t$  the equation use only the prior knowledge,  $\psi(\tau)$  for  $\tau \leq t$ , of the wave  $\psi$ . In the time Fourier domain this equation can be recast as a Schrödinger equation of the form

$$i \frac{\omega}{c_0} \partial_z \check{\psi} + \frac{1}{2} \Delta_{\mathbf{x}} \check{\psi} + \tilde{r}_0 \omega |\omega|^{1+\gamma} \check{\psi} = 0 \quad \text{with} \quad \check{\psi}(\omega, \mathbf{x}, z) = \int \psi(t, \mathbf{x}, z) e^{i\omega t} dt,$$

where  $\tilde{r}_0$  is a constant with positive imaginary part. For  $\gamma = 1$ , the above fractional derivative turns to a classical third order derivative. This equation provides a frequency-dependent attenuation power law of the form (1) with  $\lambda = 1 + \gamma \in (1, 2]$ .

The analysis developed in this paper relies on an approximation-diffusion theorem, which is usually used for mixing fluctuations. Despite some restrictions on the noise model, the proof of this theorem requires a very careful attention due to the nonintegrability of the correlation function. Also, our restriction does not allow us to capture frequency power law with  $\lambda \in (0, 1)$ , and further investigations would be required to capture these cases.

The paper is organized as follows. In Section 2 we describe the physical model under consideration and introduce the main assumptions we need to derive the limiting fractional paraxial wave equation, which

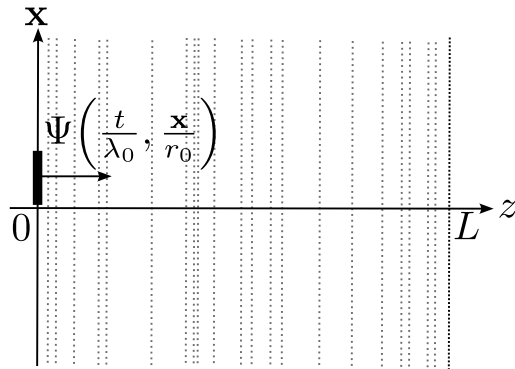


Figure 1: Illustration of the physical setting.

is presented and discussed in Section 3. Section 4 presents the influence of the slow decay of the random medium correlations on the wave-front travel time. Section 5 reformulate the propagation problem in terms of a proper random ordinary differential equation, and Section 6 describes the asymptotic behavior of this equation. Sections 7, 8, and 9 are dedicated to the proof of the main results.

## 2 The wave model

In this section we describe the physical model under consideration, the random medium fluctuations, and introduce some assumptions that are necessary to derive the main results stated in Section 3.

### 2.1 Random wave equation

In this paper a three-dimensional linear wave equation model is considered

$$\Delta p - \frac{1}{c^2(z)} \partial_t^2 p = \mathbf{F}(t, \mathbf{x}, z) \quad (t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}.$$

equipped with null initial conditions

$$p(t = 0, \mathbf{x}, z) = \partial_t p(t = 0, \mathbf{x}, z) = 0, \quad (\mathbf{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \quad (2)$$

meaning that the system is initially at rest. The  $z$ -coordinate represents the main propagation axis, while the  $\mathbf{x}$ -coordinate represents the transverse section (cf. Figure 1 for an illustration of the setting). The forcing term  $\mathbf{F}(t, \mathbf{x}, z)$  is assumed to be of the form

$$\mathbf{F}(t, \mathbf{x}, z) := \Psi\left(\frac{t}{\lambda_0}, \frac{\mathbf{x}}{r_0}\right) \delta(z),$$

and represents a source emitting a pulse located in the plan  $z = 0$ , with central wavelength  $\lambda_0$  and beam radius  $r_0$ . To be consistent with (2) the source profile  $\Psi$  needs to be supported in time in  $(0, \infty)$ . For mathematical convenience we also assume that the frequency  $\omega = 0$  is not supported by the Fourier transform in time of  $\Psi$ . This assumption will be made more precise later in the paper. Finally, the wave-speed profile is assumed to be of the form

$$\frac{1}{c^2(z)} := \frac{1}{c_0^2} \left(1 + \nu \left(\frac{z}{l_c}\right) \mathbf{1}_{]0, L[}(z)\right), \quad (3)$$

where  $c_0$  stands for the background wave speed,  $\nu$  represents the fluctuations of the wave speed, and  $l_c$  is the correlation length of these fluctuations. In other words, the correlation length can be seen as the typical scale of variation for the random fluctuations. In practice, the wave-speed variations are almost impossible to determine exactly, and it is therefore reasonable to consider these fluctuations as random. The indicator function indicates that the random perturbations take place only in the slab  $z \in ]0, L[$  with respect to the  $z$ -variable. The terminology "*layered medium*" comes from the fact that the wave-speed variations appear in only one direction, here the  $z$ -direction.

Our goal is to give a description of the wave at  $z = L$  through a paraxial wave equation. This wave is referred to as the *transmitted wave*, in contrast with the one observed at  $z = 0$ , which is referred to as the *reflected wave*.

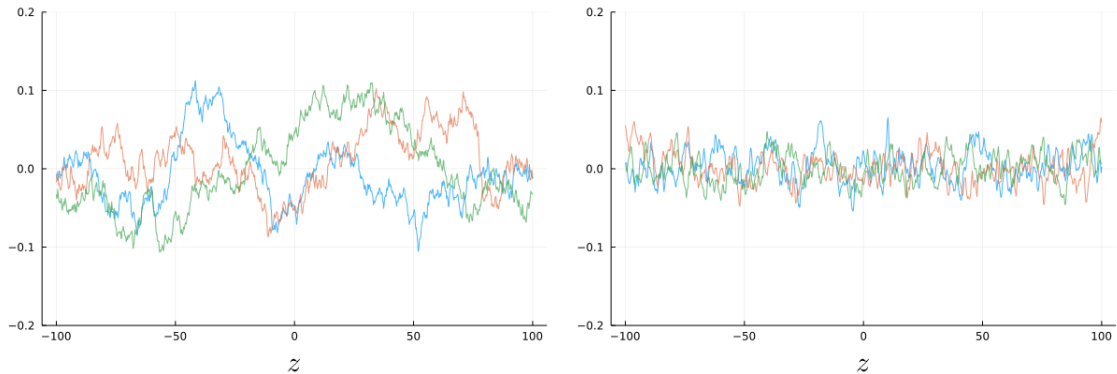


Figure 2: Illustration of three realizations of the random process defined by (4) with long-range correlations for the left-picture, and short-range correlations for the right-picture. Here,  $\mu = 1$ ,  $\alpha = 1/4$ ,  $a(p) = \mathbf{1}_{(-10,10)}(p)$ ,  $\beta = 1/2$  (that is  $\gamma = 1/2$ ) for the left-picture illustrating the long-range correlations, and  $\beta = 1/6$  (that is  $\gamma = 3/2$ ) for the right-picture illustrating short-range correlations.

## 2.2 The random fluctuations

In this paper the random fluctuations are assumed to be given by

$$\nu(z) := \Theta(\sigma V(z)),$$

where  $\Theta$  is an odd smooth bounded function with

$$\theta'_0 := \Theta'(0) \neq 0 \quad \text{and} \quad \sup |\Theta| < 1,$$

and  $V$  is a mean-zero stationary Gaussian random process. Here, the function  $\Theta$  plays no significant role. Gaussian random processes being not bounded, this function just guarantees, for a modelization purpose, that  $c^2(z)$  is actually always positive (see (3)). Also, for  $\sigma$  small enough, the Taylor expansion

$$\Theta(\sigma V(z)) = \sigma V(z)\Theta'(0) + \mathcal{O}(\sigma^3),$$

since  $\Theta''(0) = 0$ , indicates that the medium fluctuations are driven by  $V$ .

In this paper, to derive the fractional behavior of the paraxial wave equation, or equivalently the frequency-dependent power law attenuation, we consider random fluctuations with long-range correlations, or in other words with slowly decaying correlations. Behind this terminology we assume that the two-point correlation function is not integrable:

$$\int_0^\infty |\mathbb{E}[V(z+s)V(z)]| ds = \infty.$$

In Figure 2 we illustrate the difference of statistical behavior between long-range correlations and short-range correlations (that is with rapidly decaying correlations), the later having an integrable two-point correlation function. From these pictures, one can observe that slowly decaying correlations produce longer excursions of the random trajectories due to the persistence of the correlations than for rapidly decaying correlations. In this later case the trajectories cannot produce correlation patterns and look almost like the ones of white-noises. These simulations are random trajectories of a stochastic process we now introduce precisely.

The method we propose to analyze the transmitted and reflected waves is based on the perturbed-test-function method [29, 21], and to use this technique under the context of long-range correlations we need some assumptions on  $V$ . The following construction has already been used to study the impact of random fluctuations with long-range-correlations on the Schrödinger equation [2, 17, 18] and nonlinear oscillators [21]. More precisely, we consider  $V$  as being a linear superposition of Ornstein-Uhlenbeck type processes by setting

$$V(z) := \int_{-\infty}^z \int_S e^{-\mu|p|^{2\beta}(z-u)} B(du, dp) = \int_S V(z, dp), \quad (4)$$

with

$$V(z, dp) := \int_{-\infty}^z e^{-\mu|p|^{2\beta}(z-u)} B(du, dp), \quad (5)$$

where  $\mu, \beta > 0$ , and  $S = (-r_S, r_S) \subset \mathbb{R}$  is a bounded interval containing 0 for  $r_S > 0$ . Also,  $B$  is a Gaussian random measure with covariance function

$$\mathbb{E}[B(du, dp)B(dv, dq)] := 2\mu r(p)|p|^{2\beta} \delta(u-v) \delta(p-q) du dv dp dq,$$

with

$$r(p) := \frac{a(p)}{|p|^{2\alpha}} \quad \alpha < 1/2 \quad \text{and} \quad p \in S \setminus \{0\},$$

where  $a$  is a nonnegative smooth bounded function with  $a(0) > 0$ . From this definition, we obtain the autocorrelation function for  $V$

$$R(z) := \mathbb{E}[V(z_0 + z)V(z_0)] = \int_S e^{-\mu|p|^{2\beta}|z|} r(p) dp. \quad (6)$$

Note that we actually need  $\alpha < 1/2$  for the process  $V$  to be well-defined. After some algebra, we can exhibit the long-range behavior of the autocorrelation function:

$$R(z) \underset{|z| \rightarrow \infty}{\sim} \frac{R_0}{|z|^\gamma}, \quad (7)$$

with

$$R_0 := a(0) \int_{-\infty}^{\infty} \frac{e^{-\mu|p|^{2\beta}}}{|p|^{2\alpha}} dp \quad \text{and} \quad \gamma := \frac{1 - 2\alpha}{2\beta} > 0. \quad (8)$$

We refer to Figure 2 for illustrations of this process.

Using the fact that an odd bounded function is of Hermite-rank one, we have (see [31] for more details)

$$\mathbb{E}[\nu(z_0 + z)\nu(z_0)] \underset{|z| \rightarrow \infty}{\sim} \frac{R_0 \Theta_1}{|z|^\gamma}$$

with

$$\Theta_1 := \left( \frac{1}{\sqrt{2\pi}} \int \Theta(\sigma u) u e^{-u^2/2} du \right)^2.$$

As a result, depending on the value of  $\gamma$  the two-point correlation function for  $\nu$  can be integrable or not, that is the medium fluctuations  $\nu$  is having short-range correlations (for  $\gamma > 1$ ) or long-range correlations (for  $\gamma \in (0, 1)$ ) respectively.

We chose a specific form for  $V$  in this paper for a mathematical tractability purpose. More general fluctuation models have been considered in [31, 32], but the method they proposed, based on the rough-path theory, does not seem to apply in the scaling regime described below. Despite this method can be used to analyze the competition between randomness and periodicity in random differential equation emanating from wave propagation problems [30], in our scaling the structure of the periodic component turns out to involve the randomness itself in a way that cannot be controlled easily by this strategy. Also, in [12, 14], the authors use more general fluctuation models but in a scaling regime similar to the one presented here. Nevertheless, the method they use to analyze the problem cannot be applied for 3D propagation media, it is designed for 1D propagation media, that is without the transverse variable  $\mathbf{x}$ . Another remark is that our choice for  $V$  provides mixing conditions for  $\gamma > 1$  corresponding to short-range correlations. Unfortunately, with our choice we cannot exhibit finner results for short-range correlations as the one obtained in [14], with  $\lambda \in (0, 1)$  in (1), that does not necessarily restrict to mixing conditions.

## 2.3 The scaling regime

The asymptotic analysis we provide here is based on a separation of the characteristic scales of the problem. The scales of interest are the propagation distance  $L$  into the random medium, the central wavelength  $\lambda_0$ , the correlation length  $l_c$  of the medium fluctuations, the beam radius  $r_0$ , and the fluctuation strength  $\sigma$ . Our scaling regime is based on the four following assumptions. First, we consider a high-frequency regime, that is the central wavelength is small compared to the propagation distance:

$$\varepsilon := \frac{\lambda_0}{L} \ll 1.$$

Second, we assume that the correlation length is of order the central wavelength

$$l_c \sim \lambda_0,$$

providing a full interaction between the random fluctuations and the propagating wave. Third, we assume that the beam width satisfies

$$r_0^2 \sim \lambda_0,$$

so that the Rayleigh length is of order one, which is crucial to obtain the paraxial approximation. In fact, the Rayleigh length is defined as the distance from the beam waist to the place where its cross-section is doubled by diffraction, and in homogeneous media it is of order  $r_0^2/\lambda_0$ . Finally, the strength of the fluctuations is assumed to be small, so that we place ourselves in a weak coupling regime:

$$\sigma \ll 1.$$

To fix the ideas, we set

$$L \sim 1, \quad \lambda_0 = l_c = \varepsilon, \quad \text{and} \quad r_0 = \sigma = \sqrt{\varepsilon}.$$

The choice of  $\sigma$  allows us to derive a nontrivial limit, for both short-range and long-range correlations, to describe the effects of the propagation medium fluctuations on the propagating wave.

Our choice on the parameter scalings leads to the system

$$\Delta p_\varepsilon - \frac{1}{c_\varepsilon^2(z)} \partial_{tt}^2 p_\varepsilon = \Psi\left(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\sqrt{\varepsilon}}\right) \delta(z) \quad (t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \quad (9)$$

with

$$\frac{1}{c_\varepsilon^2(z)} = \frac{1}{c_0^2} \left(1 + \nu_\varepsilon\left(\frac{z}{\varepsilon}\right) \mathbf{1}_{]0, L[}(z)\right) \quad \text{and} \quad \nu_\varepsilon(z) = \Theta(\sqrt{\varepsilon} V(z)).$$

### 3 The main results

To state our main result we follow the strategy of [12, 14] and introduce the random travel time

$$T_\varepsilon^0(L) := \frac{L}{c_0} + \frac{1}{2c_0} \int_0^L \nu_\varepsilon(z/\varepsilon) dz, \quad (10)$$

corresponding to the expected travel time  $L/c_0$  with some random correction, and the wave-front

$$p_{tr, \varepsilon}^L(s, \mathbf{y}) := p_\varepsilon(T_\varepsilon^0(L) + \varepsilon s, \sqrt{\varepsilon} \mathbf{y}, L) \quad (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^2. \quad (11)$$

This wave-front corresponds to the wave observed at the end of the random section  $z = L$  on a time window corresponding to the pulse width  $\varepsilon$  and centered at the random travel time  $T_\varepsilon^0(L)$ .

Before stating our first result, which is proved in Section 7, let us introduce some notations. Below  $\mathcal{S}'_{s, \mathbf{y}}(\mathbb{R} \times \mathbb{R}^2)$  stands for the set of tempered distribution w.r.t. the variables  $s$  and  $\mathbf{y}$ , and  $\mathcal{C}_z^1$  stands for  $\mathcal{C}^1$ -function w.r.t. the  $z$ -variable. We also use the following Fourier transform convention

$$\hat{f}(\omega, \kappa) := \iint f(s, \mathbf{y}) e^{i\omega(s - \kappa \cdot \mathbf{y})} ds d\mathbf{y},$$

and

$$f(s, \mathbf{y}) := \frac{1}{(2\pi)^3} \iint \hat{f}(\omega, \kappa) e^{-i\omega(s - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\mathbf{y},$$

which is convenient to study space-time problems.

**Theorem 3.1** *The family  $(p_{tr, \varepsilon}^L)_\varepsilon$  converges in probability in  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$  to*

$$p_{tr}^L(s, \mathbf{y}) = \frac{1}{2} \mathcal{K}(\cdot, \cdot, L) * \Psi(s, \mathbf{y}), \quad (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^2, \quad (12)$$

where, in the Fourier domain,

$$\hat{\mathcal{K}}(\omega, \kappa, z) := e^{-\theta_0^2 \omega^2 (\Gamma_c(\omega) + i\Gamma_s(\omega))z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2}, \quad (13)$$

with

$$\Gamma_c(\omega) := 2 \int_0^\infty R(s) \cos\left(\frac{2\omega s}{c_0}\right) ds \quad \text{and} \quad \Gamma_s(\omega) := 2 \int_0^\infty R(s) \sin\left(\frac{2\omega s}{c_0}\right) ds. \quad (14)$$

Here,  $R$  is the correlation function of the medium fluctuations (6). The convolution kernel  $\mathcal{K}$  is the unique solution on  $\mathcal{C}_z^0([0, \infty), \mathcal{S}'_{s, \mathbf{y}}(\mathbb{R} \times \mathbb{R}^2)) \cap \mathcal{C}_z^1((0, \infty), \mathcal{S}'_{s, \mathbf{y}}(\mathbb{R} \times \mathbb{R}^2))$  to the paraxial wave equation

$$\partial_{sz}^2 \mathcal{K} + \frac{c_0}{2} \Delta_{\mathbf{y}} \mathcal{K} - \mathcal{I}(\mathcal{K}) = 0, \quad (15)$$

with  $\mathcal{K}(s, \mathbf{y}, z = 0) = \delta(s)\delta(\mathbf{y})$ , and

$$\mathcal{I}(\psi)(s) := \frac{\theta_0^2}{8c_0^2} \int_{-\infty}^s R\left(\frac{c_0(s-\tau)}{2}\right) \partial_{sss}^3 \psi(\tau) d\tau \quad s \in \mathbb{R}. \quad (16)$$

The asymptotic transmitted wave-front  $p_{tr}^L$  at the end of the random section ( $z = L$ ) can be written in term of a convolution where  $\mathcal{K}$  represents the pulse deformation. From the explicit formulation in the Fourier domain of the kernel  $\mathcal{K}$ , the pulse shape, after traveling through the random slab  $z \in (0, L)$ , is affected in a way which is consistent with the standard ODA theory (see [10, Chapter 8] and references therein for more details) even if we are not considering mixing fluctuations. Typically, according to this theory, the propagating pulse exhibits a deterministic spreading characterized by a frequency-dependent attenuation and phase modulation. In our context we observe these effects that are respectively produced by  $\omega^2 \Gamma_c(\omega)$

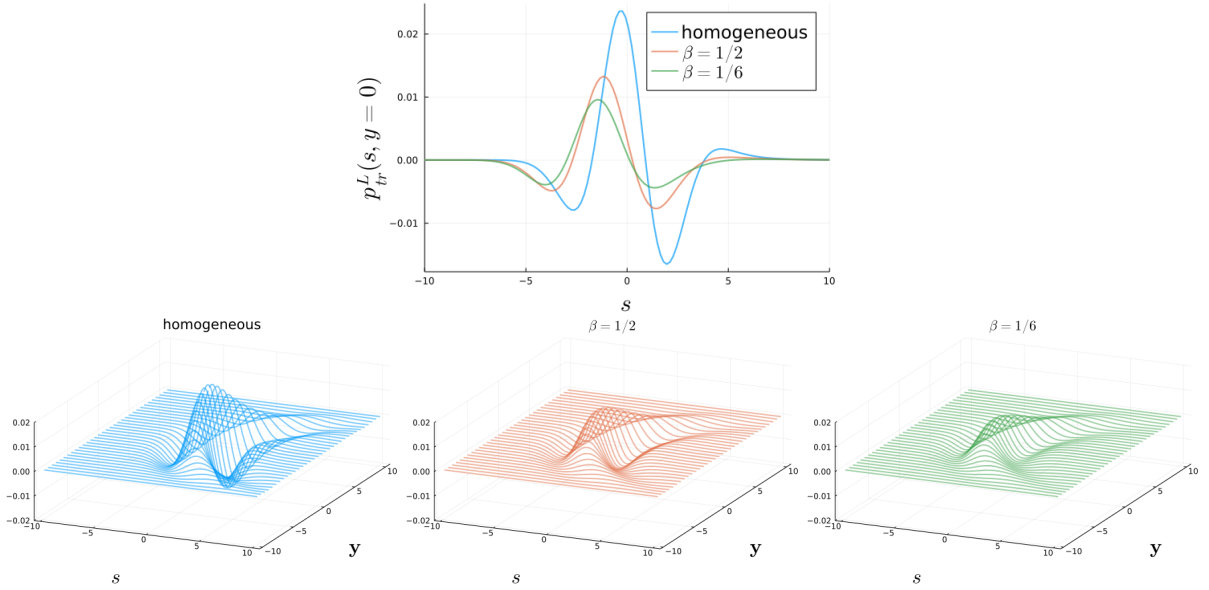


Figure 3: Illustrations of the profile  $p_{tr}^L$  in the homogeneous case (blue lines) and for two values of  $\beta$  ( $\beta = 1/2$  for the orange curves and  $\beta = 1/6$  for the green curves). We take  $\alpha = 1/4$ ,  $\mu = 2$ ,  $a(p) = \mathbf{1}_{(-10,10)}(p)$ ,  $L = 5$ ,  $c_0 = \theta'_0 = 1$  and a source profile given by  $\hat{\Psi}(\omega, \kappa) = 2\omega^2 e^{-\omega^2(1+\kappa^2)/2}$  centered at  $s = 0$  in the time domain for simplicity.

