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# ON THE GEOMETRY OF DIFFEOLOGICAL VECTOR PSEUDO-BUNDLES AND INFINITE DIMENSIONAL VECTOR BUNDLES: AUTOMORPHISMS, CONNECTIONS AND COVARIANT DERIVATIVES

JEAN-PIERRE MAGNOT

**ABSTRACT.** We consider here the category of diffeological vector pseudo-bundles, and study a possible extension of classical differential geometric tools on finite dimensional vector bundles, namely, the group of automorphisms, the frame bundle, the space of connection 1-forms and the space of covariant derivatives. Substantial distinctions are highlighted in this generalized framework, among which the non-isomorphism between connection 1-forms and covariant derivatives. Applications not only include finite dimensional examples with singularities, but also infinite dimensional vector bundles whose base is not a smooth manifold with atlas, pushing the definitions over the classical framework of infinite dimensional geometry.

*Keywords:* diffeological spaces, vector pseudo-bundles, automorphisms, differential operators, connexions.

*MSC (2020):* 58A05, 53C15

## 1. INTRODUCTION

The extension of the classical framework of differential geometry to examples of interest is a long-term goal since Sophus Lie, Elie Cartan and Emmy Noether, methods got deeper and deeper insight in order to deal with more and more complex frameworks.

We consider here the setting of vector pseudo-bundles, a terminology which first appears in [22] to our knowledge but the objects under consideration are of interest since a much longer time. The point is to consider “bundles” which typical fiber can depend on the basepoint, in the category of vector spaces. Ekatarina Pervova [47, 49, 48, 50] considered finite dimensional vector pseudo-bundles in the category of diffeological spaces. Diffeological spaces, first defined by Souriau and Chen, is actually a not so badly developed framework for differential geometry, see the textbook [25] which is now a not-so-up-to-date treatise of the basic concepts of the subject. This category includes, as subcategories, all the other categories of interest for differential geometry: finite dimensional manifolds, Hilbert, Banach or Fréchet manifolds, orbifolds. This is the category that we choose to deal with.

This work starts with the necessary preliminaries on diffeologies, in which we give some examples that appear to us of interest, some of them are already disseminated in the recent literature, and few of them are new to our knowledge. Then it is necessary to extend first the notion of group of automorphisms to the category of diffeological vector pseudo-bundles. The classical short exact sequence that decomposes the group of automorphisms of a finite dimensional vector bundle, see e.g.

[45], extends to the group of automorphisms of a diffeological vector pseudo-bundle. It enables us to reduce the diffeology  $\mathcal{P}$  of the diffeological vector pseudo-bundle  $E$  to a weaker diffeology  $\mathcal{P}_{Aut(E)}$  which separates in different connected components the various typical fibers of  $E$ , that is, a connected component of  $E$  in  $\mathcal{P}_{Aut(E)}$  is a diffeological vector bundle (with typical fiber, but maybe without local trivialization).

With this new diffeology, we show that the definition of the frame bundle and of connections 1-forms extends quite straightway, as well as the bundle of vector space endomorphisms  $End(E)$ . But there is still a problem: recover a diffeology on all these objects that depends on  $\mathcal{P}$  and not on the restricted diffeology  $\mathcal{P}_{Aut(E)}$ . In this perspective, we only manage to “link again”  $End(E)$  by bridging together the disconnected parts of  $\mathcal{P}_{Aut(E)}$  through the plots of  $\mathcal{P}$ , and we remark that we can define a generalized notion of covariant derivatives, that for an affine space parametrized by  $End(E)$ , while connection 1-forms of the frame bundle have very different aspects.

After all these descriptions, an outlook section seems very necessary to us, in order to highlight how our new objects fit with existing constructions. In this last section, we do not have the ambition to be exhaustive, but only to show how this generalized framework fits with a selected family of examples of interest for us.

## 2. PRELIMINARIES ON DIFFEOLOGIES AND VECTOR PSEUDO-BUNDLES

This section provides the necessary background on diffeology and vector pseudo-bundles, mostly following [38, 41] in the exposition while the main reference for diffeologies remains [25]. A complementary non exhaustive bibliography is [3, 4, 5, 8, 9, 10, 12, 14, 15, 24, 30, 47, 53].

