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# ON PARABOLIC SUBGROUPS OF ARTIN-TITS GROUPS

EDDY GODELLE

ABSTRACT. We address the conjecture which states that an intersection of parabolic subgroups of an Artin-Tits group is a parabolic subgroup. We prove that the conjecture is equivalent to a, a priori, weaker conjecture. We also prove the conjecture in a specific case. Along the way, we provide short and almost self-contained algebraical proofs of several classical results on Artin-Tits groups, such as those of Van der Lek on intersection of standard parabolic subgroups.

## INTRODUCTION

We fix a finite simplicial labelled graph  $\Gamma = (I, E, m)$  with vertex set  $I$ , edge set  $E$  and label map  $m : E \rightarrow \mathbb{N}_{\geq 2}$ . We also fix two sets  $\Sigma_I = \{\sigma_i \mid i \in I\}$  and  $S_I = \{s_i \mid i \in I\}$  that are indexed by  $I$ . If  $X$  is a subset of  $I$ , by  $\Gamma_X = (X, E_X, m_X)$  we denote the full (labelled) subgraph of  $\Gamma$  spanned by  $X$ . We set  $\Sigma_X = \{\sigma_i \in \Sigma_I \mid i \in X\}$  and  $S_X = \{s_i \in S_I \mid i \in X\}$ . If  $S$  is a set, by  $S^*$  we denote the monoid of words on  $S$ . The length of a word  $w$  on  $S$  is denoted by  $\ell_S(w)$ . For convenience, for  $\{i, j\}$  in  $E$ , we will often write  $m(i, j)$  for  $m(\{i, j\})$ . The Artin-Tits group  $A_I$  generated by  $\Sigma_I$  and associated with  $\Gamma$  is defined by the following presentation of group:

$$(E.1) \quad A_I = \left\langle \Sigma_I \left| \underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m(e) \text{ terms}} = \underbrace{\sigma_j \sigma_i \sigma_j \cdots}_{m(e) \text{ terms}} \text{ for } e = \{i, j\} \in E \right. \right\rangle$$

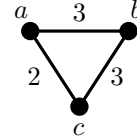
The Coxeter group  $W_I$  generated by  $S_I$  and associated with  $\Gamma$  is defined by the following presentation:

$$(E.2) \quad W_I = \left\langle S \left| \begin{array}{ll} s^2 = 1 & \text{for } s \in S \\ \underbrace{s_i s_j s_i \cdots}_{m(e) \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m(e) \text{ terms}} & \text{for } e = \{i, j\} \in E \end{array} \right. \right\rangle$$

To speak about the relations that appear in the presentation of  $A_I$  or to their corresponding relations in the presentation of  $W_I$ , we will speak of the *braid relations* of the presentation.

**Example 0.1.** Consider  $I = \{a, b, c\}$  and  $\Gamma_I$  as below. Then,  $A_I$  is the braid group on four strands. The groupe  $W_I$  is the permutation group on 4 elements where  $S_I$  consists on the elementary transpositions.

$$W_I = \left\langle s_a, s_b, s_c \left| \begin{array}{l} s_a^2 = s_b^2 = s_c^2 = 1 \\ s_a s_b s_a = s_b s_a s_b \\ s_b s_c s_b = s_c s_b s_c \\ s_a s_c = s_c s_a \end{array} \right. \right\rangle$$



Note that  $\Gamma$  is not the classical Coxeter graph associated with the above presentation (here no edge means no relation, and a commutation relation corresponds to an edge labelled with 2). If  $w$  is a word on  $S_I$  or on  $\Sigma_I \cup \Sigma_I^{-1}$ , by  $\mathbf{w}$  (in bold) we denote the corresponding element in  $W_I$ , or  $A_I$ , respectively. We will simply write  $\ell_I$  for both  $\ell_{S_I}$  and  $\ell_{\Sigma_I \cup \Sigma_I^{-1}}$ . If  $w'$  is another word on the same set, we write  $w \equiv_I w'$  when  $\mathbf{w} = \mathbf{w}'$  in the corresponding group. Let  $\theta_I^* : (\Sigma_I \cup \Sigma_I^{-1})^* \rightarrow S_I^*$  be the morphism of monoids that sends both  $\sigma_i$  and  $\sigma_i^{-1}$  onto  $s_i$ . Clearly, the morphism  $\theta_I^*$  induces a morphism of groups  $\theta_I : A_I \rightarrow W_I$ .

The kernel of  $\theta_I$  is called the colored Artin-Tits group and will be denoted by  $CA_I$ . The length  $\ell_I(\mathbf{w})$  of  $\mathbf{w}$  in  $W_I$  is defined as  $\ell_I(\mathbf{w}) = \min\{\ell_I(v) \mid v \in S_I^* \text{ with } v = \mathbf{w}\}$ . When  $\ell_I(w) = \ell_I(\mathbf{w})$ , we say that  $w$  is a *reduced* word. If  $X \subseteq I$ , it is immediate that the inclusion maps  $\Sigma_X \rightarrow \Sigma$  and  $S_X \rightarrow S$  extend to morphisms of groups  $\iota_X : A_X \rightarrow A_I$  and  $\tau_X : W_X \rightarrow W_I$ . This is well-known that these morphisms are into [12]. But we are going here to provide a new direct and short algebraical proof that  $\iota_X$  is into. So, in the sequel we identify  $W_X$  with its image by  $\tau_X$  in  $W_I$ . But, at this stage, we do not consider as known that  $\iota_X$  is into. We denote by  $A_I^+$  the submonoid of  $A_I$  generated by  $S_I$ . This is known by [8] that  $A_I^+$  and  $A_I$ , possess the same presentation, but considered as a presentation of monoid for the former, and considered as a presentation of group for the latter. However, we do not need to assume this result known when proving our results. We define  $\theta_X^*$ ,  $\theta_X$ ,  $A_X^+$ ,  $\ell_X$  and  $\equiv_X$  similarly to  $\theta_I^*$ ,  $\theta_I$ ,  $A_I^+$ ,  $\ell_I$  and  $\equiv_I$ , respectively. Our first objective is to prove Proposition 0.2. This proposition claims the existence of two maps  $\pi_{I,X}^*$  and  $\pi_{I,X}$  and states several of their properties. We will see (Proposition 2.7 and Corollary 2.8) that these two maps are indeed the map  $\hat{\pi}_X$  and  $\pi_X$  defined in [1] (see also [3, 6]).

**Proposition 0.2.** *Let  $X \subseteq I$ . There exists a map  $\pi_{I,X}^* : (\Sigma_I \cup \Sigma_I^{-1})^* \rightarrow (\Sigma_X \cup \Sigma_X^{-1})^*$  such that*

- (i) *For any word  $\omega$  on  $\Sigma_I \cup \Sigma_I^{-1}$ , we have  $\ell_X(\pi_{I,X}^*(\omega)) \leq \ell_I(\omega)$ . The equality holds if and only if  $\omega$  is a word on  $\Sigma_X \cup \Sigma_X^{-1}$ . In the latter case,  $\pi_{I,X}^*(\omega) = \omega$ .*
- (ii) *if  $\omega$  and  $\omega'$  are words on  $\Sigma_I \cup \Sigma_I^{-1}$ , then  $\pi_{I,X}^*(\omega)$  is a prefix of  $\pi_{I,X}^*(\omega\omega')$ .*
- (iii) [1] *The map  $\pi_{I,X}^*$  induces a map  $\pi_{I,X} : A_I \rightarrow A_X$ .*
- (iv) *If  $\omega$  lies in  $A_I$  then  $\left(\theta_X(\pi_{I,X}(\omega))\right)^{-1} \theta_I(\omega)$  is  $(X, \emptyset)$ -reduced and*

$$\ell_I(\theta_I(\omega)) = \ell_I(\theta_X(\pi_{I,X}(\omega))) + \ell_I\left(\left(\theta_X(\pi_{I,X}(\omega))\right)^{-1} \theta_I(\omega)\right)$$

- (v) [1] *The set map  $\pi_{I,X} : A_I \rightarrow A_X$  restricts to an homomorphism  $\pi_{I,X} : CA_I \rightarrow CA_X$ .*
- (vi) *We have  $\pi_{I,X}(A_I^+) = A_X^+$ .*
- (vii) *For any word  $\omega$  on  $\Sigma_X \cup \Sigma_X^{-1}$  and any  $\omega'$  in  $A_I$ , one has  $\pi_{I,X}(\omega\omega') = \pi_{I,X}(\omega)\pi_{I,X}(\omega')$ .*
- (viii) *If  $\omega$  lies in  $CA_I$  then for any  $\omega'$  in  $A_I$ , one has  $\pi_{I,X}(\omega\omega') = \pi_{I,X}(\omega)\pi_{I,X}(\omega')$ .*
- (ix) *If  $Y \subseteq I$  and  $\omega$  is a word on  $\Sigma_Y \cup \Sigma_Y^{-1}$ , then  $\pi_{I,X}^*(\omega)$  is a word on  $\Sigma_{Y \cap X} \cup \Sigma_{Y \cap X}^{-1}$  and the restriction of  $\pi_{I,X}^*$  to  $(\Sigma_Y \cup \Sigma_Y^{-1})^*$  is  $\pi_{Y, Y \cap X}^*$ .*
- (x) *Let  $X, Y \subseteq I$  and  $\omega$  be a word on  $\Sigma_I \cup \Sigma_I^{-1}$ , then  $\pi_{I,Y}^*(\pi_{I,X}^*(\omega)) = \pi_{I,X}^*(\pi_{I,Y}^*(\omega)) = \pi_{I, X \cap Y}^*(\omega)$*
- (xi) *Let  $X \subseteq Y \subseteq I$  and  $\omega$  be a word on  $\Sigma_I \cup \Sigma_I^{-1}$ , then  $\pi_{Y,X}^*(\pi_{I,Y}^*(\omega)) = \pi_{I,X}^*(\omega)$ .*

Note that, since at this stage we do not identify  $A_X$  with its image  $\iota_X(A_X)$  in  $A_I$ , in Point (vii) of the above proposition we distinguish  $\omega$  and  $\pi_{I,X}(\omega)$  (see Corollary 0.4 below). Some points in the above proposition, such as Point (vi), are not explicitly stated in [1], but can be deduced from the definition of  $\hat{\pi}_X$  given in [1]. As a consequence of Proposition 0.2, we will deduce :

