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ABSTRACT. We determine the all-genus Hodge–Gromov–Witten theory of a smooth hypersurface in weighted projective space defined by a chain or loop polynomial. In particular, we obtain the first genus-zero computation of Gromov–Witten invariants for hypersurfaces in non-Gorenstein ambiant spaces, where the convexity property fails. We eventually extend it to any weighted projective hypersurface defined by an invertible polynomial.

CONTENTS

0. INTRODUCTION

Gromov–Witten theory has known a tremendous development in the last thirty years. Originated from theoretical physics, it is mathematically formulated as an intersection theory of complex curves traced on a complex smooth projective variety, and provides invariants that one thinks of as a virtual count of these curves. The most famous example is a full computation of the genus-zero invariants enumerating rational curves on the quintic threefold [8, 22, 39].

Gromov–Witten theory is well understood in all genus for toric varieties, or even toric Deligne–Mumford (DM) stacks [24, 40]. Precisely, the moduli space of stable maps inherits a torus action from the target space and the computation essentially reduces to a calculation on the moduli space to the fixed locus. This is the content of the virtual localization formula [24], which is an enhancement of the classical Atiyah–Bott localization formula [3]. We also refer to [16] for an algebraic proof.

Smooth hypersurfaces in toric DM stacks are the next class of spaces to consider, but little is known in this situation. The difficulty comes from the non-invariance of the hypersurface by the torus action in general, so that there is no direct way to apply a localization formula to decrease the complexity of the problem. Consider the famous example of the quintic hypersurface in $\mathbb{P}^4$. As we mention above, the genus-zero theory is fully determined [8, 22, 39]. The genus-one case is completely proven by Zinger [44] after a great deal of hard work, and nowadays several approaches are solving it up to genus three [10, 18, 29]. It is worth noticing that physicists have predictions up to genus 52 [32] and that Maulik–Pandharipande [41] described a proposal working in any genus, although it is too hard to implement for practical use. We also mention a recent breakthrough proving the BCOV holomorphic anomaly conjecture [5], see [10, 29].

Even in genus zero the problem of computing Gromov–Witten invariants of smooth hypersurfaces in toric DM stacks is far from being completely solved. Consider the special case of hypersurfaces in weighted projective spaces. The genus-zero theory is only known under a restrictive condition: the degree of the hypersurface is a multiple of every weight. One refers to it as the Gorenstein condition, as it is the condition for the coarse space of the hypersurface to have Gorenstein singularities. We recall that Gromov–Witten theory is invariant under smooth deformations, hence we can choose any defining polynomial of degree d as long as the associated hypersurface is a smooth DM stack. As a consequence, one can also rephrase the Gorenstein condition as the existence of a Fermat hypersurface of degree d, that is defined by a Fermat polynomial of the form $x_1^{a_1} + \cdots + x_N^{a_N}$.

There is a substantial simplification for the genus-zero theory of hypersurfaces in weighted projective spaces under Gorenstein condition; it is called the convexity property, see [26, Introduction]. It implies that the virtual cycle of the theory, which is the crucial object to handle, equals the top Chern class of a vector bundle over the moduli space of stable maps to the weighted projective space. It is then calculated by a Grothendieck–Riemann–Roch formula [13, 33, 37, 43] and genus-zero Gromov–Witten theory of the hypersurface is deduced from genus-zero Gromov–Witten theory of the weighted projective space; one calls it Quantum Lefschetz Principle [13, 33, 37, 43]. Without the Gorenstein condition, the convexity property may fail and the virtual cycle is not computable. Although Fan and Lee [18] obtain a version of Quantum Lefschetz Principle in higher genus for projective hypersurfaces, a general statement is false [12].

In this paper, we work on smooth hypersurfaces in weighted projective spaces, under a mild condition. Precisely, we relax the existence of a Fermat hypersurface to the existence of a chain hypersurface, that is defined by a chain polynomial of the form $x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}$, or to the existence of a loop hypersurface, that is defined by a loop polynomial of the form $x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}x_1$. This is more general than the Gorenstein condition and non-convex cases appear. In genus zero, we are even more general as we can relax the condition on weights and degree to the existence of an hypersurface defined by an invertible polynomial, see the beginning of Section 3.3. We then prove two results for these hypersurfaces:

- a genus-zero Quantum Lefchetz Principle, Corollaries 3.6, 3.11, and Theorem 3.13,
- a Hodge Quantum Lefchetz Principle in arbitrary genus, Theorems 3.3 and 3.8.

In the first, we express genus-zero Gromov–Witten theory of the hypersurface in terms of genus-zero Gromov–Witten theory of the weighted projective space. In the second, we do the same in arbitrary genus, once we cap virtual cycles with the Hodge class, that is the top Chern class of the Hodge bundle, see Definition 1.1. As a consequence, this paper gives the first computation of genus-zero Gromov–Witten theory of hypersurfaces in a range of cases where the convexity property fails, see for instance Remark 3.17 for a discussion on Calabi–Yau 3-folds with Euler characteristic equal to ± 6. It also gives the first comprehensive computation of Hodge integrals, that are Gromov–Witten invariants involving the Hodge class, in arbitrary genus for chain or loop hypersurfaces.

In order to tackle non-convexity issues, we develop in this paper a method that we phrase in a general framework, opening the way to further new results in Gromov–Witten theory. We call it *Regular Specialization* Theorem 1.18 as it consists of deforming a given smooth DM stack into a singular one in a regular way. It can be understood as an enhancement of the invariance of Gromov–Witten theory under smooth deformations. Precisely, given a regular $\mathbb{A}^1$-family $\mathcal{X}$ of DM stacks, that is a flat morphism $\mathcal{X} \to \mathbb{A}^1$ with $\mathcal{X}$ smooth, the perfect obstruction theory on the moduli space of stable maps to the total space $\mathcal{X}$ pulls back to a perfect obstruction theory on every fiber and the associated virtual cycle is independent of the fiber, we call it *regularized* virtual cycle. Furthermore, on smooth fibers, it equals the cap product of the Gromov–Witten cycle with the Hodge class. Provided we have a global torus action on the $\mathbb{A}^1$-family $\mathcal{X}$, the regularized virtual cycle localizes to the fixed locus in the central fiber, see Theorem 1.24.

Under special assumptions listed at the beginning of Section 2, we prove a version of an equivariant Quantum Lefschetz theorem, see Theorem 2.6, relating for a $\mathbb{C}^*$-equivariant embedding $\mathcal{X} \hookrightarrow \mathcal{P}$ of DM stacks the equivariant virtual cycles associated to $\mathcal{X}$ and to $\mathcal{P}$. Moreover, if the ambient space $\mathcal{P}$ carries a torus action of $T = (\mathbb{C}^*)^r$, e.g. if it is a toric DM stack, then we can relate the $\mathbb{C}^*$-virtual cycle associated to $\mathcal{X}$ to the T-virtual cycle associated to $\mathcal{P}$. Together with the regular specialization theorem, it yields Theorem 2.7.

Genus zero is a special interesting case, as the Hodge class equals the fundamental class and the regularized virtual cycle equals the Gromov–Witten virtual cycle. Let us call a DM stack regularizable, see Definition 3.18, if we can embed it as a fiber of a regular affine family of DM stacks. Although a regularizable DM stack may have bad singularities, we provide it a genus-zero Gromov–Witten theory via the regularized virtual cycle, and we prove invariance of the genus-zero theory under regular deformations, see Proposition 3.20. As a consequence, we can apply the localization formula whenever we have a torus action on the fiber, not necessarily on the total family. One strategy to compute genus-zero Gromov–Witten theory of a DM stack is thus to take a regular specialization to another DM stack admitting a torus action with sufficiently nice fixed locus, see below for more details.

At last, we highlight this paper is the Gromov–Witten counterpart of our previous results [26–28] on the quantum singularity (FJRW, [19, 20, 42]) theory of Landau–Ginzburg orbifolds defined by chain polynomials. It enters the big picture of the Landau–Ginzburd/Calabi–Yau (LG/CY) correspondence [11]. In particular, Theorem 3.6 should lead to a computation of the I-function using Givental's formalism [23] and eventually to a genus-zero mirror symmetry theorem without convexity. Comparing with results in [26], we should then obtain the LG/CY correspondence, extending the work of Chiodo–Iritani–Ruan [11]. We will discuss it in another paper. We also observe that the knowledge of Hodge integrals is crucial for a computation of the hamiltonians of the Double Ramification (DR) hierarchy introduced by Buryak [6] and may lead to new insights on the structure of Gromov–Witten invariants.

A note on tautological classes. A fundamental question about virtual cycles is whether their pushforward to the moduli space of stable curves lies in the tautological Chow ring. This question is largely open, e. g. it is unknown in the case of the quintic threefold. A straightforward and yet noticeable consequence of our results is that the product of the Hodge class λ_g with the virtual cycle is tautological in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ in all the cases we study, e. g. for smooth hypersurfaces defined by chain or loop polynomials. It follows from the virtual localization formula and from the fact that fixed loci in the target space are isolated points.

Future works. This paper is the foundation stone of a strategy aiming at computing all-genus Gromov–Witten invariants of projective hypersurfaces, and possibly other projective varieties. The idea is the following: by Costello's theorem [14], genus-g Gromov–Witten invariants of a projective variety X are explicitely expressed in terms of genus-0 Gromov–Witten invariants of the symmetric product $S^{g+1}X$.

Let X be a projective variety and assume we have an $\mathbb{A}^1$-family $\mathcal{X}$ of DM stacks admitting a torus action and whose fiber at $1 \in \mathbb{A}^1$ is X. Taking the symmetric fibered product over $\mathbb{A}^1$, we obtain an $\mathbb{A}^1$-family $\mathcal{X}_g$ of DM stacks admitting a torus action and whose fiber at $1 \in \mathbb{A}^1$ is $S^{g+1}X$. Precisely, we have

$$\mathcal{X}_g = [\mathcal{X} \times_{\mathbb{A}^1} \cdots \times_{\mathbb{A}^1} \mathcal{X} / \mathfrak{S}_{g+1}].$$

By Hironaka's theorem [31] and its equivariant version (see e.g. [35]), there exists a resolution of singularities $\widetilde{\mathcal{X}_g}$ of the DM stack $\mathcal{X}_g$, which is an isomorphism outside the singular locus of $\mathcal{X}_g$ and which preserves the torus action. In particular, we get a morphism $\widetilde{\mathcal{X}_g} \to \mathbb{A}^1$ and the fiber at $1 \in \mathbb{A}^1$ is still $S^{g+1}X$. Moreover, the birational map $\widetilde{\mathcal{X}_g} \to \mathcal{X}_g$ is obtained by a sequence of blow-ups and the morphism $\mathcal{X}_g \to \mathbb{A}^1$ is flat, hence the morphism $\widetilde{\mathcal{X}_g} \to \mathbb{A}^1$ is flat as well, see for instance [21, Appendix B.6.7]. As a consequence, the DM stack $\widetilde{\mathcal{X}_g}$ is a regular $\mathbb{A}^1$-family admitting a torus action and whose fiber at $1 \in \mathbb{A}^1$ is the symmetric product $S^{g+1}X$. According to our genus-0 Regular Specialization Theorem, genus-0 Gromov–Witten invariants of $S^{g+1}X$, and thus genus-g Gromov–Witten invariants of X, are expressed by the localization formula in terms of genus-0 Gromov–Witten invariants of the torus-fixed loci in (the fiber at $0 \in \mathbb{A}^1$ of) $\widetilde{\mathcal{X}_g}$.

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1. Hodge–Gromov–Witten theory

In this section, we prove a general theorem on Hodge–Gromov–Witten theory, that we call 'Regular Specialization Theorem'. The context is the following.

Definition 1.1. Given a smooth DM stack $\mathcal{Y}$, Gromov–Witten theory provides a virtual fundamental cycle for the moduli space $\mathcal{M}_{\mathcal{Y}}$ of stable maps to $\mathcal{Y}$. We call Hodge virtual cycle the cup product of the virtual fundamental cycle with the top Chern class of the Hodge bundle[1]. Hodge–Gromov–Witten theory is then intersection theory on $\mathcal{M}_{\mathcal{Y}}$ against this cycle.

Definition 1.2. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ between two DM stacks is called a family when it is flat. We also say that $\mathcal{X}$ is a $\mathcal{Y}$-family. Inverse images of geometric points $y \in \mathcal{Y}$ are called fibers. A regular family is a family for which the DM stack $\mathcal{X}$ is smooth.

Let $p\colon \mathcal{X} \to \mathbb{A}^1$ be a regular $\mathbb{A}^1$-family of DM stacks, and denote by X_0 and X_1 its fibers at $0 \in \mathbb{A}^1$ and at $1 \in \mathbb{A}^1$. We assume X_0 and X_1 to be proper, and X_1 to be smooth, but we do not impose any restriction on singularities of X_0. Depending on the purpose, we may also assume the $\mathbb{A}^1$-family $\mathcal{X}$ is equipped with a torus action leaving X_0 invariant.

Let $\mathcal{M}_{X_0}$ and $\mathcal{M}_{X_1}$ be the moduli spaces of stable maps to X_0 and to X_1, with arbitrary genus, degree, number of markings, and isotropy type at markings. Gromov–Witten theory for smooth DM stacks provides a perfect obstruction theory and a virtual fundamental cycle for the moduli space $\mathcal{M}_{X_1}$, but not for $\mathcal{M}_{X_0}$.

In the first subsection, we construct perfect obstruction theories on the moduli spaces $\mathcal{M}_{X_0}$ and $\mathcal{M}_{X_1}$, and we call the associated virtual fundamental cycles 'regularized virtual cycles'.

The Regular Specialization Theorem can be phrased as an equality between regularized virtual cycles of $\mathcal{M}_{X_0}$ and $\mathcal{M}_{X_1}$. Moreover, whenever the target space is smooth, e.g. for X_1, we show the regularized virtual cycle equals the Hodge–Gromov–Witten virtual cycle, up to a sign.

Graber–Pandharipande's virtual localization formula [24] applies to regularized virtual cycles. Therefore, provided we have a torus action on the $\mathbb{A}^1$-family preserving the central fiber X_0 and since the fixed locus in X_0 is smooth, we can decompose the Hodge–Gromov–Witten cycle of $\mathcal{M}_{X_1}$ into Hodge–Gromov–Witten cycles of the fixed loci in $\mathcal{M}_{X_0}$.

