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► **To cite this version:**

Felisia Angela Chiarello, Paola Goatin. A non-local system modeling bi-directional traffic flows. 2021 Young Researchers Conference in Numerical Aspects of Hyperbolic Balance Laws and Related Problems, Dec 2021, Verona, Italy. pp.49-66, 10.1007/978-3-031-29875-2_3 . hal-03722225

HAL Id: hal-03722225

<https://hal.science/hal-03722225>

Submitted on 13 Jul 2022

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A non-local system modeling bi-directional traffic flows

Felisia Angela Chiarello and Paola Goatin

Abstract We present a non-local model describing the dynamics of two groups of agents moving in opposite directions. The model consists of a 2×2 system of conservation laws with non-local fluxes, coupled in the speed functions. We prove local in time existence of weak solutions and present some numerical tests illustrating their behaviour.

1 Introduction

Conservation laws with non-local flux are suitable to describe several phenomena arising in many fields of application. In this paper, we are interested in describing two groups of agents moving in opposite directions using this class of equations.

The first macroscopic traffic flow model based on fluid-dynamics equations was introduced in the transportation literature in the mid-fifties of last century, with the Lighthill, Whitham and Richards (LWR) model [10, 11]. In recent years, “non-local” versions of the LWR model have been proposed in [2, 3, 6, 8]. In most of these models, the speed depends on a weighted mean of the downstream traffic density, describing the behaviour of agents that adapt their velocity with respect to what happens in front of them. Therefore, the flux function depends on a “downstream” convolution term of the density of agents with a kernel function. In [4, 5], the authors consider a multi-class traffic model expressed by a system of conservation laws with non-local fluxes obtained generalizing the n -populations model for traffic flow introduced in [1], where each equation of the system describes the evolution of the density ρ_i of the vehicles belonging to the i -th class. In particular, the non-local multi-class model takes into account the distribution of heterogeneous agents characterized by their maximal speeds and look-ahead visibility in a traffic stream.

In this paper, we consider two non-local conservation laws describing two classes of agents moving in opposite directions. The resulting system is a non-local version of the model presented in [7], where the authors study a mixed type system of conservation laws describing two populations moving in opposite directions. The latter model is not hyperbolic for certain density values, because the Jacobian matrix of the flux exhibits complex eigenvalues in a subset of the phase space, and oscillations arise in the elliptic region. In particular, existence and uniqueness of solutions are still open problems. On the contrary, we will show that introducing a non-local dependence in the speed function allows to prove existence of solutions through the convergence of a suitable finite volume scheme, at least for sufficiently small times.

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The paper is organized as follows. In Section 2 we present a non-local version of the mixed system studied in [7] and we discretize it with an upwind scheme; in Section 3 we recover uniform \mathbf{L}^∞ and \mathbf{BV} bounds on the sequence of approximate solutions, which allow to prove the existence of weak solutions locally in time. Finally, in Section 4, we exhibit some numerical simulations showing the behaviour of weak solutions of our system considering different kernel and velocity functions. In particular, we compare non-local solutions to their local counterpart presented in [7].

2 A non-local bi-directional traffic flow model

We consider the following system

$$\begin{cases} \partial_t \rho_1 + \partial_x (\rho_1 v_1(r * \omega_1)) = 0, \\ \partial_t \rho_2 - \partial_x (\rho_2 v_2(r * \omega_2)) = 0, \end{cases} \quad (1)$$

where

$$\begin{aligned} r &= \rho_1 + \rho_2, \\ r * \omega_1(t, x) &= \int_x^{x+\eta_1} \omega_1(y-x)r(t, y)dy, \\ r * \omega_2(t, x) &= \int_{x-\eta_2}^x \omega_2(y-x)r(t, y)dy. \end{aligned}$$

In (1), $\rho_1 = \rho_1(t, x)$ is the density of the population moving in the direction of increasing space coordinate with speed $v_1 = v_1(r * \omega_1)$ depending on the downstream weighed mean of the total density r , while $\rho_2 = \rho_2(t, x)$ is the density of the population moving in the opposite direction, with non-local speed v_2 .

In the following, we will assume

(H1) $\omega_1 \in \mathbf{C}^1([0, \eta_1]; \mathbb{R}^+)$, $\omega_2 \in \mathbf{C}^1([- \eta_2, 0]; \mathbb{R}^+)$, $\eta_i > 0$ for $i \in \{1, 2\}$, $\omega_1' \leq 0$ and $\omega_2' \geq 0$, $\int_0^{\eta_1} \omega_1(x)dx = \int_{-\eta_2}^0 \omega_2(x)dx = 1$; we set $\omega_{\max} := \max\{\omega_1(0), \omega_2(0)\}$;

(H2) $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, $i \in \{1, 2\}$, are smooth non-increasing functions such that $v_i(0) = V_i^{\max} > 0$ and $v_i(r) = 0$ for $r \geq 1$.

We couple (1) with an initial datum

$$\rho_i(0, x) = \rho_i^0(x), \quad i = 1, 2, \quad (2)$$

and we construct a sequence of finite volume approximate solutions as follows. We fix a space step Δx and a time step Δt subject to a CFL condition that will be specified later. Let $x_{j+1/2} = j\Delta x$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ be the cells interfaces, $x_j = (j - 1/2)\Delta x$ the cells centers and $t^n = n\Delta t$ the time mesh. We aim at constructing finite volume approximate solutions of (1) of the form $\rho^{\Delta x} = (\rho_1^{\Delta x}, \rho_2^{\Delta x})$ with $\rho_i^{\Delta x}(t, x) = \rho_{i,j}^n$ for $i = 1, 2$, $(t, x) \in C_j^n = [t^n, t^{n+1}[\times [x_{j-1/2}, x_{j+1/2}[$. To this end, we approximate the initial data with piece-wise constant functions

$$\rho_{i,j}^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_i^0(x)dx, \quad \forall j \in \mathbb{Z}, i = 1, 2. \quad (3)$$

Similarly, for the kernels, we set

$$\omega_{i,k} := \frac{1}{\Delta x} \int_{k\Delta x}^{(k+1)\Delta x} \omega_i(x)dx, \quad \forall k \in \mathbb{Z}, i = 1, 2, \quad (4)$$

so that $\Delta x \sum_{k=-\infty}^{+\infty} \omega_{i,k} = \int_{-\infty}^{+\infty} \omega_i(x)dx = 1$ (extending $\omega_i(x) = 0$ outside their domain).

We use the following upwind scheme:

$$\rho_{1,j}^{n+1} = \rho_{1,j}^n - \frac{\Delta t}{\Delta x} \left[\rho_{1,j}^n v_1(R_{1,j+1}^n) - \rho_{1,j-1}^n v_1(R_{1,j}^n) \right], \quad (5a)$$

$$\rho_{2,j}^{n+1} = \rho_{2,j}^n + \frac{\Delta t}{\Delta x} \left[\rho_{2,j+1}^n v_2(R_{2,j}^n) - \rho_{2,j}^n v_2(R_{2,j-1}^n) \right], \quad (5b)$$

where

$$R_{1,j}^n = \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k}^n, \quad R_{2,j}^n = \Delta x \sum_{k=-\infty}^0 \omega_{2,k} r_{j+k}^n, \quad (6)$$

where the sums are indeed finite due to the compact supports of ω_1 and ω_2 .

3 Existence of weak solutions

In this section, we prove existence of weak solutions for sufficiently small times.

Theorem 1 *Let $\rho_i^0(x) \in (\mathbf{BV} \cap \mathbf{L}^\infty)(\mathbb{R}; \mathbb{R}^+)$, for $i = 1, 2$, and assumptions **(H1)** - **(H2)** hold. Then the Cauchy problem (1), (2) admits a weak solution on $[0, T] \times \mathbb{R}$, for some $T > 0$ sufficiently small.*

The proof relies on the following estimates, which ensure the convergence of the numerical scheme (5).