**2.1. Basics of Diffeology.** In this subsection we review the basics of the theory of diffeological spaces; in particular, their definition, categorical properties, as well as their induced topology. The main idea of diffeologies (and Frölicher spaces defined shortly after) is to replace the atlas of a classical manifold by other intrinsic objects that enable to define smoothness of mappings in a safe way, considering manifolds as a restricted class of examples. Many such settings have been developed independently. We choose these two settings because they carry nice properties such as cartesian closedness, carrying the necessary fundamental properties of e.g. calculus of variations, and also because they are very easy to use in a differential geometric way of thinking. The fundamental idea of these two settings consists in defining families of smooth maps, with mild conditions on them that ensure technical features of interest.

**Definition 2.1** (Diffeology). *Let  $X$  be a set. A **parametrisation** of  $X$  is a map of sets  $p: U \rightarrow X$  where  $U$  is an open subset of Euclidean space (no fixed dimension). A **diffeology**  $\mathcal{P}$  on  $X$  is a set of parametrisations satisfying the following three conditions:*

- (1) (Covering) *For every  $x \in X$  and every non-negative integer  $n$ , the constant function  $p: \mathbb{R}^n \rightarrow \{x\} \subseteq X$  is in  $\mathcal{P}$ .*
- (2) (Locality) *Let  $p: U \rightarrow X$  be a parametrisation such that for every  $u \in U$  there exists an open neighbourhood  $V \subseteq U$  of  $u$  satisfying  $p|_V \in \mathcal{P}$ . Then  $p \in \mathcal{P}$ .*

(3) (*Smooth Compatibility*) Let  $(p: U \rightarrow X) \in \mathcal{P}$ . Then for every  $n$ , every open subset  $V \subseteq \mathbb{R}^n$ , and every smooth map  $F: V \rightarrow U$ , we have  $p \circ F \in \mathcal{P}$ .

A set  $X$  equipped with a diffeology  $\mathcal{P}$  is called a **diffeological space**, and is denoted by  $(X, \mathcal{P})$ . When the diffeology is understood, we will drop the symbol  $\mathcal{P}$ . The parametrisations  $p \in \mathcal{P}$  are called **plots**.

**Notation.** We recall that  $\mathbb{N}^* = \{n \in \mathbb{N} \mid n \neq 0\}$  and that  $\forall m \in \mathbb{N}^*, \mathbb{N}_m = \{1, \dots, m\} \subset \mathbb{N}$ .

**Definition 2.2** (Diffeologically Smooth Map). Let  $(X, \mathcal{P}_X)$  and  $(Y, \mathcal{P}_Y)$  be two diffeological spaces, and let  $F: X \rightarrow Y$  be a map. Then we say that  $F$  is **diffeologically smooth** if for any plot  $p \in \mathcal{P}_X$ ,

$$F \circ p \in \mathcal{P}_Y.$$

Diffeological spaces with diffeologically smooth maps form a category. This category is complete and co-complete, and forms a quasi-topos (see [2]).

**Proposition 2.3.** [51, 25] Let  $(X', \mathcal{P})$  be a diffeological space, and let  $X$  be a set. Let  $f: X \rightarrow X'$  be a map. We define  $f^*(\mathcal{P})$  the **pull-back diffeology** as

$$f^*(\mathcal{P}) = \{p: O_p \rightarrow X \mid f \circ p \in \mathcal{P}\}.$$

**Proposition 2.4.** [51, 25] Let  $(X, \mathcal{P})$  be a diffeological space, and let  $X'$  be a set. Let  $f: X \rightarrow X'$  be a map. We define  $f_*(\mathcal{P})$  the **push-forward diffeology** as the coarsest (i.e. the smallest for inclusion) among the diffeologies on  $X'$ , which contains  $f \circ \mathcal{P}$ .

**Definition 2.5.** Let  $(X, \mathcal{P})$  and  $(X', \mathcal{P}')$  be two diffeological spaces. A map  $f: X \rightarrow X'$  is called a **subduction** if  $\mathcal{P}' = f_*(\mathcal{P})$ .

. In particular, we have the following constructions.