1.1. Perfect obstruction theories.

Notation 1.3. For a DM stack $\mathcal{Y}$, we denote by $\mathcal{M}_{\mathcal{Y}}$ the moduli space of stable maps to $\mathcal{Y}$, by $\pi_{\mathcal{Y}}\colon \mathcal{C}_{\mathcal{Y}} \to \mathcal{M}_{\mathcal{Y}}$ the universal curve, by $f_{\mathcal{Y}}\colon \mathcal{C}_{\mathcal{Y}} \to \mathcal{Y}$ the universal map, and by $\omega_{\pi_{\mathcal{Y}}}$ the relative dualizing sheaf. In the special cases of $\mathcal{X}$, X_0, and X_1, we simplify notations of the maps as

$$\pi = \pi_{\mathcal{X}}\ ,\ \pi_0 = \pi_{X_0}\ ,\ \pi_1 = \pi_{X_1}\ ,\ f = f_{\mathcal{X}}\ ,\ f_0 = f_{X_0}\ ,\ f_1 = f_{X_1}.$$

The flat morphism $p\colon \mathcal{X} \to \mathbb{A}^1$ induces a flat morphism

$$q\colon \mathcal{M}_{\mathcal{X}} \to \mathcal{M}_{\mathbb{A}^1} \simeq \mathbb{A}^1 \times \overline{\mathcal{M}}_{g,n}.$$

Furthermore, we have fiber diagrams

$$\begin{array}{ccc} \mathcal{M}_{X_0} & \xrightarrow{j_0} & \mathcal{M}_{\mathcal{X}} \\ {\scriptstyle q_0}\downarrow & \square & \downarrow{\scriptstyle q} \\ \overline{\mathcal{M}}_{g,n} & \longrightarrow & \mathcal{M}_{\mathbb{A}^1} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{M}_{X_1} & \xrightarrow{j_1} & \mathcal{M}_{\mathcal{X}} \\ {\scriptstyle q_1}\downarrow & \square & \downarrow{\scriptstyle q} \\ \overline{\mathcal{M}}_{g,n} & \longrightarrow & \mathcal{M}_{\mathbb{A}^1} \end{array}$$

[1]For a family $\pi\colon \mathcal{C} \to S$ of genus-g curves, the Hodge bundle is a rank-g vector bundle on S defined by the push-forward $\pi_*\omega_{\mathcal{C}/S}$ of the relative canonical sheaf.

where bottom arrows are inclusions of $0 \in \mathbb{A}^1$ and $1 \in \mathbb{A}^1$. In particular, the maps j_0 and j_1 are closed immersions, hence proper. We also introduce notations for maps in the following fiber diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & \mathcal{X} \\ {\scriptstyle p_0}\downarrow & \square & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \mathbb{A}^1 \end{array} \qquad\qquad \begin{array}{ccc} X_1 & \xrightarrow{i_1} & \mathcal{X} \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle p} \\ 1 & \longrightarrow & \mathbb{A}^1 \end{array}$$

The map i_0 yields an exact triangle of cotangent complexes

$$i_0^* L_{\mathcal{X}} \to L_{X_0} \to L_{X_0/\mathcal{X}} \to i_0^* L_{\mathcal{X}}[1].$$

From the construction of obstruction theories on moduli spaces of maps, we obtain a commutative diagram

$$\begin{array}{ccccccc} j_0^* E_{\mathcal{X}} & \longrightarrow & E_{X_0} & \longrightarrow & E_{X_0/\mathcal{X}} & \longrightarrow & j_0^* E_{\mathcal{X}}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_0^* L_{\mathcal{M}_{\mathcal{X}}} & \longrightarrow & L_{\mathcal{M}_{X_0}} & \longrightarrow & L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}} & \longrightarrow & j_0^* L_{\mathcal{M}_{\mathcal{X}}}[1] \end{array} \tag{1}$$

where each row is an exact triangle and where obstruction theories are defined as

$$\begin{aligned} E_{\mathcal{X}} &:= R\pi_* \left(f^* L_{\mathcal{X}} \otimes \omega_{\pi_{\mathcal{X}}} \right) \simeq \left(R\pi_* f^* T_{\mathcal{X}} \right)^\vee , \\ E_{X_0} &:= R\pi_{0*} \left(f_0^* L_{X_0} \otimes \omega_{\pi_{X_0}} \right) , \\ E_{X_0/\mathcal{X}} &:= R\pi_{0*} \left(f_0^* L_{X_0/\mathcal{X}} \otimes \omega_{\pi_{X_0}} \right) \simeq \mathbb{E}[2] \oplus \mathcal{O}[1]. \end{aligned}$$

Note that for the second equality of the first line, we use that $\mathcal{X}$ is smooth. For the second equality of the third line, we use that p is flat to compute $L_{X_0/\mathcal{X}} \simeq p_0^* L_{0/\mathbb{A}^1} = \mathcal{O}[1]$ and then $\mathbb{E} := \pi_{0*}(\omega_{\pi_{X_0}})$ is the Hodge bundle[2].

In the exact same way, we use the map i_1 to obtain a commutative diagram

$$\begin{array}{ccccccc} j_1^* E_{\mathcal{X}} & \longrightarrow & E_{X_1} & \longrightarrow & E_{X_1/\mathcal{X}} & \longrightarrow & j_1^* E_{\mathcal{X}}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_1^* L_{\mathcal{M}_{\mathcal{X}}} & \longrightarrow & L_{\mathcal{M}_{X_1}} & \longrightarrow & L_{\mathcal{M}_{X_1}/\mathcal{M}_{\mathcal{X}}} & \longrightarrow & j_1^* L_{\mathcal{M}_{\mathcal{X}}}[1] \end{array} \tag{2}$$

where each row is an exact triangle and where obstruction theories are defined as

$$\begin{aligned} E_{X_1} &:= R\pi_{1*} \left(f_1^* L_{X_1} \otimes \omega_{\pi_{X_1}} \right) \simeq \left(R\pi_{1*} f_1^* T_{X_1} \right)^\vee , \\ E_{X_1/\mathcal{X}} &:= R\pi_{1*} \left(f_1^* L_{X_1/\mathcal{X}} \otimes \omega_{\pi_{X_1}} \right) \simeq \mathbb{E}[2] \oplus \mathcal{O}[1], \end{aligned}$$

where smoothness of X_1 is used in the second equality of the first line.

Remark 1.4. Since the stacks $\mathcal{X}$ and X_1 are smooth, obstruction theories $E_{\mathcal{X}}$ and E_{X_1} are perfect, i.e. of amplitude in $[-1, 0]$. On the other hand, obstruction theories $E_{X_0/\mathcal{X}}$ and $E_{X_1/\mathcal{X}}$ are of amplitude $[-2, -1]$, hence they are not perfect, and we do not know whether the obstruction theory E_{X_0} is perfect, as X_0 is not assumed to be smooth.

Definition 1.5. The regularized obstruction theory for $\mathcal{M}_{X_0}$ is defined as follows. We first take the cone

$$F_{X_0} := \mathrm{Cone} \left(\mathcal{O} \to j_0^* E_{\mathcal{X}} \right),$$

[2]We do not specify the subscript X_0 for the Hodge bundle because it is a pull-back from the moduli space of stable curves $\overline{\mathcal{M}}_g$.

where the map is the composition of the inclusion $\mathcal{O} \to E_{X_0/\mathcal{X}}[-1] = \mathbb{E}[1] \oplus \mathcal{O}$ with the connecting morphism $E_{X_0/\mathcal{X}}[-1] \to j_0^* E_{\mathcal{X}}$ from the exact triangle (1). We then obtain a map of cones

$$F_{X_0} \to \mathrm{Cone}\left(L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}[-1] \to j_0^* L_{\mathcal{M}_{\mathcal{X}}}\right).$$

At last, from the exact triangle (1), we observe that the right-hand side above is quasi-isomorphic to $L_{\mathcal{M}_{X_0}}$, giving us a morphism

$$F_{X_0} \to L_{\mathcal{M}_{X_0}}.$$

Remark 1.6. In genus zero, the Hodge bundle is the zero vector bundle and the regularized obstruction theory F_{X_0} is quasi-isomorphic to the Gromov–Witten obstruction theory E_{X_0}.

Definition 1.5 works as well for $\mathcal{M}_{X_1}$. However, smoothness of X_1 yields the following equivalent definition.

Definition 1.7. The regularized obstruction theory for $\mathcal{M}_{X_1}$ is defined as follows. We first take

$$F_{X_1} := E_{X_1} \oplus \mathbb{E}[1],$$

and then use the map $E_{X_1} \to L_{\mathcal{M}_{X_1}}$ and the composed morphism

$$\mathbb{E} \to (j_1^* E_{\mathcal{X}})^{-1} \to (j_1^* L_{\mathcal{M}_{\mathcal{X}}})^{-1} \to L^{-1}_{\mathcal{M}_{X_1}}$$

to get $F_{X_1} \to L_{\mathcal{M}_{X_1}}$. Clearly, it is a perfect obstruction theory on $\mathcal{M}_{X_1}$.

Lemma 1.8. *The regularized obstruction theory $F_{X_0} \to L_{\mathcal{M}_{X_0}}$ defines a perfect obstruction theory on $\mathcal{M}_{X_0}$.*

Proof. Since the complex $E_{\mathcal{X}}$ has amplitude in $[-1, 0]$, so does $j_0^* E_{\mathcal{X}}$ and thus so does F_{X_0}.

Since the map $j_0 \colon \mathcal{M}_{X_0} \to \mathcal{M}_{\mathcal{X}}$ is a closed immersion, then the cohomologies of the relative cotangent complex are

$$h^{-1}(L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}) = I/I^2 \quad \text{and} \quad h^0(L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}) = 0,$$

where I is the coherent sheaf of ideals defining j_0. Since $E_{\mathcal{X}} \to L_{\mathcal{M}_{\mathcal{X}}}$ is an obstruction theory, then we have

$$h^{-1}(j_0^* E_{\mathcal{X}}) \twoheadrightarrow h^{-1}(j_0^* L_{\mathcal{M}_{\mathcal{X}}}) \quad \text{and} \quad h^0(j_0^* E_{\mathcal{X}}) \simeq h^0(j_0^* L_{\mathcal{M}_{\mathcal{X}}}).$$

Moreover, we have a surjection

$$\mathcal{O} \twoheadrightarrow I/I^2$$

between the (pullback of the) conormal sheaf of $0 \hookrightarrow \mathbb{A}^1$ and the conormal sheaf of the closed immersion j_0.

Furthermore, by unicity of the cone, we have the following commutative diagram

$$\begin{array}{ccc} h^{-1}(j_0^* E_{\mathcal{X}}) \oplus \mathcal{O} & \xrightarrow{f} & h^0(j_0^* E_{\mathcal{X}}) \\ \downarrow\!\!\downarrow & \circlearrowleft & \downarrow \simeq \\ h^{-1}(j_0^* L_{\mathcal{M}_{\mathcal{X}}}) \oplus I/I^2 & \xrightarrow[g]{} & h^0(j_0^* L_{\mathcal{M}_{\mathcal{X}}}) \end{array}$$

where we introduce notations $f\colon \mathcal{O} \to h^0(j_0^* E_{\mathcal{X}})$ and $g\colon I/I^2 \to h^0(j_0^* L_{\mathcal{M}_{\mathcal{X}}})$.

Let U be an open subset of $\mathcal{M}_{X_0}$ and $x \in h^{-1}(j_0^* L_{\mathcal{M}_{\mathcal{X}}})$ and $y \in I/I^2$ be two sections over U, such that $g(y) = 0$. Then, there exist $x' \in h^{-1}(j_0^* E_{\mathcal{X}})$ and $y' \in \mathcal{O}$ such that x' is sent to x and y' is sent to y by the second vertical map from the diagram. Then by the commutativity of the diagram, we have $f(y') = 0$ and thus $f(x' + y') = 0$, which proves surjectivity of $\ker(f) \to \ker(g)$.

To prove that $\mathrm{coker}(f) \simeq \mathrm{coker}(g)$, we apply the five lemma to the diagram

$$\begin{array}{ccccccccc}
\mathcal{O} & \xrightarrow{f} & h^0(j_0^* E_{\mathcal{X}}) & \longrightarrow & \mathrm{coker}(f) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow\simeq & & \downarrow & & \downarrow\simeq & & \downarrow \\
I/I^2 & \xrightarrow[g]{} & h^0(j_0^* L_{\mathcal{M}_{\mathcal{X}}}) & \longrightarrow & \mathrm{coker}(g) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

As a consequence, we have proved that the morphism

$$F_{X_0} \to \mathrm{Cone}\left(L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}[-1] \to j_0^* L_{\mathcal{M}_{\mathcal{X}}}\right) \simeq L_{\mathcal{M}_{X_0}}$$

is an obstruction theory. □

Definition 1.9. We call regularized virtual cycle of $\mathcal{M}_{X_0}$ (resp. of $\mathcal{M}_{X_1}$) the virtual fundamental cycle $[\mathcal{M}_{X_0}, F_{X_0}] \in A_*(\mathcal{M}_{X_0})$ (resp. $[\mathcal{M}_{X_1}, F_{X_1}] \in A_*(\mathcal{M}_{X_1})$) obtained by Behrend–Fantechi [4] from the perfect obstruction theory F_{X_0} (resp. F_{X_1}).

We also call Gromov–Witten virtual cycle of $\mathcal{M}_{X_1}$ the virtual fundamental cycle $[\mathcal{M}_{X_1}, E_{X_1}] \in A_*(\mathcal{M}_{X_1})$ obtained by Behrend–Fantechi from the perfect obstruction theory E_{X_1}.

Lemma 1.10. *In the smooth case, the regularized virtual cycle equals the Hodge–Gromov–Witten virtual cycle up to a sign. Precisely, for the DM stack X_1, we have the relation*

$$[\mathcal{M}_{X_1}, F_{X_1}] = (-1)^g \lambda_g \cdot [\mathcal{M}_{X_1}, E_{X_1}] \in A_*(\mathcal{M}_{X_1}),$$

where $\lambda_g := c_{\mathrm{top}}(\mathbb{E})$ is the top Chern class of the Hodge bundle and g is the genus of curves involved in a given connected component of the moduli space.

Proof. The virtual fundamental class $[\mathcal{M}_{X_1}, E_{X_1}]$ is the intersection of the intrinsic normal cone $\mathfrak{C}_{\mathcal{M}_{X_1}}$ of $\mathcal{M}_{X_1}$ with the zero section of $h^1/h^0(E_{X_1}^\vee)$, and similarly for $[\mathcal{M}_{X_1}, F_{X_1}]$. Since $F_{X_1} := E_{X_1} \oplus \mathbb{E}[1]$, we get

$$h^1/h^0(F_{X_1}^\vee) \simeq h^1/h^0(E_{X_1}^\vee) \times \mathrm{Spec}(\mathrm{Sym}\mathbb{E}).$$

Therefore, we have

$$\begin{aligned}
[\mathcal{M}_{X_1}, F_{X_1}] &= 0^!_{h^1/h^0(F_{X_1}^\vee)}[\mathfrak{C}_{\mathcal{M}_{X_1}}] \\
&= 0^!_{\mathrm{Spec}(\mathrm{Sym}\mathbb{E})} 0^!_{h^1/h^0(E_{X_1}^\vee)}[\mathfrak{C}_{\mathcal{M}_{X_1}}] \\
&= 0^!_{\mathrm{Spec}(\mathrm{Sym}\mathbb{E})}[\mathcal{M}_{X_1}, E_{X_1}] \\
&= c_{\mathrm{top}}(\mathbb{E}^\vee) \cap [\mathcal{M}_{X_1}, E_{X_1}].
\end{aligned}$$

□

1.2. **Regular Specialization Theorem.** First, we compare regularized virtual cycles of $\mathcal{M}_{X_0}$ and of $\mathcal{M}_{X_1}$.