(which is positive thanks to the Bochner Theorem) and  $\omega^2 \Gamma_s(\omega)$ , and are similar to the ones obtained in [12, 14]. Note that these two coefficients are well defined even for slowly decaying correlations thanks to the oscillatory functions. We refer to Figure 3 for illustrations regarding the influence of the kernel  $\mathcal{K}$  on the propagating pulse spreading in the case of a 1D variable  $\mathbf{y}$ .

The ODA theory for mixing fluctuations also provides a random time shift, driven by a standard Brownian motion, meaning that the pulse has a random arrival time at  $z = L$  which is of order the pulse width. In the context of long-range correlations the situation is more delicate. The aforementioned random time shift is already compensated in Theorem 3.1 by considering the random travel time  $T_\varepsilon^0$  in the definition of the transmitted wave-front (11). This allows to remove pathological behavior when studying the wave-front in the case of long-range correlations. As we will see in Section 4, in the case of rapidly decaying correlations,  $T_\varepsilon^0$  can be approximated by a Brownian motion with mean  $L/c_0$  and a standard deviation of order the pulse width, which then respects the standard ODA theory. For the case of long-range correlations, terms that was leading to an effective random time shift in the context of rapidly decaying correlations would now blow up. In fact, as we will see in Section 4,  $T_\varepsilon^0$  can be approximated by a fractional Brownian motion, with Hurst index ranging from 1/2 to 1, and a standard deviation very large w.r.t. the pulse width. This is the reason we compensate this term in (11) and then avoid this blow up in the derivation of the pulse spreading. These facts will be made more precise in Section 4, in which we also remark that the travel time  $T_\varepsilon^0$  does not correspond to the travel time along random characteristics to reach depth  $z = L$

$$\int_0^L \frac{dz}{c_\varepsilon(z)}.$$

The difference between these two random travel times is analyzed in Proposition 4.2.

Regarding the backscattered signal at  $z = 0$ , it can be shown that

$$p_{bk,\varepsilon}(s, \mathbf{y}) := p_\varepsilon(\varepsilon s, \sqrt{\varepsilon} \mathbf{y}, 0)$$

converges in probability to 0 in  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$  as  $\varepsilon \rightarrow 0$ . It is a consequence of Theorem 6.1 and is consistent with [10, Chapter 9] in which the authors show that the backscatter signal is made of a *small* incoherent signal that can be described through a random field. We will not go in this direction here since it requires involved mathematical developments that are beyond the scope of this paper.

In the time domain the pulse deformation is described by the paraxial wave equation (15) involving an integral term similar to the one obtained in [12, 14]. It is interesting to note that this integro-differential operator preserves the causality since at a fixed time  $s$  it involves only the knowledge of  $\mathcal{K}(\tau, \cdot, \cdot)$  for  $\tau \leq s$ . Also, in the Fourier domain (15) can be written as the following Schrödinger equation

$$i \frac{\omega}{c_0} \partial_z \check{\mathcal{K}}(\omega, \mathbf{y}, z) + \frac{1}{2} \Delta_{\mathbf{y}} \check{\mathcal{K}}(\omega, \mathbf{y}, z) + i \frac{\theta_0'^2 \omega^3}{8c_0^3} (\Gamma_c(\omega) + i\Gamma_s(\omega)) \check{\mathcal{K}}(\omega, \mathbf{y}) = 0, \quad (17)$$



where

$$\check{\mathcal{K}}(\omega, \mathbf{y}, z) := \int \mathcal{K}(s, \mathbf{y}, z) e^{i\omega s} ds. \quad (18)$$

Moreover, from (13) one can see that (17) satisfies the Kramers-Kronig relations [27, 28] as stated more precisely in the following proposition and proved in Appendix A. Both the causality and the Kramers-Kronig relations contribute to the physical relevance of our paraxial wave equation.

**Proposition 3.1** *The effective frequency-dependent attenuation  $\omega^2 \Gamma_c(\omega)$  and dispersion  $\omega^2 \Gamma_s(\omega)$  are analytic function w.r.t.  $\omega$  on the complex upper half-plane, and satisfy the following Kramers-Kronig relations in  $\mathcal{S}'(\mathbb{R})$ ,*

$$\mathcal{H}(\omega^2 \Gamma_c(\omega'))(\omega) = \omega^2 \Gamma_s(\omega) \quad \text{and} \quad \mathcal{H}(\omega^2 \Gamma_s(\omega'))(\omega) = -\omega^2 \Gamma_c(\omega),$$

where  $\mathcal{H}$  stands for the Hilbert transform, and  $\mathcal{S}'(\mathbb{R})$  is the set of tempered distributions on  $\mathbb{R}$ .

In case of long-range correlations  $\mathcal{I}$  can be approximated by a Weyl fractional derivative whose order depends on the decay rate  $\gamma$  of the correlation function  $R$  at infinity (see (7)). Before stating the result, let us briefly introduce the notion of Weyl derivative, which is given for  $\gamma \in (0, 1)$  by

$$D^\gamma f(s) := \frac{\gamma}{\Gamma(1-\gamma)} \int_{-\infty}^s \frac{f(s) - f(\tau)}{(s-\tau)^{1+\gamma}} d\tau \quad s \in \mathbb{R},$$

whenever this quantity is well-defined, and  $\Gamma$  stands for the Gamma function. For instance,  $f$  can be a bounded  $\gamma'$ -Hölder function with  $\gamma < \gamma'$ . However, for  $\mathcal{C}^1$ -functions with fast enough decay at  $-\infty$ , the Weyl derivative can be rewritten as

$$D^\gamma f(s) = \frac{1}{\Gamma(1-\lambda)} \int_{-\infty}^s \frac{f'(\tau)}{(s-\tau)^\gamma} d\tau.$$

To define higher order derivatives, one can just set

$$D^{j+\gamma} f(s) := D^\gamma f^{(j)}(s) = \frac{1}{\Gamma(1-\gamma)} \int_{-\infty}^s \frac{f^{(j+1)}(\tau)}{(s-\tau)^\gamma} d\tau \quad j \in \mathbb{N}, \quad (19)$$

assuming  $f$  smooth enough with enough decay at  $-\infty$  of its derivatives  $f^{(j)}$ . These later requirements are true for the kernel  $\mathcal{K}$  as soon as  $z > 0$  thanks to the damping term  $\omega^2 \Gamma_c(\omega)$  in (13). Therefore, in view of (7) and (16), one can think that the integro-differential operator  $\mathcal{I}$  in (15) can be approximated as follows

$$\mathcal{I}(\mathcal{K}) \propto D_s^{2+\gamma} \mathcal{K},$$

where we next emphasize that the fractional derivative  $D_s^{2+\gamma}$  acts on the  $s$ -variable with the notation  $D_s^{2+\gamma}$ . To derive properly this observation we rescale the correlation function as follows. We replace the correlation function  $R$  with the following scaled version,

$$\sigma(l_0) R(z/l_0), \quad (20)$$

where  $l_0$  will be sent to 0, and with

$$\sigma(l_0) := \begin{cases} \frac{1}{l_0^\gamma} & \text{if } \gamma \in (0, 1), \\ \frac{1}{l_0 |\ln(l_0)|} & \text{if } \gamma = 1, \\ \frac{1}{l_0} & \text{if } \gamma > 1. \end{cases} \quad (21)$$

In other words, we assume that the correlation length  $l_0$  is small compared to the pulse duration. Under this scaling, the attenuation and dispersion coefficients now read

$$\Gamma_c(\omega, l_0) := 2\sigma(l_0) \int_0^\infty \mathbb{E}[V(0)V(s/l_0)] \cos(2\omega s/c_0) ds$$

and

$$\Gamma_s(\omega, l_0) := 2\sigma(l_0) \int_0^\infty \mathbb{E}[V(0)V(s/l_0)] \sin(2\omega s/c_0) ds.$$

We can define accordingly, following (12) and (13), the transmitted wave-front  $p_{tr, l_0}^L$  for which we have the following result proved in Section 8.

**Theorem 3.2** *The family  $(p_{tr,l_0}^L)_{l_0}$  converges on  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$  as  $l_0 \rightarrow 0$  to*

$$p_{tr,0}^L(s, \mathbf{y}) := \frac{1}{2} \mathcal{K}_0(\cdot, \cdot, L) * \Psi(s, \mathbf{y}) \quad (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^2,$$

where  $\mathcal{K}_0$  is defined in the Fourier domain by

$$\hat{\mathcal{K}}_0(\omega, \kappa, z) := \begin{cases} e^{-\theta_0'^2 \omega^2 \Gamma_0 z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2} & \text{if } \gamma \geq 1, \\ e^{-\theta_0'^2 R_0 |\omega|^{1+\gamma} (\Gamma_{c,0}(\omega) + i\Gamma_{s,0}(\omega)) z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2} & \text{if } \gamma \in (0, 1), \end{cases} \quad (22)$$

with

$$\Gamma_{c,0}(\omega) = \Gamma(1 - \gamma) \cos\left(\frac{(1 - \gamma)\pi}{2}\right) \left(\frac{2}{c_0}\right)^{\gamma-1}, \quad \Gamma_{s,0}(\omega) = \Gamma(1 - \gamma) \sin\left(\frac{(1 - \gamma)\pi}{2}\right) \left(\frac{2}{c_0}\right)^{\gamma-1} \text{sign}(\omega),$$

and

$$\Gamma_0 := \begin{cases} \frac{2a(0)}{\mu\beta} & \text{if } \gamma = 1, \\ \Gamma_c(0) & \text{if } \gamma > 1. \end{cases} \quad (23)$$

Also,  $\mathcal{K}_0$  is the unique solution on  $\mathcal{C}^0([0, \infty), \mathcal{S}'_{s,\mathbf{y}}(\mathbb{R} \times \mathbb{R}^2)) \cap \mathcal{C}^1((0, \infty), \mathcal{S}'_{s,\mathbf{y}}(\mathbb{R} \times \mathbb{R}^2))$  to the paraxial wave equation

$$\partial_{sz}^2 \mathcal{K}_0 + \frac{c_0}{2} \Delta_{\mathbf{y}} \mathcal{K}_0 - \mathcal{I}_0(\mathcal{K}_0) = 0,$$

with  $\mathcal{K}_0(s, \mathbf{y}, z = 0) = \delta(s)\delta(\mathbf{y})$  and

$$\mathcal{I}_0(\psi) := \begin{cases} \frac{\theta_0'^2 R_0}{8c_0^2} \partial_{sss}^3 \psi & \text{if } \gamma \geq 1, \\ \frac{\theta_0'^2 R_0 \Gamma(1 - \gamma)}{2^{3-\gamma} c_0^{1+\gamma}} D_s^{2+\gamma} \psi & \text{if } \gamma \in (0, 1). \end{cases}$$

In this result, we can easily observe the difference between the case  $\gamma \geq 1$  and  $\gamma \in (0, 1)$ . The first remark is that in the former case  $\mathcal{I}_0$  is a classical third order differential operator, while for  $\gamma \in (0, 1)$  we have a fractional derivative of order  $2 + \gamma \in (2, 3)$ . Also, even if the case  $\gamma = 1$  corresponds to slowly decaying correlations, the kernel  $\mathcal{K}_0$  behaves as if  $\gamma > 1$ , and somehow plays the role of a continuity point w.r.t. the order of derivation in  $\mathcal{I}_0$ . Moreover, as we will see in the next section, the random travel time (10) has a very large standard deviation w.r.t. the pulse width for  $\gamma = 1$ . Therefore, the case  $\gamma = 1$  do have the behavior of long-range correlations.

Another remark is that for  $\gamma \geq 1$  there is no effective dispersion anymore in the limit  $l_0 \rightarrow 0$ , it remains only an effective frequency-dependent attenuation in  $\omega^2$ . Nevertheless, the effective dispersion is still present for  $\gamma \in (0, 1)$ , and as  $\gamma \nearrow 1$  this dispersion remains of order 1 while the attenuation becomes strong. Finally, for long-range correlations, one can observe in (22) the frequency-dependent attenuation given by the power law

$$|\omega|^{1+\gamma} \quad \gamma \in (0, 1],$$

with exponent depending on the decay rate at infinity of the correlation of the medium fluctuations (7).

## 4 Travel time analysis

In this section we discuss the asymptotic behavior of the random travel time (10) and its consequences. This analysis has already been carried out in [12, 14] under more general fluctuation models, but we remind here the main lines under our setting to provide a complete picture regarding the impact of long-range correlations on the propagating pulse.

As already noticed the travel time  $T_\varepsilon^0(L)$  for the stable wave-front to reach the plan  $z = L$  is random. Its precise behavior can be described through the following result which is proved in Appendix B.

**Proposition 4.1** *Let us defined*

$$W_\varepsilon(L) := \frac{1}{\sigma_\varepsilon} \left( T_\varepsilon^0(L) - \frac{z}{c_0} \right),$$

where

$$\sigma_\varepsilon := \begin{cases} \varepsilon^{(1+\gamma)/2} & \text{if } \gamma \in (0, 1), \\ \varepsilon |\ln(\sqrt{\varepsilon})|^{1/2} & \text{if } \gamma = 1, \\ \varepsilon & \text{if } \gamma > 1. \end{cases}$$

The family  $(W_\varepsilon(L))_\varepsilon$  converges in distribution to a limit  $W_0(L)$ , where:

- for  $\gamma \in (0, 1)$ ,  $W_0$  is a fractional Brownian motion with Hurst index

$$H = 1 - \frac{\gamma}{2} \in (1/2, 1),$$

and

$$\mathbb{E}[W_0(L)^2] = L^{2H} \frac{\theta_0'^2 R_0}{H(2H-1)}$$

with  $R_0$  defined by (8);

- for  $\gamma \geq 1$ ,  $W_0$  is a Brownian motion with

$$\mathbb{E}[W_0(L)^2] = L \theta_0'^2 \Gamma_0,$$

with  $\Gamma_0$  defined by (23).

In other words, the random travel time for the wave-front can be formally expressed as follows for  $\gamma > 1$

$$T_\varepsilon^0(L) = \frac{L}{c_0} + \varepsilon W_0(L) + o(\varepsilon).$$

We can observe an effective random time shift, w.r.t. the expected travel time  $L/c_0$ , given by a Brownian motion of order the pulse width  $\varepsilon$ , which is consistent with the standard ODA theory. For  $\gamma = 1$ , we now have

$$T_\varepsilon^0(L) = \frac{L}{c_0} + \varepsilon |\ln(\sqrt{\varepsilon})|^{1/2} W_0(L) + o(\varepsilon),$$

with still a random time shift given by a Brownian motion, but with a standard deviation (sd) larger, by a factor  $|\ln(\sqrt{\varepsilon})|^{1/2}$ , than the pulse width. In other words, we have

$$\text{sd}\left[\frac{T_\varepsilon^0(L)}{\varepsilon}\right] \propto |\ln(\sqrt{\varepsilon})|^{1/2} \gg 1. \quad (24)$$

This becomes more significant for slowly decaying correlations (that is for  $\gamma \in (0, 1)$ ), since we have

$$T_\varepsilon^0(L) = \frac{L}{c_0} + \varepsilon \cdot \varepsilon^{-(1-\gamma)/2} W_0(L) + o(\varepsilon^{(1+\gamma)/2}),$$

where the random time shift is now given by a fractional Brownian motion. The standard deviation of the random time shift is larger than the pulse width by a factor  $\varepsilon^{-(1-\gamma)/2} \gg 1$ , that is

$$\text{sd}\left[\frac{T_\varepsilon^0(L)}{\varepsilon}\right] \propto \varepsilon^{-(1-\gamma)/2} \gg 1. \quad (25)$$

To sum up, for short-range correlations we observe a time-shift w.r.t. the expected travel time  $L/c_0$  of order the pulse width. But for long-range correlations, even if this shift remains small compared to the expected travel time  $L/c_0$ , it becomes very large compared to the pulse width.

Also, it is interesting to remark that the random travel time (10) for the stable wave-front does not correspond to the travel time along random characteristics

$$T_\varepsilon(L) := \int_0^L \frac{du}{c_\varepsilon(u)} = \frac{1}{c_0} \int_0^L \sqrt{1 + \nu_\varepsilon(u/\varepsilon)} du,$$

representing the arrival time at  $z = L$ . Hence, we can observe an arrival delay

$$\Delta T_\varepsilon(L) := T_\varepsilon^0(L) - T_\varepsilon(L),$$

for the wave-front which can be roughly expressed after some algebra as

$$\Delta T_\varepsilon(L) = \frac{1}{8c_0} \int_0^L V_\varepsilon^2(z/\varepsilon) dz + o(1).$$

From this expression, one can see that the arrival delay is positive, for  $\varepsilon$  small enough, meaning that compared to the travel time  $T_\varepsilon(L)$  the stable wave-front exhibit a delay to reach the plan  $z = L$ . The comparison of the arrival times w.r.t. the pulse width  $\varepsilon$  can be characterized precisely as follows.