**Definition 2.6** (Product Diffeology). Let  $\{(X_i, \mathcal{P}_i)\}_{i \in I}$  be a family of diffeological spaces. Then the **product diffeology**  $\mathcal{P}$  on  $X = \prod_{i \in I} X_i$  contains a parametrisation  $p: U \rightarrow X$  as a plot if for every  $i \in I$ , the map  $\pi_i \circ p$  is in  $\mathcal{P}_i$ . Here,  $\pi_i$  is the canonical projection map  $X \rightarrow X_i$ .

In other words, in last definition,  $\mathcal{P} = \bigcap_{i \in I} \pi_i^*(\mathcal{P}_i)$  and each  $\pi_i$  is a subduction.

**Definition 2.7** (Subset Diffeology). Let  $(X, \mathcal{P})$  be a diffeological space, and let  $Y \subseteq X$ . Then  $Y$  comes equipped with the **subset diffeology**, which is the set of all plots in  $\mathcal{P}$  with image in  $Y$ .

If  $X$  is a smooth manifold, finite or infinite dimensional, modelled on a complete locally convex topological vector space, we define the **nebulae diffeology**

$$\mathcal{P}(X) = \{p \in C^\infty(O, X) \text{ (in the usual sense)} \mid O \text{ is open in } \mathbb{R}^d, d \in \mathbb{N}^*\}.$$

## 2.2. Frölicher spaces.

**Definition 2.8.** • A **Frölicher space** is a triple  $(X, \mathcal{F}, \mathcal{C})$  such that

- $\mathcal{C}$  is a set of paths  $\mathbb{R} \rightarrow X$ ,
- A function  $f: X \rightarrow \mathbb{R}$  is in  $\mathcal{F}$  if and only if for any  $c \in \mathcal{C}$ ,  $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ ;
- A path  $c: \mathbb{R} \rightarrow X$  is in  $\mathcal{C}$  (i.e. is a **contour**) if and only if for any  $f \in \mathcal{F}$ ,  $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ .

• Let  $(X, \mathcal{F}, \mathcal{C})$  et  $(X', \mathcal{F}', \mathcal{C}')$  be two Frölicher spaces, a map  $f : X \rightarrow X'$  is **differentiable** (=smooth) if and only if one of the following equivalent conditions is fulfilled:

- $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$
- $f \circ \mathcal{C} \subset \mathcal{C}'$
- $\mathcal{F}' \circ f \subset \mathcal{F}$

Any family of maps  $\mathcal{F}_g$  from  $X$  to  $\mathbb{R}$  generate a Frölicher structure  $(X, \mathcal{F}, \mathcal{C})$ , setting [29]:

- $\mathcal{C} = \{c : \mathbb{R} \rightarrow X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \rightarrow \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$ .

One easily see that  $\mathcal{F}_g \subset \mathcal{F}$ . This notion will be useful in the sequel to describe in a simple way a Frölicher structure. A Frölicher space carries a natural topology, which is the pull-back topology of  $\mathbb{R}$  via  $\mathcal{F}$ . In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

Let us now compare Frölicher spaces with diffeological spaces, with the following diffeology  $\mathcal{P}_\infty(\mathcal{F})$  called "nebulae": Let  $O$  be an open subset of a Euclidian space;

$$\mathcal{P}_\infty(\mathcal{F})_O = \coprod_{p \in \mathbb{N}} \{f : O \rightarrow X; \mathcal{F} \circ f \subset C^\infty(O, \mathbb{R}) \text{ (in the usual sense)}\}$$

and

$$\mathcal{P}_\infty(\mathcal{F}) = \bigcup_O \mathcal{P}_\infty(\mathcal{F})_O,$$

where the latter union is extended over all open sets  $O \subset \mathbb{R}^n$  for  $n \in \mathbb{N}^*$ . With this construction, we get a natural diffeology when  $X$  is a Frölicher space. In this case, one can easily show the following:

**Proposition 2.9.** [32] *Let  $(X, \mathcal{F}, \mathcal{C})$  and  $(X', \mathcal{F}', \mathcal{C}')$  be two Frölicher spaces. A map  $f : X \rightarrow X'$  is smooth in the sense of Frölicher if and only if it is smooth for the underlying nebulae diffeologies.*

Thus, we can also state intuitively:

$$\text{smooth manifold} \Rightarrow \text{Frölicher space} \Rightarrow \text{Diffeological space}$$