1.2.1. *Pull-backs from the regular family.*

Proposition 1.11. *The regularized virtual cycle associated to a fiber of a regular $\mathbb{A}^1$-family does not depend on the fiber. Precisely, we have equalities*

$$\begin{aligned} j_0^![\mathcal{M}_{\mathcal{X}}, E_{\mathcal{X}}] &= [\mathcal{M}_{X_0}, F_{X_0}] \in A_*(\mathcal{M}_{X_0}), \\ j_1^![\mathcal{M}_{\mathcal{X}}, E_{\mathcal{X}}] &= [\mathcal{M}_{X_1}, F_{X_1}] \in A_*(\mathcal{M}_{X_1}). \end{aligned}$$

Proof. Since the varieties 0 and $\mathbb{A}^1$ are smooth, by [4, Proposition 5.10], it is enough to find a compatibility datum relative to $0 \to \mathbb{A}^1$ for $E_{\mathcal{X}}$ and F_{X_0}, see [4, Definition 5.8], that is a triple (ϕ, ψ, χ) of derived morphisms giving rise to a morphism of exact triangles

$$\begin{array}{ccccccc} j_0^* E_{\mathcal{X}} & \xrightarrow{\phi} & F_{X_0} & \xrightarrow{\psi} & q_0^* L_{0/\mathbb{A}^1} & \xrightarrow{\chi} & j_0^* E_{\mathcal{X}}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_0^* L_{\mathcal{M}_{\mathcal{X}}} & \longrightarrow & L_{\mathcal{M}_{X_0}} & \longrightarrow & L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}} & \longrightarrow & j_0^* L_{\mathcal{M}_{\mathcal{X}}}[1] \end{array}$$

The existence of the compatibility datum follows from the exact triangles of cones

$$\mathcal{O} \to j_0^* E_{\mathcal{X}} \to \mathrm{Cone}\,(\mathcal{O} \to j_0^* E_{\mathcal{X}}) \to \mathcal{O}[1]$$

$$L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}[-1] \to j_0^* L_{\mathcal{M}_{\mathcal{X}}} \to \mathrm{Cone}\left(L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}[-1] \to j_0^* L_{\mathcal{M}_{\mathcal{X}}}\right) \to L_{\mathcal{M}_{X_0}/\mathcal{M}_{\mathcal{X}}}$$

and from the quasi-isomorphism $q_0^* L_{0/\mathbb{A}^1} \simeq \mathcal{O}[1]$. The same holds for X_1. □

Lemma 1.12. *The morphism $q\colon \mathcal{M}_{\mathcal{X}} \to \mathbb{A}^1$ is proper. Moreover, we have commutative diagrams*

$$\begin{array}{ccc} \mathcal{M}_{X_1} & \xrightarrow{j_1} & \mathcal{M}_{\mathcal{X}} \\ {\scriptstyle \mathrm{ev}_{X_1}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{X}}} \\ X_1 & \xrightarrow[i_1]{} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} \mathcal{M}_{X_1} & \xrightarrow{j_1} & \mathcal{M}_{\mathcal{X}} \\ & {\scriptstyle r_{X_1}}\searrow \ \circlearrowleft \ \swarrow{\scriptstyle r_{\mathcal{X}}} & \\ & \bigcup_{g,n} \overline{\mathcal{M}}_{g,n} & \end{array}$$

where the maps $\mathrm{ev}_{\mathcal{X}}$ and ev_{X_1} are the evaluation maps and the maps $r_{\mathcal{X}}$ and r_{X_1} remember only the coarse curve and stabilize it. We also have the same commutative diagrams when we replace X_1 by X_0.

Proof. Since every morphism from a nodal curve to the affine line is a contraction to a point, then we have an isomorphism between the moduli space $\mathcal{M}_{\mathcal{X}}$ of stable maps to $\mathcal{X}$ and the moduli space $\mathcal{M}_{g,n}(\mathcal{X}/\mathbb{A}^1)$ of relative stable maps to the $\mathbb{A}^1$-family $p\colon \mathcal{X} \to \mathbb{A}^1$. Therefore, by [2, Section 8.3], the morphism $q\colon \mathcal{M}_{\mathcal{X}} \to \mathbb{A}^1$ is proper. Commutativities of the diagrams are obvious. □

Remark 1.13. In all moduli spaces above, we consider curves with arbitrary genus, degree, number of markings, and isotropy type at markings. Hence, these moduli spaces are disconnected. We write subscripts to indicate restrictions to a (bunch of connected) component of the moduli space. For instance, the moduli space of stable maps to $\mathcal{X}$ from genus-g n-marked curves is $(\mathcal{M}_{\mathcal{X}})_{g,n}$.

Proposition 1.11 works as well when adding the subscript (g, n). Furthermore, we can also add isotropies, since we have closed immersions of inertia stacks

$$IX_0 \subset I\mathcal{X} \quad \text{and} \quad IX_1 \subset I\mathcal{X} \quad \text{with } I\mathcal{X} = \bigsqcup_{\rho} \mathcal{X}_\rho.$$

It does not compare isotropies for X_0 and for X_1. Nevertheless, in the case when an isotropy ρ of $\mathcal{X}$ is contained in X_1 but not in X_0 (or in X_0 but not in X_1), then the moduli space $(\mathcal{M}_{X_0})_\rho$ is empty and its regularized virtual cycle $[\mathcal{M}_{X_0}, F_{X_0}]_\rho$ is zero. Proposition 1.11 is still valid.

Remark 1.14. It is not straightforward to compare curve classes for $\mathcal{X}$, X_0, and X_1, because we can have vanishing cycles. To solve this issue, we introduce an ambient space which contains every fiber of $\mathcal{X}$.

1.2.2. *Ambient space.* From now on, we assume we have a smooth proper DM stack $\mathcal{P}$ with an embedding of $\mathbb{A}^1$-families $\mathcal{X} \hookrightarrow \mathcal{P} \times \mathbb{A}^1$, i.e. every fiber of $\mathcal{X}$ lies in $\mathcal{P}$. In particular, we have push-forward maps

$$H_2(X_t) \to H_2(\mathcal{P}) \ , \ \text{for every } t \in \mathbb{A}^1.$$

We also have maps $\mathcal{M}_{X_t} \to \mathcal{M}_{\mathcal{P}}$, that we can decompose in terms of curve classes. Precisely, for every $\beta \in H_2(\mathcal{P})$, we have

$$\bigsqcup_{\substack{\beta' \in H_2(X_t) \text{ with} \\ \beta' = \beta \in H_2(\mathcal{P})}} \mathcal{M}_{X_t}(\beta') \to \mathcal{M}_{\mathcal{P}}(\beta).$$

As a consequence, Proposition 1.11 becomes the following.

Proposition 1.15. *For every genus g, number of markings n, curve class $\beta \in H_2(\mathcal{P})$, and isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathcal{X}$, we have*

$$\begin{aligned} j_0^![\mathcal{M}_{\mathcal{X}}, E_{\mathcal{X}}]_{g,n,\beta,\underline{\rho}} &= \sum_{\substack{\beta_0 \in H_2(X_0) \text{ with} \\ \beta_0 = \beta \in H_2(\mathcal{P})}} [\mathcal{M}_{X_0}, F_{X_0}]_{g,n,\beta_0,\underline{\rho}} \in A_*(\mathcal{M}_{X_0}), \\ j_1^![\mathcal{M}_{\mathcal{X}}, E_{\mathcal{X}}]_{g,n,\beta,\underline{\rho}} &= \sum_{\substack{\beta_1 \in H_2(X_1) \text{ with} \\ \beta_1 = \beta \in H_2(\mathcal{P})}} [\mathcal{M}_{X_1}, F_{X_1}]_{g,n,\beta_1,\underline{\rho}} \in A_*(\mathcal{M}_{X_1}). \end{aligned}$$

1.2.3. *Gromov–Witten cycles.* In this subsection, we fix a genus g, a number of markings n, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathcal{X}$, and curve classes $\beta \in H_2(\mathcal{P})$, $\beta_0 \in H_2(X_0)$, and $\beta_1 \in H_2(X_1)$ satisfying

$$\beta_0 = \beta \in H_2(\mathcal{P}) \quad \text{and} \quad \beta_1 = \beta \in H_2(\mathcal{P}).$$

We also fix $\alpha_1, \ldots, \alpha_n \in A^*(I\mathcal{X})$ such that

$$\alpha_i \in A^*(\mathcal{X}_{\rho_i}) \subset A^*(I\mathcal{X}),$$

where $\mathcal{X}_{\rho_i}$ is the component of the inertia stack of $\mathcal{X}$ with isotropy ρ_i. Furthermore, we denote by ψ_i the usual psi-class on the moduli space of stable curves, i.e. the first Chern class of the cotangent line of the curve at the i-th marking, and by λ_g the top Chern class of the (pull-back of the) Hodge bundle.

Definition 1.16. A Gromov–Witten cycle of X_1 is

$$[\alpha_1, \ldots, \alpha_n]^{X_1}_{g,n,\beta_1} := q_{1*} \left([\mathcal{M}_{X_1}, E_{X_1}]_{g,n,\beta_1,\underline{\rho}} \cdot \prod_{i=1}^n j_1^* \mathrm{ev}_{\mathcal{X}}^*(\alpha_i) \right) \in A_* \left(\overline{\mathcal{M}}_{g,n} \right).$$

A Hodge–Gromov–Witten cycle of X_1 is

$$[\alpha_1, \ldots, \alpha_n]^{X_1,\lambda_g}_{g,n,\beta_1} := q_{1*} \left(\lambda_g \cdot [\mathcal{M}_{X_1}, E_{X_1}]_{g,n,\beta_1,\underline{\rho}} \cdot \prod_{i=1}^n j_1^* \mathrm{ev}_{\mathcal{X}}^*(\alpha_i) \right) \in A_* \left(\overline{\mathcal{M}}_{g,n} \right).$$

A relative Gromov–Witten cycle of $p\colon \mathcal{X} \to \mathbb{A}^1$ is

$$[\alpha_1, \ldots, \alpha_n]^{\mathcal{X},\mathrm{rel}}_{g,n,\beta} := q_* \left([\mathcal{M}_{\mathcal{X}}, E_{\mathcal{X}}]_{g,n,\beta,\underline{\rho}} \cdot \prod_{i=1}^{n} \mathrm{ev}^*_{\mathcal{X}}(\alpha_i) \right) \in A_*(\mathcal{M}_{\mathbb{A}^1}) \simeq A_*\left(\overline{\mathcal{M}}_{g,n}\right).$$

A regularized Gromov–Witten cycle of X_0 is

$$[\alpha_1, \ldots, \alpha_n]^{X_0,\mathrm{reg}}_{g,n,\beta_0} := q_0 * \left([\mathcal{M}_{X_0}, F_{X_0}]_{g,n,\beta_0,\underline{\rho}} \cdot \prod_{i=1}^{n} j_0^* \mathrm{ev}^*_{\mathcal{X}}(\alpha_i) \right) \in A_*\left(\overline{\mathcal{M}}_{g,n}\right).$$

A regularized Gromov–Witten cycle of X_1 is

$$[\alpha_1, \ldots, \alpha_n]^{X_1,\mathrm{reg}}_{g,n,\beta_1} := q_1 * \left([\mathcal{M}_{X_1}, F_{X_1}]_{g,n,\beta_1,\underline{\rho}} \cdot \prod_{i=1}^{n} j_1^* \mathrm{ev}^*_{\mathcal{X}}(\alpha_i) \right) \in A_*\left(\overline{\mathcal{M}}_{g,n}\right).$$

Definition 1.17. We call ambient theory the special case where we take isotropies $\underline{\rho}$ in $\mathcal{P}$ and insertions

$$\alpha_i \in A^*(\mathcal{P}_{\rho_i}) \subset A^*(I\mathcal{P}) \to A^*(I\mathcal{X}),$$

where pull-back is taken under the map $\mathcal{X} \hookrightarrow \mathcal{P} \times \mathbb{A}^1 \to \mathcal{P}$.

From Lemma 1.10, we see that

$$[\alpha_1, \ldots, \alpha_n]^{X_1,\mathrm{reg}}_{g,n,\beta_1} = (-1)^g \, [\alpha_1, \ldots, \alpha_n]^{X_1,\lambda_g}_{g,n,\beta_1},$$

and from Proposition 1.15, we obtain

$$\begin{aligned}[\alpha_1, \ldots, \alpha_n]^{\mathcal{X},\mathrm{rel}}_{g,n,\beta} &= \sum_{\substack{\beta_1 \in H_2(X_1) \text{ with} \\ \beta_1 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_1,\mathrm{reg}}_{g,n,\beta_1} \\ &= \sum_{\substack{\beta_0 \in H_2(X_0) \text{ with} \\ \beta_0 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_0,\mathrm{reg}}_{g,n,\beta_0}. \end{aligned} \tag{3}$$

We sum up with the following statement.

Theorem 1.18 (Regular Specialization Theorem)**.** *Let $\mathcal{X}$ be a regular $\mathbb{A}^1$-family whose fibers are embedded in a smooth proper DM stack $\mathcal{P}$. For every genus g, number of markings n such that $2g-2+n>0$, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in X_1, curve class $\beta \in H_2(\mathcal{P})$, and insertions $\alpha_1, \ldots, \alpha_n \in A^*(I\mathcal{X})$ with $\alpha_i \in A^*(\mathcal{X}_{\rho_i})$, we have in $A_*\left(\overline{\mathcal{M}}_{g,n}\right)$*

$$\sum_{\substack{\beta_1 \in H_2(X_1) \textit{ with} \\ \beta_1 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_1,\lambda_g}_{g,n,\beta_1} = (-1)^g \sum_{\substack{\beta_0 \in H_2(X_0) \textit{ with} \\ \beta_0 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_0,\mathrm{reg}}_{g,n,\beta_0}.$$

Remark 1.19. We might not have access to all insertions for X_1, as it is possible that the pull-back map $A^*(\mathcal{X}) \to A^*(X_1)$ is not surjective. However, it is easy to work with the ambient theory, i.e. insertions pulled-back from $A^*(I\mathcal{P})$.

Corollary 1.20 (Regular Specialization Theorem in genus zero)**.** *Under the same assumptions as before, we have in $A_*\left(\overline{\mathcal{M}}_{0,n}\right)$*

$$\sum_{\substack{\beta_1 \in H_2(X_1) \textit{ with} \\ \beta_1 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_1}_{0,n,\beta_1} = \sum_{\substack{\beta_0 \in H_2(X_0) \textit{ with} \\ \beta_0 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]^{X_0,\mathrm{reg}}_{0,n,\beta_0}.$$

1.2.4. *Torus action.* In this subsection, we assume we have a torus action on the $\mathbb{A}^1$-family $\mathcal{X}$ leaving the fiber X_0 invariant. Denote by T the torus. Then, we get a T-action on the moduli spaces $\mathcal{M}_\mathcal{X}$ and $\mathcal{M}_{X_0}$, and the perfect obstruction theories $E_\mathcal{X}$ on $\mathcal{M}_\mathcal{X}$ and F_{X_0} on $\mathcal{M}_{X_0}$ are also T-equivariant.

Notation 1.21. For a DM stack $\mathcal{Y}$ with a T-action, we denote by $\iota\colon \mathcal{Y}_T \hookrightarrow \mathcal{Y}$ the fixed locus. For a T-equivariant perfect obstruction theory $E_\mathcal{Y}$ on $\mathcal{Y}$, we denote by N_T^{vir} the moving part of the dual of its restriction to the fixed locus $\mathcal{Y}_T$, and by E_T the fixed part, which is a perfect obstruction theory on the fixed locus $\mathcal{Y}_T$.