**Proposition 4.2** *We have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\Delta T_\varepsilon(L)}{\varepsilon} = \frac{\theta_0'^2 R(0)L}{8c_0}$$

*in probability.*

The details of the proof are provided in Appendix C, and this result tells us that the wave-front exhibits a deterministic delay of order the pulse width w.r.t. the travel time  $T_\varepsilon(L)$ .

## 5 Modal decomposition

The stochastic analysis provided in this paper is based on a modal decomposition of the wave field in the space-time frequency domain. To study (9) we introduce the following specific Fourier transform

$$\hat{f}_\varepsilon(\omega, \kappa) := \iint f(t, \mathbf{x}) e^{i\omega(t/\varepsilon - \kappa \cdot \mathbf{x}/\sqrt{\varepsilon})} dt d\mathbf{x},$$

and its corresponding inverse formulation

$$f(t, \mathbf{x}) := \frac{1}{(2\pi)^3 \varepsilon^2} \iint \hat{f}_\varepsilon(\omega, \kappa) e^{-i\omega(t/\varepsilon - \kappa \cdot \mathbf{x}/\sqrt{\varepsilon})} \omega^2 d\omega d\mathbf{x}, \quad (26)$$

which are scaled according to the source term. Applying this Fourier transform to (9) gives

$$\partial_{zz}^2 \hat{p}_\varepsilon + \frac{\omega^2 \lambda_\varepsilon^2(\kappa)}{\varepsilon^2} \hat{p}_\varepsilon + \frac{\omega^2}{\varepsilon^2 c_0^2} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) \mathbf{1}_{]0, L[}(z) \hat{p}_\varepsilon = \varepsilon^2 \hat{\Psi}(\omega, \kappa) \delta(z) \quad (t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}. \quad (27)$$

with

$$\hat{\Psi}(\omega, \kappa) := \iint \Psi(t, \mathbf{x}) e^{i\omega(t - \kappa \cdot \mathbf{x})} dt d\mathbf{x},$$

which is the unscaled Fourier transform of the source profile  $\Psi$ , and

$$\lambda_\varepsilon(\kappa) := \frac{\sqrt{1 - \varepsilon c_0^2 |\kappa|^2}}{c_0}. \quad (28)$$

Throughout this paper we assume, for simplicity, that  $\hat{\Psi}$  is compactly supported within a ball centered at 0 and radius of order 1, that is not depending on  $\varepsilon$ . We also assume for technical reasons that  $\omega = 0$  does not belong to the support of the source:

$$\text{supp}_\omega \hat{\Psi} \subset (-\infty, -\omega_c) \cup (\omega_c, \infty),$$

for some cutoff frequency  $\omega_c > 0$ . This assumption allows to avoid unnecessary complications to define (17) and in the proof of Theorem 3.2. Therefore, we have

$$\hat{p}_\varepsilon(\omega, \kappa, z) = 0 \quad \text{for} \quad |\kappa| \geq \frac{1}{c_0 \sqrt{\varepsilon}}.$$

These assumptions are not restrictive and does not change the overall result, but simplify greatly the presentation. In fact, for  $|\kappa| < 1/(c_0 \sqrt{\varepsilon})$  we only deal with the oscillatory components of the solution to (27). The components associated to  $|\kappa| > 1/(c_0 \sqrt{\varepsilon})$  correspond to the evanescent modes that decay exponentially w.r.t the  $z$ -variable. Because of this decay they do not contribute in a significant way in the limit  $\varepsilon \rightarrow 0$ , these modes become negligible.

Note also that the source term in (27) produces the following jump conditions at the source location  $z = 0$ :

$$\begin{aligned} \hat{p}_\varepsilon(\omega, \kappa, z = 0^+) - \hat{p}_\varepsilon(\omega, \kappa, z = 0^-) &= 0, \\ \partial_z \hat{p}_\varepsilon(\omega, \kappa, z = 0^+) - \partial_z \hat{p}_\varepsilon(\omega, \kappa, z = 0^-) &= \varepsilon^2 \hat{\Psi}(\omega, \kappa). \end{aligned} \quad (29)$$

### 5.1 Mode coupling in random media

In the random section, that is for  $z \in (0, L)$ , we decompose the solution to the second order equation (27) as right- and left-going modes,

$$\hat{p}_\varepsilon(\omega, \kappa, z) := \frac{1}{\sqrt{\omega \lambda_\varepsilon(\kappa)}} \left( \hat{a}_\varepsilon(\omega, \kappa, z) e^{i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} + \hat{b}_\varepsilon(\omega, \kappa, z) e^{-i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} \right), \quad (30)$$

with the additional condition

$$\frac{d}{dz} \hat{a}_\varepsilon(\omega, \kappa, z) e^{i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} + \frac{d}{dz} \hat{b}_\varepsilon(\omega, \kappa, z) e^{-i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} = 0,$$

so that

$$\partial_z \hat{p}_\varepsilon(\omega, \kappa, z) = \frac{i\sqrt{\omega \lambda_\varepsilon(\kappa)}}{\varepsilon} \left( \hat{a}_\varepsilon(\omega, \kappa, z) e^{i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} - \hat{b}_\varepsilon(\omega, \kappa, z) e^{-i\omega \lambda_\varepsilon(\kappa) z/\varepsilon} \right).$$

Hence, both  $\hat{a}_\varepsilon$  and  $\hat{b}_\varepsilon$  can be expressed in terms of  $\hat{p}_\varepsilon$  and  $\partial_z \hat{p}_\varepsilon$ :

$$\begin{aligned} \hat{a}_\varepsilon(\omega, \kappa, z) &= \frac{1}{2} \left( \sqrt{\omega \lambda_\varepsilon(\kappa)} \hat{p}_\varepsilon(\omega, \kappa, z) + \frac{\varepsilon}{i\sqrt{\omega \lambda_\varepsilon(\kappa)}} \partial_z \hat{p}_\varepsilon(\omega, \kappa, z) \right) e^{-i\omega \lambda_\varepsilon(\kappa) z/\varepsilon}, \\ \hat{b}_\varepsilon(\omega, \kappa, z) &= \frac{1}{2} \left( \sqrt{\omega \lambda_\varepsilon(\kappa)} \hat{p}_\varepsilon(\omega, \kappa, z) - \frac{\varepsilon}{i\sqrt{\omega \lambda_\varepsilon(\kappa)}} \partial_z \hat{p}_\varepsilon(\omega, \kappa, z) \right) e^{i\omega \lambda_\varepsilon(\kappa) z/\varepsilon}. \end{aligned}$$

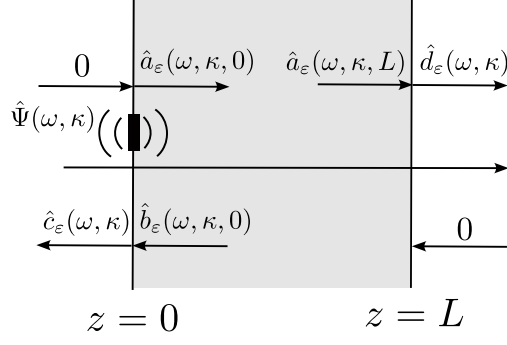


Figure 4: Illustration of the mode amplitudes associated to the incoming and outgoing waves at  $z = 0$  and  $z = L$ .

Here,  $\hat{a}_\varepsilon$  represents the amplitude of the right-going mode, while  $\hat{b}_\varepsilon$  the one of the left-going mode. Differentiating in  $z$  these two last expressions, and using (27) gives

$$\frac{d}{dz} \begin{pmatrix} \hat{a}_\varepsilon(\omega, \kappa, z) \\ \hat{b}_\varepsilon(\omega, \kappa, z) \end{pmatrix} = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) H_\varepsilon \left( \omega, \kappa, \frac{z}{\varepsilon} \right) \begin{pmatrix} \hat{a}_\varepsilon(\omega, \kappa, z) \\ \hat{b}_\varepsilon(\omega, \kappa, z) \end{pmatrix}, \quad (31)$$

where

$$H_\varepsilon(\omega, \kappa, z) = \frac{i\omega}{2\lambda_\varepsilon(\kappa)c_0^2} \begin{pmatrix} 1 & e^{-2i\omega\lambda_\varepsilon(\kappa)z} \\ -e^{2i\omega\lambda_\varepsilon(\kappa)z} & -1 \end{pmatrix}. \quad (32)$$

This differential equation describes how a wave is affected while going through the scattering slab  $(0, L)$ . More precisely, it describes how the medium fluctuations produce the scattering effects on the propagating wave through the exchange between the modes. Note that there is no coupling between any two distinct  $\kappa$ -modes since we consider a randomly layered propagation medium. Moreover, the scattering slab is surrounded by two homogeneous half-spaces, so that we need to complement this system with boundary conditions representing the incoming waves in the slab and the outgoing waves from the slab. This is the aim of the next section.

## 5.2 Boundary conditions

In this section we depict the propagation mechanism in absence of random fluctuations of the wave-speed profile, and corresponding to the situation where  $z < 0$  or  $z > L$ . Considering a decomposition similar to (30), in absence of random fluctuations, leads to constant mode amplitudes in view of (31). Therefore, if we assume that no waves are coming from the left- and the right-hand side of the random slab  $z \in (0, L)$  (see Figure 4 for an illustration), we necessarily have

$$\hat{p}_\varepsilon(\omega, \kappa, z) := \frac{\hat{c}_\varepsilon(\omega, \kappa)}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} e^{-i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \quad \text{for } z < 0,$$

and

$$\hat{p}_\varepsilon(\omega, \kappa, z) := \frac{\hat{d}_\varepsilon(\omega, \kappa)}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \quad \text{for } z > L.$$

As we will see below, on the left-hand-side of the source term ( $z < 0$ ) we only have a left-going mode produced by the source and the backscattered field at  $z = 0$  with amplitude  $\hat{b}_\varepsilon(\omega, \kappa, 0)$ , no right-going mode coming from the left. In the same way, on the right-hand-side of the random section ( $z > L$ ), we only have a right-going mode produced by the wave outgoing the random section at  $z = L$ , and no left-going mode coming from the right. To be more precise, reminding the expression of  $\hat{p}_\varepsilon$  at each side of the interface  $z = L$

$$\begin{aligned} \hat{p}_\varepsilon(\omega, \kappa, z) &= \frac{1}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} \left( \hat{a}_\varepsilon(\omega, \kappa, z) e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} + \hat{b}_\varepsilon(\omega, \kappa, z) e^{-i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \right) \mathbf{1}_{(0, L)}(z) \\ &+ \frac{\hat{d}_\varepsilon(\omega, \kappa)}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \mathbf{1}_{(L, \infty)}(z), \end{aligned}$$

and by continuity w.r.t.  $z$  of  $\hat{p}_\varepsilon$  and  $\partial_z \hat{p}_\varepsilon$  at  $z = L$ , we have

$$\hat{d}_\varepsilon(\omega, \kappa) := \hat{a}_\varepsilon(\omega, \kappa, z = L) \quad \text{and} \quad \hat{b}_\varepsilon(\omega, \kappa, z = L) = 0.$$

To see how the source term charges the modes we remind the expression of  $\hat{p}_\varepsilon$  at each side of the source position  $z = 0$

$$\begin{aligned}\hat{p}_\varepsilon(\omega, \kappa, z) &= \frac{\hat{c}_\varepsilon(\omega, \kappa)}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} e^{-i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \mathbf{1}_{(-\infty, 0)}(z) \\ &\quad + \frac{1}{\sqrt{\omega\lambda_\varepsilon(\kappa)}} \left( \hat{a}_\varepsilon(\omega, \kappa, z) e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} + \hat{b}_\varepsilon(\omega, \kappa, z) e^{-i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \right) \mathbf{1}_{(0, L)}(z),\end{aligned}$$

and use the jump condition (29) to obtain

$$\hat{a}_\varepsilon(\omega, \kappa, z = 0) = \frac{\varepsilon^2 \sqrt{\omega\lambda_\varepsilon(\kappa)}}{2} \hat{\Psi}(\omega, \kappa) \quad \text{and} \quad \hat{c}_\varepsilon(\omega, \kappa) = \hat{b}_\varepsilon(\omega, \kappa, z = 0) + \frac{\varepsilon^2 \sqrt{\omega\lambda_\varepsilon(\kappa)}}{2} \hat{\Psi}(\omega, \kappa).$$

### 5.3 Wave propagation in homogeneous media

Here we provide a derivation of the paraxial approximation in the case of a homogeneous propagation medium ( $\nu_\varepsilon \equiv 0$ ). In this context  $\hat{b}_\varepsilon \equiv 0$  from (31), and we simply have

$$\hat{p}_\varepsilon(\omega, \kappa, z) = \frac{\varepsilon^2}{2} \hat{\Psi}(\omega, \kappa) e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} \quad \text{for } z > 0,$$

so that taking the inverse Fourier transform (26) gives

$$p_\varepsilon(t, \mathbf{x}, z) = \frac{1}{2(2\pi)^3} \int \hat{\Psi}(\omega, \kappa) e^{i\omega\lambda_\varepsilon(\kappa)z/\varepsilon} e^{-i\omega(t/\varepsilon - \kappa \cdot \mathbf{x}/\sqrt{\varepsilon})} \omega^2 d\omega d\kappa.$$

Let us remark that

$$\lambda_\varepsilon(\kappa) = \frac{1}{c_0} - \frac{\varepsilon c_0}{2} |\kappa|^2 + \mathcal{O}(\varepsilon^2), \quad (33)$$

thanks to (28), so that considering

$$t = \frac{z}{c_0} + \varepsilon s \quad \text{and} \quad \mathbf{x} = \sqrt{\varepsilon} \mathbf{y},$$

that is we look at the wave in the frame of the source term, we obtain

$$\lim_{\varepsilon \rightarrow 0} p_\varepsilon(z/c_0 + \varepsilon s, \sqrt{\varepsilon} \mathbf{y}, z) = \psi(s, \mathbf{y}, z) := \frac{1}{2(2\pi)^3} \int \hat{\Psi}(\omega, \kappa) e^{-i\omega s} e^{-i\omega(c_0|\kappa|^2 z/2 - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\kappa.$$

Here  $\psi$  satisfies the paraxial wave equation

$$\partial_{sz}^2 \psi(s, \mathbf{y}, z) + \frac{c_0}{2} \Delta_{\mathbf{y}} \psi(s, \mathbf{y}, z) = 0 \quad \text{with} \quad \psi(s, \mathbf{y}, z = 0) = \frac{1}{2} \Psi(s, \mathbf{y}).$$

In other words the pulse front can be described through the so-called paraxial wave equation with initial condition at  $z = 0$  given by half the source profile. Note that if we take the Fourier transform of  $\psi$  w.r.t. time (the  $s$ -variable) we obtain the following Schrödinger equation

$$i \frac{\omega}{c_0} \partial_z \check{\psi}(\omega, \mathbf{y}, z) + \frac{1}{2} \Delta_{\mathbf{y}} \check{\psi}(\omega, \mathbf{y}, z) = 0 \quad \text{with} \quad \check{\psi}(\omega, \mathbf{y}, z = 0) = \frac{1}{2} \check{\Psi}(\omega, \mathbf{y}),$$

where

$$\check{\psi}(\omega, \mathbf{y}, z) := \int \psi(s, \mathbf{y}, z) e^{i\omega s} ds.$$

## 6 Propagator matrix

The system (31) is a boundary value problem with

$$\hat{a}_\varepsilon(\omega, \kappa, z = 0) = \frac{\varepsilon^2 \sqrt{\omega\lambda_\varepsilon(\kappa)}}{2} \hat{\Psi}(\omega, \kappa) \quad \text{and} \quad \hat{b}_\varepsilon(\omega, \kappa, z = L) = 0, \quad (34)$$

which is not convenient for our analysis based on a martingale technique and diffusion processes corresponding to initial value problems. In this section, we introduce initial value problems that can be related to (31). To rewrite the random differential equation (31) we introduce the associated propagator matrix  $\mathbf{P}_\varepsilon$  which is the solution to

$$\frac{d}{dz} \mathbf{P}_\varepsilon(\omega, \kappa, z) = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) H_\varepsilon \left( \omega, \kappa, \frac{z}{\varepsilon} \right) \mathbf{P}_\varepsilon(\omega, \kappa, z) \quad \text{with} \quad \mathbf{P}_\varepsilon(\omega, \kappa, z = 0) = \mathbf{I}_2,$$

where  $\mathbf{I}_2$  is the  $2 \times 2$  identity matrix. The relation between the left- and -right-going modes with the propagator is given by

$$\begin{pmatrix} \hat{a}_\varepsilon(\omega, \kappa, z) \\ \hat{b}_\varepsilon(\omega, \kappa, z) \end{pmatrix} = \mathbf{P}_\varepsilon(\omega, \kappa, z) \begin{pmatrix} \hat{a}_\varepsilon(\omega, \kappa, 0) \\ \hat{b}_\varepsilon(\omega, \kappa, 0) \end{pmatrix}. \quad (35)$$

From the symmetries of  $H_\varepsilon$ , given by (32), the propagator matrix can be recast as

$$\mathbf{P}_\varepsilon(\omega, \kappa, z) = \begin{pmatrix} \alpha_\varepsilon(\omega, \kappa, z) & \overline{\beta_\varepsilon(\omega, \kappa, z)} \\ \beta_\varepsilon(\omega, \kappa, z) & \alpha_\varepsilon(\omega, \kappa, z) \end{pmatrix},$$

where  $(\alpha_\varepsilon, \beta_\varepsilon)$  being the solution to

$$\frac{d}{dz} \begin{pmatrix} \alpha_\varepsilon(\omega, \kappa, z) \\ \beta_\varepsilon(\omega, \kappa, z) \end{pmatrix} = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) H_\varepsilon \left( \omega, \kappa, \frac{z}{\varepsilon} \right) \begin{pmatrix} \alpha_\varepsilon(\omega, \kappa, z) \\ \beta_\varepsilon(\omega, \kappa, z) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \alpha_\varepsilon(\omega, \kappa, 0) \\ \beta_\varepsilon(\omega, \kappa, 0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (36)$$

From this equation, using that  $H_\varepsilon$  has null trace, the determinant of the propagator is then constant in  $z$ ,

$$\det \mathbf{P}_\varepsilon(\omega, \kappa, z) = \det \mathbf{P}_\varepsilon(\omega, \kappa, 0) = 1,$$

yielding the conservation relation

$$|\alpha_\varepsilon(\omega, \kappa, z)|^2 - |\beta_\varepsilon(\omega, \kappa, z)|^2 = 1. \quad (37)$$

From these new variables,  $\alpha_\varepsilon$  and  $\beta_\varepsilon$ , one can describe the transmitted mode amplitude at  $z = L$  and the reflected mode amplitude at  $z = 0$  using (34) and (35):

$$\hat{a}_\varepsilon(\omega, \kappa, z = L) = \frac{1}{\alpha_\varepsilon(\omega, \kappa, L)} \hat{a}_\varepsilon(\omega, \kappa, z = 0) \quad \text{and} \quad \hat{b}_\varepsilon(\omega, \kappa, 0) = -\frac{\beta_\varepsilon(\omega, \kappa, L)}{\alpha_\varepsilon(\omega, \kappa, L)} \hat{a}_\varepsilon(\omega, \kappa, z = 0).$$

One can also remark from (37), that we have the following conservation relation for the right- and left-going modes

$$|\hat{a}_\varepsilon(\omega, \kappa, L)|^2 + |\hat{b}_\varepsilon(\omega, \kappa, 0)|^2 = |\hat{a}_\varepsilon(\omega, \kappa, 0)|^2,$$

telling us that the input energy at  $z = 0$  equals the sum of the transmitted energy at  $z = L$  and the reflected one at  $z = 0$ .