With this construction, any complete locally convex topological vector space is a diffeological vector space, that is, a vector space for which addition and scalar multiplication is smooth. The same way, any finite or infinite dimensional manifold  $X$  has a nebulae diffeology, which fully determines smooth functions from or with values in  $X$ . We now finish the comparison of the notions of diffeological and Frölicher space following mostly [32, 53]:

**Theorem 2.10.** *Let  $(X, \mathcal{P})$  be a diffeological space. There exists a unique Frölicher structure  $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P})$  on  $X$  such that for any Frölicher structure  $(X, \mathcal{F}, \mathcal{C})$  on  $X$ , these two equivalent conditions are fulfilled:*

- (i) *the canonical inclusion is smooth in the sense of Frölicher  $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P}) \rightarrow (X, \mathcal{F}, \mathcal{C})$*
- (ii) *the canonical inclusion is smooth in the sense of diffeologies  $(X, \mathcal{P}) \rightarrow (X, \mathcal{P}_\infty(\mathcal{F}))$ .*

Moreover,  $\mathcal{F}_{\mathcal{P}}$  is generated by the family

$$\mathcal{F}_0 = \{f : X \rightarrow \mathbb{R} \text{ smooth for the usual diffeology of } \mathbb{R}\}.$$

We call **Frölicher completion** of  $\mathcal{P}$  the Frölicher structure  $(X, \mathcal{F}_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}})$ .

A deeper analysis of these implications has been given in [53]. The next remark is inspired on this work; it is based on [29, p.26, Boman's theorem].

**Remark 2.11.** We notice that the set of contours  $\mathcal{C}$  of the Frölicher space  $(X, \mathcal{F}, \mathcal{C})$  **does not** give us a diffeology, because a diffeology needs to be stable under restriction of domains. In the case of paths in  $\mathcal{C}$  the domain is always  $\mathbb{R}$ . However,  $\mathcal{C}$  defines a “minimal diffeology”  $\mathcal{P}_1(\mathcal{F})$  whose plots are smooth parameterizations which are locally of the type  $c \circ g$ , where  $g \in \mathcal{P}_{\infty}(\mathbb{R})$  and  $c \in \mathcal{C}$ . Within this setting, a map  $f : (X, \mathcal{F}, \mathcal{C}) \rightarrow (X', \mathcal{F}', \mathcal{C}')$  is smooth if and only if it is smooth  $(X, \mathcal{P}_{\infty}(\mathcal{F})) \rightarrow (X', \mathcal{P}_{\infty}(\mathcal{F}'))$  or equivalently smooth  $(X, \mathcal{P}_1(\mathcal{F})) \rightarrow (X', \mathcal{P}_1(\mathcal{F}'))$

We apply the results on product diffeologies to the case of Frölicher spaces and we derive very easily, (compare with e.g. [29]) the following:

**Proposition 2.12.** Let  $(X, \mathcal{F}, \mathcal{C})$  and  $(X', \mathcal{F}', \mathcal{C}')$  be two Frölicher spaces equipped with their natural diffeologies  $\mathcal{P}$  and  $\mathcal{P}'$ . There is a natural structure of Frölicher space on  $X \times X'$  which contours  $\mathcal{C} \times \mathcal{C}'$  are the 1-plots of  $\mathcal{P} \times \mathcal{P}'$ .

We can even state the result above for the case of infinite products; we simply take cartesian products of the plots or of the contours. We also remark that given an algebraic structure, we can define a corresponding compatible diffeological structure. For example, a  $\mathbb{R}$ -vector space equipped with a diffeology is called a diffeological vector space if addition and scalar multiplication are smooth (with respect to the canonical diffeology on  $\mathbb{R}$ ), see [25, 47, 49]. An analogous definition holds for Frölicher vector spaces. Other examples will arise in the rest of the text.

**Remark 2.13.** Frölicher,  $c^{\infty}$  and Gateaux smoothness are the same notion if we restrict to a Fréchet context, see [29, Theorem 4.11]. Indeed, for a smooth map  $f : (F, \mathcal{P}_1(F)) \rightarrow \mathbb{R}$  defined on a Fréchet space with its 1-dimensional diffeology, we have that  $\forall(x, h) \in F^2$ , the map  $t \mapsto f(x + th)$  is smooth as a classical map in  $C^{\infty}(\mathbb{R}, \mathbb{R})$ . And hence, it is Gateaux smooth. The converse is obvious.