Lemma 1 (Positivity)

For any $T > 0$, under the CFL stability condition

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{\max\{\|v_1\|_\infty, \|v_2\|_\infty\}}, \quad (1)$$

the scheme (5) is positivity preserving on $[0, T] \times \mathbb{R}$.

Proof Let us assume that $\rho_{i,j}^n \geq 0$ for all $j \in \mathbb{Z}$ and $i \in \{1, 2\}$. Let us show that the same holds for $\rho_{i,j}^{n+1}$. Let us compute

$$\begin{aligned} \rho_{1,j}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x} v_1(R_{1,j+1}^n)\right) \rho_{1,j}^n + \rho_{1,j-1}^n \frac{\Delta t}{\Delta x} v_1(R_{1,j}^n), \\ \rho_{2,j}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x} v_2(R_{2,j-1}^n)\right) \rho_{2,j}^n + \rho_{2,j+1}^n \frac{\Delta t}{\Delta x} v_2(R_{2,j}^n). \end{aligned}$$

By assumption (1), we ensure $\rho_{i,j}^{n+1} \geq 0$ for $i \in \{1, 2\}$ and for all $j \in \mathbb{Z}$. □

Corollary 1 (\mathbf{L}^1 -bound)

If $\rho_{i,j}^0 \geq 0$ for all $j \in \mathbb{Z}$ and $i \in \{1, 2\}$ and the CFL condition (1) holds, the approximate solutions constructed via the scheme (5) satisfy

$$\|\rho_i^n\|_1 = \|\rho_i^0\|_1, \quad i = 1, \dots, M, \quad (2)$$

for any $n \in \mathbb{N}$, where $\|\rho_i^n\|_1 := \Delta x \sum_j |\rho_{i,j}^n|$ denotes the \mathbf{L}^1 norm of the i -th component of $\rho^{\Delta x}$.

Proof Thanks to Lemma 1, we have

$$\begin{aligned} \|\rho_1^{n+1}\|_1 &= \Delta x \sum_j \rho_{1,j}^{n+1} = \Delta x \sum_j \rho_{1,j}^n - \frac{\Delta t}{\Delta x} \left[\rho_{1,j}^n v_1(R_{1,j+1}^n) - \rho_{1,j-1}^n v_1(R_{1,j}^n) \right] = \Delta x \sum_j \rho_{1,j}^n, \\ \|\rho_2^{n+1}\|_1 &= \Delta x \sum_j \rho_{2,j}^{n+1} = \Delta x \sum_j \rho_{2,j}^n + \frac{\Delta t}{\Delta x} \left[\rho_{2,j+1}^n v_2(R_{2,j}^n) - \rho_{2,j}^n v_2(R_{2,j-1}^n) \right] = \Delta x \sum_j \rho_{2,j}^n, \end{aligned}$$

proving (2). □

Lemma 2 (\mathbf{L}^∞ -bound)

If $\rho_{i,j}^0 \geq 0$ for all $j \in \mathbb{Z}$ and $i \in \{1, 2\}$ and (1) holds, then the approximate solution $\rho^{\Delta x}$ constructed by the algorithm (5) is uniformly bounded on $[0, T] \times \mathbb{R}$ for any T such that

$$T \leq \frac{1}{4 \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} w_{\max} \|\rho^0\|_\infty}.$$

Proof Let us define $\bar{\rho} = \max\{\rho_{1,j-1}^n, \rho_{1,j}^n, \rho_{2,j}^n, \rho_{2,j+1}^n\}$. Then we get

$$\begin{aligned}\rho_{1,j}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x} v_1(R_{1,j+1}^n)\right) \rho_{1,j}^n + \frac{\Delta t}{\Delta x} v_1(R_{1,j}^n) \rho_{1,j-1}^n \\ &\leq \left(1 - \frac{\Delta t}{\Delta x} (v_1(R_{1,j+1}^n) - v_1(R_{1,j}^n))\right) \bar{\rho} \\ \rho_{2,j}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x} v_2(R_{2,j-1}^n)\right) \rho_{2,j}^n + \frac{\Delta t}{\Delta x} v_2(R_{2,j}^n) \rho_{2,j+1}^n \\ &\leq \left(1 - \frac{\Delta t}{\Delta x} (v_2(R_{2,j-1}^n) - v_2(R_{2,j}^n))\right) \bar{\rho},\end{aligned}$$

and

$$\begin{aligned}\left|v_1(R_{1,j}^n) - v_1(R_{1,j+1}^n)\right| &= \left|v_1'(\xi_{j+1/2}) \left(\Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+1}^n - \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k}^n\right)\right| \\ &= \Delta x \left|v_1'(\xi_{j+1/2})\right| \left|\sum_{k=1}^{+\infty} (\omega_{1,k-1} - \omega_{1,k}) r_{j+k}^n - \omega_{1,0} r_j^n\right| \\ &\leq 4 \Delta x \omega_1(0) \|v_1'\|_{\infty} \|\rho^n\|_{\infty},\end{aligned}\tag{3}$$

$$\begin{aligned}\left|v_2(R_{2,j-1}^n) - v_2(R_{2,j}^n)\right| &= \left|v_2'(\xi_{j-1/2}) \left(\Delta x \sum_{k=-\infty}^0 \omega_{2,k} r_{j+k-1}^n - \Delta x \sum_{k=-\infty}^0 \omega_{2,k} r_{j+k}^n\right)\right| \\ &= \Delta x \left|v_2'(\xi_{j-1/2})\right| \left|\sum_{k=-\infty}^{-1} (\omega_{2,k+1} - \omega_{2,k}) r_{j+k}^n - \omega_{2,0} r_j^n\right| \\ &\leq 4 \Delta x \omega_2(0) \|v_2'\|_{\infty} \|\rho^n\|_{\infty},\end{aligned}\tag{4}$$

where $\xi_{j+1/2} \in i(R_{1,j}^n, R_{1,j+1}^n)$, with $I(a, b) = [\min\{a, b\}, \max\{a, b\}]$, and $\|\rho^n\|_{\infty} = \|(\rho_1^n, \rho_2^n)\|_{\infty} = \max_{i,j} |\rho_{i,j}^n|$. So, until $\|\rho^n\|_{\infty} \leq K$ for some $K \geq \|\rho^0\|_{\infty}$, we get

$$\|\rho^{n+1}\|_{\infty} \leq \|\rho^n\|_{\infty} \left(1 + 4K \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} \Delta t w_{\max}\right)$$

with $w_{\max} = \max\{\omega_1(0), \omega_2(0)\}$. This implies

$$\|\rho^n\|_{\infty} \leq \|\rho^0\|_{\infty} e^{Cn\Delta t},$$

where $C = 4K \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max}$. Therefore we get that $\|\rho(t, \cdot)\|_{\infty} \leq K$ for

$$t \leq \frac{1}{4K \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max}} \ln \left(\frac{K}{\|\rho^0\|_{\infty}}\right) \leq \frac{1}{4e \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max} \|\rho^0\|_{\infty}},$$

where the maximum is attained for $K = e \|\rho^0\|_{\infty}$. Let us iterate the procedure: at time t^m , $m \geq 1$, we set $K = e^m \|\rho^0\|_{\infty}$ and we get that the solution is bounded by K until t^{m+1} such that

$$t^{m+1} \leq t^m + \frac{m}{4e^m \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max} \|\rho^0\|_{\infty}}.$$

Therefore, the approximate solution remains bounded, uniformly in Δx , at least for $t \leq T$ with

$$T \leq \frac{1}{4 \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max} \|\rho^0\|_{\infty}} \sum_{m=1}^{+\infty} \frac{m}{e^m} \leq \frac{1}{4 \max\{\|v_1'\|_{\infty}, \|v_2'\|_{\infty}\} w_{\max} \|\rho^0\|_{\infty}}.$$

Lemma 3 (Spatial BV-bound)

Let $\rho_i^0 \in (\mathbf{BV} \cap \mathbf{L}^{\infty})(\mathbb{R}; \mathbb{R}^+)$ for $i \in \{1, 2\}$. If (1) holds, then the approximate solution $\rho^{\Delta x}(t, \cdot)$ constructed by the algorithm (5) has uniformly bounded total variation for $t \in [0, T]$ for any T such that

$$T \leq \min_{i=1,2} \frac{1}{\mathcal{H}(TV(\rho_i^0) + 1)}, \quad (5)$$

where $\mathcal{H} = 2\|\rho^n\|_\infty w_{\max} \left(12\|\rho^n\|_\infty \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} + \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} \right)$.