Proposition 1.22 (Localization formula, [24, Equation (8)])**.** *Let $\mathcal{Y}$ be a DM stack with a T-action and a T-equivariant perfect obstruction theory $E \to L_\mathcal{Y}$. Let $A_*^T(\mathcal{Y})$ denote the T-equivariant Chow ring*[3] *of $\mathcal{Y}$ and $\underline{t}$ denote the T-equivariant parameters. Introduce the ring*

$$A_*^T(\mathcal{Y})_{\mathrm{loc}} := A_*^T(\mathcal{Y}) \otimes_{\mathbb{Q}[\underline{t}]} \mathbb{Q}[\underline{t}^{\pm 1}]$$

obtained by inverting equivariant parameters $\underline{t}$. Then the virtual localization formula is

$$[\mathcal{Y}, E] = \iota_* \left(\frac{[\mathcal{Y}_T, E_T]}{e_T(N_T^{\mathrm{vir}})} \right),$$

where e_T denotes the T-equivariant Euler class.

Remark 1.23. In our situation, the fixed locus $\mathcal{X}_T$ lies in the central fiber X_0 and we have $\mathcal{X}_T = X_{0T}$. Moreover, it is a smooth DM stack and we denote it by X_T.

Theorem 1.24 (Equivariant Regular Specialization Theorem)**.** *Let $\mathcal{X}$ be a T-equivariant regular $\mathbb{A}^1$-family whose fibers are embedded in a smooth proper DM stack $\mathcal{P}$ and where the torus action leaves the central fiber invariant. For every genus g, number of markings n such that $2g-2+n>0$, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in X_1, curve class $\beta \in H_2(\mathcal{P})$, and insertions $\alpha_1, \ldots, \alpha_n \in A^*(I\mathcal{X})$ with $\alpha_i \in A^*(\mathcal{X}_{\rho_i})$ admitting a T-equivariant lifting, we have*

$$\sum_{\substack{\beta_1 \in H_2(X_1) \text{ with} \\ \beta_1 = \beta \in H_2(\mathcal{P})}} [\alpha_1, \ldots, \alpha_n]_{g,n,\beta_1}^{X_1, \lambda_g} = \lim_{\underline{t} \to 0} \sum_{\substack{\beta_0 \in H_2(X_0) \text{ with} \\ \beta_0 = \beta \in H_2(\mathcal{P})}} (-1)^g \times \int_{[\mathcal{M}_{X_T}, E_T]_{g,n,\beta_0}} \frac{\prod_{i=1}^n \mathrm{ev}_T^*(\alpha_i)}{e(N_T^{\mathrm{vir}})},$$

where $\mathrm{ev}_T = \mathrm{ev}_\mathcal{X} \circ j_0 \circ \iota_{\mathcal{M}_{X_T}}$ and $r_T = r_{X_0} \circ \iota_{\mathcal{M}_{X_T}}$, and $[\mathcal{M}_{X_T}, E_T]$ is the Gromov–Witten virtual fundamental cycle of the moduli space of stable maps to the smooth DM stack X_T.

Remark 1.25. The Regular Specialization Theorem and its equivariant version work as well with a regular family defined over an affine basis $\mathbb{A}^m$. Precisely, we then have to multiply the virtual cycle by the m-th power of the Hodge class. Note that it is only interesting in genus zero, see Section 3.3, as for positive genus the power of the Hodge class vanishes.

2. Equivariant quantum Lefschetz theorem

In this section, we state an 'equivariant quantum Lefschetz' theorem which will be useful for computations in the next section.

[3] We refer to [15] for a detailed construction of the equivariant Chow ring.

2.1. **$\mathbb{C}^*$-localization versus torus-localization.** Let $\mathcal{X} \hookrightarrow \mathcal{P}$ be an embedding of smooth DM stacks equipped with a $\mathbb{C}^*$-action. We assume that

- the $\mathbb{C}^*$-fixed loci of $\mathcal{X}$ and of $\mathcal{P}$ are equal,
- $\mathcal{P}$ is equipped with a torus action $T = (\mathbb{C}^*)^N$, extending the $\mathbb{C}^*$-action by an embedding $\mathbb{C}^* \hookrightarrow T$,
- the normal bundle of $\mathcal{X} \hookrightarrow \mathcal{P}$ is the pull-back of a T-equivariant vector bundle $\mathcal{N}$ over $\mathcal{P}$,
- the vector bundle $\mathcal{N}$ is convex up to two markings, i.e. for every stable map $f : \mathcal{C} \to \mathcal{P}$ where $\mathcal{C}$ is a smooth genus-0 orbifold curve with at most two markings we have $H^1(\mathcal{C}, f^*\mathcal{N}) = 0$.

First, we look at the $\mathbb{C}^*$-fixed loci of the moduli spaces of stable maps and we find the following fibered diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{X})^{\mathbb{C}^*} & \xrightarrow{j} & \mathcal{M}(\mathcal{P})^{\mathbb{C}^*} \\ \iota\downarrow & \square & \downarrow\widetilde{\iota} \\ \mathcal{M}(\mathcal{X}) & \xrightarrow[\widetilde{j}]{} & \mathcal{M}(\mathcal{P}) \end{array}$$

Writing $i : \mathcal{X} \hookrightarrow \mathcal{P}$, we have a $\mathbb{C}^*$-equivariant short exact sequence

$$0 \to T_{\mathcal{X}} \to i^*T_{\mathcal{P}} \to i^*\mathcal{N} \to 0$$

and it induces a distinguished triangle for the dual of the perfect obstruction theories of $\mathcal{M}(\mathcal{X})$ and of $\mathcal{M}(\mathcal{P})$

$$R\pi_* f^*T_{\mathcal{X}} \to R\pi_* f^*T_{\mathcal{P}} \to R\pi_* f^*\mathcal{N} \to (R\pi_* f^*T_{\mathcal{X}})\,[1]. \tag{4}$$

The term $\mathcal{E} := R\pi_* f^*\mathcal{N}$, pulled-back to $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$, has a fixed and a moving part, that we denote respectively by $\mathcal{E}_{\mathrm{fix}}$ and $\mathcal{E}_{\mathrm{mov}}$.

Proposition 2.1. *The fixed part $\mathcal{E}_{\mathrm{fix}}$ is a vector bundle over the fixed moduli space $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$.*

Proof. Let $f : \mathcal{C} \to \mathcal{P}$ be a stable map belonging to $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$. We denote by $\rho : \mathcal{C} \to C$ the coarse map. It is enough to prove

$$H^1(C, \rho_* f^*\mathcal{N})^{\mathrm{fix}} = 0.$$

Take the normalization $\nu : C^\nu \to C$ of the curves at all their nodes. We have

$$C^\nu = \bigsqcup_{i \in I} C_i^{\mathrm{fix}} \sqcup \bigsqcup_{j \in J} C_j^{\mathrm{nf}},$$

where the upperscripts refer respectively to fixed/non-fixed components of C^ν under the map f and the $\mathbb{C}^*$-action. In particular, a non-fixed component is unstable and maps to a one-dimensional $\mathbb{C}^*$-orbit. By the normalization exact sequence, we obtain an exact sequence

$$\bigoplus_{\text{nodes}} H^0(\text{node}, f^*\mathcal{N}_{|\text{node}}) \to H^1(C, f^*\mathcal{N}) \to H^1(C^\nu, \nu^* f^*\mathcal{N}) \to 0,$$

with

$$H^1(C^\nu, \nu^* f^*\mathcal{N}) = \bigoplus_{i \in I} H^1(C_i^{\mathrm{fix}}, \nu^* f^*\mathcal{N}) \oplus \bigoplus_{j \in J} H^1(C_j^{\mathrm{nf}}, \nu^* f^*\mathcal{N}).$$

Since the normal bundle has a non-trivial $\mathbb{C}^*$-action once restricted to the fixed locus of $\mathcal{X}$ (or equivalently of $\mathcal{P}$), then we have

$$H^0(\text{node}, f^*\mathcal{N}_{|\text{node}})^{\text{fix}} = 0 \quad \text{and} \quad H^1(C_i^{\text{fix}}, \nu^* f^*\mathcal{N})^{\text{fix}} = 0.$$

Therefore, it remains to see the vanishing of H^1 for non-fixed unstable curves C_j^{nf}, $j \in J$. The curve C_j^{nf} is isomorphic to $\mathbb{P}^1$ with either one or two markings, hence $H^1(C_j^{\text{nf}}, \nu^* f^*\mathcal{N}) = 0$ by our assumption of convexity up to two markings. □

Denote by $\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\text{vir}}$ the virtual fundamental cycle obtained by the $\mathbb{C}^*$-fixed part of the perfect obstruction theory $R\pi_* f^* T_{\mathcal{P}}$.

Proposition 2.2. *In the Chow ring of $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$, we have*

$$j_* \left[\mathcal{M}(\mathcal{X})^{\mathbb{C}^*}\right]^{\text{vir}} = e_{\mathbb{C}^*}\left(\mathcal{E}_{\text{fix}}\right) \cdot \left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\text{vir}}.$$

Furthermore, in the localized equivariant Chow ring, we have

$$e_{\mathbb{C}^*}(N_\iota^{\text{vir}})^{-1} = j^* \left(\frac{e_{\mathbb{C}^*}(\mathcal{E}_{\text{mov}})}{e_{\mathbb{C}^*}(N_{\tilde{\iota}}^{\text{vir}})}\right).$$

Proof. It follows from the standard proof using convexity and we recall here the main arguments.

The variety $\mathcal{X}$ is the zero locus of a section of the vector bundle $\mathcal{N}$ over the ambient space $\mathcal{P}$. This section induces a map s from the moduli space of stable maps to $\mathcal{P}$ to the direct image cone $\pi_* f^*\mathcal{N}$, see [9, Definition 2.1]. Since the moduli space $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$ is fixed by the action of $\mathbb{C}^*$, then it maps to the fixed part of the direct image cone, that is the vector bundle $\mathcal{E}_{\text{fix}}$. Hence we have the fibered diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{X})^{\mathbb{C}^*} & \xrightarrow{\ j\ } & \mathcal{M}(\mathcal{P})^{\mathbb{C}^*} \\ \downarrow & \square & \downarrow s \\ \mathcal{M}(\mathcal{P})^{\mathbb{C}^*} & \xrightarrow[\ 0\ ]{} & \mathcal{E}_{\text{fix}} \end{array}$$

where the bottom map is the embedding as the zero section. The fixed part of the distinguished triangle (4) gives a compatibility datum of perfect obstruction theories for the fixed moduli spaces. Functoriality of the virtual fundamental cycle gives

$$0^! \left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\text{vir}} = \left[\mathcal{M}(\mathcal{X})^{\mathbb{C}^*}\right]^{\text{vir}}$$

which is the desired result once we push-forward via the map j on both sides. The second part of the statement follows from the moving part of the distinguished triangle (4). □

Eventually, by the virtual localization formula, the $\mathbb{C}^*$-equivariant virtual cycle satisfies

$$\begin{aligned}
\widetilde{j}_* \left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir},\mathbb{C}^*} &= \widetilde{j}_* \iota_* \left(\frac{\left[\mathcal{M}(\mathcal{X})^{\mathbb{C}^*}\right]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_\iota^{\mathrm{vir}})} \right) \\
&= \widetilde{\iota}_* j_* \left(\left[\mathcal{M}(\mathcal{X})^{\mathbb{C}^*}\right]^{\mathrm{vir}} \cdot j^* \left(\frac{e_{\mathbb{C}^*}(\mathcal{E}_{\mathrm{mov}})}{e_{\mathbb{C}^*}(N_{\widetilde{\iota}}^{\mathrm{vir}})} \right) \right) \\
&= \widetilde{\iota}_* \left(e_{\mathbb{C}^*}\left(\mathcal{E}_{\mathrm{fix}}\right) \cdot \left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir}} \cdot \frac{e_{\mathbb{C}^*}(\mathcal{E}_{\mathrm{mov}})}{e_{\mathbb{C}^*}(N_{\widetilde{\iota}}^{\mathrm{vir}})} \right) \\
&= \widetilde{\iota}_* \left(e_{\mathbb{C}^*}\left(\mathcal{E}\right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_{\widetilde{\iota}}^{\mathrm{vir}})} \right),
\end{aligned}$$

where equalities happen in the $\mathbb{C}^*$-localized equivariant Chow ring $A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))_{\mathrm{loc}}$.

Remark 2.3. If it were defined, the right-hand side would equal

$$e_{\mathbb{C}^*}\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},\mathbb{C}^*},$$

using the virtual localization formula, but it is not clear that the $\mathbb{C}^*$-equivariant Euler class of $R\pi_* f^*\mathcal{N}$ is defined in $A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))_{\mathrm{loc}}$. However, we say that $e_{\mathbb{C}^*}\left(R\pi_* f^*\mathcal{N}\right)$ is defined after localization[4] to mean that its pull-back to the fixed locus is defined.

Now, we aim to extend the right-hand side of the equality to the torus-T action. We denote by $t_1, \ldots, t_N$ the T-equivariant parameters and we have a push-forward ring map

$$\xi_* \colon A_*^T(\mathcal{M}(\mathcal{P})) \to A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))$$

expressing each $t_1, \ldots, t_N$ in terms of t using the embedding $\mathbb{C}^* \hookrightarrow T$. Clearly, we get

$$\xi_*\left(\left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T}\right) = \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},\mathbb{C}^*}.$$

Unfortunately, the map ξ_* is only partially defined when we invert equivariant parameters: it is defined as long as the denominators are non-zero when expressed in terms of the variable t. It is easier to work out this issue on the fixed loci of the moduli space.

Let $\mathcal{M}(\mathcal{P})^T \hookrightarrow \mathcal{M}(\mathcal{P})$ denote the T-fixed locus of the moduli space. In particular, we have the inclusion $\widehat{\iota} \colon \mathcal{M}(\mathcal{P})^T \hookrightarrow \mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$. We notice that the moduli space $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$ is stable under the T-action from $\mathcal{M}(\mathcal{P})$ and that the map $\widehat{\iota}$ is T-equivariant. Moreover, we have a T-equivariant virtual cycle

$$\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir},T} \in A_*^T(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})$$

and the equality

$$\xi_*\left(\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir},T}\right) = \left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir}} \in A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}).$$

By the virtual localization formula, we have

$$\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir},T} = \widehat{\iota}_*\left(\frac{\left[\mathcal{M}(\mathcal{P})^T\right]^{\mathrm{vir}}}{e_T(N_{\widehat{\iota}}^{\mathrm{vir}})}\right) \in A_*^T(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}.$$

---

[4] We find this definition for the formal quintic, see [38].

Furthermore, we have the following equality in K-theory on the space $\mathcal{M}(\mathcal{P})^T$

$$N^{\mathrm{vir}}_{\widetilde{\iota}\circ\widehat{\iota}} = \widehat{\iota}^* N^{\mathrm{vir}}_{\widetilde{\iota}} + N^{\mathrm{vir}}_{\widehat{\iota}}. \tag{5}$$

Indeed, let $\mathcal{F}$ be the pull-back of $R\pi_* f^* T\mathcal{P}$ to $\mathcal{M}(\mathcal{P})^T$. By definition, the virtual normal bundle $N^{\mathrm{vir}}_{\widetilde{\iota}\circ\widehat{\iota}}$ is the T-moving part $\mathcal{F}^{\mathrm{mov}}$, which decomposes as $\mathcal{F}^{\mathrm{mov}} = \mathcal{F}^{\mathrm{mov}}_{\mathrm{fix}} + \mathcal{F}^{\mathrm{mov}}_{\mathrm{mov}}$, where the subscript denotes the $\mathbb{C}^*$-fixed/moving part. By definition, the virtual normal bundle $\widehat{\iota}^* N^{\mathrm{vir}}_{\widetilde{\iota}}$ is the $\mathbb{C}^*$-moving part of $\mathcal{F}$, i.e. $\mathcal{F}^{\mathrm{mov}}_{\mathrm{mov}}$ since there is no $\mathbb{C}^*$-moving T-fixed part in $\mathcal{F}$. Eventually, the virtual normal bundle $N^{\mathrm{vir}}_{\widehat{\iota}}$ identifies with $\mathcal{F}^{\mathrm{mov}}_{\mathrm{fix}}$.