**2.3. Quotient and subsets.** We give here only the results that will be used in the sequel.

We have now the tools needed to describe the diffeology on a quotient:

**Proposition 2.14.** let  $(X, \mathcal{P})$  be a diffeological space and  $\mathcal{R}$  an equivalence relation on  $X$ . Then, there is a natural diffeology on  $X/\mathcal{R}$ , denoted by  $\mathcal{P}/\mathcal{R}$ , defined as the push-forward diffeology on  $X/\mathcal{R}$  by the quotient projection  $X \rightarrow X/\mathcal{R}$ .

Given a subset  $X_0 \subset X$ , where  $X$  is a Frölicher space or a diffeological space, we can define on subset structure on  $X_0$ , induced by  $X$ .

- If  $X$  is equipped with a diffeology  $\mathcal{P}$ , we can define a diffeology  $\mathcal{P}_0$  on  $X_0$ , called **subset diffeology** [51, 25] setting

$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0\}.$$

- If  $(X, \mathcal{F}, \mathcal{C})$  is a Frölicher space, we take as a generating set of maps  $\mathcal{F}_g$  on  $X_0$  the restrictions of the maps  $f \in \mathcal{F}$ . In that case, the contours (resp. the induced diffeology) on  $X_0$  are the contours (resp. the plots) on  $X$  which image is a subset of  $X_0$ .

**Example 2.15.** Let  $X$  be a diffeological space. Let us denote by

$$X^\infty = \{(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \mid \{n \mid x_n \neq 0\} \text{ is a finite set}\}$$

Then this is a diffeological space, as a subset of  $X^\mathbb{N}$ . The quotient space  $X^\mathbb{N}/X^\infty$  is a diffeological space related to the settings of non standard analysis, see e.g. [1].

**2.4. Vector pseudo-bundles.** Let us now have a precise look at the notion of fiber bundle in classical (finite dimensional) fiber bundles. Fiber bundles, in the context of smooth finite dimensional manifolds, are defined by

- a smooth manifold  $E$  called total space
- a smooth manifold  $X$  called base space
- a smooth submersion  $\pi : E \rightarrow X$  called fiber bundle projection
- a smooth manifold  $F$  called typical fiber, because  $\forall x \in X, \pi^{-1}(x)$  is a smooth submanifold of  $E$  diffeomorphic to  $F$ .
- a smooth atlas on  $X$ , with domains  $U \subset X$  such that  $\pi^{-1}(U)$  is an open submanifold of  $E$  diffeomorphic to  $U \times F$ . We then get a system of local trivializations of the fiber bundle.

By the way, in order to be complete, a smooth fiber bundle should be the quadruple data  $(E, X, F, \pi)$  (because the definition of  $\pi$  and of  $X$  enables to find systems of local trivializations). For short, this quadruple setting is often denoted by the projection map  $\pi : E \rightarrow X$ .

There exists some diffeological spaces which carry no atlas, so that, the condition of having a system of smooth trivializations in a generalization of the notion of fiber bundles is not a priori necessary, even if this condition, which is additional, enables interesting technical aspects [37, pages 194-195]. So that, in a general setting, we do not need to assume the existence of local trivializations. Now, following [47], in which the ideas from [51, last section] have been developed to vector spaces, the notion of quantum structure has been introduced in [51] as a generalization of principal bundles, and the notion of vector pseudo-bundle in [47]. The common idea consists in the description of fibered objects made of a total (diffeological) space  $E$ , over a diffeological space  $X$  and with a canonical smooth bundle projection  $\pi : E \rightarrow X$  such as,  $\forall x \in X, \pi^{-1}(x)$  is endowed with a (smooth) algebraic structure, but for which we do not assume the existence of a system of local trivialization.