**Proposition 0.3.** *Let  $X$  be a subset of  $I$ .*

- (i) [12] *Let  $\omega, \omega'$  be in  $(\Sigma_X \cup \Sigma_X^{-1})^*$ , if  $\omega \equiv_I \omega'$ , then  $\omega \equiv_X \omega'$ . In particular,  $\iota_X$  is into. So one can identify  $A_X$  with its image in  $A_I$ .*
- (ii) [12] *If  $Y$  is another subset of  $I$  then  $A_X \cap A_Y = A_{X \cap Y}$  in  $A_I$ .*
- (iii) *We have  $A_X^+ = A_I^+ \cap A_X$ .*
- (iv) [3] *The subgroup  $A_X$  is convex : if  $\omega$  is in  $A_X$ , any of its representative words of minimal length on  $\Sigma_I \cup \Sigma_I^{-1}$  is actually a word on  $\Sigma_X \cup \Sigma_X^{-1}$ .*

The statements in Proposition 0.3 are already known. However previous proofs were long and based on topological arguments. The interest of the present proof is that it is short, only uses algebraical arguments and is only based on elementary prerequisites. Indeed we only refer to the first pages of [2,

Chapter 4], where Coxeter groups are defined, and to a result of [10], which is a easy consequence of the same pages in [2, Chapter 4]. With Proposition 0.3(i) at hand, we deduce from Proposition 0.2 that

**Corollary 0.4.** *Let  $X \subseteq I$  and identify  $A_X$  with its image in  $A_I$ .*

- (i) [3] *The morphism  $\pi_{I,X} : A_I \rightarrow A_X$  is a retraction.*
- (ii) *For any  $\omega$  in  $A_X$  and any  $\omega'$  in  $A_I$ , one has  $\pi_{I,X}(\omega\omega') = \omega \pi_{I,X}(\omega')$ .*
- (iii) *Let  $X, Y \subseteq I$ . For any  $\omega$  in  $A_I$ ,  $\pi_{I,Y}(\pi_{I,X}(\omega)) = \pi_{I,X}(\pi_{I,Y}(\omega)) = \pi_{I,X \cap Y}(\omega)$ .*
- (iv) *Let  $X \subseteq Y \subseteq I$ .*
  - (a) *The restriction of  $\pi_{I,X}$  to  $A_Y$  is  $\pi_{Y,X}$ ;*
  - (b) *For any  $\omega$  in  $A_I$ ,  $\pi_{Y,X}(\pi_{I,Y}(\omega)) = \pi_{I,X}(\omega)$ .*

In Point (iii) above, the equalities can be written as  $\pi_{X,X \cap Y}(\pi_{I,X}(\omega)) = \pi_{Y,Y \cap X}(\pi_{I,Y}(\omega)) = \pi_{I,X \cap Y}(\omega)$  by (iv)(a).

If  $X$  is a subset of  $I$ , then the subgroups of  $A_I$  and  $W_I$  generated by  $\Sigma_X$  and  $S_X$ , respectively, are called *standard parabolic subgroups*. A *parabolic subgroup* of  $A_I$  is a subgroup that is conjugated to one of its standard parabolic subgroups. In the framework of Artin-Tits groups, a main open conjecture states that the family of parabolic subgroups of an Artin-Tits group is closed under intersection. This conjecture can be formulated as it follows:

**Conjecture 1.** Let  $A_I$  be an Artin-Tits group,  $X, Y$  be in  $I$  and  $\omega$  be in  $A_I$ . Then,  $(\omega A_Y \omega^{-1}) \cap A_X$  is a parabolic subgroup of  $A_I$ .

By [1], when the conjecture holds, then  $(\omega A_Y \omega^{-1}) \cap A_X$  is a parabolic subgroup of  $\omega A_Y \omega^{-1}$  and a parabolic subgroup of  $A_X$ . Conjecture 1 has been proved to hold when one restricts to some particular families such as the family of Artin-Tits groups of spherical type [4], of FC type [7] or of large type [5]. However, the question in the general case remains open. Our second objective is to prove that Conjecture 1 is equivalent to the following conjecture:

**Conjecture 2.** Let  $A_I$  be an Artin-Tits group. Let  $X$  be in  $I$  and  $\omega$  be in  $CA_I$ . Then  $(\omega A_X \omega^{-1}) \cap A_X$  is a parabolic subgroup of  $A_I$ .

Clearly, Conjecture 1 implies Conjecture 2, so we address the other implication.

**Theorem 0.5.** *Conjecture 2 implies Conjecture 1.*

In order to prove Theorem 0.5 we need to prove Conjecture 1 in the case of specific elements  $\omega$  of  $A_I$ . The morphism  $\theta_I : A_I \rightarrow W_I$  possesses a well-defined and well-known set section  $\kappa_I : W_I \rightarrow A_I$  (see Section 1). We will prove that

**Theorem 0.6.** *Let  $X, Y$  be in  $I$  and  $w$  be in  $W_I$ . Set  $\omega = \kappa_I(w)$ . Then, there exists  $w_1$  in  $W_X$ ,  $Y_1 \subseteq Y$ , and  $X_1 \subseteq X$  so that  $(\omega A_Y \omega^{-1}) \cap A_X = (\omega A_{Y_1} \omega^{-1}) = \kappa_I(w_1) A_{X_1} \kappa_I(w_1)^{-1}$ .*

Section 1 is devoted to the necessary backgrounds on Coxeter groups. In Section 2, we turn to the proof of Propositions 0.2 and 0.3. We also prove that the retraction  $\pi_{I,X}$  is the same as the retraction  $\pi_X$  defined in [1]. Finally, in Section 3 we prove Theorems 0.5 and 0.6.

## 1. BACKGROUND ON COXETER GROUPS

We start with the elementary properties on Coxeter groups that we shall need. They can all be found in [2, chap. IV, §1], except Proposition 1.8 that can be found in [10] and which is a direct consequence of Proposition 1.7 below.

**Definition 1.1.** Let  $w = s_{i_1} \cdots s_{i_k}$  be a word on  $S$ .

- (i) By  $r_j(w)$ , or  $r_i$  when no confusion is possible, we denote the word  $s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_2} s_{i_1}$ . We set  $R(w) = (r_1(w), r_2(w), \dots, r_k(w))$ .

- (ii) By  $\mathbf{R}(w)$  we denote the associated sequence  $(\mathbf{r}_1(w), \mathbf{r}_2(w), \dots, \mathbf{r}_k(w))$  of elements of  $W_I$ .
- (iii) By  $\mathbf{N}(w)$  we denote the set of elements of  $W_I$  that appear an odd number of times in  $\mathbf{R}(w)$ .

**Example 1.2.** Consider Example 0.1 and the word  $w = s_a s_b s_c s_a s_c s_b$ . Then  $\mathbf{N}(w) = \{\mathbf{s}_b; \mathbf{s}_a \mathbf{s}_b \mathbf{s}_a\}$  because  $R(s) = \{s_a; s_a s_b s_a; s_a s_b s_c s_b s_a; s_a s_b s_c s_a s_c s_b s_a; s_a s_b s_c s_a s_c s_b s_a; s_a s_b s_c s_a s_c s_b s_c s_a s_c s_b s_a\}$  with  $\mathbf{s}_a \mathbf{s}_b \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_b \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_b \mathbf{s}_a = \mathbf{s}_a$  and  $\mathbf{s}_a \mathbf{s}_b \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_b \mathbf{s}_a = \mathbf{s}_a \mathbf{s}_b \mathbf{s}_c \mathbf{s}_b \mathbf{s}_a$ , and  $\mathbf{s}_a \mathbf{s}_b \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_b \mathbf{s}_a = \mathbf{s}_b$ .

**Remark 1.3.** *It follows from Definition 1.1 that for any two words  $w, w'$  on  $S$ , the sequence  $\mathbf{R}(ww')$  is the concatenation of the two sequences  $\mathbf{R}(ww')$  and  $\mathbf{w} \mathbf{R}(w') \mathbf{w}^{-1}$ .*

**Proposition 1.4.** [2, chap. IV §1] *Let  $w$  and  $w'$  be two words on  $S_I$ .*

- (i) *If  $w$  and  $w'$  represent the same element in  $W$ , then  $\mathbf{N}(w) = \mathbf{N}(w')$ .*
- (ii) *The word  $w$  is reduced if and only if the cardinality of  $\mathbf{N}(w)$  is equal to the length of  $w$ , in other words when all the elements of  $\mathbf{R}(w)$  are distinct.*

**Example 1.5.** Consider Example 1.2. Since  $\mathbf{N}(w) = \{\mathbf{s}_b; \mathbf{s}_a \mathbf{s}_b \mathbf{s}_a\}$ , the length on  $S$  of  $\mathbf{w}$  is 2. Indeed,  $\mathbf{w} = \mathbf{s}_a \mathbf{s}_b \mathbf{s}_c \mathbf{s}_a \mathbf{s}_c \mathbf{s}_b = \mathbf{s}_b \mathbf{s}_a$ .