Remark 2.4. The virtual normal bundle $N^{\mathrm{vir}}_{\widetilde{\iota}}$ is defined on $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$ and we have a well-defined equality

$$\xi_* \left(e_T \left(N^{\mathrm{vir}}_{\widetilde{\iota}} \right)^{-1} \right) = e_{\mathbb{C}^*} \left(N^{\mathrm{vir}}_{\widetilde{\iota}} \right)^{-1} \in A^{\mathbb{C}^*}_* (\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}.$$

We also have seen the $\mathbb{C}^*$-decomposition $\mathcal{E} = \mathcal{E}_{\mathrm{fix}} + \mathcal{E}_{\mathrm{mov}}$ over $\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}$ with $\mathcal{E}_{\mathrm{fix}}$ being a T-equivariant vector bundle. Indeed, the vector bundle $\mathcal{N}$ over $\mathcal{P}$ is T-equivariant, thus so are $\mathcal{E}$ and $\mathcal{E}_{\mathrm{fix}}$. As a consequence, the following equality is well-defined

$$\xi_* \left(e_T(\mathcal{E}) \right) = e_{\mathbb{C}^*}(\mathcal{E}) \in A^{\mathbb{C}^*}_* (\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}.$$

Proposition 2.5. *Consider the well-defined class*

$$C_T := \widehat{\iota}^* \left(e_T \left(\mathcal{E} \right) \right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^T \right]^{\mathrm{vir}}}{e_T(N^{\mathrm{vir}}_{\widetilde{\iota}\circ\widehat{\iota}})} \in A^T_*(\mathcal{M}(\mathcal{P})^T)_{\mathrm{loc}}.$$

Then its push-forward under the inclusion $\widehat{\iota}$ equals

$$\widehat{\iota}_* \left(C_T \right) = e_T \left(\mathcal{E} \right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*} \right]^{\mathrm{vir},T}}{e_T(N^{\mathrm{vir}}_{\widetilde{\iota}})} \in A^T_*(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}.$$

In particular, we have

$$\xi_* \left(\widehat{\iota}_* \left(C_T \right) \right) = e_{\mathbb{C}^*} \left(\mathcal{E} \right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*} \right]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\widetilde{\iota}})} \in A^{\mathbb{C}^*}_*(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}.$$

Proof. By the virtual localization above and Equation (5), we have

$$\begin{aligned}
\widehat{\iota}_* \left(C_T \right) &= e_T \left(\mathcal{E} \right) \cdot \widehat{\iota}_* \left(\frac{\left[\mathcal{M}(\mathcal{P})^T \right]^{\mathrm{vir}}}{e_T(N^{\mathrm{vir}}_{\widetilde{\iota}\circ\widehat{\iota}})} \right) \\
&= e_T \left(\mathcal{E} \right) \cdot \widehat{\iota}_* \left(\frac{\left[\mathcal{M}(\mathcal{P})^T \right]^{\mathrm{vir}}}{\widehat{\iota}^* \left(e_T(N^{\mathrm{vir}}_{\widetilde{\iota}}) \right) \cdot e_T(N^{\mathrm{vir}}_{\widehat{\iota}})} \right) \\
&= \frac{e_T \left(\mathcal{E} \right)}{e_T(N^{\mathrm{vir}}_{\widetilde{\iota}})} \cdot \widehat{\iota}_* \left(\frac{\left[\mathcal{M}(\mathcal{P})^T \right]^{\mathrm{vir}}}{e_T(N^{\mathrm{vir}}_{\widehat{\iota}})} \right) \\
&= \frac{e_T \left(\mathcal{E} \right) \cdot \left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*} \right]^{\mathrm{vir},T}}{e_T(N^{\mathrm{vir}}_{\widetilde{\iota}})}.
\end{aligned}$$

The last sentence follows from the following property of ξ_*. For any space Z with a T-action and any localized classes $A, B \in A^T_*(Z)_{\mathrm{loc}}$ and $a, b \in A^{\mathbb{C}^*}_*(Z)_{\mathrm{loc}}$, if $\xi_*(A) = a$ and $\xi_*(B) = b$ are well-defined equalities, then $\xi_*(AB)$ is well-defined and equals the localized class ab. $\square$

Eventually, the push-forward maps $\widetilde{\iota}_*$ and ξ_* commute when the later is well-defined. Precisely, the map $\widetilde{\iota}$ is T-equivariant and for any localized class $C \in A_*^T(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}$ such that $\xi_*(C)$ is well-defined in $A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P})^{\mathbb{C}^*})_{\mathrm{loc}}$, then the localized class $\widetilde{\iota}_*(C)$ is well-defined under ξ_* and we have

$$\widetilde{\iota}_*\xi_*(C) = \xi_*\widetilde{\iota}_*(C) \in A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))_{\mathrm{loc}}.$$

2.2. **Equivariant quantum Lefschetz formula.** Summarizing our discussion, we obtain the following.

Theorem 2.6 (Equivariant quantum Lefschetz)**.** *Let $\mathcal{X} \hookrightarrow \mathcal{P}$ be a $\mathbb{C}^*$-equivariant embedding of smooth DM stacks satisfying assumptions listed at the beginning of this section. Then we have*

$$\widetilde{j}_* \left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir},\mathbb{C}^*} = \xi_* \left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T}\right) \in A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))_{\mathrm{loc}},$$

where $\widetilde{j}$ is the embedding of moduli spaces and ξ_* is the specialization of T-equivariant parameters into the $\mathbb{C}^*$-equivariant parameter. Here, the T-equivariant Euler class $e_T(R\pi_* f^*\mathcal{N})$ is defined after localization, see Remark 2.3.

Proof. Using previous equalities, we get

$$\begin{aligned}\widetilde{j}_* \left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir},\mathbb{C}^*} &= \widetilde{\iota}_* \left(e_{\mathbb{C}^*}\left(\mathcal{E}\right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^{\mathbb{C}^*}\right]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_{\widetilde{\iota}}^{\mathrm{vir}})}\right), \\ &= \xi_*\widetilde{\iota}_*\widehat{\iota}_*\left(\widehat{\iota}^*\left(e_T\left(\mathcal{E}\right)\right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^T\right]^{\mathrm{vir}}}{e_T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}})}\right).\end{aligned}$$

Following Remark 2.3, the meaning of 'defined after localization' is precisely

$$\begin{aligned}\xi_*\left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T}\right) &= \xi_*\left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \widetilde{\iota}_*\widehat{\iota}_*\left(\frac{\left[\mathcal{M}(\mathcal{P})^T\right]^{\mathrm{vir}}}{e_T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}})}\right)\right) \\ &= \xi_*\widetilde{\iota}_*\widehat{\iota}_*\left(\widehat{\iota}^*\left(e_T\left(\mathcal{E}\right)\right) \cdot \frac{\left[\mathcal{M}(\mathcal{P})^T\right]^{\mathrm{vir}}}{e_T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}})}\right).\end{aligned}$$

□

Eventually, we summarize Sections 1 and 2 in the following theorem.

Theorem 2.7. *Let $\mathcal{X} \hookrightarrow \mathcal{P}$ be a $\mathbb{C}^*$-equivariant embedding of regular $\mathbb{A}^1$-families. We assume the fixed loci of $\mathcal{X}$ and of $\mathcal{P}$ are equal, the ambient space $\mathcal{P}$ carries a $T := (\mathbb{C}^*)^N$-action, e.g. it is a toric DM stack, extending the $\mathbb{C}^*$-action via an embedding $\mathbb{C}^* \hookrightarrow T$, the normal bundle $\mathcal{N}$ of $\mathcal{X} \hookrightarrow \mathcal{P}$ is a pull-back from a T-equivariant vector bundle over $\mathcal{P}$, and is convex up to two markings, e.g. it satisfies some positivity condition.*

Let X be a generic smooth fiber of $\mathcal{X}$. Fix a genus g, a number of markings n such that $2g-2+n>0$, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in X, a curve class $\beta \in H_2(\mathcal{P})$, and ambient insertions $\alpha_1, \ldots, \alpha_n \in A^*(I\mathcal{P})$ with $\alpha_i \in A^*(\mathcal{P}_{\rho_i})$ admitting T-equivariant liftings. We set $\alpha := \prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)$ to be the product of insertions.

Then we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{g,n}$

$$\left[\mathcal{M}_{g,\underline{\rho}}(\mathcal{P},\beta)\right]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^*\mathcal{N}) \cdot \alpha \xrightarrow[t\to 0]{} e(\mathbb{E}^\vee) \cdot \left[\mathcal{M}_{g,\underline{\rho}}(X,\beta)\right]^{\mathrm{vir}} \cdot \alpha,$$

where on the right hand side we take the sum over curve classes $\beta_1 \in H_2(X)$ such that $\beta_1 = \beta \in H_2(\mathcal{P})$. Precisely, the class $e_T(R\pi_* f^*\mathcal{N})$ is only defined after localization, so we first apply the virtual localization formula to the left-hand side, then we compute it in $A_*^T(\overline{\mathcal{M}}_{g,n} \times \mathbb{A}^1)$ as a formal series in the T-equivariant parameters and their inverse, then we specialize them to a single variable t using $\mathbb{C}^* \hookrightarrow T$ and obtain a well-defined polynomial in t, and eventually we take the constant coefficient and pull-it back from $A_*(\overline{\mathcal{M}}_{g,n} \times \mathbb{A}^1)$ to $A_*(\overline{\mathcal{M}}_{g,n})$.

Furthermore, if the regular family $\mathcal{P}$ is trivial, i.e. $\mathcal{P} = P \times \mathbb{A}^1$ for some T-equivariant smooth DM stack P, then the formula simplifies as

$$e_T(\mathbb{E}^\vee) \cdot [\mathcal{M}_{g,\underline{\rho}}(P,\beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^*\mathcal{N}) \cdot \alpha \xrightarrow[t\to 0]{} e(\mathbb{E}^\vee) \cdot [\mathcal{M}_{g,\underline{\rho}}(X,\beta)]^{\mathrm{vir}} \cdot \alpha,$$

with the same meaning as above, except we work directly in $A_*^T(\overline{\mathcal{M}}_{g,n})$ instead of $A_*^T(\overline{\mathcal{M}}_{g,n} \times \mathbb{A}^1)$.

Proof. By Theorem 2.6, we obtain

$$\widetilde{j}_* \left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir},\mathbb{C}^*} = \xi_* \left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T}\right) \in A_*^{\mathbb{C}^*}(\mathcal{M}(\mathcal{P}))_{\mathrm{loc}},$$

where $\widetilde{j}$ is the embedding of moduli spaces and ξ_* is the specialization of T-equivariant parameters into the $\mathbb{C}^*$-equivariant parameter, corresponding to the specialization induced by $\mathbb{C}^* \hookrightarrow T$. We recall that the T-equivariant Euler class $e_T(R\pi_* f^*\mathcal{N})$ is only defined after localization, see Remark 2.3.

By Proposition 1.15 and Lemma 1.10, we get

$$e(\mathbb{E}^\vee) \cdot q_{1*}\left(\left[\mathcal{M}(X)\right]^{\mathrm{vir}} \cdot \alpha\right) = 1^* q_*\left(\left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir}} \cdot \alpha\right),$$

where $q\colon \mathcal{M}(\mathcal{X}) \to \mathbb{A}^1 \times \overline{\mathcal{M}}_{g,n}$, $q_1\colon \mathcal{M}(\mathcal{X}_{s=1}) \to 1 \times \overline{\mathcal{M}}_{g,n}$, and $1\colon 1 \times \overline{\mathcal{M}}_{g,n} \to \mathbb{A}^1 \times \overline{\mathcal{M}}_{g,n}$ form a fibered diagram, see the beginning of Section 1.1. Using the T-equivariant map $0\colon 0 \times \overline{\mathcal{M}}_{g,n} \to \mathbb{A}^1 \times \overline{\mathcal{M}}_{g,n}$, we have in $A_*(\overline{\mathcal{M}}_{g,n})$ the equality

$$1^* q_*\left(\left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir}} \cdot \alpha\right) = 0^* q_*\left(\left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir}} \cdot \alpha\right)$$

and the right-hand side equals the non-equivariant limit of $0^* q_*\left(\left[\mathcal{M}(\mathcal{X})\right]^{\mathrm{vir},\mathbb{C}^*} \cdot \alpha\right)$. Denoting $\widetilde{q}\colon \mathcal{M}(\mathcal{P}) \to \mathbb{A}^1 \times \overline{\mathcal{M}}_{g,n}$, we obtain $e(\mathbb{E}^\vee) \cdot q_{1*}\left(\left[\mathcal{M}(X)\right]^{\mathrm{vir}} \cdot \alpha\right)$ as the non-equivariant limit of

$$0^* \widetilde{q}_*\left(\xi_*\left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T}\right) \cdot \alpha\right), \tag{6}$$

where we recall that α consists of ambient insertions, so that it is a pull-back from $\mathcal{M}(\mathcal{P})$. Since there exists a T-equivariant lift of α, it can be written as $\xi_*(\alpha)$, and by ring properties of the pushforward ξ_* (see the end of proof of Proposition 2.5), and by commutativity of push-forwards, Equation (6) equals

$$0^* \xi_*\left(\widetilde{q}_*\left(e_T\left(R\pi_* f^*\mathcal{N}\right) \cdot \left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T} \cdot \alpha\right)\right), \tag{7}$$

yielding the first part of the statement.