Proof Let us consider the component ρ_1 and subtract the identities

$$\begin{aligned} \rho_{1,j+1}^{n+1} &= \rho_{1,j+1}^n - \frac{\Delta t}{\Delta x} \left[\rho_{1,j+1}^n v_1(R_{1,j+2}^n) - \rho_{1,j}^n v_1(R_{1,j+1}^n) \right] \\ \rho_{1,j}^{n+1} &= \rho_{1,j}^n - \frac{\Delta t}{\Delta x} \left[\rho_{1,j}^n v_1(R_{1,j+1}^n) - \rho_{1,j-1}^n v_1(R_{1,j}^n) \right], \end{aligned}$$

analogously for the second component ρ_2 ,

$$\begin{aligned} \rho_{2,j+1}^{n+1} &= \rho_{2,j+1}^n + \frac{\Delta t}{\Delta x} \left[\rho_{2,j+2}^n v_2(R_{2,j+1}^n) - \rho_{2,j+1}^n v_2(R_{2,j}^n) \right] \\ \rho_{2,j}^{n+1} &= \rho_{2,j}^n + \frac{\Delta t}{\Delta x} \left[\rho_{2,j+1}^n v_2(R_{2,j}^n) - \rho_{2,j}^n v_2(R_{2,j-1}^n) \right]. \end{aligned}$$

Setting $\Delta_{i,j+1/2}^n = \rho_{i,j+1}^n - \rho_{i,j}^n$ for $i \in \{1, 2\}$, we get

$$\begin{aligned} \Delta_{1,j+1/2}^{n+1} &= \Delta_{1,j+1/2}^n - \frac{\Delta t}{\Delta x} \left(\rho_{1,j+1}^n v_1(R_{1,j+2}^n) - 2\rho_{1,j}^n v_1(R_{1,j+1}^n) + \rho_{1,j-1}^n v_1(R_{1,j}^n) \right), \\ \Delta_{2,j+1/2}^{n+1} &= \Delta_{2,j+1/2}^n + \frac{\Delta t}{\Delta x} \left(\rho_{2,j+2}^n v_2(R_{2,j+1}^n) - 2\rho_{2,j+1}^n v_2(R_{2,j}^n) + \rho_{2,j}^n v_2(R_{2,j-1}^n) \right). \end{aligned}$$

We can write

$$\Delta_{1,j+1/2}^{n+1} = \left(1 - \frac{\Delta t}{\Delta x} v_1(R_{1,j+2}^n) \right) \Delta_{1,j+1/2}^n \quad (6)$$

$$\begin{aligned} &- \frac{\Delta t}{\Delta x} \rho_{1,j}^n \left(v_1(R_{1,j+2}^n) - 2v_1(R_{1,j+1}^n) + v_1(R_{1,j}^n) \right) \\ &+ \frac{\Delta t}{\Delta x} \Delta_{1,j-1/2}^n v_1(R_{1,j}^n), \end{aligned} \quad (7)$$

and

$$\Delta_{2,j+1/2}^{n+1} = \left(1 - \frac{\Delta t}{\Delta x} v_2(R_{2,j-1}^n) \right) \Delta_{2,j+1/2}^n \quad (8)$$

$$\begin{aligned} &+ \frac{\Delta t}{\Delta x} \rho_{2,j+1}^n \left(v_2(R_{2,j+1}^n) - 2v_2(R_{2,j}^n) + v_2(R_{2,j-1}^n) \right) \\ &+ \frac{\Delta t}{\Delta x} \Delta_{2,j+3/2}^n v_2(R_{2,j+1}^n). \end{aligned} \quad (9)$$

Observe that assumption (1) guarantees the positivity of (6) and (8). The term (7) can be estimated as

$$\begin{aligned} &v_1(R_{1,j+2}^n) - 2v_1(R_{1,j+1}^n) + v_1(R_{1,j}^n) \\ &= v_1 \left(\Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+2}^n \right) - 2v_1 \left(\Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+1}^n \right) + v_1 \left(\Delta x \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k}^n \right) \\ &= v_1'(\xi_{1,j+3/2}^n) \Delta x \left(\sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+2}^n - \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+1}^n \right) \\ &\quad - v_1'(\xi_{1,j+1/2}^n) \Delta x \left(\sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k+1}^n - \sum_{k=0}^{+\infty} \omega_{1,k} r_{j+k}^n \right) \\ &= v_1'(\xi_{1,j+3/2}^n) \Delta x \left(\sum_{k=1}^{+\infty} (\omega_{1,k-1} - \omega_{1,k}) r_{j+k+1}^n - \omega_{1,0} r_{j+1}^n \right) \\ &\quad - v_1'(\xi_{1,j+1/2}^n) \Delta x \left(\sum_{k=1}^{+\infty} (\omega_{1,k-1} - \omega_{1,k}) r_{j+k}^n - \omega_{1,0} r_j^n \right) \end{aligned}$$

$$\begin{aligned}
&= v_1''(\tilde{\xi}_{1,j+1}^n)(\xi_{1,j+3/2}^n - \xi_{1,j+1/2}^n)\Delta x \left(\sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \Delta_{i,j+k+3/2}^n \right) \\
&\quad + v_1'(\xi_{1,j+1/2}^n)\Delta x \left(\sum_{k=1}^{+\infty} \sum_{i=1}^2 (\omega_{1,k-1} - \omega_{1,k}) \Delta_{i,j+k+1/2}^n - \omega_{1,0} \Delta_{i,j+1/2}^n \right),
\end{aligned}$$

with $\xi_{1,j+1/2}^n \in \mathcal{I}(R_{1,j}^n, R_{1,j+1}^n)$ and $\tilde{\xi}_{1,j+1}^n \in \mathcal{I}(\xi_{1,j+1/2}^n, \xi_{1,j+3/2}^n)$. For some $\theta, \mu \in [0, 1]$, we compute

$$\begin{aligned}
\xi_{1,j+3/2}^n - \xi_{1,j+1/2}^n &= \theta \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k+2}^n + (1-\theta) \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k+1}^n \\
&\quad - \mu \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k+1}^n - (1-\mu) \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k}^n \\
&= \theta \Delta x \sum_{k=1}^{+\infty} \omega_{1,k-1} \sum_{i=1}^2 \rho_{i,j+k+1}^n + (1-\theta) \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k+1}^n \\
&\quad - \mu \Delta x \sum_{k=0}^{+\infty} \omega_{1,k} \sum_{i=1}^2 \rho_{i,j+k+1}^n - (1-\mu) \Delta x \sum_{k=-1}^{+\infty} \omega_{1,k+1} \sum_{i=1}^2 \rho_{i,j+k+1}^n \\
&= \Delta x \sum_{k=1}^{+\infty} [\theta \omega_{1,k-1} + (1-\theta) \omega_{1,k} - \mu \omega_{1,k} - (1-\mu) \omega_{1,k+1}] \sum_{i=1}^2 \rho_{i,j+k+1}^n \\
&\quad + \Delta x (1-\theta) \omega_{1,0} \sum_{i=1}^2 \rho_{i,j+1}^n - \Delta x \mu \omega_{1,0} \sum_{i=1}^2 \rho_{i,j+1}^n \\
&\quad - \Delta x (1-\mu) \left(\omega_{1,0} \sum_{i=1}^2 \rho_{i,j}^n + \omega_{1,1} \sum_{i=1}^2 \rho_{i,j+1}^n \right).
\end{aligned}$$