**Proposition 1.6.** [2, chap. IV, §1] *Let  $w$  and  $w'$  be two words on  $S_I$ .*

- (i) *Let  $s_{i_0}$  be in  $S_I$  and write  $w = s_{i_1} \cdots s_{i_k}$ . If  $w$  is a reduced word, but  $s_{i_0} w$  is not, then there exists  $j$  so that  $s_{i_1} \cdots s_{i_k} \equiv_I s_{i_0} s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k}$ .*
- (ii) *If  $w$  and  $w'$  are reduced with  $w \equiv_I w'$ , then the word  $w$  can be transformed into the word  $w'$  by using the braid relations of the presentation of  $W_I$ , only.*
- (iii) *Let  $w$  be a word on  $S_I$  and  $s, t$  be in  $S_I$  so that both words  $sw$  and  $wt$  are reduced, but the word  $swt$  is not. Then  $sw \equiv_I wt$ . In other words  $\mathbf{sw} = \mathbf{wt}$  in  $W_I$ .*

The above proposition implies several important properties. First, the map  $S_I \rightarrow W_I, s_i \mapsto \mathbf{s}_i$  is into and we can identify  $S_I$  with its image  $\mathbf{S}_I$  in  $W_I$ . Depending on the situation, we will write  $s_i$  or  $\mathbf{s}_i$ . Moreover for  $X \subseteq I$  we have  $S_I \cap W_X = S_X$ , and for any two subsets  $X, Y$  of  $I$ , we have  $W_X \cap W_Y = W_{X \cap Y}$ . It also follows that the morphism  $\Theta_I : A_I \rightarrow W_I$  possesses a well-defined set section  $\kappa_I : W_I \rightarrow A_I$  defined in the following way : for  $\mathbf{w}$  in  $W_I$  and any reduced representative word  $w = s_{i_1} \cdots s_{i_k}$  on  $S_I$  of  $\mathbf{w}$ , we have  $\kappa_I(\mathbf{w}) = \sigma_{i_1} \cdots \sigma_{i_k}$ . Indeed, two reduced words on  $S$  represent the same element in  $W_I$  if and only if one can be transformed into the other by using braid relations only. As a consequence, for any  $X$  included in  $I$ , the subgroup  $W_X$  is convex : if  $\mathbf{w}$  is in  $W_X$  and  $w$  is one of its reduced representative word, then  $w$  is a word on  $S_X$ . In other words, if the product  $\mathbf{u} \mathbf{v}$  lies in  $W_X$  and  $\ell_I(\mathbf{u} \mathbf{v}) = \ell_I(\mathbf{u}) + \ell_I(\mathbf{v})$  then both  $\mathbf{u}$  and  $\mathbf{v}$  lie in  $W_X$ , too. In addition, the equality  $\ell_I(\mathbf{u} \mathbf{v}) = \ell_I(\mathbf{u}) + \ell_I(\mathbf{v})$  holds if and only if the equality  $\kappa_I(\mathbf{u} \mathbf{v}) = \kappa_I(\mathbf{u}) \kappa_I(\mathbf{v})$  holds. Finally, we recall that for  $i, j$  in  $I$  and distinct, then the order of the product  $\mathbf{s}_i \mathbf{s}_j$  in  $W_I$  is  $m(i, j)$ , the label of the edge  $\{i, j\}$  in the graph  $\Gamma$  (see [2, Chap V §4]).

**Proposition 1.7.** [2, chap. IV, §1] *Let  $X, Y$  be included in  $I$  and  $\mathbf{w}$  be in  $W_I$ . The double-coset  $W_X \mathbf{w} W_Y$  possesses a unique element  $\mathbf{w}_0$  of minimal length in  $W_I$ . Moreover:*

- (i) *For any  $\mathbf{w}_1$  in  $W_X \mathbf{w} W_Y$  there exist  $\mathbf{x}$  in  $W_X$  and  $\mathbf{y}$  in  $W_Y$  so that  $\mathbf{w}_1 = \mathbf{x} \mathbf{w}_0 \mathbf{y}$  with  $\ell_I(\mathbf{w}_1) = \ell_I(\mathbf{x}) + \ell_I(\mathbf{w}_0) + \ell_I(\mathbf{y})$ . One says that  $\mathbf{w}_0$  is  $(X, Y)$ -reduced.*
- (ii) *For any  $\mathbf{x}$  in  $W_X$  and any  $\mathbf{y}$  in  $W_Y$ , one has  $\ell_I(\mathbf{x} \mathbf{w}_0) = \ell_I(\mathbf{x}) + \ell_I(\mathbf{w}_0)$  and  $\ell_I(\mathbf{w}_0 \mathbf{y}) = \ell_I(\mathbf{w}_0) + \ell_I(\mathbf{y})$ .*
- (iii) *The element  $\mathbf{w}$  is  $(X, Y)$ -reduced if and only if it is both  $(X, \emptyset)$ -reduced. and  $(\emptyset, Y)$ -reduced.*

The following result corresponds to Proposition 0.2, but in the context of Coxeter groups. It was announced in [10] and has been proved in [11] and [9]. This is an almost direct consequence of Proposition 1.7.

**Proposition 1.8.** [9] *Let  $X, Y \subseteq I$  and  $w$  be in  $W_I$ . Then, there exist  $w_1$  in  $W_X$ ,  $Y_1 \subseteq Y$ , and  $X_1 \subseteq X$  so that  $(wW_Y w^{-1}) \cap W_X = (wW_{Y_1} w^{-1}) = w_1 W_{X_1} w_1^{-1}$ . Moreover, if  $w$  is  $(X, Y)$ -reduced, then  $w_1 = 1$  and  $wS_{Y_1} w^{-1} = wS_Y w^{-1} \cap S_X = S_{X_1}$ .*

**Remark 1.9.** *In [9], the author considered the case  $(X, Y)$ -reduced, only. If  $w$  is not  $(X, Y)$ -reduced, then we can write  $w = w_1 w' w_2$  with  $w_1$  in  $W_X$ ,  $w_2$  in  $W_Y$  and  $w'$  that is  $(X, Y)$ -reduced. Then  $(wW_Y w^{-1}) \cap W_X = w_1 (w' W_Y w'^{-1} \cap W_X) w_1^{-1}$ .*

**Corollary 1.10.** *Let  $X \subseteq I$ ,  $i, j$  be in  $I$  and distinct and  $w$  be in  $W$ . There are only three possibilities:  $wW_{\{i,j\}} w^{-1} \cap W_X$  is trivial, or  $wW_{\{i,j\}} w^{-1}$  is included in  $W_X$ , or  $wW_{\{i,j\}} w^{-1} \cap W_X$  contains only one non trivial element.*

**Remark 1.11.** *In the above corollary consider the case  $wW_{\{i,j\}} w^{-1}$  is included in  $W_X$ . Let  $X_1 = \{i', j'\}$  be as in Proposition 1.8. Then,  $m(i', j') = m(i, j)$  because  $m(i, j)$  and  $m(i', j')$  are the orders of  $s_i s_j$  and  $s_{i'} s_{j'}$ , respectively, as recalled above.*

**Lemma 1.12.** *Let  $X \subseteq I$ . Let  $u$  be in  $W_I$  and  $s_i$  be in  $S_I$ . Write  $u = v w$  such that  $v$  lies in  $W_X$  and  $w$  is  $(X, \emptyset)$ -reduced. Then*

$$u s_i u^{-1} \in W_X \iff w s_i w^{-1} \in S_X \iff w s_i \text{ is not } (X, \emptyset)\text{-reduced}$$

Moreover, in this case,  $w$  is  $(X, \{i\})$ -reduced.

*Proof.* The first equivalence is clear. Implication “ $w s_i w^{-1} \in S_X \Rightarrow w s_i$  is not  $(X, \emptyset)$ -reduced” follows from Proposition 1.8, considering  $Y = \{i\}$ . Assume  $w s_i$  is not  $(X, \emptyset)$ -reduced. If  $w$  were not  $(X, \{i\})$ -reduced, by Proposition 1.7(iii) it would be not  $(\emptyset, \{i\})$ -reduced, and we could write  $w = (w s_i) s_i$  with  $\ell_I(w) = \ell_I(w s_i) + 1$ , which contradicts the fact that  $w$  is  $(X, \emptyset)$ -reduced. The last implication then follows from Proposition 1.6(iii) : let  $w$  be a reduced representative word of  $w$ . if  $w$  is  $(X, \{i\})$ -reduced and  $w s_i$  is not  $(X, \emptyset)$ -reduced, then there is  $s_j$  in  $S_X$  so that  $s_j w$  and  $w s_i$  are reduced words, but  $s_j w s_i$  is not. Then  $w s_i = s_j w$ .  $\square$

The following result is implicit in [2, §1 Sec. 4] and is well-known from specialists. As we will need it when proving Proposition 0.2, we prove it for completeness.

**Lemma 1.13.** *Let  $i, j$  be in  $I$  distinct with  $\{i, j\}$  in  $E$ . Set  $m = m(i, j)$ . Let  $\underline{s}_{i,j} = \underbrace{s_i s_j s_i \cdots}_{m \text{ terms}}$  and*

$$\underline{s}_{j,i} = \underbrace{s_j s_i s_j \cdots}_{m \text{ terms}}. \text{ Then, for all } n \text{ with } 1 \leq n \leq m \text{ we have } \mathbf{r}_n(\underline{s}_{i,j}) = \mathbf{r}_{m-n+1}(\underline{s}_{j,i}).$$

*Proof.* By formula (15) in [2, §1 Sec. 4] we have  $\mathbf{r}_n(\underline{s}_{i,j}) = (\mathbf{s}_i \mathbf{s}_j)^{n-1} \mathbf{s}_i$  and  $\mathbf{r}_{m-n+1}(\underline{s}_{j,i}) = (\mathbf{s}_j \mathbf{s}_i)^{m-n} \mathbf{s}_j$ . So  $\mathbf{r}_n(\underline{s}_{i,j}) (\mathbf{r}_{m-n+1}(\underline{s}_{j,i}))^{-1} = (\mathbf{s}_i \mathbf{s}_j)^{n-1} \mathbf{s}_i \mathbf{s}_j (\mathbf{s}_i \mathbf{s}_j)^{m-n} = (\mathbf{s}_i \mathbf{s}_j)^m = 1$ .  $\square$

## 2. THE RETRACTION MAP

**Definition 2.1.** Let  $w = s_{i_1} \cdots s_{i_k}$  be a word on  $S$ . Consider the notation in Definition 1.1. Let  $X \subseteq I$ .

- (i) We set  $r_{j,X}(w) = r_j(w)$  when  $r_j(w)$  lies in  $W_X$ , and  $r_{j,X}(w) = \varepsilon$ , the empty word, otherwise. When no confusion is possible, we will write  $r_{j,X}$  for  $r_{j,X}(w)$ . By  $R_X(w)$ , we denote the  $k$ -uple of words on  $S$  defined by  $R_X(w) = (r_{1,X}, r_{2,X}, \dots, r_{k,X})$ . By  $\mathbf{R}_X(w) = (\mathbf{r}_{1,X}(w), \mathbf{r}_{2,X}(w), \dots, \mathbf{r}_{k,X}(w))$  we denote the corresponding  $k$ -uple of elements of  $W_X$ .
- (ii) By  $\widehat{R}_X(w)$  we denote the subsequence of  $R_X(w)$  obtained by keeping the nonempty words only and by  $\widehat{\mathbf{R}}_X(w)$  we denote the corresponding subsequence of  $\mathbf{R}_X(w)$ .

