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On Cutting Planes for Signomial Programming

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Abstract The generation of valid inequalities is central to solving non-convex Mixed-Integer Nonlinear Programming (MINLP) problems to global epsilon-optimality. Signomial Programming (SP), a.k.a. Generalized Geometric Programming, is an important class of MINLP problems. In this work, we study two kinds of cutting planes for the SP: intersection cuts and outer approximation cuts. The core ingredient in the derivation of the intersection cuts is so-called \mathcal{S} -free sets. Earlier studies considered \mathcal{S} to be various non-convex sets such as the lattice and outer-product sets. Here we consider the signomial set that appeared in the SP. We first construct signomial-free sets, and we show that these signomial-free sets are maximal in the non-negative orthant so that they generate strong intersection cuts. Secondly, we explore a combinatorial structure in the signomial set, namely supermodularity. Using supermodularity, we construct a convex relaxation of the signomial set and derive valid outer approximation cuts. We implement these cuts in the global solver SCIP and test their performance on the instances from the MINLPLib.

Keywords: MINLP, signomial programming, intersection cuts, supermodular inequalities

1. Introduction

Given an n -dimensional multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$, a signomial term is defined as $x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}$. We assume the variables are positive, thus allowing the multi-index α to have negative entries.

Signomial Programming (SP) problems generalize Polynomial Programming (PP) problems with positive variables. The SP in general form is as follows:

$$\min \quad c \cdot x \quad (1a)$$

$$\forall i \in [m] \quad \sum_{k \in \mathcal{K}_i} a_{ik} x^{\alpha^k} \leq 0 \quad (1b)$$

$$\forall j \in [n] \quad x_j \in [\underline{x}_j, \bar{x}_j] \subseteq \mathbb{R}_{++} \quad (1c)$$

where m is the number of signomial constraints, n is the number of decision variables, $[p] = \{1, \dots, p\}$ for any integer p , $[\underline{x}_j, \bar{x}_j]$ is a range constraint on variable x_j , \mathcal{K} is the index set of all signomial terms in the SP, $\mathcal{K}_i \subseteq \mathcal{K}$ is the index set of signomial terms in the i -th constraint, $\alpha^k \in \mathbb{R}^n$, and $a_{ik} \in \mathbb{R}$ for any $k \in \mathcal{K}_i$, for any $i \in [m]$.

The global solvers such as SCIP [1] use a factorization approach to transform the SP into an extended formulation. The transformation replaces x^{α^k} with an auxiliary variable y_k and adds the equality constraint $y_k = x^{\alpha^k}$. We study the convexification of the following *signomial set* for some $\alpha \in \mathbb{R}_{++}^N$:

$$\mathcal{S}_s := \{z := (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+ : y = x^\alpha\}.$$

2. Intersection Cuts

Intersection Cuts

We assume that there is an initial Linear Programming (LP) relaxation of the SP. The LP relaxation gives a polyhedral outer approximation \mathcal{P} of \mathcal{S}_s . We propose the intersection cuts to tighten \mathcal{P} . We first introduce the following concepts.

Definition 1. *Given a signomial set \mathcal{S}_s , a closed set \mathcal{C} is called \mathcal{S}_s -free or signomial-free, if the following conditions are satisfied: \mathcal{C} is convex; and $\text{int}(\mathcal{C}) \cap \mathcal{S}_s = \emptyset$.*

Usually, in the LP relaxation solution, we have the relaxation point \tilde{z} not in \mathcal{S}_s . The separation requires: a translated simplicial cone \mathcal{R} (containing \mathcal{S}_s) with apex \tilde{z} such that $\mathcal{S}_s \subseteq \mathcal{R}$, and an \mathcal{S}_s -free set \mathcal{C} containing \tilde{z} in its

interior. Then, an intersection cut separates \tilde{z} from $\text{conv}(\mathcal{R} \setminus \text{int}(\mathcal{C}))$ (a set which, we note, contains \mathcal{S}_s) as follows.

Since \tilde{z} is a vertex of \mathcal{P} \tilde{z} is defined by a basis of the simplex tableau of the LP relaxation. The translated simplicial cone \mathcal{R} is given in the following half-space representation:

$$\mathcal{R} = \{z \in \mathbb{R}^p : B(z - \tilde{z}) \leq 0\},$$

where B is a $p \times p$ invertible matrix extracted from the basis of the simplex tableau. Define the *step length* $\eta_j^* \in (0, +\infty)$ as $\eta_j^* := \sup_{\eta_j \geq 0} \{\eta_j : \tilde{z} + \eta_j r^j \in \mathcal{C}\}$.