By monotonicity we have

$$\theta \omega_{1,k-1} + (1-\theta) \omega_{1,k} - \mu \omega_{1,k} - (1-\mu) \omega_{1,k+1} \geq 0.$$

Taking the absolute values, we get

$$\begin{aligned}
\left| \xi_{1,j+3/2}^n - \xi_{1,j+1/2}^n \right| &\leq \Delta x \left\{ \sum_{k=1}^{+\infty} [\theta \omega_{1,k-1} + (1-\theta) \omega_{1,k} - \mu \omega_{1,k} - (1-\mu) \omega_{1,k+1}] + 4 \omega_{1,0} \right\} 2 \|\rho^n\|_\infty \\
&\leq 12 \Delta x \omega_1(0) \|\rho^n\|_\infty.
\end{aligned}$$

Applying the same argument to the second component, we compute

$$\begin{aligned}
&v_2(R_{2,j+1}^n) - 2v_2(R_{2,j}^n) + v_2(R_{2,j-1}^n) \\
&= v_2''(\tilde{\xi}_{2,j}^n)(\xi_{2,j+1/2}^n - \xi_{2,j-1/2}^n)\Delta x \left(\sum_{k=-\infty}^0 \omega_{2,k} \sum_{i=1}^2 \Delta_{i,j+k+1/2}^n \right) \\
&\quad + v_2'(\xi_{2,j-1/2}^n)\Delta x \left(\sum_{k=-\infty}^0 \sum_{i=1}^2 (\omega_{2,k-1} - \omega_{2,k}) \Delta_{i,j+k-1/2}^n - \omega_{2,0} \Delta_{i,j-1/2}^n \right),
\end{aligned}$$

with $\xi_{2,j+1/2}^n \in \mathcal{I}(R_{2,j}^n, R_{2,j+1}^n)$ and $\tilde{\xi}_{2,j}^n \in \mathcal{I}(\xi_{2,j-1/2}^n, \xi_{2,j+1/2}^n)$, and

$$\left| \xi_{2,j+1/2}^n - \xi_{2,j-1/2}^n \right| \leq 12 \Delta x \omega_2(0) \|\rho^n\|_\infty.$$

Let now $K_1 > 0$ be such that $\sum_j |\Delta_{i,j}^\ell| \leq K_1$ for $i \in \{1, 2\}$, $\ell = 0, \dots, n$. Taking the absolute values and rearranging the indexes, we have

$$\sum_j \left| \Delta_{1,j+1/2}^{n+1} \right| \leq \sum_j \left| \Delta_{1,j+1/2}^n \right| \left(1 - \lambda \left(v_1(R_{1,j+2}^n) - v_1(R_{1,j+1}^n) \right) \right) + \Delta t \mathcal{H} K_1,$$

$$\sum_j \left| \Delta_{2,j+1/2}^{n+1} \right| \leq \sum_j \left| \Delta_{2,j+1/2}^n \right| \left(1 - \lambda \left(v_2(R_{2,j-1}^n) - v_2(R_{2,j}^n) \right) \right) + \Delta t \mathcal{H} K_1,$$

where $\mathcal{H} = 2 \|\rho^n\|_\infty w_{\max} \left(12 \|\rho^n\|_\infty \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} + \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} \right)$. Therefore, by (3)-(4) we get

$$\sum_j \left| \Delta_{i,j+1/2}^{n+1} \right| \leq \sum_j \left| \Delta_{i,j+1/2}^n \right| (1 + \Delta t \mathcal{G}) + \Delta t \mathcal{H} K_1,$$

with $\mathcal{G} = 4 \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} w_{\max} \|\rho^n\|_\infty$. We thus obtain

$$\sum_j \left| \Delta_{i,j+1/2}^n \right| \leq e^{\mathcal{G} n \Delta t} \sum_j \left| \Delta_{i,j+1/2}^0 \right| + e^{\mathcal{H} K_1 n \Delta t} - 1,$$

that we can rewrite as

$$\begin{aligned} TV(\rho_i^{\Delta x})(n \Delta t, \cdot) &\leq e^{\mathcal{G} n \Delta t} TV(\rho_i^0) + e^{\mathcal{H} K_1 n \Delta t} - 1 \\ &\leq e^{\mathcal{H} K_1 n \Delta t} (TV(\rho_i^0) + 1) - 1, \end{aligned}$$

since $\mathcal{H} \geq \mathcal{G}$ and it is not restrictive to assume $K_1 \geq 1$.

Therefore, we have that $TV(\rho_i^{\Delta x}) \leq K_1$ for

$$t \leq \frac{1}{\mathcal{H} K_1} \ln \left(\frac{K_1 + 1}{TV(\rho_i^0) + 1} \right),$$

where the maximum is attained for some $K_1 < e (TV(\rho_i^0) + 1) - 1$ such that

$$\ln \left(\frac{K_1 + 1}{TV(\rho_i^0) + 1} \right) = \frac{K_1}{K_1 + 1}.$$

Therefore the total variation is uniformly bounded for

$$t \leq \frac{1}{\mathcal{H} e (TV(\rho_i^0) + 1)}.$$

Iterating the procedure, at time t^m , $m \geq 1$ we set $K_1 = e^m (TV(\rho_i^0) + 1) - 1$ and we get that the solution is bounded by K_1 until t^{m+1} such that

$$t^{m+1} \leq t^m + \frac{m}{\mathcal{H} e^m (TV(\rho_i^0) + 1)}. \quad (10)$$

Therefore, the approximate solution has bounded total variation for $t \leq T$ with

$$T \leq \frac{1}{\mathcal{H} (TV(\rho_i^0) + 1)}.$$

Corollary 2 Let $\rho_i^0 \in (\mathbf{BV} \cap \mathbf{L}^\infty)(\mathbb{R}; \mathbb{R}^+)$. If (1) holds, then the approximate solution $\rho^{\Delta x}$ constructed by the algorithm (5) has uniformly bounded total variation on $[0, T] \times \mathbb{R}$, for any T satisfying (5).

Proof If $T \leq \Delta t$, then $TV(\rho_i^{\Delta x}; [0, T] \times \mathbb{R}) \leq T TV(\rho_i^0)$. Let us assume now that $T > \Delta t$. Let $n_T \in \mathbb{N} \setminus \{0\}$ such that $n_T \Delta t < T \leq (n_T + 1) \Delta t$. Then, for $i \in \{1, 2\}$,

$$\begin{aligned} &TV(\rho_i^{\Delta x}; [0, T] \times \mathbb{R}) \\ &= \underbrace{\sum_{n=0}^{n_T-1} \sum_{j \in \mathbb{Z}} \Delta t \left| \rho_{i,j+1}^n - \rho_{i,j}^n \right| + (T - n_T \Delta t) \sum_{j \in \mathbb{Z}} \left| \rho_{i,j+1}^{n_T} - \rho_{i,j}^{n_T} \right| + \sum_{n=0}^{n_T-1} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right|}_{\leq T \sup_{t \in [0, T]} TV(\rho_i^{\Delta x})(t, \cdot)}. \end{aligned}$$

We then need to bound the term

$$\sum_{n=0}^{n_T-1} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right|.$$