Note that for  $w$  in  $X^*$ , one has  $\widehat{\mathbf{R}}_X(w) = \mathbf{R}_X(w) = \mathbf{R}(w)$ . Let  $w = s_{i_1} \cdots s_{i_k}$  be a word on  $S$  and  $1 < j_1 < \cdots < j_n \leq k$ . In the following proposition, by  $s_{i_1} \cdots \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_2}} \cdots \hat{s}_{i_{j_n}} \cdots s_{i_k}$  we denote the word obtained from  $w$  by removing the letters with a hat. For instance in Example 1.2  $s_a s_b \hat{s}_c s_a s_c \hat{s}_b = s_a s_b s_a s_c$ .

The first crucial result is the following:

**Proposition 2.2.** *Let  $X \subseteq I$  and  $w = s_{i_1} \cdots s_{i_k}$  be a non-empty word on  $S$ . Write  $\widehat{R}_X(w) = (r_{j_1, X}, r_{j_2, X}, \dots, r_{j_p, X})$ . Set  $\mathbf{w}_0 = 1$  and for  $1 \leq j \leq k$  write  $\mathbf{s}_{i_1} \cdots \mathbf{s}_{i_j} = \mathbf{v}_j \mathbf{w}_j$  with  $\mathbf{v}_j$  in  $W_X$  and  $\mathbf{w}_j$  that is  $(X, \emptyset)$ -reduced. For  $1 \leq n \leq p$ , let  $\mathbf{t}_n = \mathbf{w}_{j_n-1} \mathbf{s}_{j_n} \mathbf{w}_{j_n-1}^{-1}$ .*

- (i) (a) For  $1 \leq n \leq p$ ,  $\mathbf{t}_n$  lies in  $\mathbf{S}_X$ .
- (b) Set  $j_0 = 0$  and  $j_{p+1} = k + 1$ . For  $n$  in  $\{1, \dots, p+1\}$  and  $j$  in  $\{j_{n-1}, \dots, j_n - 1\}$ , we have  $\mathbf{w}_j = \mathbf{s}_{i_1} \cdots \hat{\mathbf{s}}_{i_{j_1}} \cdots \hat{\mathbf{s}}_{i_{j_2}} \cdots \hat{\mathbf{s}}_{i_{j_{n-1}}} \cdots \mathbf{s}_{i_j}$  and  $\mathbf{v}_j = \mathbf{t}_1 \cdots \mathbf{t}_{n-1}$ .
- (ii) Set  $w_X = t_1 \cdots t_p$ . Then,  $\mathbf{R}(w_X) = \widehat{\mathbf{R}}_X(w)$ .

*Proof.* (i) (a) and (b) are consequences of Lemma 1.12: (a) is clear. Now, for any  $j$  in  $\{1, \dots, k\}$  but not in  $\{j_1, \dots, j_p\}$ , the element  $\mathbf{w}_j$  is  $(X, \emptyset)$ -reduced and  $\mathbf{w}_{j-1} \mathbf{s}_j \mathbf{w}_{j-1}^{-1}$  is not in  $W_X$ , because  $\mathbf{v}_{j-1} \mathbf{w}_{j-1} \mathbf{s}_j \mathbf{w}_{j-1}^{-1} \mathbf{v}_{j-1}^{-1}$  is not. So,  $\mathbf{w}_{j-1} \mathbf{s}_j$  is  $(X, \emptyset)$ -reduced. In particular,  $\mathbf{v}_j = \mathbf{v}_{j-1}$  and  $\mathbf{w}_j = \mathbf{w}_{j-1} \mathbf{s}_j$ . If  $j = j_n$  for  $n$  in  $\{1, \dots, p\}$ , then  $\mathbf{v}_{j_n} \mathbf{w}_{j_n} = \mathbf{v}_{j_n-1} \mathbf{w}_{j_n-1} \mathbf{s}_{i_{j_n}} = \mathbf{v}_{j_n-1} \mathbf{t}_n \mathbf{w}_{j_n-1}$ . So,  $\mathbf{v}_{j_n} = \mathbf{v}_{j_n-1} \mathbf{t}_n$  and  $\mathbf{w}_{j_n} = \mathbf{w}_{j_n-1}$ . Point (ii) follows directly from (i): write  $\mathbf{u}_n = \mathbf{s}_{i_{j_{n-1}+1}} \cdots \mathbf{s}_{i_{j_n-1}}$ . By (i)(b), for all  $n$  we have  $\mathbf{w}_{j_n-1} = \mathbf{u}_1 \cdots \mathbf{u}_n$  and  $\mathbf{t}_1 \mathbf{t}_2 \cdots \mathbf{t}_n = (\mathbf{u}_1 \mathbf{s}_{j_1} \mathbf{u}_1^{-1}) (\mathbf{u}_1 \mathbf{u}_2 \mathbf{s}_{j_2} \mathbf{u}_2^{-1} \mathbf{u}_1^{-1}) \cdots (\mathbf{u}_1 \cdots \mathbf{u}_n \mathbf{s}_{j_n} \mathbf{u}_n^{-1} \cdots \mathbf{u}_1^{-1}) = \mathbf{u}_1 \mathbf{s}_{j_1} \mathbf{u}_2 \mathbf{s}_{j_2} \cdots \mathbf{s}_{j_{n-1}} \mathbf{u}_n \mathbf{s}_{j_n} \mathbf{u}_n^{-1} \cdots \mathbf{u}_1^{-1} = \mathbf{s}_1 \cdots \mathbf{s}_{j_n} \mathbf{u}_n^{-1} \cdots \mathbf{u}_1^{-1}$ . By symmetry, we have  $\mathbf{t}_{n-1} \cdots \mathbf{t}_2 \mathbf{t}_1 = \mathbf{u}_1 \cdots \mathbf{u}_{n-1} \mathbf{s}_{j_{n-1}} \cdots \mathbf{s}_1$ . Therefore, we deduce that  $\mathbf{r}_n(w_X) = \mathbf{s}_1 \cdots \mathbf{s}_{j_n} \mathbf{u}_n^{-1} \cdots \mathbf{u}_1^{-1} \mathbf{u}_1 \cdots \mathbf{u}_{n-1} \mathbf{s}_{j_{n-1}} \cdots \mathbf{s}_1 = \mathbf{s}_1 \cdots \mathbf{s}_{j_n} \mathbf{u}_n^{-1} \mathbf{s}_{j_{n-1}} \cdots \mathbf{s}_1 = \mathbf{r}_{j_n, X}(w)$ .  $\square$

**Remark 2.3.** *In Point (ii),  $\mathbf{t}_n$  is uniquely defined by  $\mathbf{t}_n$ . Moreover,  $w_X$  is uniquely defined by  $\widehat{\mathbf{R}}_X(w)$  since  $\mathbf{t}_1 = \mathbf{r}_{j_1, X}$ ;  $\mathbf{t}_2 = \mathbf{r}_{j_1, X} \mathbf{r}_{j_2, X} \mathbf{r}_{j_1, X}^{-1}$ ;  $\dots$ ;  $\mathbf{t}_n = \mathbf{r}_{j_1, X} \cdots \mathbf{r}_{j_{n-1}, X} \mathbf{r}_{j_n, X} \mathbf{r}_{j_{n-1}, X}^{-1} \cdots \mathbf{r}_{j_1, X}^{-1}$ .*

**Lemma 2.4.** *Let  $X \subseteq I$  and  $w = s_{i_1} \cdots s_{i_k}$ ,  $w' = s_{i'_1} \cdots s_{i'_k}$  be two non-empty words on  $S$ .*

- (i) Let  $1 \leq n \leq \min(k, k')$ . If for all  $1 \leq m \leq n$ ,  $i_m = i'_m$  then  $r_{n, X}(w) = r_{n, X}(w')$ .
- (ii) Let  $1 \leq n \leq k$  and  $1 \leq n' \leq k'$ . Assume  $\mathbf{s}_{i_1} \cdots \mathbf{s}_{i_{n-1}} = \mathbf{s}_{i'_1} \cdots \mathbf{s}_{i'_{n'-1}}$  and  $i_n = i'_{n'}$ . Then  $\mathbf{r}_{n, X}(w) = \mathbf{r}_{n', X}(w')$ .

*Proof.* (i)  $r_{n, X}(w)$  and  $r_{n, X}(w')$  only depend on the prefixes  $s_{i_1} \cdots s_{i_n}$  and  $s_{i'_1} \cdots s_{i'_n}$ , respectively. (ii)  $\mathbf{r}_{n, X}(w)$  and  $\mathbf{r}_{n', X}(w')$  only depend on the pair  $(\mathbf{s}_{i_1} \cdots \mathbf{s}_{i_{n-1}}, s_{i_n})$  and the pair  $(\mathbf{s}_{i'_1} \cdots \mathbf{s}_{i'_{n'-1}}, s_{i'_{n'}})$ , respectively.  $\square$

**Definition 2.5.** Let  $X \subseteq I$ . We define the map  $\pi_{I, X}^* : (\Sigma_I \cup \Sigma_I^{-1})^* \rightarrow (\Sigma_X \cup \Sigma_X^{-1})^*$  in the following way. Let  $\omega = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k}$  be a word on  $\Sigma_I \cup \Sigma_I^{-1}$ , where  $\varepsilon_i$  lies in  $\{\pm 1\}$ . Set  $w = \theta_I^*(\omega)$  and let  $w_X = s_{q_1} \cdots s_{q_p}$  be the word on  $S_X$  provided by Proposition 2.2 so that  $\mathbf{R}(w_X) = \widehat{\mathbf{R}}_X(w) = (r_{j_1, X}(w), \dots, r_{j_p, X}(w))$ . We set  $\pi_{I, X}^*(\omega) = \sigma_{q_1}^{\varepsilon_{j_1}} \cdots \sigma_{q_p}^{\varepsilon_{j_p}}$ .