- (1) For a diffeological vector pseudo-bundle, the fibers  $\pi^{-1}(x)$  are assumed diffeological vector spaces, i.e. vector spaces where addition and multiplication over a diffeological field of scalars (e.g.  $\mathbb{R}$  or  $\mathbb{C}$ ) is smooth. We notice that [47] only deals with finite dimensional vector spaces.
- (2) For a so-called “structure quantique” (i.e. “quantum structure”) following the terminology of [51], a diffeological group  $G$  is acting on the right, smoothly and freely on a diffeological space  $E$ . The space of orbits  $X = E/G$  defines the base of the quantum structure  $\pi : E \rightarrow X$ , which generalizes the notion of principal bundle by not assuming the existence of local trivialization. In this picture, each fiber  $\pi^{-1}(x)$  is isomorphic to  $G$ .

From these two examples, we can generalize the picture.

**Definition 2.16.** Let  $E$  and  $X$  be two diffeological spaces and let  $\pi : E \rightarrow X$  be a smooth surjective map. Then  $(E, \pi, X)$  is a **diffeological fiber pseudo-bundle** if and only if  $\pi$  is a subduction.

Let us precise that we do not assume that there exists a typical fiber, in coherence with Pervova’s diffeological vector pseudo-bundles. We can give the following definitions along the lines of [41]:

**Definition 2.17.** *Let  $\pi : E \rightarrow X$  be a diffeological fiber pseudo-bundle. Then:*

- (1) *Let  $\mathbb{K}$  be a diffeological field.  $\pi : E \rightarrow X$  is a **diffeological  $\mathbb{K}$ -vector pseudo-bundle** if there exists:*

- *a smooth fiberwise map  $\cdot : \mathbb{K} \times E \rightarrow E$ ,*
- *a smooth fiberwise map  $+$  :  $E^{(2)} \rightarrow E$  where*

$$E^{(2)} = \coprod_{x \in X} \{(u, v) \in E^2 \mid (u, v) \in \pi^{-1}(x)\}$$

*equipped by the pull-back diffeology of the canonical map  $E^{(2)} \rightarrow E^2$ , such that  $\forall x \in X$ ,  $(\pi^{-1}(x), +, \cdot)$  is a diffeological  $\mathbb{K}$ -vector bundle. We say that  $E$  is a **diffeological vector bundle** if  $E$  is a diffeological vector pseudo-bundle whose fibers are all isomorphic as diffeological vector spaces.*

- (2)  *$\pi : E \rightarrow X$  is a **Souriau quantum structure** if it is a diffeological principal pseudo-bundle with diffeological gauge (pseudo-)bundle  $X \times G \rightarrow X$ .*

**2.5. Tangent and cotangent spaces.** There are actually two main definitions, (2) and (3) below, of the tangent space of a diffeological space  $(X, \mathcal{P})$ , while definition (1) of the tangent cone will be used in the sequel. These definitions are very similar to the definitions of the two tangent spaces in the  $c^\infty$ -setting given in [29].

The **internal tangent cone** extends straightway the definition of the tangent space of a smooth manifold by germs of paths (compare with the kinematic tangent space in [29]). This defines the internal tangent cone first described for Frölicher spaces in [14]. This approach seems sufficient for application purpose [21].

But there are other tangent spaces in the category of diffeological spaces: the diff-tangent space which is defined in [41] and that will be described in next parts of the exposition, the **internal tangent space** is defined in [23, 10], based on germs of paths. This second definition is necessary, and the internal tangent space differs from the internal tangent cone. Indeed, spaces of germs do not carry intrinsically a structure of abelian group. This remark was first formulated in the context of Frölicher spaces, see [14], and see [10] for the generalization to diffeologies. For this reason, one can complete the tangent cone into a vector space, called internal tangent space. This was performed in [10] via mild considerations on colimits in categories, and the **external tangent space** spanned by derivations. For finite dimensional manifolds these tangent spaces coincide.

The definition of the cotangent space  $T^*X$ , as well as the definition of the algebra of differential forms  $\Omega(X, \mathbb{K})$  does not carry so many ambiguities and is defined through pull-back on each domain of plots, see [25] and e.g. [34] for a comprehensive exposition.

**2.6. On examples of interest.** We give here a non-exhaustive list of examples where one can exhibit structures of diffeological vector bundles, and also diffeological vector pseudo-bundles, that we hope motivating for the reader.