From the definition of the numerical scheme (5a)-(5b), we obtain

$$\begin{aligned} \rho_{1,j}^{n+1} - \rho_{1,j}^n &= -\frac{\Delta t}{\Delta x} \left[\rho_{1,j}^n v_1(R_{1,j+1}^n) - \rho_{1,j-1}^n v_1(R_{1,j}^n) \right], \\ &= \frac{\Delta t}{\Delta x} \left[\rho_{1,j-1}^n \left(v_1(R_{1,j}^n) - v_1(R_{1,j+1}^n) \right) + v_1(R_{1,j+1}^n) \left(\rho_{1,j-1}^n - \rho_{1,j}^n \right) \right], \\ \rho_{2,j}^{n+1} - \rho_{2,j}^n &= \frac{\Delta t}{\Delta x} \left[\rho_{2,j+1}^n v_2(R_{2,j}^n) - \rho_{2,j}^n v_2(R_{2,j-1}^n) \right] \\ &= \frac{\Delta t}{\Delta x} \left[\rho_{2,j+1}^n \left(v_2(R_{2,j}^n) - v_2(R_{2,j-1}^n) \right) + v_2(R_{2,j-1}^n) \left(\rho_{2,j+1}^n - \rho_{2,j}^n \right) \right]. \end{aligned}$$

Taking the absolute values and using (3)- (4) we get

$$\begin{aligned} \left| \rho_{1,j}^{n+1} - \rho_{1,j}^n \right| &\leq \frac{\Delta t}{\Delta x} \left(4 \Delta x \omega_1(0) \|v_1'\|_\infty \|\rho^n\|_\infty \left| \rho_{1,j-1}^n \right| + \|v_1\|_\infty \left| \rho_{1,j-1}^n - \rho_{1,j}^n \right| \right) \\ \left| \rho_{2,j}^{n+1} - \rho_{2,j}^n \right| &\leq \frac{\Delta t}{\Delta x} \left(4 \Delta x \omega_2(0) \|v_2'\|_\infty \|\rho^n\|_\infty \left| \rho_{2,j+1}^n \right| + \|v_2\|_\infty \left| \rho_{2,j+1}^n - \rho_{2,j}^n \right| \right). \end{aligned}$$

Summing on j , we get

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{1,j}^{n+1} - \rho_{1,j}^n \right| &\leq 4 \omega_1(0) \|v_1'\|_\infty \|\rho^n\|_\infty \Delta t \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{1,j-1}^n \right| + \|v_1\|_\infty \Delta t \sum_{j \in \mathbb{Z}} \left| \rho_{1,j-1}^n - \rho_{1,j}^n \right| \\ \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{2,j}^{n+1} - \rho_{2,j}^n \right| &\leq 4 \omega_2(0) \|v_2'\|_\infty \|\rho^n\|_\infty \Delta t \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{2,j+1}^n \right| + \|v_2\|_\infty \Delta t \sum_{j \in \mathbb{Z}} \left| \rho_{2,j+1}^n - \rho_{2,j}^n \right|. \end{aligned}$$

which yields, for $i \in \{1, 2\}$,

$$\begin{aligned} &\sum_{n=0}^{n_T-1} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right| \\ &\leq \max\{\|v_1\|_\infty, \|v_2\|_\infty\} T \sup_{t \in [0, T]} TV(\rho_i^{\Delta x})(t, \cdot) \\ &\quad + 4 \max\{\omega_1(0), \omega_2(0)\} \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\} \|\rho^n\|_\infty T \sup_{t \in [0, T]} \|\rho_i^{\Delta x}(t, \cdot)\|_1, \end{aligned}$$

that is bounded by Corollary 1, Lemma 2 and Lemma 3. \square

Proof (of Theorem 1) To complete the proof of the existence of solutions to the Cauchy problem (1)-(2), we apply Helly's theorem and follow a Lax-Wendroff type argument as in [2, 3, 4], see also [9], to show that the approximate solutions constructed by scheme (5) converge to a weak solution of (1)-(2). \square

4 Numerical tests

4.1 Kernel support tending to zero

In this subsection, we observe the solution behaviour as the length of kernel support diminishes. We consider the space domain given by the interval $[-1, 1]$ and the space discretization mesh size is set to $\Delta x = 0.001$. We impose absorbing conditions at the boundaries, adding $N = \eta_i/\Delta x$ ghost cells at the right boundary for the first population ρ_1 and at the left boundary for the second population ρ_2 , where we extend the solution constantly equal to the last value inside the domain.

We take the following linear decreasing kernel and velocity functions:

$$\omega_1(x) = \frac{2}{\eta_1} \left(1 - \frac{x}{\eta_1}\right), \quad \omega_2(x) = \frac{2}{\eta_2} \left(1 + \frac{x}{\eta_2}\right); \quad (1a)$$

$$v_1(\rho) = 1 - \rho, \quad v_2(\rho) = 1 - \rho. \quad (1b)$$

In Figures 1 and 2, we consider the same initial data of [7, Figures 5 and 6], observing that the solutions have a similar behaviour, especially when the kernel supports $\eta_1 = \eta_2$ are taken sufficiently small. Moreover, comparing Figures 3 and 4 with the numerical tests in [7, Figures 14 and 15], we note that when we consider initial data in the elliptic region of the local system, oscillations increase as the kernel support diminish.

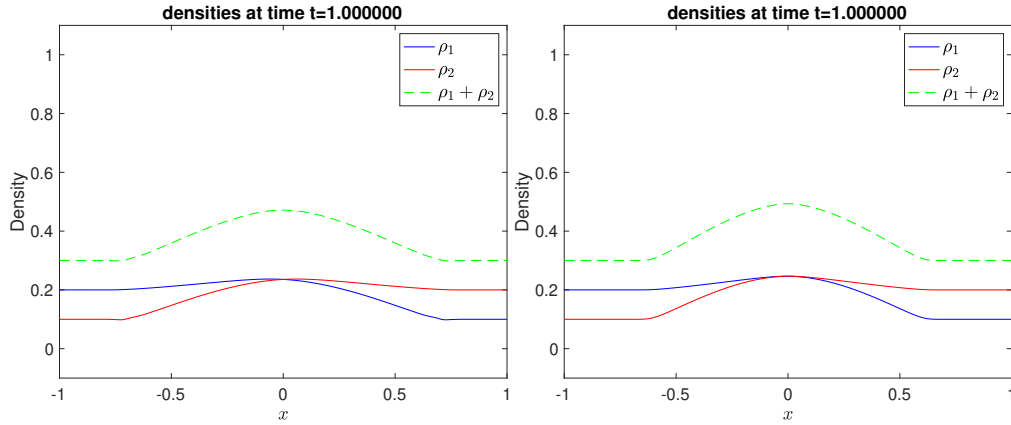


Fig. 1: Test 1. Solution to (1) with (1) and $\rho_1^0 = 0.2 * (x < 0) + 0.1 * (x \geq 0)$ and $\rho_2^0 = 0.1 * (x < 0) + 0.2 * (x \geq 0)$ with $\eta_1 = \eta_2 = 0.1$ on the left and $\eta_1 = \eta_2 = 0.01$ on the right.