Note that  $(\varepsilon_1, \dots, \varepsilon_k)$  and  $\widehat{\mathbf{R}}_X(w)$  do not fix  $\pi_{I, X}^*(\omega)$  because they do not fix the exponents. However,  $\pi_{I, X}^*(\omega)$  is fixed by  $(\varepsilon_1, \dots, \varepsilon_k)$  and  $\mathbf{R}_X(w)$ . So the sequence of steps that define  $\pi_{I, X}^*$  can be summarized as it follows :

$$\omega \rightarrow w \rightarrow R(w) \rightarrow \mathbf{R}_X(w) \rightarrow \widehat{\mathbf{R}}_X(w) = \mathbf{R}(w_X) \rightarrow w_X \rightarrow \pi_{I, X}^*(\omega)$$

**Proposition 2.6.** *Let  $X \subseteq I$  and  $\omega, \omega'$  be two words on  $\Sigma_I \cup \Sigma_I^{-1}$ . If  $\omega \equiv_I \omega'$ , then  $\pi_{I, X}^*(\omega) \equiv_X \pi_{I, X}^*(\omega')$ .*

*Proof.* Let  $\omega = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k}$  and  $\omega'$  be two words on  $\Sigma_I \cup \Sigma_I^{-1}$  so that  $\omega \equiv_I \omega'$ . Set  $w = \theta_I^*(\omega) = s_{i_1} \cdots s_{i_k}$  and  $w' = \theta_I^*(\omega')$ . Consider the notation of Definition 2.5. In particular,  $\widehat{\mathbf{R}}_X(w) = (r_{j_1, X}(w), \dots, r_{j_p, X}(w))$ ;

$w_X = s_{q_1} \cdots s_{q_p}$  and  $\pi_{I,X}^*(\omega) = \sigma_{q_1}^{\varepsilon_{j_1}} \cdots \sigma_{q_p}^{\varepsilon_{j_p}}$ . Since  $\omega \equiv_I \omega'$ , there exists a finite sequence  $\omega_0 = \omega, \omega_1, \dots, \omega_n = \omega'$  of words on  $\Sigma_I \cup \Sigma_I^{-1}$  so that  $\omega_{i+1}$  is obtained from  $\omega_i$  by using either a braid relation or one of the two relations  $\sigma_i \sigma_i^{-1} = \varepsilon$  and  $\sigma_i^{-1} \sigma_i = \varepsilon$ . Clearly it is enough to consider the case  $n = 1$ . Case 1:  $\omega$  is transformed into  $\omega'$  by using a relation  $\sigma_i \sigma_i^{-1} = \varepsilon$  or  $\sigma_i^{-1} \sigma_i = \varepsilon$ . Up to exchanging  $\omega$  and  $\omega'$  we may assume that  $\omega' = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_{n-1}}^{\varepsilon_{n-1}} \sigma_i^\rho \sigma_i^{-\rho} \sigma_{i_n}^{\varepsilon_n} \cdots \sigma_{i_k}^{\varepsilon_k}$  with  $\rho \in \{-1, 1\}$ . Then,  $w' = s_{i_1} \cdots s_{i_{n-1}} s_i s_i s_{i_n} \cdots s_{i_k}$ . By Lemma 2.4(i), for  $m \leq n-1$  we have  $\mathbf{r}_{m,\mathbf{X}}(w') = \mathbf{r}_{m,\mathbf{X}}(w)$  and, by Lemma 2.4(ii), for  $m \geq n$ , we have  $\mathbf{r}_{m,\mathbf{X}}(w) = \mathbf{r}_{m+2,\mathbf{X}}(w')$ . Moreover,  $\mathbf{r}_n(w) = \mathbf{r}_{n+1}(w')$ . So either both  $\mathbf{r}_n(w')$  and  $\mathbf{r}_{n+1}(w')$  are in  $W_X$  or both of them are not. In the second case,  $\mathbf{R}_X(w) = \mathbf{R}_X(w')$  and  $\pi_{I,X}^*(\omega) = \pi_{I,X}^*(\omega')$ . Assume the first case. Let  $w'_X \in S_X^*$  so that  $\mathbf{R}(w'_X) = \widehat{\mathbf{R}}_X(w')$ . Since  $\mathbf{r}_{n,\mathbf{X}}(w') = \mathbf{r}_{n+1,\mathbf{X}}(w') \neq 1$  there is  $0 \leq m \leq p$  so that  $\widehat{\mathbf{R}}_X(w') = (\mathbf{r}_{j_1,\mathbf{X}}(w), \dots, \mathbf{r}_{j_m,\mathbf{X}}(w), \mathbf{r}_{n,\mathbf{X}}(w'), \mathbf{r}_{n,\mathbf{X}}(w'), \mathbf{r}_{j_{m+1},\mathbf{X}}(w), \dots, \mathbf{r}_{j_p,\mathbf{X}}(w))$ . In particular, there is  $j$  in  $X$  so that  $w'_X = s_{q_1} \cdots s_{q_m} s_j s_j s_{q_{m+1}} \cdots s_{q_p}$  (see Remark 2.3) and  $\pi_{I,X}^*(\omega') = \sigma_{q_1}^{\varepsilon_{j_1}} \cdots \sigma_{q_m}^{\varepsilon_{j_m}} \sigma_j^\rho \sigma_j^{-\rho} \sigma_{q_{m+1}}^{\varepsilon_{j_{m+1}}} \cdots \sigma_{q_p}^{\varepsilon_{j_p}}$ . Thus,  $\pi_{I,X}^*(\omega) \equiv_X \pi_{I,X}^*(\omega')$ . Case 2:  $\omega'$  is obtained from  $\omega$  by using a braid relation. For  $i, j$  in  $I$ , write  $\underline{\sigma}_{i,j} = \underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m(i,j) \text{ terms}}$  and  $\underline{s}_{i,j} = \underbrace{s_i s_j s_i \cdots}_{m(i,j) \text{ terms}}$ . We can write  $\omega = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_{n-1}}^{\varepsilon_{n-1}} \underline{\sigma}_{i,j} \sigma_{i_{n+m}}^{\varepsilon_{n+m}} \cdots \sigma_{i_k}^{\varepsilon_k}$  and  $\omega' = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_{n-1}}^{\varepsilon_{n-1}} \underline{\sigma}_{j,i} \sigma_{i_{n+m}}^{\varepsilon_{n+m}} \cdots \sigma_{i_k}^{\varepsilon_k}$  with  $i = i_n, j = i_{n+1}$  and  $m = m(i, j)$ . It follows that we have  $w = s_{i_1} \cdots s_{i_{n-1}} \underline{s}_{i,j} s_{i_{n+m}} \cdots s_{i_k}$  and  $w' = s_{i_1} \cdots s_{i_{n-1}} \underline{s}_{j,i} s_{i_{n+m}} \cdots s_{i_k}$ . As in Case 1, by Lemma 2.4,  $\mathbf{r}_h(w) = \mathbf{r}_h(w')$  when either  $1 \leq h \leq n-1$  or  $n+m \leq h \leq k$ . Since  $\underline{\sigma}_{i,j}$  is a reduced word, all the elements of  $\mathbf{R}(\underline{\sigma}_{i,j})$  are distinct (see Proposition 1.4). It follows that  $\mathbf{r}_n(w), \mathbf{r}_{n+1}(w), \dots, \mathbf{r}_{n+m-1}(w)$  are all distinct. Moreover, by Lemma 1.13, for  $h$  in  $\{1, \dots, m\}$  we have  $\mathbf{r}_h(\underline{\sigma}_{i,j}) = \mathbf{r}_{m-h+1}(\underline{\sigma}_{j,i})$ , and  $\mathbf{r}_{n-1+h}(w) = \mathbf{r}_{n+m-h}(w')$ . Therefore, if none of the elements  $\mathbf{r}_n(w), \mathbf{r}_{n+1}(w), \dots, \mathbf{r}_{n+m-1}(w)$  lie in  $W_X$  then  $\mathbf{R}_X(w) = \mathbf{R}_X(w')$  and  $\pi_{I,X}^*(\omega) \equiv_X \pi_{I,X}^*(\omega')$ . If only one of them lies in  $W_X$ , say  $\mathbf{r}_{n-1+\tilde{h}}(w)$ , then  $\mathbf{R}_X(w)$  and  $\mathbf{R}_X(w')$  are not equal, but differ only at positions  $n-1+\tilde{h}$  and  $n-1+m-\tilde{h}$  and  $\widehat{\mathbf{R}}_X(w') = \widehat{\mathbf{R}}_X(w)$ . Then  $\pi_{I,X}^*(\omega)$  and  $\pi_{I,X}^*(\omega')$  because the exponents in position  $n-1+\tilde{h}$  and  $n-1+m-\tilde{h}$ , respectively, are both equal to 1. Assume finally that at least two terms among  $\mathbf{r}_n(w), \mathbf{r}_{n+1}(w), \dots, \mathbf{r}_{n+m-1}(w)$  lie in  $W_X$ . By construction, all the terms of the previous list belong to  $(s_{i_1} \cdots s_{i_{n-1}}) W_{i,j} (s_{i_1} \cdots s_{i_{n-1}})^{-1}$ . Since  $(s_{i_1} \cdots s_{i_{n-1}}) W_{i,j} (s_{i_1} \cdots s_{i_{n-1}})^{-1} \cap W_X$  contains at least two distinct elements different from 1, Corollary 1.10 implies that  $(s_{i_1} \cdots s_{i_{n-1}}) W_{i,j} (s_{i_1} \cdots s_{i_{n-1}})^{-1}$  is included in the parabolic subgroup  $W_X$ , and  $\mathbf{r}_n(w), \mathbf{r}_{n+1}(w), \dots, \mathbf{r}_{n+m-1}(w)$  all belong to  $W_X$ . Recall the notations given in the introduction for  $\widehat{\mathbf{R}}_X(w)$ ,  $w_X$  and  $\pi_{I,X}^*(\omega)$ . Let  $p'$  be such that  $j_{p'} = i_n$ . Set  $i' = q_{p'}$  and  $j' = q_{p'+1}$ . If  $\mathbf{v}_j$  and  $\mathbf{w}_j$  are defined like in Proposition 2.2, then, by Point(i)(b) of this proposition,  $\mathbf{w}_n = \mathbf{w}_{n+1} = \cdots = \mathbf{w}_{n+m-1}$ . Now by Remark 1.11 we have  $m(i', j') = m(i_n, i_{n+1})$ . It follows that  $s_{q_{p'}} \cdots s_{q_{p'+m-1}} = \underline{s}_{i',j'}$ ,  $w_X = s_{q_1} \cdots s_{q_{p'-1}} \underline{s}_{i',j'} s_{q_{p'+m}} \cdots s_{q_p}$  and  $w'_X = s_{q_1} \cdots s_{q_{p'-1}} \underline{s}_{j',i'} s_{q_{p'+m}} \cdots s_{q_p}$ . Thus,  $\pi_{I,X}^*(\omega) = \sigma_{q_1}^{\varepsilon_{j_1}} \cdots \sigma_{q_{p'-1}}^{\varepsilon_{j_{p'-1}}} \underline{\sigma}_{i',j'} \sigma_{j_{p'+m}}^{\varepsilon_{j_{p'+m}}} \cdots \sigma_p^{\varepsilon_{j_p}}$  and  $\pi_{I,X}^*(\omega') = \sigma_{q_1}^{\varepsilon_{j_1}} \cdots \sigma_{q_{p'-1}}^{\varepsilon_{j_{p'-1}}} \underline{\sigma}_{j',i'} \sigma_{j_{p'+m}}^{\varepsilon_{j_{p'+m}}} \cdots \sigma_p^{\varepsilon_{j_p}}$  and we are done.  $\square$