As already discussed in Section 4, the random travel time of the wave-front produces terms that blow up in our scaling regime. To overcome this difficulty in our analysis, we reformulate (36) through the variables

$$A_\varepsilon(\omega, \kappa, z) := \alpha_\varepsilon(\omega, \kappa, z) e^{-i\omega\phi_\varepsilon(\kappa, z)/\varepsilon} \quad \text{and} \quad B_\varepsilon(\omega, \kappa, z) := \beta_\varepsilon(\omega, \kappa, z) e^{i\omega\phi_\varepsilon(\kappa, z)/\varepsilon}, \quad (38)$$

where

$$\phi_\varepsilon(\kappa, z) := \frac{1}{2\lambda_\varepsilon(\kappa)c_0^2} \int_0^z \nu_\varepsilon(s/\varepsilon^2) ds. \quad (39)$$

While this quantity provides an effective limit as  $\varepsilon \rightarrow 0$  for mixing fluctuations, it blows up in case of long-range correlations, and this is the reason why we remove its contribution from (36). In this later context, this term is responsible of the large standard deviation of the random travel time  $T_\varepsilon^0$  w.r.t. to the pulse width (24, 25).

The variables  $A_\varepsilon$  and  $B_\varepsilon$  are solution to the system

$$\frac{d}{dz} \begin{pmatrix} A_\varepsilon(\omega, \kappa, z) \\ B_\varepsilon(\omega, \kappa, z) \end{pmatrix} = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) \mathcal{H}_\varepsilon \left( \omega, \kappa, \frac{\tau_\varepsilon(\kappa, z)}{\varepsilon} \right) \begin{pmatrix} A_\varepsilon(\omega, \kappa, z) \\ B_\varepsilon(\omega, \kappa, z) \end{pmatrix}, \quad (40)$$

with

$$\mathcal{H}_\varepsilon(\omega, \kappa, \tau) = \frac{i\omega}{2\lambda_\varepsilon(\kappa)c_0^2} \begin{pmatrix} 0 & e^{-i\omega\tau} \\ -e^{i\omega\tau} & 0 \end{pmatrix}.$$

and where

$$\tau_\varepsilon(\kappa, z) := 2\lambda_\varepsilon(\kappa)z + \phi_\varepsilon(\kappa, z). \quad (41)$$

In (40) the fast periodic components are now random and involve a random blowing terms. It is then not clear how the strategy proposed by [30] based on the rough-path theory could apply in this context. This is the reason why we do not follow this route in this paper. Note also that from (37) and (38) we still have the conservation relation

$$|A_\varepsilon(\omega, \kappa, z)|^2 - |B_\varepsilon(\omega, \kappa, z)|^2 = 1. \quad (42)$$

As a result, to study the asymptotic behavior of the mode amplitude  $\hat{a}_\varepsilon(\omega, \kappa, L)$  and  $\hat{b}_\varepsilon(\omega, \kappa, 0)$ , one can study the one of  $(A_\varepsilon, B_\varepsilon)$  as  $\varepsilon \rightarrow 0$ , which is given by the following result.

**Theorem 6.1** *Let  $n \geq 1$ , and set*

$$X_\varepsilon(\omega, \kappa, z) = \begin{pmatrix} A_\varepsilon(\omega, \kappa, z) \\ B_\varepsilon(\omega, \kappa, z) \end{pmatrix}.$$

*For any  $(\omega_1, \dots, \omega_n)$  and  $(\kappa_1, \dots, \kappa_n)$ , the process  $X_\varepsilon$  defined by*

$$\mathcal{X}_\varepsilon(z) := (X_\varepsilon(\omega_1, \kappa_1, z), \dots, X_\varepsilon(\omega_n, \kappa_n, z))$$

*converges in distribution in  $\mathcal{C}((0, \infty), \mathbf{C}^{2n})$  to a process*

$$\mathcal{X}_0(z) := (X_0(\omega_1, z), \dots, X_0(\omega_n, z)),$$

*independent of the  $\kappa$ -variables, and where all its components are statistically independent. Here,  $X_0$  is solution to the following stochastic differential equation,*

$$\begin{aligned} dX_0(\omega, z) = & -\sqrt{\frac{\theta_0^2 \omega^2 \Gamma_c(\omega)}{4c_0^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_0(\omega, z) \circ dW_1(z) \\ & -i\sqrt{\frac{\theta_0^2 \omega^2 \Gamma_c(\omega)}{4c_0^2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_0(\omega, z) \circ dW_2(z) \\ & -i\frac{\theta_0^2 \omega^2 \Gamma_s(\omega)}{8c_0^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X_0(\omega, z) dz \end{aligned} \quad (43)$$

*where  $W_1$  and  $W_2$  are two independent real-valued standard Brownian motions,  $\circ$  stands for the Stratonovich integral, and*

$$\Gamma_c(\omega) := 2 \int_0^\infty R(s) \cos\left(\frac{2\omega s}{c_0}\right) ds \quad \text{and} \quad \Gamma_s(\omega) := 2 \int_0^\infty R(s) \sin\left(\frac{2\omega s}{c_0}\right) ds,$$

*with  $R$  given by (6).*

This result is consistent with the one of [10, Section 7.1] but without the term involving  $\Gamma_c(0)$  which is responsible for the Brownian arrival time shift in the standard ODA theory, and as exhibited through the random travel time of the wave-front in Section 4. In Theorem 6.1, even if we consider short-range correlations ( $\gamma > 1$ ), this term has disappeared because we have considered the compensated mode amplitudes (38) to remove the impact of the random travel time. Note also that in [10, Section 7.1] the Brownian time shift is the same for each of the frequencies  $\omega_j$ , which then correlates all the mode amplitudes at different frequencies. This is in contrast with our result where the contribution of the random travel time have been compensated. This statistical independence for any distinct frequencies is responsible for the pulse stabilization, that is the convergence in probability toward a deterministic limit in Theorem 3.1. In the context of long-range correlations ( $\gamma \in (0, 1]$ ) these compensations (38) are mandatory since we remind that  $\Gamma_c(0) = \infty$  in this case.

Before going into the proof of Theorem 6.1, which is given in Section 9, we show how this result plays a role in the proof of our main result Theorem 3.1.

## 7 The transmitted waves and proof of Theorem 3.1

The transmitted wave on the time frame of the random travel time, and scaled according to the source profile (9), is given by

$$\begin{aligned} p_{tr,\varepsilon}^L(s, \mathbf{y}) &:= p_\varepsilon(T_\varepsilon^0(L) + \varepsilon s, \sqrt{\varepsilon} \mathbf{y}, L) \\ &= \frac{1}{(2\pi)^3 \varepsilon^2} \int \frac{\hat{a}_\varepsilon(\omega, \kappa, L)}{\sqrt{\omega \lambda_\varepsilon(\kappa)}} e^{i\omega(\lambda_\varepsilon(\kappa)L - T_\varepsilon^0(L))/\varepsilon} e^{-i\omega(s - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\kappa \\ &= \frac{1}{2(2\pi)^3} \int \frac{\hat{\Psi}(\omega, \kappa)}{A_\varepsilon(\omega, \kappa, L)} e^{i\omega\Phi_\varepsilon(\kappa, L)/\varepsilon} e^{-i\omega(s - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\kappa, \end{aligned}$$

where

$$\Phi_\varepsilon(\kappa, L) := \lambda_\varepsilon(\kappa)L - T_\varepsilon^0(L) + \phi_\varepsilon(\kappa, L),$$

with  $T_\varepsilon^0(L)$  defined by (10), and  $\phi_\varepsilon(\kappa, L)$  defined by (39). For  $\Phi_\varepsilon(\kappa, L)$  we have the following result.

**Lemma 7.1** *We have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E} \left[ \left| \Phi_\varepsilon(\kappa, L) + \frac{Lc_0}{2} |\kappa|^2 \right|^2 \right] = 0.$$



**Proof** By definition we have

$$\Phi_\varepsilon(\kappa, L) = L\left(\lambda_\varepsilon(\kappa) - \frac{1}{c_0}\right) + \frac{1}{2c_0}\left(\frac{1}{c_0\lambda_\varepsilon(\kappa)} - 1\right) \int_0^L \nu_\varepsilon\left(\frac{s}{\varepsilon}\right) ds,$$

with from (28)

$$\lambda_\varepsilon(\kappa) = \frac{1}{c_0} - \varepsilon \frac{c_0|\kappa|^2}{2} + \mathcal{O}(\varepsilon^2),$$

where  $\mathcal{O}$  is uniform in  $\kappa$  since  $\hat{\Psi}$  is compactly supported in both  $\omega$  and  $\kappa$  in a ball with radius of order 1. As a result,

$$\frac{1}{\varepsilon} \mathbb{E} \left[ \left| \Phi_\varepsilon(\kappa, L) + \frac{Lc_0}{2} |\kappa|^2 \right| \right] = \sqrt{\varepsilon} \sup_{s \in [0, L]} \mathbb{E}[|V(s/\varepsilon)|] \sup |\Theta'| + \mathcal{O}(\varepsilon),$$

which concludes the proof of the lemma according to (50).  $\square$

The conservation relation (42) implies that  $1/\overline{A_\varepsilon}$  is uniformly bounded by 1 in all its variables, so that thanks to Lemma 7.1,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{s, \mathbf{y}} |p_{tr, \varepsilon}^L(s, \mathbf{y}) - q_{tr, \varepsilon}^L(s, \mathbf{y})| \right] = 0,$$

with

$$q_{tr, \varepsilon}^L(s, \mathbf{y}) := \frac{1}{2(2\pi)^3} \int \frac{\hat{\Psi}(\omega, \kappa)}{A_\varepsilon(\omega, \kappa, L)} e^{-i\omega(s + Lc_0|\kappa|^2/2 - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\kappa.$$

Hence, it is enough to prove the convergence for  $q_{tr, \varepsilon}^L$  to obtain the one of  $p_{tr, \varepsilon}^L$  according to [4, Theorem 3.1 pp. 27].

**Proposition 7.1** *The family  $(q_{tr, \varepsilon}^L)_\varepsilon$  converges in probability in  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$  to*

$$p_{tr}^L(s, \mathbf{y}) := \frac{1}{2(2\pi)^3} \int \hat{\Psi}(\omega, \kappa) e^{-\theta_0'^2 \omega^2 (\Gamma_c(\omega) + i\Gamma_s(\omega))L / (8c_0^2)} e^{-i\omega(s + Lc_0|\kappa|^2/2 - \kappa \cdot \mathbf{y})} \omega^2 d\omega d\kappa.$$

**Proof** Denoting

$$\mathcal{E}(\omega) = e^{-\theta_0'^2 \omega^2 (\Gamma_c(\omega) + i\Gamma_s(\omega))L / (8c_0^2)},$$

and

$$\mathbf{E}_\varepsilon(\omega_1, \omega_2, \kappa_1, \kappa_2) = \mathbb{E} \left[ \left( \frac{1}{A_\varepsilon(\omega_1, \kappa_1, L)} - \mathcal{E}(\omega_1) \right) \left( \frac{1}{A_\varepsilon(\omega_2, \kappa_2, L)} - \overline{\mathcal{E}(\omega_2)} \right) \right],$$

we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{s, \mathbf{y}} |q_{tr, \varepsilon}^L(s, \mathbf{y}) - p_{tr}^L(s, \mathbf{y})|^2 \right] &\leq \frac{1}{4(2\pi)^6} \int |\mathbf{E}_\varepsilon(\omega_1, \omega_2, \kappa_1, \kappa_2)| \\ &\quad \times |\hat{\Psi}(\omega_1, \kappa_1) \overline{\hat{\Psi}(\omega_2, \kappa_2)}| \omega_1^2 \omega_2^2 d\omega_1 d\kappa_1 d\omega_2 d\kappa_2. \end{aligned}$$

Now, we expand the expectation in  $\mathbf{E}_\varepsilon$  so that

$$\begin{aligned} \mathbf{E}_\varepsilon(\omega_1, \omega_2, \kappa_1, \kappa_2) &= \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_1, \kappa_1, L) A_\varepsilon(\omega_2, \kappa_2, L)} \right] - \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_1, \kappa_1, L)} \right] \overline{\mathcal{E}(\omega_2)} \\ &\quad - \mathcal{E}(\omega_1) \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_2, \kappa_2, L)} \right] + \mathcal{E}(\omega_1) \overline{\mathcal{E}(\omega_2)}. \end{aligned}$$

From Theorem 6.1, with a single  $\omega$  and  $\kappa$ , together with the Itô formula [25, Theorem 3.3 pp. 149] applied to the real and imaginary parts of (43), we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega, \kappa, L)} \right] = \mathcal{E}(\omega).$$

Note that the convergence in distribution implies the convergence of the expectation thanks to [4, Theorem 3.5 pp. 31] and the fact that  $1/\overline{A_\varepsilon}$  is uniformly bounded in  $\varepsilon$  according to (42). For similar reasons, but adding the fact that the limit in  $\varepsilon$  of  $(A_\varepsilon(\omega_1, \kappa_1, L), A_\varepsilon(\omega_2, \kappa_2, L))$  for two distinct pairs of  $(\omega, \kappa)$  are independent, we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_1, \kappa_1, L) A_\varepsilon(\omega_2, \kappa_2, L)} \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_1, \kappa_1, L)} \right] \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \frac{1}{A_\varepsilon(\omega_2, \kappa_2, L)} \right] = \mathcal{E}(\omega_1) \overline{\mathcal{E}(\omega_2)}.$$

Combining these results gives

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}_\varepsilon(\omega_1, \omega_2, \kappa_1, \kappa_2) = 0,$$

and thanks to the dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{s, \mathbf{y}} |q_{tr, \varepsilon}^L(s, \mathbf{y}) - p_{tr}^L(s, \mathbf{y})|^2 \right] = 0,$$

which concludes the proof of the proposition.  $\blacksquare$

The transmitted wave-front at the end of the random slab  $z = L$  is then given by

$$p_{tr}^L(s, \mathbf{y}) = \frac{1}{2} \mathcal{K}(\cdot, \cdot, L) * \Psi(s, \mathbf{y}),$$

with the pulse deformation given in the Fourier domain by

$$\hat{\mathcal{K}}(\omega, \kappa, z) = e^{-\theta_0'^2 \omega^2 (\Gamma_c(\omega) + i\Gamma_s(\omega))z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2}.$$

From this formula, it turns out that  $\mathcal{K}$  is the unique solution to

$$\partial_{sz}^2 \mathcal{K} + \frac{c_0}{2} \Delta_{\mathbf{y}} \mathcal{K} - \mathcal{I}(\mathcal{K}) = 0,$$

with

$$\mathcal{I}(\psi)(s, \mathbf{y}) := -\frac{\theta_0'^2}{8c_0^3} \int i\omega^3 (\Gamma_c(\omega) + i\Gamma_s(\omega)) \check{\psi}(\omega, \mathbf{y}) e^{i\omega s} d\omega. \quad (44)$$

and  $\mathcal{K}(s, \mathbf{y}, z = 0) = \delta(s)\delta(\mathbf{y})$ . The operator  $\mathcal{I}$  can be recast in the time domain as

$$\mathcal{I}(\psi)(s) = \phi * \partial_{sss}^3 \psi(s) \quad \text{with} \quad \phi(s) = \frac{\theta_0'^2}{8c_0^2} R\left(\frac{c_0 s}{2}\right) \mathbf{1}_{(0, \infty)}(s), \quad (45)$$

so that

$$\mathcal{I}(\psi)(s) = \frac{\theta_0'^2}{8c_0^2} \int_{-\infty}^s R\left(\frac{c_0(s-\tau)}{2}\right) \partial_{sss}^3 \psi(\tau) d\tau.$$

Let us finish this section with a comment on the backscattered wave

$$p_{bk, \varepsilon}(s, \mathbf{y}) := p_{\varepsilon}(\varepsilon s, \sqrt{\varepsilon} \mathbf{y}, 0).$$

According to Theorem 6.1 the amplitude  $B_{\varepsilon}/\overline{A_{\varepsilon}}$  converges in distribution to some limit which has null expectation according to the Itô formula. Note that we again have the convergence of the expectation, since  $B_{\varepsilon}/\overline{A_{\varepsilon}}$  is also uniformly bounded in  $\varepsilon$  by 1 according to (42). Thanks to the independence of the limit w.r.t. to multiple frequencies  $\omega$ , the limiting expectation of the product of two  $B_{\varepsilon}/\overline{A_{\varepsilon}}$  at two distinct frequencies is the product of the limiting single expectation, which is then 0. Following the same lines as the proof of Proposition 7.1 we obtain that the backscattered wave converges to 0 as  $\varepsilon \rightarrow 0$  in probability in  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$ . This observation is consistent with [10, Chapter 9] in which the authors describe the backscattered signal as a "small" incoherent signal that can be described through a random field. This analysis is beyond the scope of this paper and will not be addressed here.

## 8 Proof of Theorem 3.2

In this section, we describe how the integral operator  $\mathcal{I}$  can be approximated by a fractional derivative or a standard or a third order derivative depending whether  $\gamma \in (0, 1)$  or  $\gamma \geq 1$  respectively. The fractional operator that derives from  $\mathcal{I}$  is given by a Weyl fractional derivative given by (19).

In view of (44) and (45), the operator  $\mathcal{I}$  is completely characterized by the correlation function  $R$  through the coefficients  $\Gamma_c$  and  $\Gamma_s$  (see (14)). Under the scaling (20), we only need to study the asymptotics to these coefficients, which is done in the following technical lemma whose proof is provided in Appendix D.