For the second part, we assume that $\mathcal{P} = P \times \mathbb{A}^1$, so that the normal bundle of $P \subset \mathcal{P}$ is the trivial line bundle $\mathcal{O}$ with a non-trivial T-action. Over the fixed moduli space $\mathcal{M}(\mathcal{P})^T$, the term $R\pi_*\mathcal{O} = [\mathcal{O} \to \mathbb{E}^\vee]$ then has no fixed part and we get

$$\left[\mathcal{M}(\mathcal{P})\right]^{\mathrm{vir},T} = e_T\left(R\pi_*\mathcal{O}\right)^{-1} \cdot \left[\mathcal{M}(P)\right]^{\mathrm{vir},T}$$

after localization. Recall that Equation (7) is meant to be after localization as well, so that the map $\widetilde{q}$ is rather a map from the T-fixed moduli space $\mathcal{M}(\mathcal{P})^T$, and thus factors through the map 0, as the fixed moduli space lies above the central fiber. Thus we write $\widetilde{q} = 0 \circ \overline{q}$ with $\overline{q}\colon \mathcal{M}(P) \to \overline{\mathcal{M}}_{g,n}$. Hence, Equation (7) becomes

$$\begin{aligned}
& 0^*\xi_*\left(0_*\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(R\pi_*\mathcal{O}\right)^{-1}\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right)\\
= \;& 0^*0_*\xi_*\left(\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(R\pi_*\mathcal{O}\right)^{-1}\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right)\\
= \;& e_{\mathbb{C}^*}(\mathcal{N}_0)\cdot\xi_*\left(\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(R\pi_*\mathcal{O}\right)^{-1}\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right)\\
= \;& \xi_*\left(e_T(\mathcal{N}_0)\cdot\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(R\pi_*\mathcal{O}\right)^{-1}\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right)\\
= \;& \xi_*\left(\frac{e_T(\mathcal{N}_0)}{e_T\left(\mathcal{O}\right)}\cdot\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(\mathbb{E}^\vee\right)\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right),
\end{aligned}$$

where $\mathcal{N}_0$ is the normal bundle of $0\colon 0\times\overline{\mathcal{M}}_{g,n}\to\mathbb{A}^1\times\overline{\mathcal{M}}_{g,n}$ and therefore equals the trivial bundle $\mathcal{O}$ with the same non-trivial T-action as before. As a consequence, we get the desired left-hand side

$$\xi_*\left(\overline{q}_*\left(e_T\left(R\pi_*f^*\mathcal{N}\right)\cdot e_T\left(\mathbb{E}^\vee\right)\cdot[\mathcal{M}(P)]^{\mathrm{vir},T}\cdot\alpha\right)\right).$$

□

3. Smooth hypersurfaces in weighted projective spaces

3.1. **Hodge–Gromov–Witten theory for chain polynomials.** Let $w_1,\ldots,w_N$ be positive integers and denote by $\mathbb{P}(\underline{w})=\mathbb{P}(w_1,\ldots,w_N)$ the weighted projective space given by these weights. In this section, we assume the chain-type arithmetic condition: there exist positive integers $a_1,\ldots,a_N$ and d such that

$$a_jw_j+w_{j+1}=d \text{ for } j<N \text{ and } a_Nw_N=d. \tag{8}$$

In particular, it gives the existence of a smooth (orbifold) hypersurface X of degree d in $\mathbb{P}(\underline{w})$. Precisely, one such example is the vanishing locus of the chain polynomial

$$x_1^{a_1}x_2+\cdots+x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}.$$

Moreover, since Gromov–Witten theory is invariant under smooth deformations, we can refer to this example for our computations.

The weighted projective space $\mathbb{P}(\underline{w})$ carries the action of a torus $T=(\mathbb{C}^*)^N$. We denote the equivariant parameters by $\underline{t}=(t_1,\ldots,t_N)$. For any integer $d\in\mathbb{Z}$ and any character $\chi\in\mathrm{Hom}(T,\mathbb{C})$, there is a T-equivariant line bundle $\mathcal{O}_\chi(d)$.

Remark 3.1. In Theorem 3.3, we take the trivial character χ on $\mathcal{O}(1)$ and then take its d-th power. It means that $\mathcal{O}(d)$ has weight $-\frac{dt_j}{w_j}$ in the affine chart $x_j=1$. We refer to Remark 3.4 below for another description of the action used on the line bundle $\mathcal{O}(d)$.

Denote the T-equivariant virtual fundamental cycle by

$$[\mathcal{M}(\mathbb{P}(\underline{w}))]^{\mathrm{vir},T}\in A_*^T(\mathcal{M}(\mathbb{P}(\underline{w}))),$$

so that its non-equivariant limit $\underline{t}\to 0$ gives back the virtual fundamental cycle. Moreover, the derived object $R\pi_*f^*\mathcal{O}_\chi(d)$, where π is the projection map from the universal curve and f is the universal stable map, is also T-equivariant. Unfortunately, the following expression

$$e_T(R\pi_* f^* \mathcal{O}_\chi(d)) \cdot [\mathcal{M}(\mathbb{P}(\underline{w}))]^{\mathrm{vir},T} \in A_*^T(\mathcal{M}(\mathbb{P}(\underline{w})))_{\mathrm{loc}},$$

which is defined after localization, does not admit a non-equivariant limit $\underline{t} \to 0$, unless the convexity condition holds and thus $R\pi_* f^* \mathcal{O}(d) = \pi_* f^* \mathcal{O}(d)$ is a vector bundle.

Remark 3.2. Convexity holds in genus zero under the Gorenstein condition: $w_j | d$ for all j. In that case, the non-equivariant limit $\underline{t} \to 0$ gives back the virtual cycle $[\mathcal{M}(X)]^{\mathrm{vir}}$ of the moduli space of stable maps to a smooth degree-d hypersurface $X \subset \mathbb{P}(\underline{w})$.

In Theorem 3.3, we overcome the difficulty of non-convexity with the help of the Hodge bundle $\mathbb{E}$. First, we pull it back to the moduli space of stable maps to $\mathbb{P}(\underline{w})$ and then we endow it with the following T-action: one rescales fibers of $\mathbb{E}$ by $t_N^{a_N}$.

Eventually, we use insertions of ambient cohomology classes of X, i.e. which are pulled-back from the ambient space $\mathbb{P}(\underline{w})$. Theses classes are naturally expressed in terms of hyperplane classes, so that there are T-invariant representatives of them.

As an application of our Regular Specialization Theorem, we prove the following.

Theorem 3.3 (Hodge–Gromov–Witten theory of chain hypersurfaces)**.** *We assume condition* (8) *and we fix $g, n \in \mathbb{N}$ such that $2g-2+n>0$, $\beta \in \mathbb{N}$, and isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathbb{P}(\underline{w})$. Let $X \subset \mathbb{P}(\underline{w})$ be a smooth hypersurface of degree d and $\alpha_1, \ldots, \alpha_n$ be ambient cohomology classes on X, i.e. pulled back from $\mathbb{P}(\underline{w})$. We set $\alpha := \prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)$ to be the product of insertions. Then we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{g,n}$*

$$e_T(\mathbb{E}^\vee) \cdot [\mathcal{M}_{g,\underline{\rho}}(\mathbb{P}(\underline{w}), \beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^* \mathcal{O}(d)) \cdot \alpha \underset{t \to 0}{\longrightarrow} e(\mathbb{E}^\vee) \cdot [\mathcal{M}_{g,\underline{\rho}}(X, \beta)]^{\mathrm{vir}} \cdot \alpha.$$

Precisely, the class $e_T(R\pi_* f^* \mathcal{O}(d))$ is only defined after localization, so we first apply the virtual localization formula to the left-hand side, then we compute it in $A_*^T(\overline{\mathcal{M}}_{g,n})$ as a formal series in the T-equivariant parameters and their inverse, then we specialize them to

$$t_{j+1} = (-a_1) \cdots (-a_j) t,$$

for all $1 \leq j \leq N$ and obtain a well-defined polynomial in t, and eventually we take the constant coefficient.

Remark 3.4. The specialization of T-equivariant parameters in terms of a single variable t can be rephrased as an embedding $\mathbb{C}^* \hookrightarrow T$. Then by equation (8), there is a $\mathbb{C}^*$-invariant (singular) hypersurface of degree d

$$X_0 = \left\{ x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N = 0 \right\} \subset \mathbb{P}(\underline{w}),$$

and the line bundle $\mathcal{O}(d)$ in Theorem 3.3 is its normal line bundle. Therefore, it comes with a $\mathbb{C}^*$-action. To be more precise, look at the weights on fibers over the fixed locus, which consists of all coordinate points in $\mathbb{P}(\underline{w})$. At the point $(0, \ldots, x_j = 1, \ldots, 0) \in \mathbb{P}(\underline{w})$, the $\mathbb{C}^*$-action has weight $-\frac{dt_j}{w_j}$, as was announced in Remark 3.1.

Remark 3.5. Theorem 3.3 yields an explicit formula for Hodge–Gromov–Witten invariants of X as a sum over dual graphs. Indeed, such a formula is known for weighted projective spaces, e.g. in [40].

Proof. As Gromov–Witten theory is invariant under smooth deformations, we can take the degree-d hypersurface X to be the zero locus of the chain polynomial

$$P = x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}.$$

Define the regular $\mathbb{A}^1$-family

$$\mathcal{X} = \left\{x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}s = 0\right\} \subset \mathbb{P}(w_1,\ldots,w_N)\times\mathbb{A}^1 =: \mathcal{P}.$$

It is endowed with a $\mathbb{C}^*$-action with weight p_j on x_j and p_{N+1} on s satisfying $p_1 = 1$ and $p_{j+1} = (-a_1)\cdots(-a_j)$ for $1 \leq j \leq N$. Moreover, the fiber X_1 at $s = 1$ equals the smooth hypersurface X and the fiber X_0 at $s = 0$ has exactly one singular point $\mathrm{Sing}(X_0) = (0,\ldots,0,1) \in \mathbb{P}(w_1,\ldots,w_N)$. It is then enough to check the assumptions of Theorem 2.7.

The $\mathbb{C}^*$-fixed loci for $\mathcal{X}$ and for $\mathcal{P}$ are the same, i.e. it is given by all N coordinate points in the central fiber $s = 0$. The DM stack $\mathcal{P}$ carries a $T = (\mathbb{C}^*)^N$ action where the action of T on $\mathbb{A}^1$ is the multiplication by $t_N^{-a_N}$. The normal bundle of $\mathcal{X} \hookrightarrow \mathcal{P}$ is the pull-back of the T-equivariant line bundle $\mathcal{O}(d)$ on $\mathbb{P}(\underline{w})$, with the trivial character, as explained in Remarks 3.1 and 3.4. It remains to prove convexity up to two markings.

Let $f\colon \mathcal{C} \to \mathcal{P}$ be a stable map where $\mathcal{C}$ is a non-contracted smooth genus-0 orbifold curve with two markings σ_1 and σ_2 (the case with one or zero markings are similar). Then $f^*\mathcal{O}(d)$ is a line bundle over $\mathcal{C}$ and can be written as

$$f^*\mathcal{O}(d) = \mathcal{O}(m + r_1\sigma_1 + r_2\sigma_2),$$

with $0 \leq r_1, r_2 < 1$ being the monodromies at the markings and $m \in \mathbb{Z}$. Moreover, we have the relation

$$m + r_1 + r_2 = d \cdot \deg(f) \geq 0.$$

Hence, we have $m > -2$, so $m \geq -1$. Therefore, denoting by $C = \mathbb{P}^1$ the coarse curve of $\mathcal{C}$, we have $H^1(\mathcal{C}, f^*\mathcal{O}(d)) = H^1(C, \mathcal{O}(m)) = 0$. □

As a special case of Theorem 3.3, we obtain a full genus-0 computation of Gromov–Witten theory of chain hypersurfaces with ambient insertions, using the simple fact that the Hodge class equals 1 in genus 0.

Corollary 3.6 (Genus-zero Gromov–Witten theory of chain hypersurfaces)**.** *Under assumptions and notations of Theorem 3.3, but with $g = 0$, we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{0,n}$*

$$[\mathcal{M}_{0,\underline{\rho}}(\mathbb{P}(\underline{w}),\beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_*f^*\mathcal{O}(d)) \cdot \alpha \xrightarrow[t\to 0]{} [\mathcal{M}_{0,\underline{\rho}}(X,\beta)]^{\mathrm{vir}} \cdot \alpha.$$

3.2. **Hodge–Gromov–Witten theory for loop polynomials.** In this section, we assume the loop-type arithmetic condition: there exist positive integers $a_1,\ldots,a_N$ and d such that

$$a_jw_j + w_{j+1} = d \text{ for } j < N \text{ and } a_Nw_N + w_1 = d. \tag{9}$$

In particular, it gives the existence of a smooth (orbifold) hypersurface X of degree d in $\mathbb{P}(\underline{w})$. Precisely, one such example is the vanishing locus of the loop polynomial

$$x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}x_1.$$

Moreover, since Gromov–Witten theory is invariant under smooth deformations, we can refer to this example for our computations.

Remark 3.7. In Theorem 3.8, we take the trivial character χ on $\mathcal{O}(1)$ and then take its d-th power. It means that $\mathcal{O}(d)$ has weight $-\frac{dt_j}{w_j}$ in the affine chart $x_j = 1$. We refer to Remark 3.9 below for another description of the action used on the line bundle $\mathcal{O}(d)$.

As in Theorem 3.3, we overcome the difficulty of non-convexity with the help of the Hodge bundle $\mathbb{E}$, but we change the T-action to the following: one rescales fibers of $\mathbb{E}$ by $t_N^{a_N} t_1$. Furthermore, we change the ambient space by performing a weighted blow-up.

Define a polynomial Q by

$$Q(x_1, \dots, x_{N-1}, s) = x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} + x_1 s.$$

Since Q is a chain polynomial, it is quasi-homogeneous with some positive weights $b_1, \dots, b_{N-1}, b_s$ and degree $\delta \in \mathbb{N}^*$.

Let $\widetilde{\mathcal{P}}$ be the weighted blow-up of $\mathbb{P}(\underline{w}) \times \mathbb{A}^1 =: \mathcal{P}$ at the point $((0, \dots, 0, 1), s = 0)$ with weights $b_1, \dots, b_{N-1}$ on the variables $x_1, \dots, x_{N-1}$ in the chart $x_N = 1$ and weight b_s on the variable $s \in \mathbb{A}^1$. We refer to [1] for the construction of the weighted blow-up.

Let $T = (\mathbb{C}^*)^{N+1}$ be the natural torus action on $\mathcal{P}$. Since the base locus $((0, \dots, 0, 1), s = 0)$ is fixed under the torus T, then the space $\widetilde{\mathcal{P}}$ also carries a T-action, it is even a toric DM stack. Moreover, the line bundle $\mathcal{O}(d)$ defined in Remark 3.7 pulls-back to a T-equivariant line bundle on $\widetilde{\mathcal{P}}$, which we again denote by $\mathcal{O}(d)$. Let E be the exceptional divisor of $\widetilde{\mathcal{P}} \to \mathcal{P}$. The line bundle $\mathcal{O}(d - E)$ is also T-equivariant.

Furthermore, over $\mathbb{A}^1 - 0$, we have $\widetilde{\mathcal{P}} \simeq \mathcal{P}$, so that we can embed the hypersurface $X \subset \mathbb{P}(\underline{w})$ defined by the loop polynomial in the fiber over $s = 1$, yielding

$$X \hookrightarrow \widetilde{\mathcal{P}}.$$

Let $\alpha \in H^*(X)$ be a cohomology class which is a pull-back from $\mathbb{P}(\underline{w})$. It can be represented by a T-equivariant cycle in $\mathbb{P}(\underline{w})$ which does not contain the point $(0, \dots, 0, 1)$. Therefore, we can view it as a T-equivariant cohomology class on $\widetilde{\mathcal{P}}$, such that the pull-back to X of its non-equivariant limit equals α.

A curve class in $\mathbb{P}(\underline{w})$ is a non-negative multiple of a line, that we can choose to avoid the point $(0, \dots, 0, 1)$, so that it gives a curve class in $\widetilde{\mathcal{P}}$. We only consider these curve classes in the following statement.