We are now ready to prove Propositions 0.2 and 0.3.

*Proof of Proposition 0.2.* Let  $\omega$  be a word on  $\Sigma_I \cup \Sigma_I^{-1}$ . Set  $w = \theta_I^*(\omega)$ . The length of  $\omega$  is equal to the length of  $w$ , which in turn is equal to the number of terms of  $\mathbf{R}(w)$ . Similarly the length of  $\pi_{I,X}^*(\omega)$  is equal to the length of  $\theta_X^*(\pi_{I,X}^*(\omega))$ . But, by definition, the latter word is  $w_X$  whom length is equal to the number of terms of  $\mathbf{R}(w_X)$ , that is of  $\widehat{\mathbf{R}}_X(w)$ . Thus,  $\ell_X(\pi_{I,X}^*(\omega)) \leq \ell_I(\omega)$ . The equality holds when  $\mathbf{R}(w_X) = \mathbf{R}(w)$ , which is equivalent to have  $w_X = w$ , which, in turn, is equivalent to have  $\pi_{I,X}^*(\omega) = \omega$ . So Point (i) holds. Let  $\omega'$  be another word on  $\Sigma_I \cup \Sigma_I^{-1}$  and set  $w' = \theta_I^*(\omega')$ . By definition,  $\mathbf{R}_X(ww')$  starts with  $\mathbf{R}_X(w)$ . Therefore,  $w_X$  is a prefix of  $(ww')_X$  and Point (ii) follows. Point (iii) follows from Proposition 2.6. Since  $\theta_X(\pi_{I,X}(\omega)) = \mathbf{w}_X$ , Point (iv) follows from Proposition 2.2 (i)(b) and (ii). Assume  $\omega$  lies in  $CA_I$ . Then,  $\mathbf{w} = 1$  in  $W_I$ . By Proposition 1.4,  $\mathbf{N}(w)$  has to be empty and all terms of



$\mathbf{R}(w)$  appear an even number of times. But in this case, all of its terms that are in  $W(X)$  appear an even number of times, too, and  $\mathbf{N}(w_X)$  is empty. Then  $\mathbf{w}_X = 1$ . But  $\theta_X^*(\pi_{I,X}^*(w)) = w_X$ . Then,  $\pi_{I,X}(\omega)$  lies in  $CA_X$  and the map  $\pi_{I,X}$  restricts to an application from  $CA_I$  to  $CA_X$ . The fact that  $\pi_{I,X} : CA_I \rightarrow CA_X$  is an homomorphism, and Point (v), will follow from Point (viii). Point (vi) is clear from the definition: with the notation of Definition 2.5, if all the  $\epsilon_n$  are equal to 1, so are the  $\epsilon_{j_n}$ . Let  $w'$  be as above, set  $k = \ell_S(w)$  and write  $(ww')_X = w_X w''$ . If  $\omega$  lies in  $CA_I$  or if  $\omega$  is on  $\Sigma_X \cup \Sigma_X^{-1}$ , then for  $n \geq k+1$ ,  $\mathbf{r}_n(ww')$  lies in  $W_X$  if and only if  $\mathbf{w} \mathbf{r}_{n-k}(w') \mathbf{w}^{-1}$  lies in  $W_X$ . So in both cases,  $\mathbf{R}_X(ww')$  is the concatenation of  $\mathbf{R}_X(w)$  and  $\mathbf{w} \mathbf{R}_X(w') \mathbf{w}^{-1}$ . It follows that  $\mathbf{R}((ww')_X)$  is the concatenation of  $\mathbf{R}(w_X)$  and  $\mathbf{w} \mathbf{R}(w'_X) \mathbf{w}^{-1}$ . By Remark 1.3, we deduce that  $\mathbf{R}(w'') = (\mathbf{w}_X^{-1} \mathbf{w}) \mathbf{R}(w'_X) (\mathbf{w}^{-1} \mathbf{w}_X)$ . But if  $\omega$  lies in  $CA_I$ , then  $\pi_{I,X}(\omega)$  lies in  $CA_X$  and  $\mathbf{w} = \mathbf{w}_X = 1$ . On the other hand, if  $\omega$  is a word on  $\Sigma_X \cup \Sigma_X^{-1}$ , then  $w_X = w$  and  $\mathbf{w}_X^{-1} \mathbf{w} = 1$ . So in both cases,  $\mathbf{R}(w'') = \mathbf{R}(w'_X)$ , which implies  $w'' = w'_X$ . This proves Points (vii) and (viii). Note that the exponents in  $\pi_{I,X}^*(\omega w')$  are the expected ones because  $\mathbf{R}_X(ww')$  is the concatenation of  $\mathbf{R}_X(w)$  and  $\mathbf{w} \mathbf{R}_X(w') \mathbf{w}^{-1}$ . Assume  $Y \subseteq I$  and  $\omega$  is a word on  $\Sigma_Y \cup \Sigma_Y^{-1}$ . Then  $w$  is a word on  $S_Y$  and the terms of  $\mathbf{R}(w)$  are in  $W_Y$ . Therefore, they are in  $W_X$  if and only if they are in  $W_X \cap W_Y$ , that is in  $W_{X \cap Y}$ . So the terms of  $\mathbf{R}_X(w)$  are in  $W_{X \cap Y}$  and  $w_X$  is a word on  $S_{X \cap Y}$ . Point (ix) follows. Finally, assume  $Y \subseteq I$  and  $\omega = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$  is a word on  $\Sigma_I \cup \Sigma_I^{-1}$ . Set  $w = \theta_I^*(\omega) = s_{i_1} \cdots s_{i_k}$  and, as in Definition 2.5,  $\widehat{R}_X(w) = (r_{j_1, X}(w), \dots, r_{j_p, X}(w))$ ;  $w_X = s_{q_1} \cdots s_{q_p}$  and  $\pi_{I,X}^*(\omega) = \sigma_{q_1}^{\epsilon_{j_1}} \cdots \sigma_{q_p}^{\epsilon_{j_p}}$ . Let us prove (x). By symmetry, it is enough to prove that  $\pi_{I,Y}^*(\pi_{I,X}^*(\omega)) = \pi_{I, X \cap Y}^*(\omega)$ . Write  $\pi_{I,Y}^*(\pi_{I,X}^*(\omega)) = \sigma_{q'_1}^{\epsilon_{j'_1}} \cdots \sigma_{q'_{p'}}^{\epsilon_{j'_{p'}}$  with  $j'_1, \dots, j'_{p'}$  in  $\{j_1, \dots, j_p\}$ . Write  $\pi_{I, Y \cap X}^*(\omega) = \sigma_{q''_1}^{\epsilon_{j''_1}} \cdots \sigma_{q''_{p''}}^{\epsilon_{j''_{p''}}}$  with  $j''_1, \dots, j''_{p''}$  in  $\{1, \dots, k\}$ . We have  $(w_X)_Y = s_{q'_1} \cdots s_{q'_{p'}}$  and  $w_{X \cap Y} = s_{q''_1} \cdots s_{q''_{p''}}$ . By construction  $\widehat{\mathbf{R}}_Y(\theta_I^*(\pi_{I,X}^*(\omega)))$  is obtained from  $\mathbf{R}(\theta_I^*(\pi_{I,X}^*(\omega)))$  by removing the terms that do not belong to  $W_Y$ . But  $\theta_I^*(\pi_{I,X}^*(\omega)) = w_X$  and  $\mathbf{R}(w_X)$  is obtained from  $\mathbf{R}(w)$  by removing the terms that do not belong to  $W_X$ . So,  $\widehat{\mathbf{R}}_Y(\theta_I^*(\pi_{I,X}^*(\omega)))$  is obtained from  $\mathbf{R}(w)$  by removing the terms that do not belong to  $W_X \cap W_Y$ , that is to  $W_{X \cap Y}$ . Thus  $\mathbf{R}((w_X)_Y) = \widehat{\mathbf{R}}_Y(\theta_I^*(\pi_{I,X}^*(\omega))) = \widehat{\mathbf{R}}_{Y \cap X}(w) = \mathbf{R}(w_{Y \cap X})$  and  $(w_X)_Y = w_{Y \cap X}$ . In particular  $p' = p''$  and  $q'_p = q''_p$  for  $\tilde{p}$  in  $1, \dots, p'$ . Now, let  $\tilde{p}$  be in  $1, \dots, p'$ . Then,  $j'_{\tilde{p}}$  is the position of the  $\tilde{p}^{\text{th}}$  term of  $\mathbf{R}(w)$  that belongs to  $W_Y \cap W_X$ , that is to  $W_{X \cap Y}$ . Therefore,  $j'_{\tilde{p}} = j''_{\tilde{p}}$  and  $\pi_{I,Y}^*(\pi_{I,X}^*(\omega)) = \pi_{I, X \cap Y}^*(\omega)$ . Hence, (x) holds. Finally, by Point (ix), Point (xi) is a special case of Point (x). Indeed, since  $\pi_{I,Y}^*(\omega)$  is a word on  $\Sigma_Y \cup \Sigma_Y^{-1}$ , we have  $\pi_{I,X}^*(\pi_{I,Y}^*(\omega)) = \pi_{I, X \cap Y}^*(\omega)$ .  $\square$