**Example 2.18.** *Let  $P \in \mathbb{R}[X_1, X_2, \dots, X_n]$  and let us consider the algebraic variety  $M = \text{Ker}P \subset \mathbb{R}^n$ . Then  $M$  is a diffeological space and its various tangent spaces, as well as its cotangent space and its space of differential forms, can be vector*

pseudo-bundles. The example of  $P(X, Y) = XY$  is discussed extensively in e.g. [41].

**Example 2.19.** Let  $\tau$  be a triangulation of a smooth paracompact manifold  $N$ . The question of the diffeology associated to a triangulation is a long story which starts by Ntumba's PhD thesis, which principal results are published in [43], and which continues with e.g. [11, 36, 41]. In any of these works, where slightly different diffeologies are defined for a same triangulation, this is a simple exercise to check that  $\Omega^*$  is only a vector pseudo-bundle. A toy example remains again on  $\text{Ker}P$ , with  $P(X, Y) = XY$  already mentioned.

**Example 2.20.** Let  $E = \mathbb{R}^{\mathbb{N}}$  and let  $F = C_0(\mathbb{N}, \mathbb{R})$  be the set of sequences that converge to 0. We equip these spaces with the topology of uniform convergence. It is well-known that  $F$  has no topological complement in  $E$ , and that  $F$  acts on  $E$  by translation. Then the quotient map  $E \rightarrow E/F$  defines a diffeological vector bundle with typical fiber  $F$ . The existence of local slices for the  $D$ -topology of  $E/F$  is actually unknown, while this setting furnishes a classical counter-example in the setting of topological vector spaces.

**Example 2.21.** Following [41], let  $X$  be a topological vector space and let  $Y$  be a Fréchet space which is the completion of  $X$ . Let  $F : X \rightarrow \mathbb{R}$  be a functional on  $X$  and let  $C(X, Y)$  be the vector space of sequences in  $X$  that converge in  $Y$ . Then we define the  $Y$ -weak solutions of the functional equation

$$F(x) = 0$$

as the set

$$C_Y = \{\lim u_n \mid (u_n) \in C(X, Y) \text{ and } \lim F(u_n) = 0\}.$$

The space  $C(X, Y)$  has a dedicated diffeology called *Cauchy diffeology* which ensures smoothness of the limit. Therefore, the limit map

$$\lim : C_Y \rightarrow Y$$

defines a diffeological pseudo-fiber bundle with total space  $C_Y$ , with base  $S_Y = \lim C_Y$  and with fibers which are all diffeomorphic to diffeological subspaces of  $C_0(X, Y)$ , the set  $X$ -sequences that converge to 0 in  $Y$ . Under mild conditions on the functional  $F$ , the fibers can be proved to be vector spaces.

**Example 2.22.** Let  $N$  be a finite dimensional Riemannian manifold. From [16], the spaces of maps  $H^s(S^1, N)$  are Hilbert manifolds when  $s > 1/2$ . Following [39],  $H^s(S^1, N)$  is a diffeological space for  $s \leq 1/2$ , with  $C^0$ -homotopy type which may depend on the value of  $s$  (compare also with [42]). Therefore, the study of their tangent and cotangent space involve diffeological vector bundles which actually have no known local trivialization. These objects are interesting in optimization theory as highlighted in [54].

### 3. TECHNICAL CONSTRUCTIONS: $G$ -DIFFEOLGY AND ISOMORPHISMS OF VECTOR PSEUDO-BUNDLES

For this section, we need to recall the following definitions from [25]:

**Definition 3.1.** Let  $(X, \mathcal{P})$  and  $(X', \mathcal{P}')$  be two diffeological spaces. Let  $S \subset C^\infty(X, X')$  be a set of smooth maps. The **functional diffeology** on  $S$  is the diffeology  $\mathcal{P}_S$  made of plots

$$\rho : D(\rho) \subset \mathbb{R}^k \rightarrow S$$

such that, for each  $p \in \mathcal{P}$ , the maps  $\bar{\Phi}_{\rho,p} : (x, y) \in D(p) \times D(\rho) \mapsto \rho(y)(x) \in X'$  are plots of  $\mathcal{P}'$ .

With this definition, we have the classical fundamental property for calculus of variations and for composition:

**Proposition 3.2.** [25] *Let  $X, Y, Z$  be diffeological spaces*

- (1)  $C^\infty(X \times Y, Z) = C^\infty(X, C^\infty(Y, Z)) = C^\infty(Y, C^\infty(X, Z))$   
 as diffeological spaces equipped with functional diffeologies.