Theorem 3.8 (Hodge–Gromov–Witten theory of loop hypersurfaces)**.** *We assume condition* (9) *and we fix $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$, $\beta \in \mathbb{N}$, and isotropies $\underline{\rho} = (\rho_1, \dots, \rho_n)$ in $\mathbb{P}(\underline{w})$. Let $X \subset \mathbb{P}(\underline{w})$ be a smooth hypersurface of degree d and $\alpha_1, \dots, \alpha_n$ be ambient cohomology classes on X, i.e. pulled back from $\mathbb{P}(\underline{w})$. We set $\alpha := \prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)$ to be the product of insertions. Then we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{g,n}$*

$$[\mathcal{M}_{g,\underline{\rho}}(\widetilde{\mathcal{P}}, \beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^* \mathcal{O}(d - E)) \cdot \alpha \xrightarrow[t \to 0]{} e(\mathbb{E}^\vee) \cdot [\mathcal{M}_{g,\underline{\rho}}(X, \beta)]^{\mathrm{vir}} \cdot \alpha.$$

Precisely, the class $e_T(R\pi_* f^* \mathcal{O}(d - E))$ is only defined after localization, so we first apply the virtual localization formula to the left-hand side, then we compute it in $A_*^T(\overline{\mathcal{M}}_{g,n} \times \mathbb{A}^1)$ as a formal series in the T-equivariant parameters and their *inverse, then we specialize them to*

$$t_{N+1} = ((-a_1)\cdots(-a_N) - 1)\,t\ ,\quad t_{j+1} = (-a_1)\cdots(-a_j)t,$$

for all $1 \leq j \leq N-1$*, and obtain a well-defined polynomial in* t*, and eventually we take the constant coefficient and pull-it back from* $A_*(\overline{\mathcal{M}}_{g,n} \times \mathbb{A}^1)$ *to* $A_*(\overline{\mathcal{M}}_{g,n})$*.*

Remark 3.9. The specialization of the N first T-equivariant parameters in terms of a single variable t is the same as in Theorem 3.3, but the last one is different, i.e. the Hodge bundle is rescaled with weight $((-a_1)\cdots(-a_N)-1)$ instead of weight $(-a_1)\cdots(-a_N)$. By equation (9), there is a $\mathbb{C}^*$-invariant (singular) hypersurface of degree d

$$X_0 = \left\{x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N = 0\right\} \subset \mathbb{P}(\underline{w}),$$

and the line bundle $\mathcal{O}(d)$ in Theorem 3.8 is its normal line bundle, with the same $\mathbb{C}^*$-action as in Theorem 3.3. The line bundle $\mathcal{O}(d-E)$ is also a normal bundle as we see in the proof below.

Remark 3.10. Theorem 3.8 yields an explicit formula for Hodge–Gromov–Witten invariants of X as a sum over dual graphs. Indeed, such a formula is known for every smooth toric DM stack, e.g. in [40], and $\widetilde{\mathcal{P}}$ is one such item.

Proof. It is similar to the proof of Theorem 3.3. We take the degree-d hypersurface X to be the zero locus of the loop polynomial

$$P = x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}x_1$$

and we define the $\mathbb{A}^1$-family

$$\mathcal{X} = \left\{x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}x_1 s = 0\right\} \subset \mathbb{P}(w_1,\ldots,w_N) \times \mathbb{A}^1.$$

It is endowed with a $\mathbb{C}^*$-action with weight p_j on x_j and p_{N+1} on s satisfying $p_1 = 1$ and $p_{j+1} = (-a_1)\cdots(-a_j)$ for $1 \leq j \leq N-1$ and $p_{N+1} = (-a_1)\cdots(-a_j) - 1$. Moreover, the fiber X_1 at $s = 1$ equals the smooth hypersurface X. However, the DM stack $\mathcal{X}$ is not smooth, as it is singular at the point $(0,\ldots,0,1)$ of the central fiber $s = 0$. We thus need to resolve the singularities and that is why we use the weighted blow-up.

Define the $\mathbb{A}^1$-family $\widetilde{\mathcal{X}}$ as the weighted blow-up of $((0,\ldots,0,1), s = 0)$ with weights $(b_1,\ldots,b_{N-1})$ on the variables $x_1,\ldots,x_{N-1}$ in the chart $x_N = 1$ and weight b_s on s. We claim that the $\mathbb{A}^1$-family $\widetilde{\mathcal{X}}$ is regular.

Indeed, in the local chart where $x_N = 1$, it is defined by the equation

$$x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}} + x_1 s = 0.$$

The choice of weights $b_1,\ldots,b_{N-1},b_s$ is such that the polynomial

$$x_2^{a_2}x_3 + \cdots + x_{N-1}^{a_{N-1}} + x_1 s$$

is quasi-homogeneous with these weights. In the blow-up chart associated to the variable x_k, we have new variables $\dot{x}_1,\ldots,\dot{x}_{N-1},u$, and $\dot{s}$ satisfying $\dot{x}_k = 1$ and

$$x_k = u^{b_k}\ ,\ x_j = u^{b_j}\dot{x}_j\ ,\ s = u^{b_s}\dot{s}$$

and the equation defining $\widetilde{\mathcal{X}}$ becomes

$$u^{a_1b_1+b_2-\delta}\dot{x}_1^{a_1}\dot{x}_2 + \dot{x}_2^{a_2}\dot{x}_3 + \cdots + \dot{x}_{N-1}^{a_{N-1}} + \dot{x}_1\dot{s} = 0,$$

where the power $a_1b_1 + b_2 - \delta = (a_1-1)b_1 + b_2$ is positive. In particular, we see that we can change chart and assume $k \neq N-1$. Therefore, the partial derivative, along $\dot{x}_{k+1}$ if $k \neq 1$ and along $\dot{s}$ if $k = 1$, does not vanish. Similarly, in the blow-up chart associated to the variable s, we take the partial derivative along $\dot{x}_1$. Hence $\widetilde{\mathcal{X}}$ is a smooth DM stack.

It remains to check the assumptions of Theorem 2.7 for the embedding $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{P}}$. The $\mathbb{C}^*$-fixed loci for $\mathcal{X}$ and for $\widetilde{\mathcal{P}}$ are the same, i.e. it is given by $2N-1$ isolated points at the central fiber. Indeed, outside the exceptional divisor E in $\widetilde{\mathcal{P}}$, fixed points are the $N-1$ coordinate points $(1,0,\dots,0),\dots,(0,\dots,0,1,0)$. The exceptional divisor E is isomorphic to the weighted projective space $\mathbb{P}(b_1,\dots,b_{N-1},b_s)$ and carries the non-trivial $\mathbb{C}^*$-action with weight $t_j - t_N$ on the j-th variable and t_{N+1} on the last variable. We check that

$$b_s(t_i - t_N) \neq b_i t_{N+1} \quad \text{and} \quad b_j(t_i - t_N) \neq b_i(t_j - t_N)\ ,\ \forall 1 \leq i < j \leq N-1,$$

which implies that the fixed locus in the exceptional divisor consists of its N coordinate points. We see that all these $2N-1$ points belong to $\mathcal{X}$.

The normal bundle of $\mathcal{X} \hookrightarrow \widetilde{\mathcal{P}}$ is the pull-back of the T-equivariant line bundle $\mathcal{O}(d-E)$ on $\widetilde{\mathcal{P}}$, see Remarks 3.7 and 3.9. It remains to prove convexity up to two markings, which follows from the same positivity argument as in the proof of Theorem 3.3. Indeed, any stable map $\widetilde{f}\colon \mathcal{C} \to \widetilde{\mathcal{P}}$ induces a stable map $f\colon \mathcal{C} \to \mathbb{P}(\underline{w}) \times \mathbb{A}^1$ and we have three cases:

- the image $f(\mathcal{C})$ does not contain the base locus $((0,\dots,0,1), s=0)$, so that it is isomorphic to the image $L := \widetilde{f}(\mathcal{C})$ and $\mathcal{O}(d-E)_L = \mathcal{O}_L(d)$,
- the image $f(\mathcal{C})$ is a curve containing the base locus $((0,\dots,0,1), s=0)$, so that the image $L := \widetilde{f}(\mathcal{C})$ is its strict transform, in which case we have $\mathcal{O}(d-E)_L = \mathcal{O}_L(d-1)$,
- the image $f(\mathcal{C})$ equals the base locus $((0,\dots,0,1), s=0)$, so that the image $L := \widetilde{f}(\mathcal{C})$ is contained in the exceptional divisor E, in which case we have $\mathcal{O}(d-E)_L = \mathcal{O}(-E)_L = \mathcal{O}_L(1)$.

The degree of the line bundle on the image $\widetilde{f}(\mathcal{C})$ is non-negative in all three cases and we have seen in the proof of Theorem 3.3 that it implies $H^1(\mathcal{C}, \widetilde{f}^*\mathcal{O}(d-E)) = 0$. $\square$

As a special case of Theorem 3.8, we obtain a full genus-0 computation of Gromov–Witten theory of loop hypersurfaces with ambient insertions.

Corollary 3.11 (Genus-zero Gromov–Witten theory of loop hypersurfaces)**.** *Under assumptions and notations of Theorem 3.8, but with $g = 0$, we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{0,n}$*

$$[\mathcal{M}_{0,\underline{\rho}}(\widetilde{\mathcal{P}},\beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^*\mathcal{O}(d-E)) \cdot \alpha \xrightarrow[t \to 0]{} [\mathcal{M}_{0,\underline{\rho}}(X,\beta)]^{\mathrm{vir}} \cdot \alpha.$$

3.3. **Hodge–Gromov–Witten theory for invertible polynomials.** A quasi-homogeneous polynomial P is called invertible if it has as many monomials as variables. By [36], an invertible polynomial has an isolated singularity at the origin if and only if it is the Thom–Sebastiani sum of chain and loop polynomials[5]. Precisely, up to renaming variables, the polynomial P equals

$$P(x_1,\dots,x_N) = P_1(x_1,\dots,x_{N_1}) + \dots + P_r(x_{N_{r-1}+1},\dots,x_N),$$

where polynomials $P_1,\dots,P_r$ are either chain or loop polynomials. In this section, we assume P is not a Fermat polynomial, i.e. we have $r < N$.

[5]A Fermat monomial is a special case of chain or loop polynomial.

We introduce r affine variables $s_1, \dots, s_r$ and we define a polynomial

$$\widehat{P}(x_1, \dots, x_N, s_1, \dots, s_r) = \widehat{P}_1(x_1, \dots, x_{N_1}, s_1) + \cdots + \widehat{P}_r(x_{N_{r-1}+1}, \dots, x_N, s_r)$$

in the following way:

- if P_i is a chain polynomial of the form $y_1^{a_1} y_2 + \cdots + y_N^{a_N}$, then we set $\widehat{P}_i = y_1^{a_1} y_2 + \cdots + y_N^{a_N} s_i$,
- otherwise, P_i is a loop polynomial of the form $y_1^{a_1} y_2 + \cdots + y_N^{a_N} y_1$ and we set $\widehat{P}_i = y_1^{a_1} y_2 + \cdots + y_N^{a_N} y_1 s_i$. In that case, we say that y_N is the last variable of $\widehat{P}_i$ and y_1 is its first variable. Of course, there are N choices to incorporate the variable s_i in P_i, and we fix one once for all.

Next, we define the $\mathbb{A}^r$-family

$$\mathcal{X} = \left\{ \widehat{P} = 0 \right\} \subset \mathbb{P}(\underline{w}) \times \mathbb{A}^r,$$

which is not regular. We first determine the singular locus of $\mathcal{X}$ and then we perform weighted blow-ups to resolve it.

Let $J \subset \{1, \dots, N\}$ be the set of indices j such that x_j is the last variable of some polynomial $\widehat{P}_i$ associated to a loop polynomial P_i. Moreover, we define a function $\Phi \colon J \to \{1, \dots, N\}$ which sends j to the index of the first variable in the loop polynomial associated to j.

For any subset $J' \subset J$, we define the subspace

$$B_{J'} \subset \mathbb{P}(\underline{w}) \times \mathbb{A}^r$$

where all variables x_j with $j \notin J'$ are zero and all variables s_j with $j \in J'$ are zero. We observe that the singular locus of $\mathcal{X}$ equals

$$\mathrm{Sing}(\mathcal{X}) = \bigcup_{J' \subset J} B_{J'} \subset \mathbb{P}(\underline{w}) \times \mathbb{A}^r.$$

Furthermore, we define the invertible polynomial

$$Q_{J'}\left(\{x_j\}_{j \notin J'}, \{s_j\}_{j \in J'} \right) = \widehat{P}(x_1, \dots, x_N, s_1, \dots, s_r)_{|\substack{x_j = 1 \ \forall j \in J' \\ s_j = 1 \ \forall j \notin J'}} - \sum_{j \in J'} x_{\Phi(j)}^{a_{\Phi(j)}} x_{\Phi(j)+1},$$

which is then quasi-homogeneous of some degree $\delta \in \mathbb{N}^*$, with respect to some positive weights $b_{J'} := (b_{J'}(1), \dots, b_{J'}(N))$ on the variables x_j, s_j.

Consider the smooth DM stack $\widetilde{\mathcal{P}}$ obtained with the following recipe:

- for all subset $J' \subset J$ of cardinality 1, we blow-up $B_{J'}$ in $\mathbb{P}(\underline{w}) \times \mathbb{A}^r$ with weights $b_{J'}$ on the respective variables,
- the strict transforms of the subspaces $B_{J'}$ with $J' \subset J$ of cardinality 2 are disjoint and we blow them up with weights $b_{J'}$ on the respective variables,
- $\dots$,
- the strict transforms of the subspaces $B_{J'}$ with $J' \subset J$ of cardinality $\#J - 1$ are disjoint and we blow them up with weights $b_{J'}$ on the respective variables,
- we blow-up the strict transform of B_J with weights $b_{J'}$ on the respective variables.

We denote by $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{P}}$ the strict transform of $\mathcal{X}$ under the birational map $\widetilde{\mathcal{P}} \to \mathbb{P}(\underline{w}) \times \mathbb{A}^r$, and by E the sum of the exceptional divisors.

Proposition 3.12. *The $\mathbb{A}^r$-family $\widetilde{\mathcal{X}}$ is regular. Moreover, the fiber at $(s_1, \dots, s_r) = (1, \dots, 1)$ equals the degree-d hypersurface X in $\mathbb{P}(\underline{w})$ defined by the invertible polynomial P.*

Proof. To simplify the exposition, we write the proof for an invertible polynomial which is a sum of two loop polynomials, i.e. we take

$$\widehat{P}(\underline{x}, \underline{y}) = x_1^{a_1} x_2 + \cdots + x_N^{a_N} x_1 s_1 + y_1^{a_1} y_2 + \cdots + y_N^{a_N} y_1 s_2.$$

The general proof follows the same arguments. Moreover, we consider only the charts when $x_N \neq 0$ or $y_N \neq 0$, because otherwise $\mathcal{X}$ is already smooth.

Let us take the chart where $x_N = 1$, and consider a point $p = (\underline{x}, \underline{y}, s_1, s_2)$ in this chart. We can assume $x_1, \dots, x_{N-1} = 0$, $y_1, \dots, y_{N-1} = 0$, $s_1 = 0$, and $y_N s_2 = 0$, otherwise $\mathcal{X}$ is smooth at p.