*Proof of Proposition 0.3.* Let  $\omega, \omega'$  be in  $(\Sigma_X \cup \Sigma_X^{-1})^*$  and such that  $\omega \equiv_I \omega'$ . By Proposition 0.2(i),  $\pi_{I,X}^*(\omega) = \omega$  and  $\pi_{I,X}^*(\omega') = \omega'$  and by Proposition 2.6,  $\pi_{I,X}^*(\omega) \equiv_X \pi_{I,X}^*(\omega')$ . So Point (i) holds. Let  $Y \subseteq I$ . Clearly  $A_{X \cap Y} \subseteq A_X \cap A_Y$ . Let  $\mathbf{w}$  be in  $A_X \cap A_Y$ . Since  $\mathbf{w}$  lies in  $A_Y$ , by Proposition 0.2(ix),  $\pi_{I,X}(\mathbf{w})$  lies in  $A_{X \cap Y}$ . Since  $\mathbf{w}$  lies in  $A_X$ , by Corollary 0.4 (i) (that we can apply as Point (i) is proved),  $\pi_{I,X}(\mathbf{w}) = \omega$ . So,  $A_X \cap A_Y \subseteq A_{X \cap Y}$  and Point (ii) holds. Similarly  $A_X^+ \subseteq A^+ \cap A_X$ . The argument to prove the inverse inclusion is similar as the one used to prove (ii), replacing Proposition 0.2(ix) with Proposition 0.2(vi). Finally, Point (iv) follows from almost the same argument : if  $\omega$  is a word representative of an element of  $A_X$  then, by Proposition 0.2(i) and (iii),  $\pi_{I,X}^*(\omega)$  is another word representative of the same element whom length is not greater than the one of  $\omega$ . If moreover  $\omega$  is a minimal word representative, then  $\omega$  and  $\pi_{I,X}^*(\omega)$  must have the same length. By Proposition 0.2(i) this imposes  $\omega = \pi_{I,X}^*(\omega)$ . Hence  $A_X$  is convex.  $\square$

**Proposition 2.7.** *the map  $\pi_{I,X}^*$  coincides with the map  $\hat{\pi}_X$  defined in [1].*

*Proof.* Let  $X \subseteq I$  and  $\omega = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$  be a word on  $\Sigma_I \cup \Sigma_I^{-1}$  where  $\epsilon_i$  lies in  $\{\pm 1\}$ . Let us recall the definition of  $\hat{\pi}_X$  given in [1]. Set  $w = \theta_I^*(\omega) = s_{i_1} \cdots s_{i_k}$ . For  $1 \leq j \leq k$ , let  $s_{i_1} \cdots s_{i_j} = \mathbf{v}_j \mathbf{w}_j$  with  $\mathbf{v}_j$  in  $W_X$  and  $\mathbf{w}_j$  that is  $(X, \emptyset)$ -reduced. If  $\epsilon_j = 1$ , set  $\mathbf{t}_j = \mathbf{w}_{j-1} \mathbf{s}_{i_j} \mathbf{w}_{j-1}^{-1}$ . If  $\epsilon_j = -1$ , set  $\mathbf{t}_j = \mathbf{w}_j \mathbf{s}_{i_j} \mathbf{w}_j^{-1}$ . If

$t_j$  lies in  $S_X$  set  $\tau_j = \sigma_{q_j}^{\varepsilon_j}$  so that  $t_j = s_{q_j}$ . Otherwise, set  $\tau_j = \varepsilon$ , the empty word. Then  $\hat{\pi}_X(\omega)$  is defined by  $\hat{\pi}_X(\omega) = \tau_1 \cdots \tau_k$ . Now, if  $\mathbf{w}_{j-1} s_{i_j} \mathbf{w}_{j-1}^{-1}$  lies in  $S_X$  then  $\mathbf{w}_j = \mathbf{w}_{j-1}$ ; otherwise  $\mathbf{w}_j = \mathbf{w}_{j-1} s_{i_j}$  (see the proof of Proposition 2.2(i)(b)). In both cases,  $\mathbf{w}_j s_{i_j} \mathbf{w}_j^{-1} = \mathbf{w}_{j-1} s_{i_j} \mathbf{w}_{j-1}^{-1}$ . So, in the above definition of  $\hat{\pi}_X$ , this is not necessary to distinguish whether  $\varepsilon_i$  is equal to 1 or to  $-1$  and  $\hat{\pi}_X(\omega) = \pi_{I,X}^*(\omega)$ .  $\square$

**Corollary 2.8.** *The map  $\pi_{I,X}$  coincides with the map  $\pi_X$  defined in [1].*

### 3. INTERSECTION OF PARABOLIC SUBGROUPS

We turn now to the proof of Theorems 0.5 and 0.6. We start with two preliminary lemmas. We recall that  $\kappa_I : W_I \rightarrow A_I$  is the set section of the morphism  $\Theta_I : A_I \rightarrow W_I$ .

**Lemma 3.1.** *Let  $X$  be a subset of  $I$  and  $\omega, \omega'$  be in  $A_I$  with  $\omega$  in  $CA_I$ . If  $\omega\omega'\omega^{-1}$  lies in  $A_X$  then  $\omega\omega'\omega^{-1} = \pi_{I,X}(\omega\omega'\omega^{-1}) = \pi_{I,X}(\omega)\pi_{I,X}(\omega')\pi_{I,X}(\omega)^{-1}$ .*

*Proof.* Since  $\omega\omega'\omega^{-1}$  lies in  $A_X$ , the first equality follows from Corollary 0.4(i). We have  $\omega\omega' = (\omega\omega'\omega^{-1})\omega$ . So  $\pi_{I,X}(\omega\omega') = \pi_{I,X}((\omega\omega'\omega^{-1})\omega)$ . By Proposition 0.2(viii)  $\pi_{I,X}(\omega\omega') = \pi_{I,X}(\omega)\pi_{I,X}(\omega')$  and by Corollary 0.4(ii),  $\pi_{I,X}((\omega\omega'\omega^{-1})\omega) = (\omega\omega'\omega^{-1})\pi_{I,X}(\omega)$ .  $\square$

Note that Lemma 3.1 extends [1, Lemma 2.4].

**Lemma 3.2.** *Let  $X, Y \subseteq I$  and  $\mathbf{w} \in W_I$  that is  $(\emptyset, Y)$ -reduced. Write  $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2$  with  $\mathbf{w}_1$  in  $W_X$  and  $\mathbf{w}_2$   $(X, Y)$ -reduced. Then,  $\pi_{I,X}(\kappa_I(\mathbf{w})A_Y) = \pi_{I,X}(\kappa_I(\mathbf{w}))A_{X_1}$  with  $S_{X_1} = S_X \cap \mathbf{w}_2 S_Y \mathbf{w}_2^{-1}$ .*

*Proof.* Let  $\omega = \kappa_I(\mathbf{w})$ ,  $\omega_1 = \kappa_I(\mathbf{w}_1)$  and  $\omega_2 = \kappa_I(\mathbf{w}_2)$ . Then,  $\ell_S(\mathbf{w}) = \ell_S(\mathbf{w}_1) + \ell_S(\mathbf{w}_2)$  and  $\omega = \omega_1 \omega_2$  with  $\omega_1$  in  $A_X$ . First, we have  $\pi_{I,X}(\omega_2) = 1$  and  $\pi_{I,X}(\omega) = \pi_{I,X}(\omega_1) = \omega_1 = \pi_{I,X}(\omega_1)\pi_{I,X}(\omega_2)$  (see Proposition 0.2(iv)). For any  $\omega'$  in  $A_Y$ , by Corollary 0.4(ii),  $\pi_{I,X}(\omega\omega') = \pi_{I,X}(\omega_1)\pi_{I,X}(\omega_2\omega')$ . So we may assume  $\mathbf{w}_1 = 1$  and  $\mathbf{w} = \mathbf{w}_2$ . Now, let  $w = s_{i_1} \cdots s_{i_k}$  be a representative word of  $\mathbf{w}$  of minimal length. Then  $\sigma_{i_1} \cdots \sigma_{i_k}$  is a word representative of  $\mathbf{w}$ . Let  $\omega'$  be in  $A_Y$  and  $\omega' = \sigma_{i'_1}^{\varepsilon_1} \cdots \sigma_{i'_k}^{\varepsilon_k}$  be a representative word of  $\omega'$  of minimal length. Set  $w' = \pi_{I,X}^*(\omega') = s_{i'_1} \cdots s_{i'_k}$ . Then all the  $i'_j$  lie in  $Y$  and  $\sigma_{i_1} \cdots \sigma_{i_k} \sigma_{i'_1}^{\varepsilon_1} \cdots \sigma_{i'_k}^{\varepsilon_k}$  is a representative word of  $\omega\omega'$ . Since  $\mathbf{w}$  is  $(X, \emptyset)$ -reduced and  $w$  is a minimal word representative of  $\mathbf{w}$ , the  $k$  first entries of  $\mathbf{R}_X(\omega\omega')$  are equal to 1 and all the next ones lie in  $\mathbf{w}W_Y\mathbf{w}^{-1} \cap W_X$ . By Proposition 1.8, there is  $X_1 \subseteq X$  and  $Y_1 \subseteq Y$  such that  $\mathbf{w}S_{Y_1} = S_{X_1}\mathbf{w}$  and  $\mathbf{w}W_{Y_1}\mathbf{w}^{-1} \cap W_X = W_{X_1} = \mathbf{w}W_{Y_1}\mathbf{w}^{-1}$ . Then,  $(\omega\omega')_X$  is a word on  $S_{X_1}$ . Since  $\theta_I^*(\omega\omega') = \omega\omega'$ , we get that  $\pi_{I,X}(\omega\omega')$  lies in  $A_{X_1}$  and  $\pi_{I,X}(\kappa_I(\mathbf{w})A_Y) \subseteq A_{X_1}$ . Conversely, let  $\omega''$  lie in  $A_{X_1}$ . Then  $\omega^{-1}\omega''\omega$  lies in  $A_{Y_1}$ . Indeed, since  $\mathbf{w}S_{Y_1} = S_{X_1}\mathbf{w}$  and  $\mathbf{w}$  is  $(X, Y)$ -reduced, for any  $i$  in  $X_1$  there is  $j$  in  $Y_1$  so that  $\mathbf{w}s_j = s_i\mathbf{w}$  with  $\ell_I(\mathbf{w}s_i) = \ell_I(s_j\mathbf{w}) = \ell_I(\mathbf{w}) + 1$ . Therefore,  $\omega\sigma_j = \sigma_i\omega$ . Now,  $\pi_{I,X}(\kappa_I(\mathbf{w})\omega^{-1}\omega''\omega) = \pi_{I,X}(\omega''\omega) = \pi_{I,X}(\omega'')\pi_{I,X}(\omega) = \omega''$ . So, the other inclusion holds.  $\square$

The following result proves Theorem 0.6.