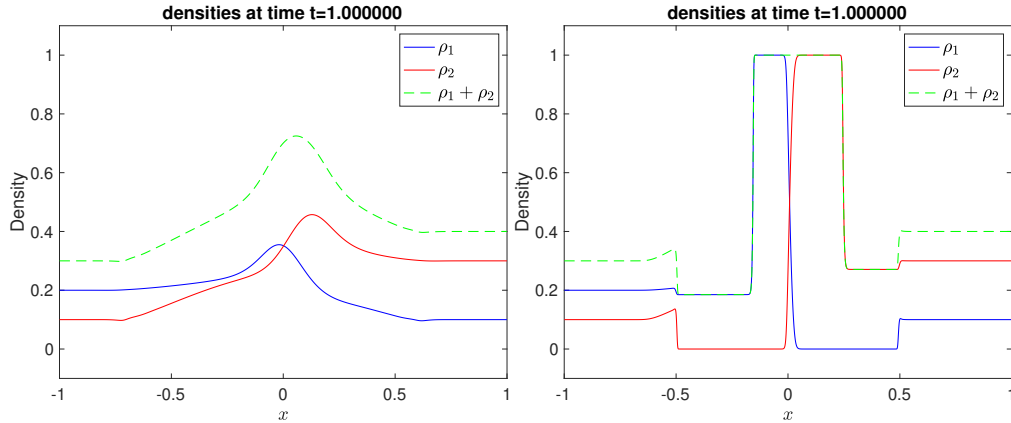


Fig. 2: Test 2. Solution to (1) with (1) and $\rho_1 = 0.2 * (x < 0) + 0.1 * (x \geq 0)$ and $\rho_2 = 0.1 * (x < 0) + 0.3 * (x \geq 0)$ with $\eta_1 = \eta_2 = 0.1$ on the left and $\eta_1 = \eta_2 = 0.01$ on the right.

4.2 Asymptotic behaviour in a periodic setting

We consider now periodic initial data with periodic boundary conditions, see Figure 5. In the following we set $\Delta x = 0.001$, the CFL condition is as in in (1). We are interested in the behaviour of solutions with different types of kernel functions and velocities.

For Test 5, let us consider the linear kernels and velocity functions

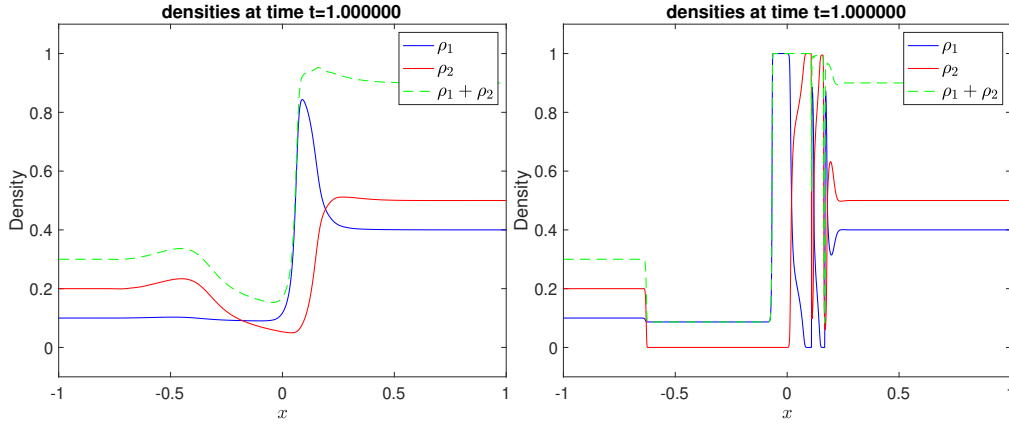


Fig. 3: Test 3. Solution to (1) with (1) and $\rho_1^0 = 0.1 * (x < 0) + 0.4 * (x \geq 0)$ and $\rho_2^0 = 0.2 * (x < 0) + 0.5 * (x \geq 0)$. With $\eta_1 = \eta_2 = 0.1$ at left, $\eta_1 = \eta_2 = 0.01$ at right.

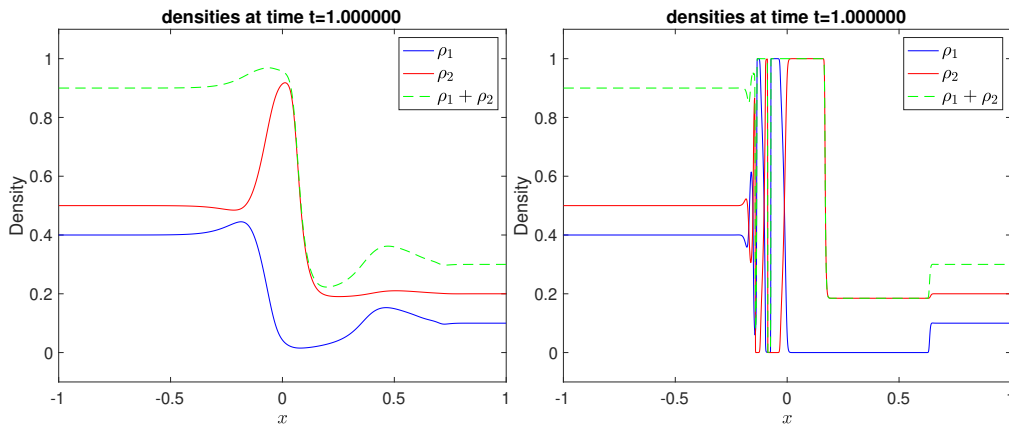


Fig. 4: Test 4. Solution to (1) with (1) and $\rho_1^0 = 0.4 * (x < 0) + 0.1 * (x \geq 0)$ and $\rho_2^0 = 0.5 * (x < 0) + 0.2 * (x \geq 0)$. With $\eta_1 = \eta_2 = 0.1$ at left, $\eta_1 = \eta_2 = 0.01$ at right.

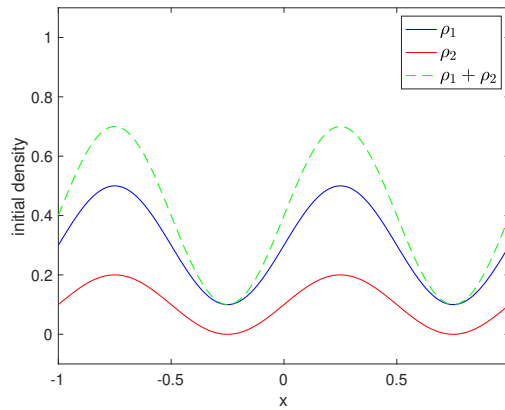


Fig. 5: Initial condition for tests 5, 6 and 7 with periodic boundary conditions. $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$.

$$\omega_1(x) = \frac{2}{\eta_1} \left(1 - \frac{x}{\eta_1} \right), \quad \omega_2(x) = \frac{2}{\eta_2} \left(1 + \frac{x}{\eta_2} \right); \quad (2a)$$

$$v_1(\rho) = 1 - \rho, \quad v_2(\rho) = 1 - \rho, \quad (2b)$$

see Figure 6. From Figure 7 we can observe that the solutions tend to a steady-state as the time increases.

For Test 6, let us consider concave kernel functions and linear decreasing function velocities with different maximum velocities $V_1^{\max} = 0.8$ and $V_2^{\max} = 1.3$:

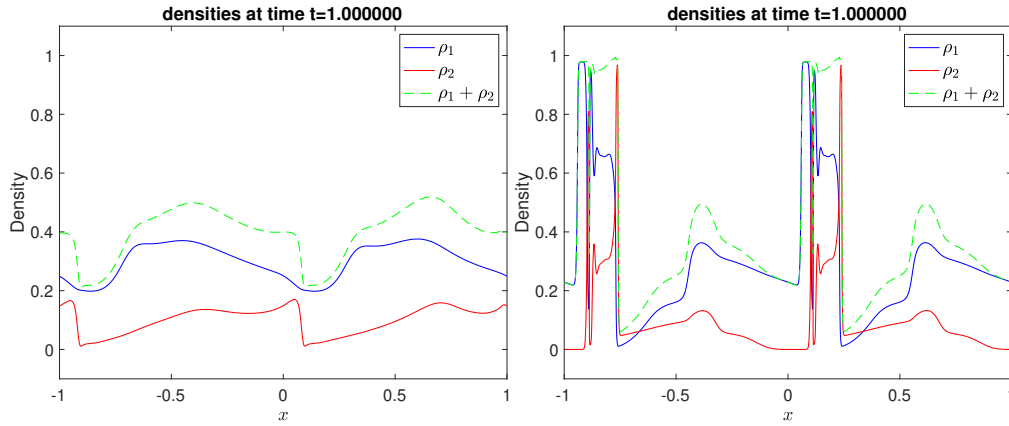


Fig. 6: Test 5. Solution to (1) with (2) and $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$ with $\eta_1 = \eta_2 = 0.1$ at left, $\eta_1 = \eta_2 = 0.01$ at right.