**Lemma 8.1** *Setting*

$$\Gamma_c(\omega, l_0) := 2\sigma(l_0) \int_0^{\infty} R\left(\frac{s}{l_0}\right) \cos\left(\frac{2\omega s}{c_0}\right) ds \quad \text{and} \quad \Gamma_s(\omega, l_0) := 2\sigma(l_0) \int_0^{\infty} R\left(\frac{s}{l_0}\right) \sin\left(\frac{2\omega s}{c_0}\right) ds,$$

where  $\sigma(l_0)$  is defined by (21), we have:

1. for  $\gamma \in [1, \infty)$ ,

$$\lim_{l_0 \rightarrow 0} \Gamma_c(\omega, l_0) = \Gamma_0 \quad \text{and} \quad \lim_{l_0 \rightarrow 0} \Gamma_s(\omega, l_0) = 0,$$

with  $\Gamma_0$  defined by (23);

2. for  $\gamma \in (0, 1)$ ,

$$\lim_{l_0 \rightarrow 0} \Gamma_c(\omega, l_0) = R_0 \Gamma_{c,0}(\omega) \quad \text{and} \quad \lim_{l_0 \rightarrow 0} \Gamma_s(\omega, l_0) = R_0 \Gamma_{s,0}(\omega),$$

with for  $\omega \neq 0$

$$\Gamma_{c,0}(\omega) = 2 \int_0^{\infty} \frac{ds}{s^{\gamma}} \cos(2\omega s/c_0) = 2\Gamma(1-\gamma) \cos\left(\frac{(1-\gamma)\pi}{2}\right) \left(\frac{2|\omega|}{c_0}\right)^{\gamma-1},$$

and

$$\Gamma_{s,0}(\omega) = 2 \int_0^{\infty} \frac{ds}{s^{\gamma}} \sin(2\omega s/c_0) = 2\Gamma(1-\gamma) \sin\left(\frac{(1-\gamma)\pi}{2}\right) \left(\frac{2|\omega|}{c_0}\right)^{\gamma-1} \text{sign}(\omega),$$

where  $R_0$  is given by (8) and  $\Gamma$  stands for the Gamma function.

Note that the two later formulas for  $\Gamma_{c,0}(\omega)$  and  $\Gamma_{s,0}(\omega)$  are given by [22, 3.761-4 and 9 pp. 436–437].  
From this lemma we obtain the following convergence in  $\mathcal{C}(\mathbb{R} \times \mathbb{R}^2)$

$$\lim_{l_0 \rightarrow 0} p_{tr,l_0}(s, \mathbf{y}, L) = p_{tr,0}(s, \mathbf{y}, L) = \frac{1}{2} \mathcal{K}_0(\cdot, \cdot, L) * \Psi(s, \mathbf{y}),$$

thanks to the dominated convergence theorem, where, in the Fourier domain,

$$\hat{\mathcal{K}}_0(\omega, \kappa, z) := \begin{cases} e^{-\theta_0'^2 \omega^2 \Gamma_0 z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2} & \text{if } \gamma \geq 1, \\ e^{-\theta_0'^2 R_0 \omega^2 (\Gamma_{c,0}(\omega) + i\Gamma_{s,0}(\omega)) z / (8c_0^2)} e^{-i\omega c_0 |\kappa|^2 z / 2} & \text{if } \gamma \in (0, 1). \end{cases}$$

From the definition of  $\mathcal{K}_0$ , we obtain for  $\gamma \geq 1$  the following paraxial wave equation with a third order derivative in  $s$ ,

$$\partial_{sz}^2 \mathcal{K}_0 + \frac{c_0}{2} \Delta_{\mathbf{y}} \mathcal{K}_0 - \frac{\theta_0'^2 \Gamma_0}{8c_0^2} \partial_{sss}^3 \mathcal{K}_0 = 0.$$

For  $\gamma \in (0, 1)$ , following the same strategy as in Section 7 but with now

$$\phi(s) = \frac{\theta_0'^2 R_0}{2^{3-\gamma} c_0^{1+\gamma}} s^{-\gamma} \mathbf{1}_{(0,\infty)}(s),$$

we obtain

$$\partial_{sz}^2 \mathcal{K}_0 + \frac{c_0}{2} \Delta_{\mathbf{y}} \mathcal{K}_0 - \frac{\theta_0'^2 R_0 \Gamma(1-\gamma)}{2^{3-\gamma} c_0^{1+\gamma}} D_s^{2+\gamma} \mathcal{K}_0 = 0,$$

where  $D_s^{2+\gamma}$  stands for the Weyl fractional derivative, w.r.t. the  $s$ -variable, of order  $2 + \gamma$ , which is given by

$$D_s^{2+\gamma} \psi(s) = \frac{1}{\Gamma(1-\gamma)} \int_{-\infty}^s (s-\tau)^{-\gamma} \partial_{sss}^3 \psi(\tau) d\tau.$$

## 9 Proof of Theorem 6.1

The convergence in distribution of the family  $(\mathcal{X}^\varepsilon)_\varepsilon$  is proved in two steps. The first step concerns the tightness of the family, and the second one concerns the characterization of all its accumulation points as being weak solutions to a well-posed stochastic differential equation. Both points are proved using the perturbed test function method [29]. These two steps make use of the notion of pseudogenerator allowing the use of martingale techniques. We introduce first this key notion below before going into the detail analysis of the convergence. Nevertheless, tightness criteria like [29, Theorem 4 pp. 48] requires uniform bound in probability that we do not have for  $(\mathcal{X}_\varepsilon)_\varepsilon$ . This appears to be also a problem to identify the subsequence limits. To bypass this problem we adopt the strategy proposed in [38, Chapter 11] and introduce a *truncated process*. This new process is related to the original one through a family of stopping times that goes to  $\infty$  when we remove the truncation.

### 9.1 The truncated process

Let us start by writing down the system satisfied by  $\mathcal{X}_\varepsilon$  in a convenient form for the forthcoming analysis,

$$\frac{d}{dz} \mathcal{X}_\varepsilon(z) = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j (-1)^l}{2\lambda_\varepsilon(\kappa_j) c_0^2} e^{i\omega_j (-1)^{l+1} \tau_\varepsilon(\kappa_j, z) / \varepsilon} \mathcal{X}_\varepsilon^{j,1-l}(z) \mathbf{e}_{jl}$$

where  $\tau_\varepsilon(\kappa, z)$  is given by (41),

$$\mathcal{X}_\varepsilon = (\mathcal{X}_\varepsilon^{1,0}, \mathcal{X}_\varepsilon^{1,1}, \dots, \mathcal{X}_\varepsilon^{n,0}, \mathcal{X}_\varepsilon^{n,1}) = \sum_{j=1}^n \sum_{l=0,1} \mathcal{X}_\varepsilon^{j,l}(z) \mathbf{e}_{jl},$$

with

$$\mathcal{X}_\varepsilon^{j,0}(z) = A_\varepsilon(\omega_j, \kappa_j, z) \quad \text{and} \quad \mathcal{X}_\varepsilon^{j,1}(z) = B_\varepsilon(\omega_j, \kappa_j, z) \quad j \in \{1, \dots, n\},$$

and  $(\mathbf{e}_{jl})_{jl}$  the canonical basis for  $\mathbb{C}^2 \times \dots \times \mathbb{C}^2$   $n$  times.

Denoting  $M > 0$  the cutoff parameter, the truncated process  $\mathcal{X}_{\varepsilon,M}$  is defined as being the solution to

$$\frac{d}{dz} \mathcal{X}_{\varepsilon,M}(z) = \frac{1}{\varepsilon} \nu_\varepsilon \left( \frac{z}{\varepsilon} \right) \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j (-1)^l}{2\lambda_\varepsilon(\kappa_j) c_0^2} e^{i\omega_j (-1)^{l+1} \tau_\varepsilon(\kappa_j, z) / \varepsilon} F_M(\mathcal{X}_{\varepsilon,M}^{j,1-l}(z)) \mathbf{e}_{jl}, \quad (46)$$

where

$$F_M(\mathcal{X}) := \mathcal{X} \phi_M(\mathcal{X}) \quad \mathcal{X} \in \mathbb{C},$$

with  $\phi_M$  a compactly supported smooth function such that  $0 \leq \phi_M \leq 1$  and

$$\phi_M(\mathcal{X}) = \begin{cases} 1 & \text{if } |\mathcal{X}| \leq M, \\ 0 & \text{if } |\mathcal{X}| \geq 2M. \end{cases}$$

Thanks to the cutoff function  $\phi_M$  the process  $\mathcal{X}_{\varepsilon, M}$  is bounded uniformly in  $\varepsilon$  by  $2M$  with probability one.

To relate the truncated process  $\mathcal{X}_{\varepsilon, M}$  with the original one  $\mathcal{X}_\varepsilon$ , let us introduce  $\Omega = \mathcal{C}([0, \infty), \mathbb{C}^{2n})$ , the space of all possible trajectories for  $\mathcal{X}_\varepsilon$  and  $\mathcal{X}_{\varepsilon, M}$ , associated to its canonical filtration

$$\mathcal{M}_z = \sigma(f(s), 0 \leq s \leq z),$$

and  $\sigma$ -field

$$\mathcal{M} := \sigma \left( \bigcup_{z \geq 0} \mathcal{M}_z \right).$$

We also introduce the stopping times

$$\eta_M(f) := \inf(z \geq 0 : \|f(z)\| \geq M) \quad f \in \Omega, \quad M > 0,$$

and the distributions  $\mathbb{P}_\varepsilon$  and  $\mathbb{P}_{\varepsilon, M}$  of respectively  $\mathcal{X}_M$  and  $\mathcal{X}_{\varepsilon, M}$ , which are defined on the measurable space  $(\Omega, \mathcal{M})$ . From the above definitions, it is clear that

$$\mathbb{P}_\varepsilon = \mathbb{P}_{\varepsilon, M} \quad \text{on } \mathcal{M}_{\eta_M},$$

so that according to [38, Lemma 11.1.1 pp.262] together with [38, Exercise 11.5.1 pp.283], to prove the convergence of  $\mathbb{P}_\varepsilon$  we only have to prove the one of  $\mathbb{P}_{\varepsilon, M}$  to  $\mathbb{P}_{0, M}$ , and show that

$$\mathbb{P}_0 = \mathbb{P}_{0, M} \quad \text{on } \mathcal{M}_{\eta_M},$$

where  $\mathbb{P}_0$  is the solution to a well-posed martingale problem. Also, to prove the convergence of  $\mathbb{P}_{\varepsilon, M}$  on  $\Omega$ , we only have to prove it on  $\mathcal{C}([0, L], \mathbb{C}^{2n})$  for any  $L > 0$ .

From the boundedness of the truncated process  $\mathcal{X}_{\varepsilon, M}$ , together with the expansion

$$\nu_\varepsilon(z/\varepsilon) = \sqrt{\varepsilon} \Theta'(0) V(z/\varepsilon) + \mathcal{O}(\varepsilon^{3/2} |V(z/\varepsilon)|^3),$$

and (46), one can write

$$\begin{aligned} \frac{d}{dz} \mathcal{X}_{\varepsilon, M}(z) &= \frac{\theta'_0}{\sqrt{\varepsilon}} V\left(\frac{z}{\varepsilon}\right) \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j (-1)^l}{2\lambda_\varepsilon(\kappa_j) c_0^2} e^{i\omega_j (-1)^{l+1} \tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \mathbf{e}_{jl} \\ &\quad + E_\varepsilon(z) \end{aligned}$$

with  $\theta'_0 = \Theta'(0)$ , and where the error term  $E_\varepsilon$  is uniformly bounded in  $\varepsilon$  and  $z \in [0, L]$  with probability one. This term provides a negligible contribution in the limit  $\varepsilon \rightarrow 0$  and does not play any role in the forthcoming analysis. For the sake of simplicity in the presentation we ignore this term and consider instead following the system:

$$\frac{d}{dz} \mathcal{X}_{\varepsilon, M}(z) = \frac{\theta'_0}{\sqrt{\varepsilon}} V\left(\frac{z}{\varepsilon}\right) \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j (-1)^l}{2\lambda_\varepsilon(\kappa_j) c_0^2} e^{i\omega_j (-1)^{l+1} \tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \mathbf{e}_{jl}. \quad (47)$$

## 9.2 Pseudogenerator

We remind the reader about the notion of pseudogenerator allowing the use of martingale technique while the underlying process is not a Markov process. Before introducing the notion of pseudogenerator let us define the  $p$ -lim.

Let

$$\mathcal{G}_z^\varepsilon = \sigma(V(s/\varepsilon, dp), \quad 0 \leq s \leq z) \quad 0 \leq z \leq L,$$

and  $\mathcal{S}^\varepsilon$  be the set of all measurable functions  $f$ , adapted to the filtration  $(\mathcal{G}_z^\varepsilon)$ , and for which  $\sup_{z \leq L} \mathbb{E}[|f(z)|] < \infty$ . Let  $f$  and  $f_h$  in  $\mathcal{S}^\varepsilon$  for all  $h > 0$ , we say that  $f = p$ -lim $_h f_h$  if

$$\sup_{z, h} \mathbb{E}[|f_h(z)|] < +\infty \quad \text{and} \quad \lim_{h \rightarrow 0} \mathbb{E}[|f_h(z) - f(z)|] = 0 \quad \forall z \geq 0.$$

Regarding the pseudogenerator itself, denoting by  $\mathcal{A}^\varepsilon$ , we say that  $f \in \mathcal{D}(\mathcal{A}^\varepsilon)$  the domain of  $\mathcal{A}^\varepsilon$ , and  $\mathcal{A}^\varepsilon f = g$  if both  $f$  and  $g$  are in  $\mathcal{S}^\varepsilon$  and

$$p - \lim_{h \rightarrow 0} \left[ \frac{\mathbb{E}_z^\varepsilon[f(z+h)] - f(z)}{h} - g(z) \right] = 0.$$

Here,  $\mathbb{E}_z^\varepsilon$  denotes the conditional expectation given  $\mathcal{G}_z^\varepsilon$ . The key property to relate pseudogenerator to martingale properties is the following.

**Proposition 9.1** For any  $f \in \mathcal{D}(\mathcal{A}^\varepsilon)$ , the process

$$M_f^\varepsilon(z) = f(z) - f(0) - \int_0^z \mathcal{A}^\varepsilon f(u) du$$

is a  $(\mathcal{G}_z^\varepsilon)$ -martingale.

This last result will allow us to characterize the limiting processes through a well-posed martingale problem with generator  $\mathcal{A}$  that has to be determined. Unfortunately, the pseudogenerator  $\mathcal{A}^\varepsilon$  associated to  $\mathcal{X}_{\varepsilon, M}$ , at some test function  $f$ , has a singular term of order  $1/\sqrt{\varepsilon}$ . The idea of the perturbed test function method is to construct a perturbation  $f^\varepsilon$  of  $f$  in order to extract an effective statistical behavior from  $\mathcal{A}^\varepsilon f^\varepsilon$ . This strategy allows  $\mathcal{A}^\varepsilon f^\varepsilon$  to converge to  $\mathcal{A}f$ , where,  $\mathcal{A}$  will be the generator describing this asymptotic statistical behavior for  $(\mathcal{X}_{\varepsilon, M})_\varepsilon$ .

### 9.3 Technical lemmas for the fluctuations $V$

Here we introduce two results that are used in the forthcoming analysis to analyze the corrections of a test function. We refer to [21] for a detailed proof of these two lemmas. The first one concerns the conditional expectation and variance.

**Lemma 9.1** *Setting*

$$\mathcal{G}_z = \sigma(V(s, dp), \quad 0 \leq s \leq z) \quad z \geq 0,$$

where  $V(\cdot, dp)$  is given by (5), we have for any  $z, h \geq 0$

$$\mathbb{E}[V(z+h, dp)|\mathcal{G}_z] = e^{-\mu|p|^{2\beta h}} V(z, dp) \quad (48)$$

and

$$\begin{aligned} \mathbb{E}[V(z+h, dp)V(z+h, dq)|\mathcal{G}_z] - \mathbb{E}[V(z+h, dp)|\mathcal{G}_z]\mathbb{E}[V(z+h, dq)|\mathcal{G}_z] \\ = (1 - e^{-2\mu|p|^{2\beta h}})r(p)\delta(p-q) dp dq. \end{aligned} \quad (49)$$

The second result concerns uniform bounds for the fluctuations  $V$ .

**Lemma 9.2** *Let  $L > 0, M > 0$ ,*

$$D_{k, M} := [0, L] \times L^\infty([0, L], W_{k, M}),$$

with

$$W_{k, M} := \{\varphi \in W^{1, k}(S) : \|\varphi\|_{W^{1, k}} \leq M\},$$

and where  $W^{1, k}(S)$  stands for the Sobolev space with  $k \in (1, \infty]$ . We have

$$\mathbb{E}\left[\sup_{(z, \varphi) \in D_{k, M}} \left|V\left(\frac{z}{\varepsilon}, \varphi(z, \cdot)\right)\right|\right] \leq C + \frac{C(\varepsilon)}{\sqrt{\varepsilon}}, \quad (50)$$

and for any  $n \in \mathbb{N}^*$

$$\sup_{\varepsilon} \sup_{z \in [0, L]} \mathbb{E}\left[\sup_{\varphi \in W_{k, M}} \left|V\left(\frac{z}{\varepsilon}, \varphi\right)\right|^n\right] \leq C_n, \quad (51)$$

where  $C, C_n$  and  $C(\varepsilon)$  are three positive constants where the latter satisfies

$$\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = 0.$$

### 9.4 Tightness

In this section we prove the tightness of  $(\mathcal{X}_{\varepsilon, M})_\varepsilon$ , which is a family of processes with continuous trajectories. According to [4, Theorem 13.4] it is enough to prove its tightness in  $\mathcal{D}([0, T], \mathbb{C}^{2n})$ , the set of càd-làg functions with values in  $\mathbb{C}^{2n}$ , and equipped with the Skorohod topology.

**Proposition 9.2** *The family  $(\mathcal{X}_{\varepsilon, M})_\varepsilon$  is tight in  $\mathcal{D}([0, T], \mathbb{C}^{2n})$ .*

This proposition can be proved using the perturbed test function method by applying [29, Theorem 4, pp. 48]. Throughout the forthcoming analysis we make use of the complex derivatives that are defined, for  $\mathcal{X} = u + iv$ , as

$$\partial_{\mathcal{X}} := \frac{1}{2}(\partial_u - i\partial_v) \quad \text{and} \quad \partial_{\bar{\mathcal{X}}} := \frac{1}{2}(\partial_u + i\partial_v).$$

These tools allow us to avoid working with (47) rewritten in terms of real and imaginary parts, and still working with complex quantities.

In what follows, let  $f$  be a smooth bounded function on  $\mathbb{C}^{2n}$  with successive bounded derivatives, and set

$$f_0^\varepsilon(z) := f(\mathcal{X}_{\varepsilon, M}(z)).$$

In order to prove the tightness we make use of the pseudogenerator and associated martingale techniques. The pseudogenerator for  $\mathcal{X}_{\varepsilon, M}$  at  $f_0^\varepsilon$  is given by

$$\begin{aligned} \mathcal{A}^\varepsilon f_0^\varepsilon(z) &= \frac{\theta'_0}{\sqrt{\varepsilon}} V\left(\frac{z}{\varepsilon}\right) \left( \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j(-1)^l}{2\lambda_\varepsilon(\kappa_j)c_0^2} e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \partial_{\mathcal{X}^{j, l}} f(\mathcal{X}_{\varepsilon, M}(z)) \right. \\ &\quad \left. + \overline{\sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j(-1)^l}{2\lambda_\varepsilon(\kappa_j)c_0^2} e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \partial_{\overline{\mathcal{X}^{j, l}}} f(\mathcal{X}_{\varepsilon, M}(z))} \right) \\ &= \frac{\theta'_0}{\sqrt{\varepsilon}} V\left(\frac{z}{\varepsilon}\right) \left( \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j(-1)^l}{2\lambda_\varepsilon(\kappa_j)c_0^2} e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \partial_{\mathcal{X}^{j, l}} f(\mathcal{X}_{\varepsilon, M}(z)) \right. \\ &\quad \left. + c.c. \right), \end{aligned}$$

where *c.c.* stands for *complex conjugate* and will be used throughout the remaining of this proof instead of rewriting quantities that only have to be conjugated.