**Proposition 3.3.** *Let  $X, Y \subseteq I$  and  $\mathbf{w} \in W_I$ . Let  $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}'_2$  with  $\mathbf{w}_1$  in  $W_X$ ,  $\mathbf{w}'_2$  in  $W_Y$  and  $\mathbf{w}_2$   $(X, Y)$ -reduced. Write  $\omega = \kappa_I(\mathbf{w})$ ,  $\omega_1 = \kappa_I(\mathbf{w}_1)$ , and  $\omega'_2 = \kappa_I(\mathbf{w}'_2)$ . Let  $X_1, Y_1$  be defined as in Proposition 1.8. Then  $\omega A_Y \omega^{-1} \cap A_X = \omega_1 A_{X_1} \omega_1^{-1}$  and  $\omega^{-1} A_X \omega \cap A_Y = \omega'^{-1}_2 A_{Y_1} \omega'_2$ .*

*Proof.* We have  $\ell_I(\mathbf{w}) = \ell_I(\mathbf{w}_1) + \ell_I(\mathbf{w}_2) + \ell_I(\mathbf{w}'_2)$  and  $\omega = \omega_1 \omega_2 \omega'_2$  with  $\omega_1$  in  $A_X$  and  $\omega'_2$  in  $A_Y$ . So  $\omega A_Y \omega^{-1} \cap A_X = \omega_1 \left( \omega_2 A_Y \omega_2^{-1} \cap A_X \right) \omega_1^{-1}$ . Similarly,  $\omega^{-1} A_X \omega \cap A_Y = \omega'^{-1}_2 \left( \omega_2^{-1} A_X \omega_2 \cap A_Y \right) \omega'_2$ . So, we may assume  $\mathbf{w}_1 = \mathbf{w}'_2 = 1$  and  $\mathbf{w}$  is  $(X, Y)$ -reduced. In this case,  $\theta_I(\omega) = \mathbf{w}$  and  $\pi_{I,X}(\omega) = 1$  (see Proposition 0.2(iv)). Let  $\omega'$  lie in  $A_Y$ . Set  $\omega'' = \omega\omega'\omega^{-1}$  and assume  $\omega''$  lies in  $A_X$ . Then  $\omega\omega' = \omega''\omega$  and  $\pi_{I,X}(\omega\omega') = \pi_{I,X}(\omega''\omega) = \pi_{I,X}(\omega'')\pi_{I,X}(\omega) = \omega''$ . But, by Lemma 3.2,  $\pi_{I,X}(\omega\omega')$  lies in  $A_{X_1}$ . So  $\omega A_Y \omega^{-1} \cap A_X \subseteq A_{X_1}$ . Conversely,  $A_{X_1}$  is included in  $A_X$  and, as seen in the end of

the proof of Lemma 3.2,  $\omega^{-1}A_{X_1}\omega = A_{Y_1}$ . So  $A_{X_1} \subseteq \omega A_Y \omega^{-1}$  and the other inclusion holds. Thus,  $\omega A_Y \omega^{-1} \cap A_X = A_{X_1}$  and  $\omega^{-1}A_X \omega \cap A_Y = \omega^{-1}(A_X \cap \omega A_Y \omega^{-1})\omega = \omega^{-1}A_{X_1}\omega = A_{Y_1}$ .  $\square$

*Proof of Theorem 0.5.* Assume Conjecture 2 holds. Let  $X, Y$  be in  $I$  and  $\omega$  be in  $A_I$ . We want to prove that  $(\omega A_Y \omega^{-1}) \cap A_X$  is a parabolic subgroup of  $A_I$ . Write  $\theta_I(\omega) = \mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}'_2$  with  $\mathbf{w}_1$  in  $W_X$ ,  $\mathbf{w}'_2$  in  $W_Y$  and  $\mathbf{w}_2$   $(X, Y)$ -reduced. Then

$$(\omega A_Y \omega^{-1}) \cap A_X = \kappa_I(\mathbf{w}_1) \left( (\kappa_I(\mathbf{w}_1)^{-1} \omega \kappa_I(\mathbf{w}'_2)^{-1}) A_Y (\kappa_I(\mathbf{w}'_2) \omega^{-1} \kappa_I(\mathbf{w}_1)) \cap A_X \right) \kappa_I(\mathbf{w}_1)^{-1}$$

and  $\theta_I((\kappa_I(\mathbf{w}_1)^{-1} \omega \kappa_I(\mathbf{w}'_2)^{-1}) = \mathbf{w}_1^{-1} \mathbf{w} \mathbf{w}'_2^{-1} = \mathbf{w}_2$ . So we can assume  $\mathbf{w}_1 = \mathbf{w}'_2 = 1$  and  $\mathbf{w}$  is  $(X, Y)$ -reduced. We set  $\omega_2 = \kappa_I(\mathbf{w})$ . Then  $\theta_I(\omega_2) = \mathbf{w}$ . Let  $\omega'$  be in  $A_Y$ . As already seen,  $\mathbf{w}$  being  $(X, Y)$ -reduced, we have  $\pi_{I,X}(\omega_2) = 1$  and, by Lemma 3.2,  $\pi_{I,X}(\omega_2 A_Y) = A_{X_1}$  where  $S_{X_1} = S_X \cap \mathbf{w} S_Y \mathbf{w}^{-1}$ . Then  $\pi_{I,X}(\omega_2 \omega')$  lies in  $A_{X_1}$ . Assume  $\omega \omega' \omega^{-1}$  belongs to  $A_X$ . Then  $\pi_{I,X}(\omega \omega' \omega^{-1}) = \omega \omega' \omega^{-1}$ . On the other hand,  $\theta_I(\omega \omega_2^{-1}) = \theta_I(\omega) \theta_I(\omega_2^{-1}) = \mathbf{w} \mathbf{w}^{-1} = 1$  and  $\omega \omega_2^{-1}$  lies in  $CA_I$ . Now,  $(\omega \omega' \omega^{-1})(\omega \omega_2^{-1})\omega_2 = (\omega \omega_2^{-1})\omega_2 \omega'$ . So, by Proposition 0.2(vii) and (viii), applying  $\pi_{I,X}$  we get,  $(\omega \omega' \omega^{-1})\pi_{I,X}(\omega \omega_2^{-1})\pi_{I,X}(\omega_2) = \pi_{I,X}(\omega \omega_2^{-1})\pi_{I,X}(\omega_2 \omega') \in \pi_{I,X}(\omega \omega_2^{-1})A_{X_1}$  and  $\omega \omega' \omega^{-1}$  lies in  $\pi_{I,X}(\omega \omega_2^{-1})A_{X_1} (\pi_{I,X}(\omega \omega_2^{-1}))^{-1}$ . Hence,  $\omega A_Y \omega^{-1} \cap A_X \subseteq \pi_{I,X}(\omega \omega_2^{-1})A_{X_1} (\pi_{I,X}(\omega \omega_2^{-1}))^{-1} \subseteq A_X$  and  $\omega A_Y \omega^{-1} \cap A_X = \omega A_Y \omega^{-1} \cap \left( \pi_{I,X}(\omega \omega_2^{-1})A_{X_1} (\pi_{I,X}(\omega \omega_2^{-1}))^{-1} \right)$ . By Proposition 3.3,  $A_{X_1} = \omega_2 A_Y \omega_2^{-1} \cap A_X = \omega_2 (A_Y \cap \omega_2^{-1} A_X \omega_2) \omega_2^{-1} = \omega_2 A_{Y_1} \omega_2^{-1}$ , so we get

$$\omega A_Y \omega^{-1} \cap A_X = \omega \left( A_Y \cap \left( (\omega^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2) A_{Y_1} (\omega^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2)^{-1} \right) \right) \omega^{-1}$$

Now,  $\omega^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2 = (\omega^{-1} \omega_2) \omega_2^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2$  and  $\omega^{-1} \omega_2$  lies in  $CA_I$ , then,  $\pi_{I,X}(\omega \omega_2^{-1})$  lies in  $CA_X$  and  $\omega^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2$  lies in  $CA_I$ . So, up to replacing  $(X, Y, \omega)$  with  $(Y, Y_1, \omega^{-1} \pi_{I,X}(\omega \omega_2^{-1}) \omega_2)$ , we may assume  $Y \subseteq X$  with  $\omega$  in  $CA_I$ . In this case  $\omega_2 = 1$  and  $A_{X_1} = A_Y$ . Thus  $\omega A_Y \omega^{-1} \cap A_X = \omega A_Y \omega^{-1} \cap \left( \pi_{I,X}(\omega) A_Y (\pi_{I,X}(\omega))^{-1} \right) = \omega \left( A_Y \cap \left( (\omega^{-1} \pi_{I,X}(\omega)) A_Y (\omega^{-1} (\pi_{I,X}(\omega))^{-1}) \right) \right) \omega^{-1}$  and we are done.  $\square$

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