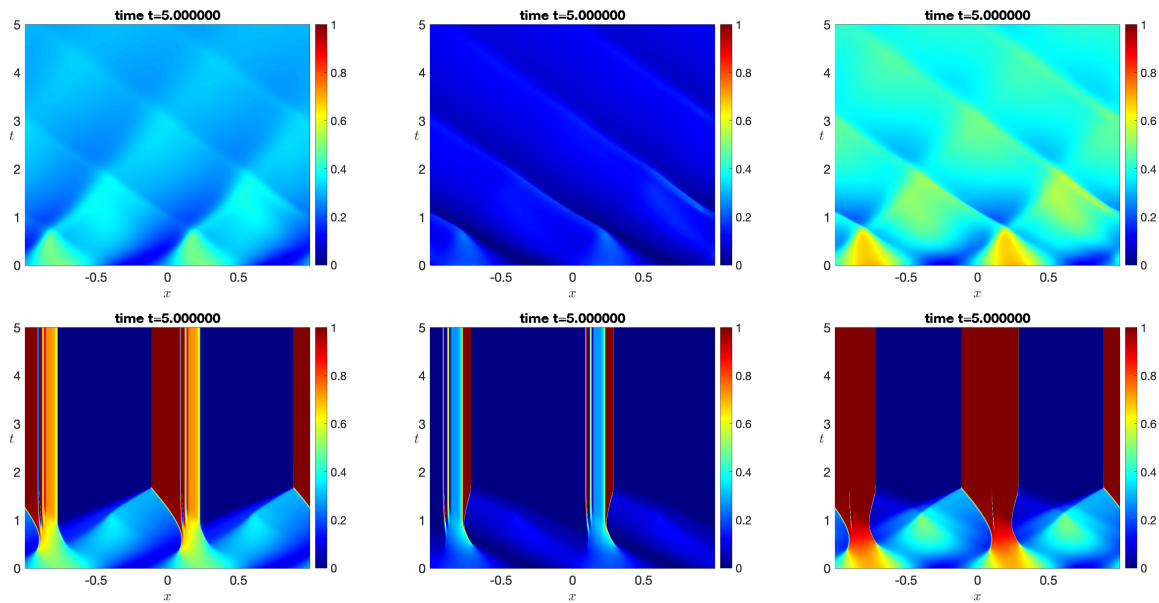


Fig. 7: (t, x) -plot solutions of (1) with (2) and initial conditions $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$, see Figure 5; from top to bottom: $\eta_1 = \eta_2 = 0.1, 0.01$; from left to right: ρ_1, ρ_2 and $\rho_1 + \rho_2$.

$$\omega_1(x) = \frac{3}{2\eta_1^3} (\eta_1^2 - x^2), \quad \omega_2(x) = \frac{3}{2\eta_2^3} (\eta_2^2 - x^2); \quad (3a)$$

$$v_1(\rho) = V_1^{\max} (1 - \rho), \quad v_2(\rho) = V_2^{\max} (1 - \rho); \quad (3b)$$

In Fig. 8 we can see the profile of the solutions where the agents of the second class are faster than the agents belonging to the first class. Also in this case, reducing the supports of the kernels, oscillations increase. Figure 9 is very similar to Figure 7; here we observe that solutions tend to a steady-state after time 2.

For Test 7, let us consider the concave kernel functions (3a) with different supports and same maximum velocities $V_1^{\max} = V_2^{\max} = 1$ in (3b). In Figure 10, we can observe the behaviour of the solutions when the agents of the two classes have different look-ahead distances, i.e. kernel supports. We can notice that, when one group of agents has a sufficiently large look-ahead distance, oscillations are not evident. From Figure 11, we see that the solutions reach the steady-state as the time increases.

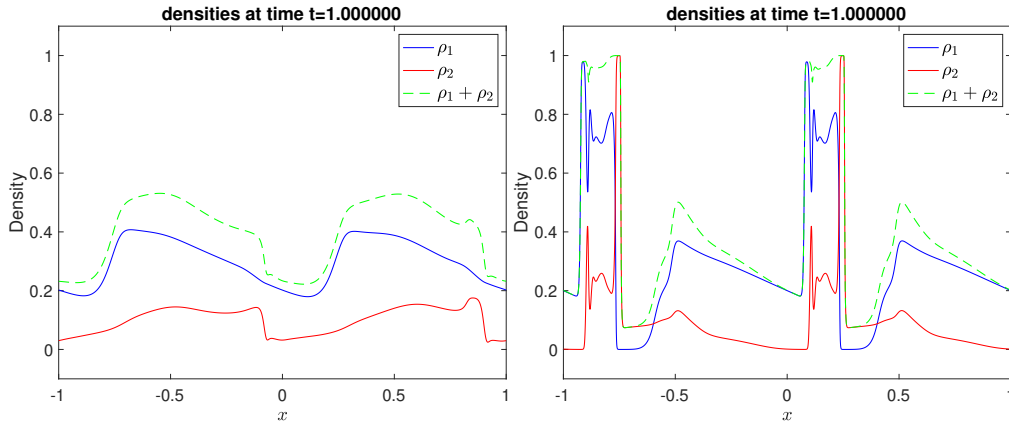


Fig. 8: Test 6. Solution to (1) with (3), $V_1^{\max} = 0.8$ and $V_2^{\max} = 1.3$, and $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$; $\eta_1 = \eta_2 = 0.1$ on the left, $\eta_1 = \eta_2 = 0.01$ on the right.

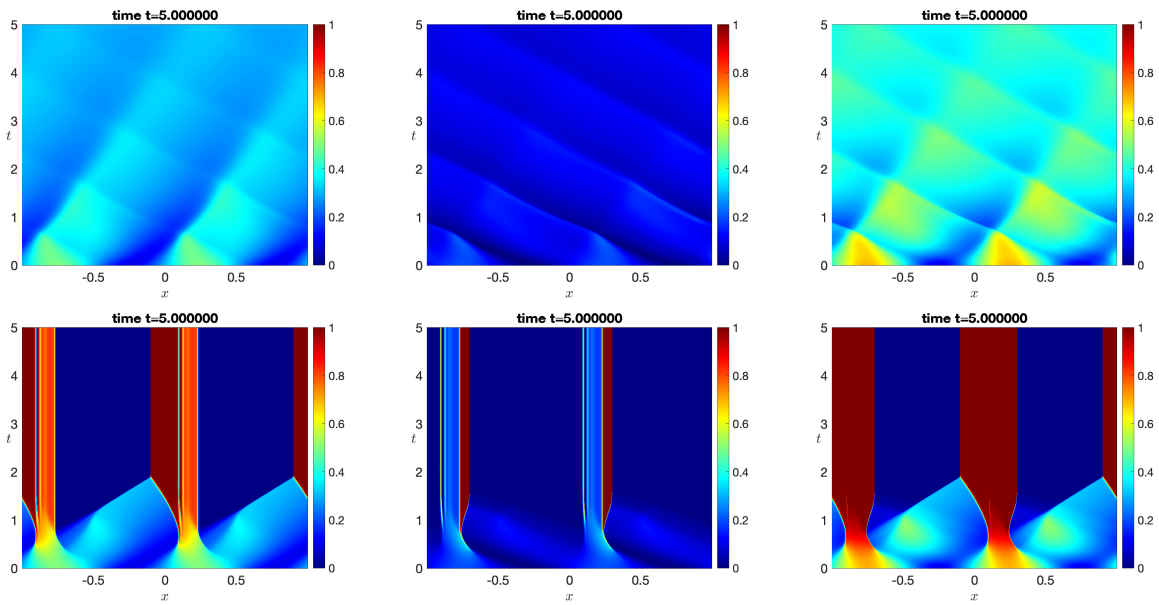


Fig. 9: (t, x) -plot solutions of (1) with (3) and initial conditions $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$, see Figure 5; from top to bottom: $\eta_1 = \eta_2 = 0.1$, 0.01 ; from left to right: ρ_1 , ρ_2 and $\rho_1 + \rho_2$.