To prove the tightness of  $(\mathcal{X}_{\varepsilon, M})_\varepsilon$  we construct a perturbation to  $f_0^\varepsilon$  in order to remove the singular term produced by  $V(\cdot/\varepsilon)/\sqrt{\varepsilon}$  so that the pseudogenerator will become of order one. More precisely, setting

$$\begin{aligned} f_1^\varepsilon(z) &= \frac{\theta'_0}{\sqrt{\varepsilon}} \int_z^\infty ds \mathbb{E}_z^\varepsilon \left[ V\left(\frac{s}{\varepsilon}\right) \right] \\ &\quad \times \left( \sum_{j=1}^n \sum_{l=0,1} \frac{i\omega_j(-1)^l}{2\lambda_\varepsilon(\kappa_j)c_0^2} e^{i\omega_j(-1)^{l+1}(\tau_\varepsilon(\kappa_j, z) + 2\lambda_\varepsilon(\kappa_j)(s-z))/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \partial_{\mathcal{X}^{j, l}} f(\mathcal{X}_{\varepsilon, M}(z)) + c.c. \right) \end{aligned}$$

we have the two following results.

**Lemma 9.3** For any  $\eta > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left( \sup_{z \in [0, L]} |f_1^\varepsilon(z)| > \eta \right) = 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sup_{z \in [0, L]} \mathbb{E}[|f_1^\varepsilon(z)|] = 0.$$

**Lemma 9.4** The family  $\{\mathcal{A}^\varepsilon(f_0^\varepsilon + f_1^\varepsilon)(z), \varepsilon \in (0, 1), 0 \leq z \leq L\}$  is uniformly integrable.

**Proof (of Lemma 9.3)** According to (48) we have

$$\mathbb{E}_z^\varepsilon \left[ V\left(\frac{z}{\varepsilon} + s\right) \right] = \int_S e^{-\mathfrak{g}(p)s} V\left(\frac{z}{\varepsilon}, dp\right),$$

where we have introduced the notation

$$\mathfrak{g}(p) = \mu|p|^{2\beta}$$

for simplicity. Making the change of variable  $s \rightarrow z + \varepsilon s$ , and integrating in  $s$ , we obtain

$$\begin{aligned} f_1^\varepsilon(z) &= \sqrt{\varepsilon} \theta'_0 \sum_{j=1}^n \sum_{l=0,1} \int \frac{V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ &\quad \times \frac{i\omega_j(-1)^l}{2\lambda_\varepsilon(\kappa_j)c_0^2} e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} F_M(\mathcal{X}_{\varepsilon, M}^{j, 1-l}(z)) \partial_{\mathcal{X}^{j, l}} f(\mathcal{X}_{\varepsilon, M}(z)) \\ &\quad + c.c.. \end{aligned}$$

It turns out that

$$|f_1^\varepsilon(z)| \leq \sqrt{\varepsilon} K \sum_{j=1}^n \left| \int_S \frac{V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j\lambda_\varepsilon(\kappa_j)} \right| + \left| \int_S \frac{V(z/\varepsilon, dp)}{\mathfrak{g}(p) - 2i\omega_j\lambda_\varepsilon(\kappa_j)} \right|,$$

for some constant  $K > 0$ . Let us denote

$$V(s, \varphi_{1, j, \varepsilon}) = \int_S \frac{V(s, dp)}{\mathfrak{g}(p) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j)} \quad \text{where} \quad \varphi_{1, j, \varepsilon}(p) := \frac{1}{\mathfrak{g}(p) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j)}, \quad (52)$$

which is Lipschitz in  $p$  if  $\beta \geq 1/2$ , or belongs to  $W^{1,k}(S)$  for  $k \in (1, 1/(1-2\beta))$  if  $\beta < 1/2$ , since

$$\int_S |\partial_p \varphi_{1,j,\varepsilon}(p)|^k dp \leq \tilde{C} \int |p|^{k(2\beta-1)} < \infty.$$

Therefore,  $\varphi_{1,j,\varepsilon} \in W_{k,C}$  for some constant  $C > 0$  and

$$|V(z/\varepsilon, \varphi_{1,j,\varepsilon})| \leq \sup_{\varphi \in W_{k,C}} |V(z/\varepsilon, \varphi)|,$$

so that

$$|f_1^\varepsilon(z)| \leq K' \sqrt{\varepsilon} \sup_{\varphi \in W_{k,C}} |V(z/\varepsilon, \varphi)|.$$

Consequently, according to (51) we obtain

$$\lim_{\varepsilon \rightarrow 0} \sup_{z \in [0,L]} \mathbb{E}[|f_1^\varepsilon(z)|] = 0.$$

Also, we have for any  $\eta' > 0$

$$\begin{aligned} \mathbb{P}\left(\sup_{z \in [0,L]} |f_1^\varepsilon(z)| > \eta\right) &\leq \mathbb{P}\left(\sup_{z \in [0,L]} |f_1^\varepsilon(z)| > \eta, \sqrt{\varepsilon} \sup_{(s,\varphi) \in [0,L/\varepsilon] \times W_{k,C}} |V(s,\varphi)| \leq \eta'\right) \\ &+ \mathbb{P}\left(\sqrt{\varepsilon} \sup_{(s,\varphi) \in [0,L/\varepsilon] \times W_{k,C}} |V(s,\varphi)| > \eta'\right) =: P_{1,\varepsilon} + P_{2,\varepsilon}, \end{aligned}$$

where  $P_{1,\varepsilon} = 0$  since  $K'\eta' \leq \eta$  for  $\eta'$  small enough but independent of  $\varepsilon$ . Finally, according to (50) together with the Markov inequality, we have

$$0 \leq P_{2,\varepsilon} \leq \frac{1}{\eta'} \mathbb{E}\left[\sqrt{\varepsilon} \sup_{(s,\varphi) \in [0,L/\varepsilon] \times W_{k,C}} |V(s,\varphi)|\right] = 0,$$

so that  $\lim_{\varepsilon \rightarrow 0} P_{2,\varepsilon} = 0$ , which concludes the proof of the lemma.  $\square$

**Proof (of Lemma 9.4)** After lengthy but straightforward algebra we obtain

$$\mathcal{A}^\varepsilon(f_0^\varepsilon + f_1^\varepsilon)(z) = \mathcal{A}_0^\varepsilon(z) + \mathcal{A}_1^\varepsilon(z), \quad (53)$$

where

$$\begin{aligned} \mathcal{A}_0^\varepsilon(z) &:= \sum_{j=1}^n \sum_{l=0,1} \theta_0^{l^2} \int \frac{V(z/\varepsilon)V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ &\times \left( \sum_{j'=1}^n \sum_{l'=0,1} \mathbf{F}_{1,j,j',l,l',\varepsilon}(z) e^{i(\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z) + \omega_{j'}(-1)^{l'+1}\tau_\varepsilon(\kappa_{j'},z))/\varepsilon} \right. \\ &\quad \left. + \mathbf{F}_{2,j,j',l,l',\varepsilon}(z) e^{i(\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z) - \omega_{j'}(-1)^{l'+1}\tau_\varepsilon(\kappa_{j'},z))/\varepsilon} \right. \\ &\quad \left. + \mathbf{G}_{1,j,\varepsilon}(z) + \mathbf{G}_{2,j,\varepsilon}(z) e^{2i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon} \right) + c.c. \\ \mathcal{A}_1^\varepsilon(z) &:= \sum_{j=1}^n \sum_{l=0,1} \frac{\theta_0^l}{\sqrt{\varepsilon}} \int \frac{\Theta(\sqrt{\varepsilon}V(z/\varepsilon))V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \mathbf{H}_{j,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon} + c.c., \end{aligned}$$

with

$$\begin{aligned} \mathbf{F}_{1,j,j',l,l',\varepsilon}(z) &:= \frac{i^2 \omega_j \omega_{j'} (-1)^{l+l'}}{4\lambda_\varepsilon(\kappa_j) \lambda_\varepsilon(\kappa_{j'}) c_0^4} F_M(\mathcal{X}_{\varepsilon,M}^{j,1-l}(z)) F_M(\mathcal{X}_{\varepsilon,M}^{j',1-l'}(z)) \partial_{\mathcal{X}^{j,l}}^2 \partial_{\mathcal{X}^{j',l'}}^2 f(\mathcal{X}_{\varepsilon,M}(z)) \\ \mathbf{F}_{2,j,j',l,l',\varepsilon}(z) &:= \frac{-i^2 \omega_j \omega_{j'} (-1)^{l+l'}}{4\lambda_\varepsilon(\kappa_j) \lambda_\varepsilon(\kappa_{j'}) c_0^4} F_M(\mathcal{X}_{\varepsilon,M}^{j,1-l}(z)) \overline{F_M(\mathcal{X}_{\varepsilon,M}^{j',1-l'}(z))} \partial_{\mathcal{X}^{j,l}}^2 \partial_{\mathcal{X}^{j',l'}}^2 f(\mathcal{X}_{\varepsilon,M}(z)) \\ \mathbf{G}_{1,j,l,\varepsilon}(z) &:= \frac{-i^2 \omega_j^2}{4\lambda_\varepsilon^2(\kappa_j) c_0^4} F_M(\mathcal{X}_{\varepsilon,M}^{j,l}(z)) \partial_{\mathcal{X}} F_M(\mathcal{X}_{\varepsilon,M}^{j,l}(z)) \partial_{\mathcal{X}^{j,l}} f(\mathcal{X}_{\varepsilon,M}(z)) \\ \mathbf{G}_{2,j,l,\varepsilon}(z) &:= \frac{i^2 \omega_j^2}{4\lambda_\varepsilon^2(\kappa_j) c_0^4} F_M(\mathcal{X}_{\varepsilon,M}^{j,l}(z)) \overline{\partial_{\mathcal{X}} F_M(\mathcal{X}_{\varepsilon,M}^{j,l}(z))} \partial_{\mathcal{X}^{j,l}} f(\mathcal{X}_{\varepsilon,M}(z)) \\ \mathbf{H}_{j,l,\varepsilon}(z) &:= \frac{-i^2 \omega_j^2}{4\lambda_\varepsilon^2(\kappa_j) c_0^4} F_M(\mathcal{X}_{\varepsilon,M}^{j,l}(z)) \partial_{\mathcal{X}^{j,l}} f(\mathcal{X}_{\varepsilon,M}(z)). \end{aligned}$$

We can see that both  $\mathcal{A}_0^\varepsilon$  and  $\mathcal{A}_1^\varepsilon$  are uniformly integrable thanks to the following lemma.

**Lemma 9.5** *We have*

$$\sup_{\varepsilon, z \in [0, L]} \mathbb{E}[|\mathcal{A}_0^\varepsilon(z)|^2] + \mathbb{E}[|\mathcal{A}_1^\varepsilon(z)|^2] < \infty.$$

**Proof (of Lemma 9.5)** Let us treat only the term  $\mathcal{A}_1^\varepsilon$  involving a term  $\Theta(\sqrt{\varepsilon}V)$ . The treatment of  $\mathcal{A}_0^\varepsilon$  follows the same lines once we get ride of the  $\Theta$  function in  $\mathcal{A}_1^\varepsilon$ .

From a second order Taylor expansion, we have

$$\begin{aligned} \mathcal{A}_1^\varepsilon(z) &= \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int \frac{V(z/\varepsilon)V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \mathbf{H}_{j,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} + c.c. \\ &+ \mathcal{O}\left(\varepsilon|V(z/\varepsilon)|^3 \sum_{j=1}^n |V(z/\varepsilon, \varphi_{1,j,\varepsilon})|\right), \end{aligned} \quad (54)$$

where  $\varphi_{1,j,\varepsilon}$  is defined by (52). Following the lines of the proof of Lemma 9.3, we obtain

$$|\mathcal{A}_1^\varepsilon(z)|^2 \leq K \left( \sup_{\varphi \in W_{k,C}} |V(z/\varepsilon, \varphi)|^4 + \varepsilon^2 \sup_{\varphi \in W_{k,C}} |V(z/\varepsilon, \varphi)|^8 \right),$$

for some appropriate positive constants  $K$  and  $C$ , with  $k = \infty$  if  $\beta \geq 1/2$ , and  $k \in (1, 1/(1-2\beta))$  if  $\beta < 1/2$ . This concludes the proof of Lemma 9.5 and then the one of Lemma 9.4 owing (53).  $\square$

$\square$

## 9.5 Identification of the limit

In this section we identify all the limit points of  $(\mathcal{X}_{\varepsilon, M})_\varepsilon$  through a martingale problem with infinitesimal generator

$$\begin{aligned} \mathcal{L}_M f(\mathcal{X}) &= \sum_{j=1}^n \sum_{l=0,1} \frac{\theta_0'^2 \omega_j^2}{4c_0^2} \int \frac{r(p)}{\mathfrak{g}(p) + 2i(\omega_j/c_0)(-1)^{l+1}} \\ &\quad \times \left( F_M(\mathcal{X}^{j,1-l}) F_M(\mathcal{X}^{j,l}) \partial_{\mathcal{X}^{j,1-l} \mathcal{X}^{j,l}}^2 f(\mathcal{X}) \right. \\ &\quad + F_M(\mathcal{X}_{\varepsilon, M}^{j,1-l}) \overline{F_M(\mathcal{X}^{j,1-l})} \partial_{\mathcal{X}^{j,l} \mathcal{X}^{j,l}}^2 f(\mathcal{X}) \\ &\quad \left. + F_M(\mathcal{X}^{j,l}) \partial_{\mathcal{X}} F_M(\mathcal{X}^{j,l}) \partial_{\mathcal{X}^{j,l}} f(\mathcal{X}) \right) \\ &+ c.c.. \end{aligned} \quad (55)$$

Let us start with the following remark. In view of (54) together with (51), we have

$$\sup_{\varepsilon, z \in [0, L]} \mathbb{E}[|\mathcal{A}_1^\varepsilon(z) - \mathcal{A}_1^{\prime\varepsilon}(z)|] = \mathcal{O}(\varepsilon),$$

with

$$\mathcal{A}_1^{\prime\varepsilon}(z) := \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int \frac{V(z/\varepsilon)V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j, z)/\varepsilon} + c.c..$$

As a result, remembering (53), we have

$$\sup_{z \in [0, L]} \mathbb{E}[|\mathcal{A}^\varepsilon(f_0^\varepsilon + f_1^\varepsilon)(z) - \mathcal{A}_0^\varepsilon(z) - \mathcal{A}_1^{\prime\varepsilon}(z)|] = \mathcal{O}(\varepsilon),$$

so that to determine the infinitesimal generator of the limit points we only have to focus on  $\mathcal{A}_0^\varepsilon + \mathcal{A}_1^{\prime\varepsilon}$ . For this term, we separate the terms which exhibit a fast phase from the others, since the former will average out and do not contribute at the limit  $\varepsilon \rightarrow 0$ . We then write

$$\mathcal{A}_0^\varepsilon(z) + \mathcal{A}_1^{\prime\varepsilon}(z) = \mathcal{B}_0^\varepsilon(z) + \mathcal{B}_1^\varepsilon(z),$$

where

$$\begin{aligned} \mathcal{B}_0^\varepsilon(z) &:= \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int \frac{V(z/\varepsilon)V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \left( \mathbf{F}_{1,j,j,l,1-l,\varepsilon}(z) + \mathbf{F}_{2,j,j,l,l,\varepsilon}(z) + \mathbf{G}_{1,j,l,\varepsilon}(z) \right) \\ &+ c.c., \end{aligned}$$



and

$$\begin{aligned} \mathcal{B}_1^\varepsilon(z) &:= \mathcal{A}_1^\varepsilon(z) + \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int \frac{V(z/\varepsilon)V(z/\varepsilon, dp)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ &\quad \times \left( \sum_{\substack{j \neq j' \\ \text{or} \\ l=l'}} \mathbf{F}_{1,j,j',l,l',\varepsilon}(z) e^{i(\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z) + \omega_{j'}(-1)^{l'+1}\tau_\varepsilon(\kappa_{j'},z))/\varepsilon} \right. \\ &\quad \left. + \sum_{\substack{j \neq j' \\ \text{or} \\ l=1-l'}} \mathbf{F}_{2,j,j',l,l',\varepsilon}(z) e^{i(\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z) - \omega_{j'}(-1)^{l'+1}\tau_\varepsilon(\kappa_{j'},z))/\varepsilon} \right. \\ &\quad \left. + \mathbf{G}_{2,j,l,\varepsilon}(z) e^{2i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon} \right) + c.c.. \end{aligned}$$

### 9.5.1 The term $\mathcal{B}_1^\varepsilon$

Because of its rapid phases this term does not contribute to the limit. To prove this we start by introducing another test function to average out the stochastic terms involving  $V$ . For notational convenience, and without loss of generality, we only treat the term involving  $\mathcal{A}_1^\varepsilon$ . The other terms are treated exactly the same way.

Setting

$$\begin{aligned} f_2^\varepsilon(z) &:= \int_z^\infty ds \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \iint \frac{\mathbb{E}_z^\varepsilon[V(s/\varepsilon, dp)V(s/\varepsilon, dq)] - \mathbb{E}[V(0, dp)V(0, dq)]}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ &\quad \times \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}(\tau_\varepsilon(\kappa_j,z) + 2\lambda_\varepsilon(\kappa_j)(s-z))/\varepsilon}, \end{aligned}$$

we have

$$\mathcal{A}^\varepsilon(f_2^\varepsilon)(z) = -\mathcal{A}_1^\varepsilon(z) + \mathcal{A}_1^{\prime\varepsilon}(z) + \sqrt{\varepsilon} \mathcal{R}_1^\varepsilon(z),$$

where

$$\mathcal{A}_1^{\prime\varepsilon}(z) := \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int \frac{r(p)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon} + c.c.,$$

and the following lemma.

**Lemma 9.6** *We have*

$$\sup_{z \in [0,L]} \mathbb{E}[|f_2^\varepsilon(t)|] = \mathcal{O}(\varepsilon) \quad \text{and} \quad \sup_{\varepsilon, z \in [0,L]} \mathbb{E}[|\mathcal{R}_1^\varepsilon(z)|] < \infty.$$

**Proof (of Lemma 9.6)** Making the change of variable  $s \rightarrow z + \varepsilon s$ , together with (49), and integrating in  $s$ , gives

$$\begin{aligned} f_2^\varepsilon(z) &= \varepsilon \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \iint \frac{V(z/\varepsilon, dp)V(z/\varepsilon, dq) - r(p)\delta(p-q)dpdq}{(\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j))(\mathfrak{g}(p) + \mathfrak{g}(q) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j))} \\ &\quad \times \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon} \\ &=: f_{21}^\varepsilon(z) + f_{22}^\varepsilon(z), \end{aligned}$$

where  $f_{21}^\varepsilon$  corresponds to the term with  $V(z/\varepsilon, dp)V(z/\varepsilon, dq)$  in  $f_2^\varepsilon$ , and  $f_{22}^\varepsilon$  the one with  $r(p)$ . It is direct to see that

$$\sup_{z \in [0,L]} \mathbb{E}[|f_{22}^\varepsilon(z)|] = \mathcal{O}(\varepsilon),$$

and we only need to focus on  $f_{21}^\varepsilon$ .