Let us first assume $y_N = 0$, so that we blow-up the locus $B_{\{x_N\}}$ first and then the locus $B_{\{x_N, y_N\}}$. After the first blow-up, the equation defining $\mathcal{X}$ becomes

$$u^{\epsilon_1} \dot{x}_1^{a_1} \dot{x}_2 + \cdots + \dot{x}_1 \dot{s}_1 + \dot{y}_1^{a_1} \dot{y}_2 + \cdots + \dot{y}_N^{a_N} \dot{y}_1 s_2 = 0,$$

in the new coordinates (see the proof of Theorem 3.8) and with ϵ_1 some non-negative integer. If one of the dotted variable other than $\dot{y}_N$ is non-zero, then this equation satisfies the smoothness criterion, and we are away the second blow-up center $B_{\{x_N, y_N\}}$. Thus, we can assume they are all zero but $\dot{y}_N = 1$. After the second blow-up, we then obtain

$$v^{\epsilon_2} u^{\epsilon_1} \ddot{x}_1^{a_1} \ddot{x}_2 + \cdots + \ddot{x}_1 \ddot{s}_1 + v^{\epsilon_3} \ddot{y}_1^{a_1} \ddot{y}_2 + \cdots + \ddot{y}_1 \ddot{s}_2 = 0,$$

in new coordinates defined in the same way as for the first blow-up and with some non-negative integers ϵ_2, ϵ_3. Then this equation satisfies the smoothness criterion.

Eventually, let us consider the case where $y_N \neq 0$ and $s_2 = 0$. Then we are away the first blow-up center $B_{\{x_N\}}$ and we only need to perform the second blow-up. The equation defining $\widetilde{\mathcal{X}}$ is then of the form

$$v'^{\epsilon_2} \left(\ddot{x}'_1\right)^{a_1} \ddot{x}'_2 + \cdots + \ddot{x}'_1 \ddot{s}'_1 + v'^{\epsilon_3} \left(\ddot{y}'_1\right)^{a_1} \ddot{y}'_2 + \cdots + y_N^{a_N} \ddot{y}'_1 \ddot{s}'_2 = 0,$$

in new coordinates and it satisfies the smoothness criterion since $y_N \neq 0$. □

Consider the natural torus action of $T = (\mathbb{C}^*)^{N+r}$ on the toric DM stack $\mathbb{P}(\underline{w}) \times \mathbb{A}^r$. Since every subspace $B_{J'}$ is stable under the torus-action, then the weighted blow-up $\widetilde{\mathcal{P}}$ is again a toric DM stack with T-action. Moreover, the normal bundle of $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{P}}$ is the pull-back of the T-equivariant line bundle $\mathcal{O}(d - E)$ on $\widetilde{\mathcal{P}}$.

Let $T' := (\mathbb{C}^*)^r$ and define an embedding $T' \hookrightarrow T$ such that the polynomial $\widehat{P}$ is T'-equivariant. Precisely, if $t_1, \dots, t_N, \tau_1, \dots, \tau_r$ denote variables of T, then we impose $M(\underline{t}, \underline{\tau}) = 1$ for every monomial M of the polynomial $\widehat{P}$. Equivalently, we let T' act separately on each $\widehat{P}_i$, where we take the $\mathbb{C}^*$-action defined for chain and loop polynomials in the previous sections. We can also consider a generic embedding $\mathbb{C}^* \hookrightarrow T'$.

As a consequence, the regular $\mathbb{A}^r$-family $\widetilde{\mathcal{X}}$ is $\mathbb{C}^*$-equivariant (even T'-equivariant), and we check easily that the $\mathbb{C}^*$-fixed loci in $\widetilde{\mathcal{X}}$ and in $\widetilde{\mathcal{P}}$ are equal and consist of isolated fixed points in the central fiber $(s_1, \dots, s_r) = (0, \dots, 0)$. Moreover, by the same positivity argument as in the proof of Theorem 3.8, we see that the normal bundle $\mathcal{O}(d - E)$ of $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{P}}$ is convex up to two markings. We then conclude with the following statement.

Theorem 3.13 (genus-0 Gromov–Witten theory of invertible hypersurfaces)**.** *We fix $n \geq 3$, $\beta \in \mathbb{N}$, and isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathbb{P}(\underline{w})$. We assume there exists a degree-d hypersurface in $\mathbb{P}(\underline{w})$ defined by an invertible polynomial. Let $X \subset \mathbb{P}(\underline{w})$ be any smooth hypersurface of degree d and $\alpha_1, \ldots, \alpha_n$ be ambient cohomology classes on X, i.e. pulled back from $\mathbb{P}(\underline{w})$. We set $\alpha := \prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)$ to be the product of insertions. Then we have the following equality in the Chow ring of $\overline{\mathcal{M}}_{0,n}$*

$$[\mathcal{M}_{0,\underline{\rho}}(\widetilde{\mathcal{P}}, \beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^*\mathcal{O}(d-E)) \cdot \alpha \underset{t \to 0}{\longrightarrow} [\mathcal{M}_{0,\underline{\rho}}(X, \beta)]^{\mathrm{vir}} \cdot \alpha.$$

Precisely, the class $e_T(R\pi_* f^*\mathcal{O}(d-E))$ is only defined after localization, so we first apply the virtual localization formula to the left-hand side, then we compute it in $A_*^T(\overline{\mathcal{M}}_{0,n} \times \mathbb{A}^r)$ as a formal series in the T-equivariant parameters and their inverse, then we specialize them via the embedding $\mathbb{C}^* \hookrightarrow T$, and obtain a well-defined polynomial in t, and eventually we take the constant coefficient and pull-it back from $A_*(\overline{\mathcal{M}}_{0,n} \times \mathbb{A}^r)$ to $A_*(\overline{\mathcal{M}}_{0,n})$.

Remark 3.14. Theorem 3.13 is also valid in higher genus, but it is not interesting as the right-hand side is multiplied by the r-th power of the Euler class of the Hodge bundle, and we know its square is zero for positive genus.

Remark 3.15. Theorem 3.13 yields an explicit formula for genus-0 Gromov–Witten invariants of X as a sum over dual graphs. Indeed, such a formula is known for every smooth toric DM stack, e.g. in [40], and $\widetilde{\mathcal{P}}$ is one such item. Moreover, the complexity of the space $\widetilde{\mathcal{P}}$ only comes from loop polynomials. In particular, when it is possible to represent the hypersurface X by a Thom–Sebastiani sum of chain polynomials, we can simplify the formula to

$$[\mathcal{M}_{0,\underline{\rho}}(\mathbb{P}(\underline{w}), \beta)]^{\mathrm{vir},T} \cdot e_T(R\pi_* f^*\mathcal{O}(d)) \cdot \alpha \underset{t \to 0}{\longrightarrow} [\mathcal{M}_{0,\underline{\rho}}(X, \beta)]^{\mathrm{vir}} \cdot \alpha,$$

in the Chow ring of $\overline{\mathcal{M}}_{0,n}$.

Remark 3.16. There is a list of all 7555 Calabi–Yau 3-folds that are hypersurfaces in weighted projective spaces, see [7, 30, 34] or on Kreuzer's webpage. Among them, there are about 800 hypersurfaces represented by a Fermat polynomial, hence satisfying the convexity assumption. There are about 6000 hypersurfaces defined by an invertible polynomial, hence computable via Theorem 3.13. The last 10% correspond to non-degenerate polynomials with more than five monomials and are not treated by this paper, e.g.

$$x_1^{15} + x_2^5 + x_3^5 x_5 + x_4^2 x_5 + x_5^9 + x_3^2 x_2 x_4 = 0 \quad \text{in } \mathbb{P}(3, 9, 8, 20, 5).$$

Observe as well that the main difficulty in Theorem 3.13 comes from the sequence of weighted blow-ups in the definition of the ambient space $\widetilde{\mathcal{P}}$. However, as we consider 3-folds, there is at most two loop polynomials in the Thom–Sebastiani sum, so that the birational map $\widetilde{\mathcal{P}} \to \mathbb{P}(\underline{w}) \times \mathbb{A}^r$ is defined by at most three blow-ups.

Remark 3.17. Calabi–Yau 3-folds with Euler characteristic equal to $\chi = \pm 6$ are especially important in string theory. The first instance is the Tian–Yau manifold, defined as the quotient of a smooth complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of degrees $(3, 0), (0, 3)$, and $(1, 1)$ by a free action of $\mathbb{Z}/3\mathbb{Z}$. It appeared as one candidate for a string theory's potential solution to the universe, see [25, 30]. Other examples are given by hypersurfaces in weighted projective space. A list of 40 items is given in [34], among which 14 hypersurfaces are defined by an invertible polynomial. None of these 14 hypersurface satisfy the convexity assumption and only one involves a loop polynomial, namely

$$x_1^7x_2 + x_2^5x_1 + x_3^17x_4 + x_4^2x_5 + x_5^3 = 0 \quad \text{in } \mathbb{P}(6,9,2,17,17).$$

3.4. **Regularizable stacks.** As a by-product of Section 1, we extend genus-0 Gromov–Witten theory to a particular set of singular DM stacks that we call regularizable and we prove invariance under regular deformations.

Definition 3.18. A DM stack X is called regularizable if there is an embedding $X \hookrightarrow \mathcal{X}$ as a fiber in a regular $\mathbb{A}^m$-family $\mathcal{X}$ for some integer m. Genus-zero Gromov–Witten theory of X is then defined using regularized virtual cycle. A substack $X \subset \mathcal{P}$ of a smooth DM stack $\mathcal{P}$ is called regularizable inside $\mathcal{P}$ if we can choose the family $\mathcal{X}$ above as a subfamily of the trivial family $\mathcal{P} \times \mathbb{A}^m$.

Examples. Smooth DM stacks are regularizable via a trivial family. The hypersurface X_0 from Remark 3.4 is singular but it is regularizable inside $\mathbb{P}(\underline{w})$. Every hypersurface in a projective space is regularizable inside the projective space. The quartic orbifold curve

$$\left\{x^4y + y^3z = 0\right\} \subset \mathbb{P}(1,1,2)$$

is not regularizable inside $\mathbb{P}(1,1,2)$.

In the following proposition, we illustrate the fundamental role of hypersurfaces defined by a Thom–Sebastiani sum of chain polynomials.

Proposition 3.19. *Let $X \subset \mathbb{P}(\underline{w})$ be a regularizable hypersurface inside a weighted projective space, such that there is a $\mathbb{C}^*$-action on $\mathbb{P}(\underline{w})$ leaving X invariant and whose fixed points are isolated. Then X is singular and there exists a smooth hypersurface in $\mathbb{P}(\underline{w})$ defined by a Thom–Sebastiani sum of chain polynomials. Conversely, if there exists a smooth hypersurface in $\mathbb{P}(\underline{w})$ defined by chain polynomials, then there is a regularizable hypersurface stable under a $\mathbb{C}^*$-action from $\mathbb{P}(\underline{w})$ with isolated fixed points.*

Proof. For every hypersurface $X = \{P = 0\} \subset \mathbb{P}(\underline{w})$, denoting by $\mathcal{M}$ the set of monomials of P, we see easily that

$$\begin{aligned} (1,0\dots,0) \notin X \quad &\Longleftrightarrow \quad \exists m \in \mathbb{N}^* \ , \ x_1^m \in \mathcal{M}, \\ (1,0\dots,0) \in X - \mathrm{Sing}(X) \quad &\Longleftrightarrow \quad \exists m \in \mathbb{N}^*, j \neq 1 \ , \ x_1^m x_j \in \mathcal{M}. \end{aligned}$$

Moreover, if $(1,0\dots,0) \in \mathrm{Sing}(X)$, then we have

$$X \text{ is regularizable inside } \mathbb{P}(\underline{w}) \implies w_1 | d.$$

Therefore, whenever X is regularizable inside $\mathbb{P}(\underline{w})$, we have, for every $1 \leq j \leq N$, either $w_j|d$ or a monomial $x_j^{a_j}x_k \in \mathcal{M}$, with possibly $k = j$.

Furthermore, assume X is invariant under a $\mathbb{C}^*$-action on $\mathbb{P}(\underline{w})$ with weights $p_1, \dots, p_N$. If we have $(1,0,\dots,0) \notin X$, then we get $w_1p_j = w_jp_1$ for all variable x_j involved in the polynomial P, and fixed points are not isolated (unless P is a Fermat monomial). Thus, if the $\mathbb{C}^*$-action has only isolated fixed points, there are no Fermat monomials in $\mathcal{M}$ (unless P is itself a Fermat monomial and then there is a smooth Fermat hypersurface in $\mathbb{P}(\underline{w})$).

As a consequence, if X is as in the statement, then it contains all coordinate points. Introduce the set $T \subset \{1,\dots,N\}^2$ defined by

$$(i,j) \in T \iff \exists a_i \in \mathbb{N}^* \ , \ x_i^{a_i} x_j \in \mathcal{M}.$$

By the discussion above, T does not intersect the diagonal and for every i such that w_i does not divide d, there is at least one j such that $(i,j) \in T$. We view T as a directed graph and we check easily that if we have a loop in T, i.e. $j_1,\dots,j_m$ such that $(j_1,j_2),\dots,(j_m,j_1)$ are in T, then the $\mathbb{C}^*$-action is trivial on $\mathbb{P}(w_{j_1},\dots,w_{j_m}) \subset \mathbb{P}(\underline{w})$. Moreover, if we have two edges colliding, i.e. i,j,k such that (i,j) and (k,j) are in T, then $w_j p_k = w_k p_j$ and the $\mathbb{C}^*$-action is trivial on $\mathbb{P}(w_j,w_k) \subset \mathbb{P}(\underline{w})$. Therefore, there is a directed subgraph in T consisting of a disjoint union of directed lines. Each line corresponds to a chain polynomial without its last Fermat monomial, thus proving the statement.

Conversely, we take a Thom–Sebastiani sum of chain polynomials $\widehat{P}$. Up to renaming variables, we can assume Fermat monomials of $\widetilde{P}$ are $y_1^{b_1},\dots,y_m^{b_m}$. If $\widehat{P}$ is not a Fermat polynomial, i.e. $m \neq N$, then we define

$$\widetilde{P} = \widehat{P} + (s_1-1)y_1^{b_1} + \cdots + (s_m-1)y_m^{b_m} \quad \text{and} \quad P = \widetilde{P}_{|s_1=\ldots=s_m=0}.$$

Then the singular hypersurface $X = \{P=0\} \subset \mathbb{P}(\underline{w})$ is invariant under a $\mathbb{C}^*$-action on $\mathbb{P}(\underline{w})$ whose fixed points are isolated and the family $\mathcal{X} = \left\{\widetilde{P}=0\right\} \subset \mathbb{P}(\underline{w}) \times \mathbb{A}^m$ is regular. If $\widehat{P}$ is the Fermat polynomial $\widehat{P} = y_1^{b_1} + \cdots + y_N^{b_N}$, then we define

$$\widetilde{P} = y_1^{b_1}s_1 + \cdots + y_{N-1}^{b_{N-1}}s_{N-1} + y_N^{b_N}$$

and $P = y_N^{b_N}$.Then the singular hypersurface $X = \{P=0\} \subset \mathbb{P}(\underline{w})$ is invariant under a $\mathbb{C}^*$-action on $\mathbb{P}(\underline{w})$ whose fixed points are isolated and the family $\mathcal{X} = \left\{\widetilde{P}=0\right\} \subset \mathbb{P}(\underline{w}) \times \mathbb{A}^{N-1}$ is regular. □

Proposition 3.20. *Let X be a regularizable DM stack and let $\mathcal{X}$ be a regular affine family containing X as a fiber. Then every fiber is a regularizable DM stack and the genus-zero Gromov–Witten theory is independent of the fiber.* □

Remark 3.21. A special feature of genus-zero Gromov–Witten theory is that we do not need a globally-defined torus action on the affine family to apply the Equivariant Regular Specialization Theorem 1.24: a torus action on the central fiber is enough, as soon as the normal bundle of the central fiber is torus-equivariant.

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