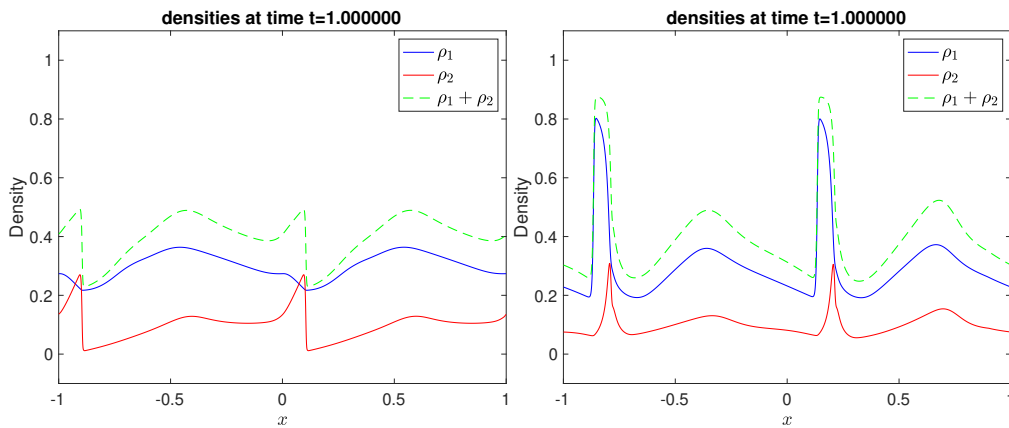


Fig. 10: Test 7. Solution to (1) with (3), $V_1^{\max} = V_2^{\max} = 1$, and $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$; $\eta_1 = 0.1$, $\eta_2 = 0.01$ on the left, $\eta_1 = 0.01$, $\eta_2 = 0.1$ on the right.

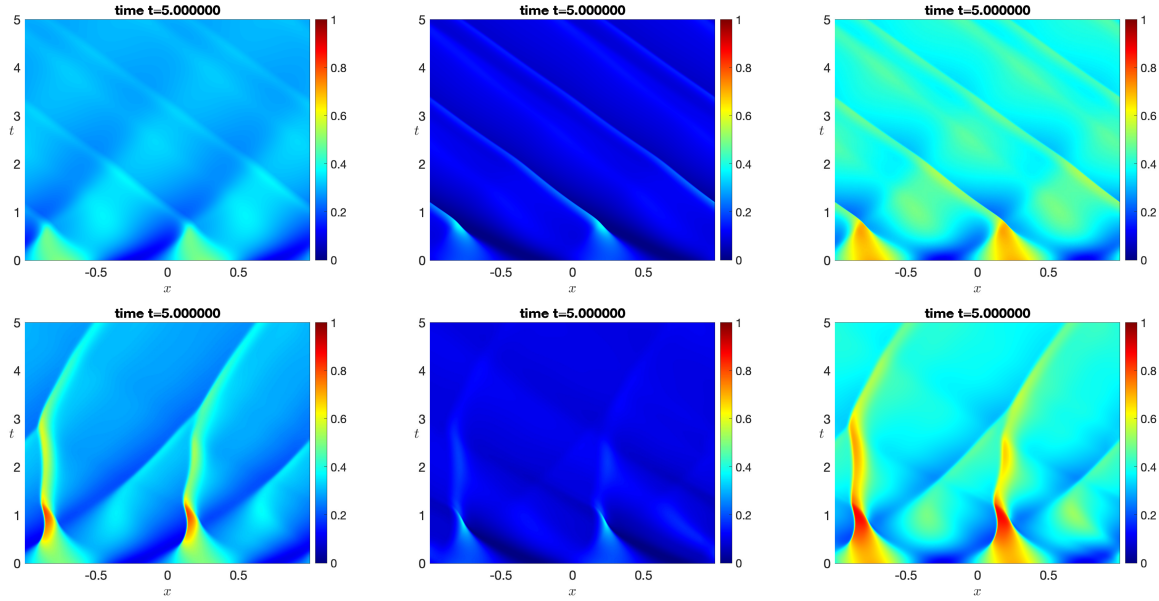


Fig. 11: (t, x) -plot solutions of (1) with (2) and initial conditions $\rho_1^0 = 0.3 + 0.2 \sin(2\pi x)$ and $\rho_2^0 = 0.1 + 0.1 \sin(2\pi x)$, see Figure 5; from top to bottom: $\eta_1 = 0.1 \eta_2 = 0.01$ and $\eta_1 = 0.01 \eta_2 = 0.1$; from left to right: ρ_1, ρ_2 and $\rho_1 + \rho_2$.

4.3 Maximum principle

The simplex

$$\mathcal{S} := \{(\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 + \rho_2 \leq 1, \rho_i \geq 0 \text{ for } i = 1, 2\},$$

is not an invariant domain, as we can see in Figure 12, as we already noticed for the multi-class system presented in [4]. In particular, we observe that the sum $r(t, x)$ can exceed 1 even if the initial condition is such that $\rho_1(0, x) + \rho_2(0, x) \leq 1$.

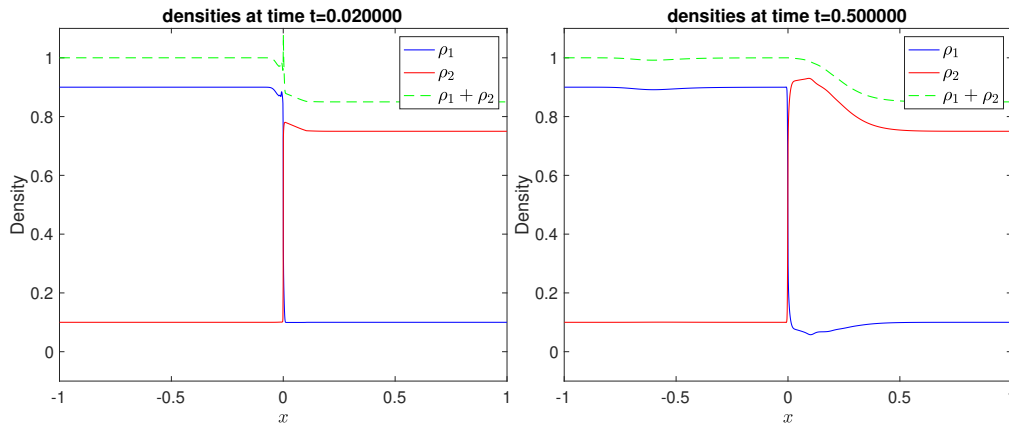


Fig. 12: Solution to (1) obtained with initial data $\rho_1^0 = 0.9 * (x < 0) + 0.1 * (x \geq 0)$ and $\rho_2^0 = 0.1 * (x < 0) + 0.75 * (x \geq 0)$, linear kernels, $\eta_1 = 0.01$ and $\eta_2 = 0.1$, $V_1^{\max} = 1.5$ and $V_2^{\max} = 0.8$ and absorbing boundary conditions.

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