Now, let us denote

$$V(s, \varphi_{2,j,s,\varepsilon}) = \int_S \int_S \frac{V(s, dp)V(s, dq)}{(\mathfrak{g}(p) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j))(\mathfrak{g}(p) + \mathfrak{g}(q) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j))},$$

where

$$\varphi_{2,j,s,\varepsilon}(p) := V(s, \varphi_{3,j,p,\varepsilon}),$$

and

$$\varphi_{3,j,p,\varepsilon}(q) := \frac{1}{(\mathfrak{g}(p) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j))(\mathfrak{g}(p) + \mathfrak{g}(q) \pm 2i\omega_j\lambda_\varepsilon(\kappa_j))}.$$

In the same way as for  $\varphi_{1,j,\varepsilon}$  in the proof of Lemma 9.3, for  $k = \infty$  if  $\beta \geq 1/2$ , or  $k \in (1, 1/(1-2\beta))$  if  $\beta < 1/2$ , we can remark that for any  $p$  the function  $\varphi_{3,j,p,\varepsilon}$  belongs to  $W_{k,C'}$  for some constant  $C' > 0$  independent to  $p$ , and so that

$$\sup_{p \in S} |\varphi_{2,j,s,\varepsilon}(p)| \leq \sup_{\varphi \in W_{k,C'}} |V(s, \varphi)|.$$

Also, we have

$$\partial_p \varphi_{2,j,s,\varepsilon}(p) = V(s, \tilde{\varphi}_{3,j,p,\varepsilon}),$$

with  $\tilde{\varphi}_{3,j,p,\varepsilon}$  belonging to  $W_{k,C''}$  where  $C'' > 0$  does not depend on  $p$ , and then

$$\sup_{p \in S} |\partial_p \varphi_{2,j,s,\varepsilon}(p)| \leq \sup_{\varphi \in W_{k,C''}} |V(s, \varphi)|.$$

Considering  $C''' = \max(C', C'')$ , we have

$$\|\varphi_{2,j,s,\varepsilon}\|_{W^{1,k}(S)} \leq \sup_{\varphi \in W_{k,C'''}} |V(s, \varphi)|,$$

and setting

$$\tilde{\varphi}_{2,j,s,\varepsilon} := \frac{\varphi_{2,j,s,\varepsilon}}{\sup_{\varphi \in W_{k,C'''}} |V(s, \varphi)|},$$

we have  $\|\tilde{\varphi}_{2,j,s,\varepsilon}\|_{W^{1,k}(S)} \leq 1$  and then  $\tilde{\varphi}_{2,j,s,\varepsilon} \in W_{k,1}$ . As a result, we obtain

$$|V(z/\varepsilon, \varphi_{2,j,z/\varepsilon,\varepsilon})| = \sup_{\varphi \in W_{k,C'''}} |V(z/\varepsilon, \varphi)| |V(z/\varepsilon, \tilde{\varphi}_{2,j,z/\varepsilon,\varepsilon})| \leq \sup_{\varphi \in W_{k,C_1}} |V(z/\varepsilon, \varphi)|^2,$$

with  $C_1 = \max(1, C''')$ , so that

$$\sup_{z \in [0,L]} \mathbb{E}[|f_{21}^\varepsilon(z)|] = \mathcal{O}(\varepsilon),$$

according to (51).

The term  $\mathcal{R}_1^\varepsilon(z)$  is treated in a similar way and we omit the precise details. This concludes the proof of Lemma 9.6.  $\square$

Now, to deal with the rapid phases of  $\mathcal{A}_1^{\prime\varepsilon}$ , we introduce the following test function

$$f_3^\varepsilon(z) := \int_z^\infty ds \sum_{j=1}^n \sum_{l=0,1} \theta_0^{\prime 2} \iint \frac{r(p)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ \times \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}(\tau_\varepsilon(\kappa_j,z) + 2\lambda_\varepsilon(\kappa_j)(s-z))/\varepsilon} e^{-\sqrt{\varepsilon}(s-z)},$$

so that making again the change of variable  $s \rightarrow z + \varepsilon s$  and integrating in  $s$ , we have

$$f_3^\varepsilon(z) := \varepsilon \sum_{j=1}^n \sum_{l=0,1} \theta_0^{\prime 2} \iint \frac{r(p) \mathbf{H}_{j,l,\varepsilon}(z) e^{i\omega_j(-1)^{l+1}\tau_\varepsilon(\kappa_j,z)/\varepsilon}}{(\varepsilon^{3/2} - 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j))(\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j))},$$

satisfying

$$\sup_{z \in [0,L]} \mathbb{E}[|f_3^\varepsilon(z)|] = \mathcal{O}(\varepsilon).$$

Now, differentiating in  $z$ , we obtain

$$\mathcal{A}^\varepsilon(f_3^\varepsilon)(z) = -\mathcal{A}_0^{\prime\varepsilon}(z) + \mathcal{R}_2^\varepsilon(z),$$

with

$$\mathcal{R}_2^\varepsilon(z) := \int_z^\infty ds \sum_{j=1}^n \sum_{l=0,1} \theta_0^{\prime 2} \iint \frac{r(p)}{\mathfrak{g}(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ \times e^{i\omega_j(-1)^{l+1}(\tau_\varepsilon(\kappa_j,z) + 2\lambda_\varepsilon(\kappa_j)(s-z))/\varepsilon} e^{-\sqrt{\varepsilon}(s-z)} \\ \times \left( \frac{d}{dz} \mathbf{H}_{j,l,\varepsilon}(z) + \frac{i\omega_j(-1)^{l+1}}{2\varepsilon\lambda_\varepsilon(\kappa_j)c_0^2} \mathbf{H}_{j,l,\varepsilon}(z) \Theta(\sqrt{\varepsilon}V(z/\varepsilon)) + \sqrt{\varepsilon} \right)$$

and for which

$$|\mathcal{R}_2^\varepsilon(z)| \leq \varepsilon K \left( \frac{|V(z/\varepsilon)|}{\sqrt{\varepsilon}} + |V(z/\varepsilon)|^2 + \sqrt{\varepsilon} \right) \frac{1}{|\varepsilon^{3/2} - 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)|},$$

after the change of variable  $s \rightarrow z + \varepsilon s$  and integrating in  $s$ . As a result, thanks to (51), we have

$$\sup_{z \in [0,L]} \mathbb{E}[|\mathcal{R}_2^\varepsilon(z)|] = \mathcal{O}(\sqrt{\varepsilon}).$$

To sum up, we have

$$\sup_{z \in [0,L]} \mathbb{E}[|\mathcal{A}^\varepsilon(f_0^\varepsilon + f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon)(z) - \mathcal{B}_0^\varepsilon(z)|] = \mathcal{O}(\sqrt{\varepsilon}),$$

and it only remains to determine the asymptotic of  $\mathcal{B}_0^\varepsilon$ .

### 9.5.2 The term $\mathcal{B}_0^\varepsilon$

This term need a careful treatment to average the random process. In fact, for this term we cannot proceed as for  $\mathcal{B}_1^\varepsilon$  with a test function like  $f_2^\varepsilon$  since there is no remaining phases and would provide a term of the form  $\int dp r(p)/g(p) = \infty$  for long-range correlations. The asymptotic of this term is given by the following result.

**Proposition 9.3** *We have*

$$\lim_{\varepsilon \rightarrow 0} \sup_{z \in [0, L]} \mathbb{E} \left[ \left| \int_0^z \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_M f(\mathcal{X}_{\varepsilon, M}(s)) ds \right| \right] = 0,$$

where  $\mathcal{L}_M$  is defined by (55).

**Proof (of Proposition 9.3)** Here, we only have to prove

$$\lim_{\varepsilon \rightarrow 0} \sup_{z \in [0, L]} \mathbb{E} \left[ \left| \int_0^z \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s)) ds \right| \right] = 0, \quad (56)$$

with

$$\begin{aligned} \mathcal{L}_{\varepsilon, M} f(\mathcal{X}) &:= \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int dp \frac{r(p)}{g(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)} \\ &\quad \times \left( \mathbf{F}_{1,j,j,l,1-l,\varepsilon}(z) + \mathbf{F}_{2,j,j,l,l,\varepsilon}(z) + \mathbf{G}_{1,j,l,\varepsilon}(z) \right), \end{aligned}$$

since passing from  $\mathcal{L}_{\varepsilon, M}$  to  $\mathcal{L}_M = \mathcal{L}_{\varepsilon=0, M}$  being straightforward owing (33). Also, in this proof, to simplify the notations let us denote

$$\mathbf{F}_{j,l,\varepsilon}(z) := \left( \mathbf{F}_{1,j,j,l,1-l,\varepsilon}(z) + \mathbf{F}_{2,j,j,l,l,\varepsilon}(z) + \mathbf{G}_{1,j,l,\varepsilon}(z) \right) / (g(p) + 2i\omega_j(-1)^{l+1}\lambda_\varepsilon(\kappa_j)), \quad (57)$$

so that

$$\int_0^z \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s)) ds = \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \int_0^z ds \mathbf{F}_{j,l,\varepsilon}(s) \int V(s/\varepsilon, dp) V(s/\varepsilon, dq) - r(p)\delta(p-q) dpdq.$$

To prove (56), we decompose the interval  $[0, z]$  over a uniform grid with a small stepsize of order  $\varepsilon$  in order to get rid of the  $s$ -dependence of  $\mathbf{F}_{j,l,\varepsilon}$ , and then average out the  $V$ 's. Let  $\eta > 0$  be an arbitrary small parameter and write

$$\begin{aligned} \int_0^z \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s)) ds &= \sum_{m=0}^{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor - 1} \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s)) ds \\ &\quad + \int_{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor \sqrt{\varepsilon}\eta}^z \mathcal{B}_0^\varepsilon(s) - \mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s)) ds \\ &=: \mathcal{R}_3^\varepsilon(z) + \mathcal{R}_4^\varepsilon(z). \end{aligned}$$

For  $\mathcal{R}_4^\varepsilon$ , we have

$$\mathbb{E}[\mathcal{R}_4^\varepsilon(z)] \leq (z - \lfloor z/(\sqrt{\varepsilon}\eta) \rfloor \sqrt{\varepsilon}\eta) \left( \sup_{s \in [0, L]} \mathbb{E}[|\mathcal{B}_0^\varepsilon(s)|] + K \right) \leq K' \sqrt{\varepsilon},$$

where  $K$  is some constant that bounds uniformly  $\mathcal{L}_{\varepsilon, M} f(\mathcal{X}_{\varepsilon, M}(s))$ . The last inequality is obtained by bounding  $\sup_{\varepsilon, s \in [0, L]} \mathbb{E}[|\mathcal{B}_0^\varepsilon(s)|]$  as a subcase of Lemma 9.5.

Regarding  $\mathcal{R}_3^\varepsilon$ , we have

$$\begin{aligned} \mathcal{R}_3^\varepsilon(z) &= \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \sum_{m=0}^{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor - 1} \mathbf{F}_{j,l,\varepsilon}(mq\eta) \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds \int (V(s/\varepsilon, dp) V(s/\varepsilon, dq) - r(p)\delta(p-q) dpdq) \\ &\quad + \sum_{j=1}^n \sum_{l=0,1} \theta_0'^2 \sum_{m=0}^{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor - 1} \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds \int (V(s/\varepsilon, dp) V(s/\varepsilon, dq) - r(p)\delta(p-q) dpdq) \\ &\quad \quad \quad \times (\mathbf{F}_{j,l,\varepsilon}(s) - \mathbf{F}_{j,l,\varepsilon}(m\sqrt{\varepsilon}\eta)) \\ &=: \mathcal{R}_{31}^\varepsilon(z) + \mathcal{R}_{32}^\varepsilon(z). \end{aligned}$$

For  $\mathcal{R}_{32}^\varepsilon(z)$ , we have from (47)

$$\mathbb{E}[|\mathcal{R}_{42}^\varepsilon(z)|] \leq \frac{K}{\sqrt{\varepsilon}} \sum_{m=0}^{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor - 1} \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds \int_{m\sqrt{\varepsilon}\eta}^s ds' \leq \eta K',$$

where the constant  $K > 0$  is obtained through similar arguments to obtain the first part of Lemma 9.6 using the denominator of (57). Finally for  $\mathcal{R}_{31}^\varepsilon(z)$ , using the Cauchy-Schwarz inequality w.r.t. the expectation, we have

$$\mathbb{E}[|\mathcal{R}_{31}^\varepsilon(z)|] \leq K \sum_{m=0}^{\lfloor z/(\sqrt{\varepsilon}\eta) \rfloor - 1} \sqrt{I_{m,\eta}^\varepsilon},$$

for some constant  $K > 0$ , with

$$I_{m,\eta}^\varepsilon := \mathbb{E} \left[ \left| \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds \iint V(s/\varepsilon, dp) V(s/\varepsilon, dq) - r(p)\delta(p-q) dp dq \right|^2 \right].$$

Therefore, using the Gaussianity of  $V$ , we have

$$I_{m,\eta}^\varepsilon = \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds_1 \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds_2 \iiint \left( \mathbb{E} \left[ V\left(\frac{s_1}{\varepsilon}, dp_1\right) V\left(\frac{s_2}{\varepsilon}, dq_1\right) \right] \mathbb{E} \left[ V\left(\frac{s_1}{\varepsilon}, dp_2\right) V\left(\frac{s_2}{\varepsilon}, dq_2\right) \right] \right. \\ \left. + \mathbb{E} \left[ V\left(\frac{s_1}{\varepsilon}, dp_1\right) V\left(\frac{s_2}{\varepsilon}, dq_2\right) \right] \mathbb{E} \left[ V\left(\frac{s_1}{\varepsilon}, dp_2\right), V\left(\frac{s_2}{\varepsilon}, dq_1\right) \right] \right)$$

and by symmetry w.r.t.  $s_1$  and  $s_2$  we obtain

$$I_{m,\eta}^\varepsilon \leq \iint dp dq r(p) r(q) \int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds_1 \int_{m\sqrt{\varepsilon}\eta}^{s_1} ds_2 e^{-(\mathfrak{g}(p)+\mathfrak{g}(q))(s_1-s_2)/\varepsilon}.$$

Now to bound  $I_{m,\eta}^\varepsilon$  properly we consider two cases. Integrating without any caution would provide a term of the form  $r(p)r(q)/(\mathfrak{g}(p)+\mathfrak{g}(q))$  which is not integrable at 0 in case of long-range correlations.

Let  $\eta' > 0$  be an arbitrary small parameter. If  $|p| \leq \eta'$ , we can bound the exponential term by 1 and obtain a term of the form

$$\varepsilon \eta^2 \int_{\{|p| \leq \eta'\}} r(p) dp,$$

upto a constant independent of  $m$ ,  $\varepsilon$ ,  $\eta$  or  $\eta'$ , and where the term  $\varepsilon \eta^2$  is compensated by the one of the above sum in  $m$ . However, this term goes to 0 as  $\eta' \rightarrow 0$  thanks to the integrability of  $r$  and the dominated convergence theorem. Now, for  $|p| > \eta'$ , that is we place ourselves away from 0, the point that can produce non integrability, we have

$$\int_{m\sqrt{\varepsilon}\eta}^{(m+1)\sqrt{\varepsilon}\eta} ds_1 \int_{m\sqrt{\varepsilon}\eta}^{s_1} ds_2 e^{-(\mathfrak{g}(p)+\mathfrak{g}(q))(s_1-s_2)/\varepsilon} \leq \frac{\varepsilon}{\mathfrak{g}(p)} \int_0^{\sqrt{\varepsilon}\eta} (1 - e^{-\mathfrak{g}(p)s_1/\varepsilon}) ds_1 \leq K_{\eta'} \varepsilon^{3/2} \eta.$$

As a result, one can finally write

$$\mathbb{E}[|\mathcal{R}_{31}^\varepsilon(z)|] \leq K' \left( \left( \int_{\{|p| \leq \eta'\}} r(p) dp \right)^{1/2} + K_{\eta'} \varepsilon^{1/4} \right),$$

which concludes the proof of Proposition 9.3.  $\blacksquare$

All the limit points are then solution to a martingale problem associated to the infinitesimal generator  $\mathcal{L}_M$  given by (55). From that definition it is direct to see that for  $\|\mathcal{X}\| \leq M$ , with  $\mathcal{X} \in \mathbb{C}^{2n}$ , we have

$$\mathcal{L}_M f(\mathcal{X}) = \mathcal{L} f(\mathcal{X}),$$

where

$$\mathcal{L} f(\mathcal{X}) := \mathcal{L} f_{M=\infty}(\mathcal{X}) = \sum_{j=1}^n \sum_{l=0,1} \frac{\theta_0^2 \omega_j^2}{4c_0^2} \int \frac{r(p)}{\mathfrak{g}(p) + 2i(\omega_j/c_0)(-1)^{l+1}} \\ \times \left( \mathcal{X}^{j,1-l} \mathcal{X}^{j,l} \partial_{\mathcal{X}^{j,1-l} \mathcal{X}^{j,l}}^2 f(\mathcal{X}) \right. \\ \left. + \mathcal{X}_{\varepsilon,M}^{j,1-l} \overline{\mathcal{X}}^{j,1-l} \partial_{\mathcal{X}^{j,l} \mathcal{X}^{j,l}}^2 f(\mathcal{X}) \right. \\ \left. + \mathcal{X}^{j,l} \partial_{\mathcal{X}} \mathcal{X}^{j,l} \partial_{\mathcal{X}^{j,l}} f(\mathcal{X}) \right) \\ + c.c..$$