The intersection cut [2] is then obtained as

$$\sum_{j=1}^p \frac{1}{\eta_j^*} B_j(z - \tilde{z}) \leq -1, \quad (2)$$

where B_j is the j -th row of B , and \tilde{z} is cut off by the above linear inequality.

We show that, after change of variables, the signomial set can be transformed to the following form:

$$\mathcal{S}_s = \{(u, v) \in \mathbb{R}_+^h \times \mathbb{R}_+^l : u^\beta \leq v^\gamma\}, \quad (3)$$

where $\beta \in \mathbb{R}_{++}^h$ and $\gamma \in \mathbb{R}_{++}^l$ are h - and l -dimensional multi-indices. After the normalization (by multiplying β and γ with a constant), we can assume that $\max(\|\beta\|_1, \|\gamma\|_1) = 1$, and both u^β and v^γ are concave.

Maximal signomial-free sets generate strong intersection cuts.

Proposition 2. *If $\max(\|\beta\|_1, \|\gamma\|_1) = 1$,*

$$\mathcal{C}_{s\check{v}} := \{(u, v) \in \mathbb{R}_+^h \times \mathbb{R}_+^l : u^\beta - \gamma \check{v}^{\gamma-1} \cdot (v - \check{v}) \geq 0\}$$

is a maximal \mathcal{S}_s -free set, where the linearization point is $\check{v} \in \mathbb{R}_+^l$.

3. Outer Approximation Cuts

In this section, we propose an alternative convex relaxation for the signomial set. W.l.o.g., we consider $\mathcal{S}_s = \{(u, v) \in \mathbb{R}_+^h \times \mathbb{R}_+^l : u^\beta - v^\gamma \leq 0\}$ in the sequel. Moreover, we assume that v belongs to a range constraint $\mathcal{U} := [\underline{u}, \bar{u}] \subseteq \mathbb{R}_+^h$. We rewrite \mathcal{S}_s as $\{(u, v) \in \mathcal{U} \times \mathbb{R}_+^l : u^\beta \leq v^\gamma\}$.

Definition 3. Given a function f defined on a closed set \mathcal{D} , the convex envelope $\text{convenv}_{\mathcal{D}}(f)$ is the tightest convex underestimating function of f on \mathcal{D} (i.e., $\text{convenv}_{\mathcal{D}}(f)(x) \leq f(x)$ for $x \in \mathcal{D}$).

Let $\text{convenv}_{\mathcal{U}}(u^\beta)$ be the convex envelope of u^β on \mathcal{U} . Then, we define

$$\overline{\mathcal{S}}_s := \{u, v\} \in \mathcal{U} \times \mathbb{R}_+^l : \text{convenv}_{\mathcal{U}}(u^\beta) \leq v^\gamma\}, \quad (4)$$

which is the convex relaxation of \mathcal{S}_s . From this relaxation, we can separate outer approximation cuts. We show that the function u^β on \mathcal{U} satisfies the following properties.

Proposition 4. The function u^β on \mathcal{U} is concave and convex-extendable from vertices $\text{vert}(\mathcal{U})$ of \mathcal{U} , i.e., $\text{convenv}_{\text{vert}(\mathcal{U})}(u^\beta) = \text{convenv}_{\mathcal{U}}(u^\beta)$. The function u^β is a supermodular function on \mathcal{U} , i.e., for every $u, u' \in \mathcal{U}$, $u^\beta + u'^\beta \leq \max\{u, u'\}^\beta + \min\{u, u'\}^\beta$, where max and min are evaluated element-wise.

After some transformation, it suffices to study the convex envelope of the following function.

Definition 5. Define $g : [0, 1]^h \rightarrow \mathbb{R} : w \rightarrow g(w) := \prod_{1 \leq j \leq h} ((\overline{x}_j - \underline{x}_j)w_j + \underline{x}_j)^{\beta_j} - \underline{u}^\beta$.

Proposition 6. Given a supermodular function g on $\{0, 1\}^h$, let $H := \{1, \dots, h\}$, and for $j \in H, S \subseteq H$ define $\rho(j, S) := g(S \cup \{j\}) - g(S)$ the increment function of g . Then,

$$\begin{aligned} g(S) + \sum_{j \in H \setminus S} \rho(j, S)w_j - \sum_{j \in S} \rho(j, H \setminus \{j\})(1 - w_j) &\leq t, \\ g(S) + \sum_{j \in H \setminus S} \rho(j, \emptyset)w_j - \sum_{j \in S} \rho(j, S \setminus \{j\})(1 - w_j) &\leq t, \end{aligned} \quad (5)$$

for $S \subseteq H$ are valid inequalities for $\text{convenv}_{\{0,1\}^h}(g)(w) \leq t$ (and $\text{convenv}_{[0,1]^h}(g)(w) \leq t$).

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