Also, one can see that there is no coupling between two components  $(\mathcal{X}^{j,0}, \mathcal{X}^{j,1})$  and  $(\mathcal{X}^{j',0}, \mathcal{X}^{j',1})$  for  $j \neq j'$  meaning that the components of the limiting process are independent. Therefore, we only have to look at the well-posedness of the (untruncated) martingale problem associated to one coordinate, that is for one frequency. The corresponding infinitesimal generator then writes

$$\begin{aligned} \mathbf{L}f(\mathcal{X}) &= \sum_{l=0,1} \frac{\theta_0^2 \omega^2}{4c_0^2} \int \frac{r(p)}{\mathbf{g}(p) + 2i(\omega/c_0)(-1)^{l+1}} \\ &\quad \times \left( \mathcal{X}^{1-l} \mathcal{X}^l \partial_{\mathcal{X}^{1-l} \mathcal{X}^l}^2 f(\mathcal{X}) + \mathcal{X}^{1-l} \overline{\mathcal{X}^{1-l}} \partial_{\mathcal{X}^l \mathcal{X}^l}^2 f(\mathcal{X}) + \mathcal{X}^l \partial_{\mathcal{X}^l} f(\mathcal{X}) \right) \\ &+ c.c. \\ &= \frac{\theta_0^2 \omega^2 \Gamma_c(\omega)}{4c_0^2} \left( \mathcal{X}^1 \mathcal{X}^0 \partial_{\mathcal{X}^1 \mathcal{X}^0}^2 f(\mathcal{X}) + \overline{\mathcal{X}^1} \overline{\mathcal{X}^0} \partial_{\overline{\mathcal{X}^1} \overline{\mathcal{X}^0}}^2 f(\mathcal{X}) + \mathcal{X}^1 \overline{\mathcal{X}^1} \partial_{\mathcal{X}^0 \overline{\mathcal{X}^0}}^2 f(\mathcal{X}) + \mathcal{X}^0 \overline{\mathcal{X}^0} \partial_{\mathcal{X}^1 \overline{\mathcal{X}^1}}^2 f(\mathcal{X}) \right) \\ &+ \frac{\theta_0^2 \omega^2 \Gamma_c(\omega)}{8c_0^2} \left( \mathcal{X}^0 \partial_{\mathcal{X}^0} f(\mathcal{X}) + \mathcal{X}^1 \partial_{\mathcal{X}^1} f(\mathcal{X}) + \overline{\mathcal{X}^0} \partial_{\overline{\mathcal{X}^0}} f(\mathcal{X}) + \overline{\mathcal{X}^1} \partial_{\overline{\mathcal{X}^1}} f(\mathcal{X}) \right) \\ &+ i \frac{\theta_0^2 \omega^2 \Gamma_s(\omega)}{8c_0^2} \left( \mathcal{X}^0 \partial_{\mathcal{X}^0} f(\mathcal{X}) - \mathcal{X}^1 \partial_{\mathcal{X}^1} f(\mathcal{X}) - \overline{\mathcal{X}^0} \partial_{\overline{\mathcal{X}^0}} f(\mathcal{X}) + \overline{\mathcal{X}^1} \partial_{\overline{\mathcal{X}^1}} f(\mathcal{X}) \right), \end{aligned}$$

for  $\mathcal{X} \in \mathbb{C}^2$ , where

$$\Gamma_c(\omega) := 2 \int_0^\infty R(s) \cos(2\omega s/c_0) ds \quad \text{and} \quad \Gamma_s(\omega) := 2 \int_0^\infty R(s) \sin(2\omega s/c_0) ds.$$

Finally, from a martingale representation theorem [25, Proposition 4.6 pp. 315], upto an extension of the underlying probability space, the solutions to this martingale problem can be represented as solutions to the following stochastic differential equation

$$\begin{aligned} d\mathcal{X}_0(z) &= -\sqrt{\frac{\omega^2 \Gamma_c(\omega)}{4c_0^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{X}_0(z) \circ dW_1(z) \\ &\quad - i \sqrt{\frac{\omega^2 \Gamma_c(\omega)}{4c_0^2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{X}_0(z) \circ dW_2(z) \\ &\quad - i \frac{\omega^2 \Gamma_s(\omega)}{8c_0^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{X}_0(z) dz \end{aligned}$$

where  $W_1$  and  $W_2$  are two independent real-valued standard Brownian motions, and  $\circ$  stands for the Stratonovich integral. This equation is readily well-posed, in the sens of probability law, since the diffusion coefficients and drift are linear [25, Theorem 2.5 pp. 287 and Proposition 3.20 pp. 309]. As a result, all the accumulation points have the same distribution, and this concludes the proof of Theorem 6.1.

## Conclusion

In this paper we have analyzed high-frequency waves propagating in randomly layered media with long-range correlations in the weak coupling regime. In this context the waves are affected in two ways. First, they exhibit a random time-shift that can be characterized as a fractional Brownian motion (or a Brownian motion for  $\gamma = 1$ ), but with a standard deviation which is very large compared to the pulse width. Second, the wave-front spreading is deterministic and can be characterized as the solution to a paraxial wave equation involving a pseudo-differential operator. This operator exhibits a frequency-depend attenuation and phase modulation, depending on the correlation function of the medium fluctuations, ensuring the causality of the limiting paraxial equation as well as the Kramers-Kronig relations that guarantees the physical relevance of this equation. The frequency-dependent attenuation is shown to be close to the form (1) for  $\lambda \in (1, 2]$ . Moreover, this pseudo-differential operator can be approximated by a fractional Weyl derivative, with order depending on  $\gamma$ , the power law decay of the correlation function of the random medium.

The noise model considered in this paper presents some restrictions that could be removed since the scattering coefficients rely on the correlation function  $R$ , not the particular structure of  $V$ . Considering more general models could also allow us to obtain exponent  $\lambda \in (0, 1)$  for the attenuation power law, and the rough path theory could be helpful to handle more general settings. Nevertheless, our method opens the road to analyze more general 3D settings with random variations with respect to the transverse section, which will be the aim of future works.

## A Proof of Proposition 3.1

The analyticity  $\omega \mapsto \omega^2 \Gamma_c(\omega)$  and  $\omega \mapsto \omega^2 \Gamma_s(\omega)$  over the upper complex half-plane is direct to see. Let us now introduce a notation and makes two remarks. Defining the Fourier inverse Fourier transform of (18)

$$\mathcal{F}^{-1}(\psi)(s) := \frac{1}{2\pi} \int e^{-i\omega s} \psi(\omega) d\omega,$$

the Fourier transform of the Hilbert transform reads

$$\mathcal{F}^{-1}(\mathcal{H}(\psi))(s) = -i \operatorname{sign}(s) \mathcal{F}^{-1}(\psi)(s).$$

From this relation, we clearly see that  $\mathcal{H}(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$ . The second remark is that by applying two integrations by part we have

$$\omega^2 (\Gamma_c(\omega) + i \Gamma_s(\omega)) = -\frac{c_0^2}{2} \int_0^\infty R''(s) e^{2i\omega s/c_0} ds,$$

so that for any test function  $\psi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} \langle \mathcal{H}(\omega'^2 \Gamma_c(\omega')), \psi \rangle_{\mathcal{S}', \mathcal{S}} &= \frac{c_0^2}{2} \int_0^\infty ds R''(s) \int d\omega \cos(2\omega s/c_0) \mathcal{H}(\psi)(\omega) \\ &= -\frac{c_0^2}{2} \int_0^\infty ds R''(s) \int d\omega \mathcal{H}(\cos(2\omega' s/c_0))(\omega) \psi(\omega) \end{aligned}$$

thanks to the Fubini theorem since  $R''$  is integrable. Now, using that

$$\mathcal{H}(\cos(2\omega' s/c_0))(\omega) = \sin(2\omega s/c_0),$$

we obtain

$$\begin{aligned} \langle \mathcal{H}(\omega'^2 \Gamma_c(\omega')), \psi \rangle_{\mathcal{S}', \mathcal{S}} &= -\frac{c_0^2}{2} \int_0^\infty ds R''(s) \int d\omega \sin(2\omega s/c_0) \psi(\omega) \\ &= \langle \omega'^2 \Gamma_s(\omega'), \psi \rangle_{\mathcal{S}', \mathcal{S}}, \end{aligned}$$

yielding the first relation. To obtain the second relations, we follow the same lines but now using

$$\mathcal{H}(\sin(2\omega' s/c_0))(\omega) = -\cos(2\omega s/c_0),$$

providing the minus sign for this relation, and then concludes the proof.

## B Proof of Propositions 4.1

Setting

$$\tilde{W}_\varepsilon(L) := \frac{\sqrt{\varepsilon} \theta'_0}{\sigma_\varepsilon} \int_0^L V(u/\varepsilon) du,$$

we have from a Taylor expansion of  $\Theta$  at the second order (remember that  $\Theta$  is odd, and then  $\Theta''(0) = 0$ ) and (51)

$$\begin{aligned} \mathbb{E}[|W_\varepsilon(L) - \tilde{W}_\varepsilon(L)|] &\leq \frac{1}{c_0 \sigma_\varepsilon} \int_0^L \mathbb{E}[|\nu_\varepsilon(u/\varepsilon) - \sqrt{\varepsilon} \theta'_0 V(u/\varepsilon)|] du \\ &\leq \frac{\sup |\Theta'''| L}{c_0 \sigma_\varepsilon} \varepsilon^{3/2} \sup_{z \in [0, L/\varepsilon]} \mathbb{E}[|V(z)|^3] \xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned}$$

so that we only need to focus on  $\tilde{W}_\varepsilon$  according to [4, Theorem 3.1 pp. 27]. For  $\gamma > 1$ , the convergence of  $\tilde{W}_\varepsilon$  to  $W_0$  is given by an invariant principle (see [31, Theorem 4] for a more advanced result), and for  $\gamma \in (0, 1]$  we refer to [21, Proposition 1.3]. This concludes the proof of the proposition.

## C Proof of Proposition 4.2

The proof is in two steps. The first step consists in approximating the time delay in a more convenient form. The second step consists in looking at the expectation and variance of this new expression to obtain the convergence in probability.

For the first step, using the Taylor expansion of  $u \mapsto \sqrt{1+u}$  at the second order, we have

$$\frac{\Delta T_\varepsilon(L)}{\varepsilon} = \frac{1}{8c_0\varepsilon} \int_0^L \nu_\varepsilon^2(z/\varepsilon) dz + E_\varepsilon^1(L),$$

with for any  $\eta > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(|E_\varepsilon^1(L)| > \eta) = 0.$$

In fact, for any  $\eta' > 0$  we have

$$\mathbb{P}(|E_\varepsilon^1(L)| > \eta) \leq \mathbb{P}\left(|E_\varepsilon^1(L)| > \eta, \sqrt{\varepsilon} \sup_{z \in [0, L/\varepsilon]} |V(z/\varepsilon)| \leq \eta'\right) + \mathbb{P}\left(\sqrt{\varepsilon} \sup_{z \in [0, L/\varepsilon]} |V(z/\varepsilon)| > \eta'\right),$$

where the second term on the r.h.s. goes to 0 as  $\varepsilon \rightarrow 0$  thanks to (50) together with the Markov inequality. Now, working on the event  $(\sqrt{\varepsilon} \sup_{z \in [0, L/\varepsilon]} |V(z/\varepsilon)| \leq \eta')$ , we have for  $\eta' \in (0, 1/\sup |\Theta'|)$

$$|E_\varepsilon^1(L)| \leq \frac{\sqrt{\varepsilon} \sup |\Theta'|^3}{4c_0(1 - \eta' \sup |\Theta'|)^{5/2}} \int_0^L |V(z/\varepsilon)|^3,$$

so that

$$\mathbb{E}\left[|E_\varepsilon^1(L)| \mathbf{1}_{(\sup_{z \in [0, L/\varepsilon]} |V(z/\varepsilon)| \leq \eta')}\right] \leq \frac{\sup |\Theta'|^3 L}{4c_0(1 - \eta' \sup |\Theta'|)^{5/2}} \sqrt{\varepsilon} \sup_{z \in [0, L/\varepsilon]} \mathbb{E}[|V(z/\varepsilon)|^3] \xrightarrow{\varepsilon \rightarrow 0} 0$$

thanks to (51). Also, we have from a second order Taylor expansion for  $\Theta$  (using that  $\Theta''(0) = 0$ )

$$\frac{1}{8c_0\varepsilon} \int_0^L \nu_\varepsilon^2(z/\varepsilon) dz = D_\varepsilon(L) + E_\varepsilon^2(L),$$

with

$$D_\varepsilon(L) := \frac{\theta_0'^2}{8c_0} \int_0^L V^2(z/\varepsilon) dz,$$

and for any  $\eta > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(|E_\varepsilon^2(L)| > \eta) = 0.$$

In fact, we have

$$\mathbb{E}\left[\left|\frac{1}{8c_0\varepsilon} \int_0^L \nu_\varepsilon^2(z/\varepsilon) dz - D_\varepsilon(L)\right|\right] \leq \varepsilon \frac{|\theta_0'| \sup |\Theta'''| L}{24c_0} \sup_{z \in [0, L/\varepsilon]} \mathbb{E}[|V(z)|^4] + \varepsilon^3 \frac{\sup |\Theta'''|^2 L}{288c_0} \sup_{z \in [0, L/\varepsilon]} \mathbb{E}[|V(z)|^6]$$

which goes to 0 as  $\varepsilon \rightarrow 0$  thanks to (51). As a result, from the Markov inequality, we have for any  $\eta > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\left(\left|\frac{\Delta T_\varepsilon(L)}{\varepsilon} - D_\varepsilon(L)\right| > \eta\right) = 0,$$

so that we only need to focus on the convergence of  $D_\varepsilon(L)$ . Regarding its expectation we have

$$\mathbb{E}[D_\varepsilon(L)] = \frac{\theta_0'^2 R(0)L}{8c_0}.$$

Now, thanks to the Chebyshev inequality, for any  $\eta > 0$  we have

$$\mathbb{P}\left(\left|D_\varepsilon(L) - \frac{\theta_0'^2 R(0)L}{8c_0}\right| > \eta\right) \leq \frac{\text{Var}[D_\varepsilon(L)]}{\eta^2},$$

and from the Gaussianity of  $V$ ,

$$\begin{aligned} \text{Var}[D_\varepsilon(L)] &= \frac{\theta_0'^4}{64c_0^2} \int_0^L \int_0^L \mathbb{E}[V^2(z_1/\varepsilon)V^2(z_2/\varepsilon)] dz_1 dz_2 - \left(\frac{\theta_0'^2 R(0)L}{8c_0}\right)^2 \\ &= \frac{\theta_0'^4}{32c_0^2} \int_0^L \int_0^L R^2\left(\frac{z_1 - z_2}{\varepsilon}\right) dz_1 dz_2 \\ &\xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned}$$

which concludes the proof of the proposition thanks to the dominated convergence theorem.

## D Proof of Lemma 8.1

**For  $\gamma > 1$**  Thanks to the integrability of the correlation function and making the change of variable  $s \rightarrow l_0 s$ , we have

$$\Gamma_c(\omega, l_0) + i\Gamma_s(\omega, l_0) = 2 \int_0^\infty R(s) e^{2i\omega s l_0 / c_0} ds \xrightarrow{l_0 \rightarrow 0} 2 \int_0^\infty R(s) ds = \Gamma_c(0),$$

using to the dominated convergence theorem.

**For  $\gamma \in (0, 1)$**  After integrating in  $s$ , we have

$$\Gamma_c(\omega, l_0) + i\Gamma_s(\omega, l_0) = \frac{2}{l_0^\gamma} \int_S \frac{a(p) dp}{|p|^{2\alpha} (\mu |p|^{2\beta} / l_0 - 2i\omega / c_0)}.$$

Now, making the change of variable  $p \rightarrow l_0^{1/(2\beta)} p$  gives

$$\begin{aligned} \Gamma_c(\omega, l_0) + i\Gamma_s(\omega, l_0) &= 2 \int_{S/l_0^{1/(2\beta)}} \frac{a(l_0^{1/(2\beta)} p) dp}{|p|^{2\alpha} (\mu |p|^{2\beta} - 2i\omega / c_0)} \\ &\xrightarrow{l_0 \rightarrow 0} 2a(0) \int_{-\infty}^\infty \frac{dp}{|p|^{2\alpha} (\mu |p|^{2\beta} - 2i\omega / c_0)}. \end{aligned}$$

This limit can be rewritten as

$$a(0) \int_{-\infty}^\infty \frac{dp}{|p|^{2\alpha} (\mu |p|^{2\beta} - 2i\omega / c_0)} = \int_0^\infty \tilde{R}(s) e^{2i\omega s} ds,$$

with

$$\tilde{R}(s) = a(0) \int_{-\infty}^\infty \frac{e^{-\mu |p|^{2\beta} s}}{|p|^{2\alpha}} dp = \frac{R_0}{s^\gamma}.$$

**For  $\gamma = 1$**  After integrating in  $s$ , we have

$$\begin{aligned} \Gamma_c(\omega, l_0) + i\Gamma_s(\omega, l_0) &= \frac{2}{l_0 |\ln(l_0)|} \int_{-r_S}^{r_S} \frac{a(p) dp}{|p|^{2\alpha} (\mu |p|^{2\beta} / l_0 - 2i\omega / c_0)} \\ &= \frac{4}{l_0 |\ln(l_0)|} \int_0^{r_S} \frac{a(p) dp}{p^{2\alpha} (\mu p^{2\beta} / l_0 - 2i\omega / c_0)} \\ &= I_{l_0} + II_{l_0} + III_{l_0} + IV_{l_0}, \end{aligned}$$

where the decomposition is defined as follows:

$$\begin{aligned} I_{l_0} &:= \frac{4}{l_0 |\ln(l_0)|} \int_0^{r_S} \frac{(a(p) - a(0)) dp}{p^{2\alpha} (\mu p^{2\beta} / l_0 - 2i\omega / c_0)}, \\ II_{l_0} &:= \frac{4a(0)}{l_0 |\ln(l_0)|} \int_0^{l_0^{1/(2\beta)}} \frac{dp}{p^{2\alpha} (\mu p^{2\beta} / l_0 - 2i\omega / c_0)}, \\ III_{l_0} &:= \frac{4a(0)}{l_0 |\ln(l_0)|} \int_{l_0^{1/(2\beta)}}^{r_S} \frac{dp}{p^{2\alpha}} \left( \frac{1}{\mu p^{2\beta} / l_0 - 2i\omega / c_0} - \frac{1}{\mu p^{2\beta} / l_0} \right), \\ IV_{l_0} &:= \frac{4a(0)}{\mu |\ln(l_0)|} \int_{l_0^{1/(2\beta)}}^{r_S} \frac{dp}{p}, \end{aligned}$$

using that  $2(\alpha + \beta) = 1$  since here  $\gamma = 1$ .

For  $I_{l_0}$ , using the triangular inequality and making the change of variable  $p \rightarrow l_0^{1/(2\beta)} p$  gives

$$|I_{l_0}| \leq \frac{4r_S \sup |a'|}{\mu |\ln(l_0)|} \xrightarrow{l_0 \rightarrow 0} 0.$$

For  $II_{l_0}$ , the idea being just to isolate the point  $p = 0$ , we have

$$|II_{l_0}| \leq \frac{4a(0)c_0}{2\mu \omega_c |\ln(l_0)|} \int_0^{l_0^{1/(2\beta)}} \frac{dp}{p^{2\alpha}} \propto \frac{l_0}{|\ln(l_0)|} \xrightarrow{l_0 \rightarrow 0} 0.$$



For  $III_{l_0}$ , we have after the change of variable  $p \rightarrow l_0^{1/(2\beta)} p$

$$|III_{l_0}| \leq \frac{4a(0)2\omega_c}{\mu^2 c_0 |\ln(l_0)|} \int_1^{r_S/l_0^{1/(2\beta)}} \frac{dp}{p^{2\alpha+4\beta}} \propto \frac{1}{|\ln(l_0)|} \xrightarrow{l_0 \rightarrow 0} 0.$$

Finally, for  $IV_{l_0}$ , integrating in  $p$  we have

$$IV_{l_0} = \frac{4a(0)}{\mu |\ln(l_0)|} \left( \ln(r_S) - \frac{1}{2\beta} \ln(l_0) \right) \xrightarrow{l_0 \rightarrow 0} \frac{2a(0)L'}{\mu\beta},$$

which concludes the proof of Lemma 8.1.

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