- (2) *The composition map*

$$C^\infty(X, Y) \times C^\infty(Y, Z) \rightarrow C^\infty(X, Z)$$

*is smooth.*

### 3.1. The $G$ -diffeology and its tangent space.

**Definition 3.3.** *Let  $G$  be a diffeological group, and let  $\rho : G \times X \rightarrow X$  be an action on a set  $X$ . Then we note by  $\mathcal{P}_\rho(X)$ , or  $\mathcal{P}_G(X)$  when there is no ambiguity, the push-forward diffeology from  $G$  on  $X$ .*

This diffeology separates the orbits of  $G$  in  $X$ , that is, two disjoint orbits of  $G$  lie in different connected components of  $X$ . Therefore, we can define:

**Definition 3.4.** • *the Frölicher completion  $(X, \mathcal{F}_\rho, \mathcal{C}_\rho)$  of  $(X, \mathcal{P}_\rho)$*

- $\Omega_\rho^*(X)$  *the de Rham complex of  $(X, \mathcal{F}_\rho, \mathcal{C}_\rho)$*
- ${}^\rho TX$  *the internal tangent cone of  $(X, \mathcal{P}_\rho)$*

**Proposition 3.5.**  ${}^\rho TX$  *is a diffeological vector pseudo-bundle.*

*Proof.* As  ${}^\rho TX$  an internal tangent cone, we only have to prove that each fiber of  ${}^\rho TX$  over  $X$  is a (diffeological) vector space. Let  $x \in X$  and let  $(c_1, c_2) \in C^\infty(\mathbb{R}, X)^2$  such that  $c_1(0) = c_2(0) = x$ . Then there exists  $\epsilon > 0$  and two local paths  $(g_1, g_2) \in C^\infty(]-\epsilon, \epsilon[; G)^2$  such that  $\forall i \in \mathbb{N}_2, c_i|_{]-\epsilon, \epsilon[} = g_i \cdot x$ . Let  $X_i = \partial_t c_i|_{t=0}$ . Then

$$X_1 + X_2 = X_2 + X_1 = \partial_t(g_1 \cdot g_2) \cdot x|_{t=0} \in {}^\rho T_x X$$

(in fact, we apply the same arguments as for the tangent space of a diffeological group along the lines of [40], which expands the arguments of [31]).  $\square$

**Example 3.6** (The  $Diff$ -diffeology and the  $Diff$ -tangent space.). *Let  $X$  be a Frölicher space and let  $G = Diff(X)$  the group of diffeomorphisms of  $X$ . Following [41], the  $Diff$ -diffeology and the  $Diff$ -tangent space coincides with our construction in Definitions 3.3 and 3.4.*

**3.2. Isomorphisms of vector pseudo-bundles.** We now rephrase vector pseudo-bundle morphisms and isomorphisms described in [55] in a non-categorical vocabulary.

**Definition 3.7.** *Let  $\pi : E \rightarrow X$  and  $\pi' : F \rightarrow Y$  be two diffeological vector pseudo-bundles.  $(\phi, \varphi)$  is a morphism of vector pseudo-bundles from  $\pi : E \rightarrow X$  to  $\pi' : F \rightarrow Y$  if:*

- $\phi : E \rightarrow F$  *is smooth*
- $\varphi : X \rightarrow Y$  *is smooth*
- $\pi' \circ \phi = \varphi \circ \pi$

- $\forall x \in X$ , the restricted map  $\phi|_{\pi^{-1}(x)} : \pi^{-1}(x) \rightarrow \pi'^{-1}(\varphi(x))$  is linear (and also smooth in the subset diffeologies).

**Remark 3.8.** Following [55], if  $(\phi, \varphi) \in \text{Aut}(E)$ , then  $\varphi$  is the restriction of  $\phi$  to the zero-section of  $E$ .

Therefore,

**Proposition 3.9.** Let  $\pi : E \rightarrow X$  and  $\pi' : F \rightarrow Y$  be two diffeological vector pseudo-bundles. Let  $(\phi, \varphi)$  be a morphism of vector pseudo-bundles from  $\pi : E \rightarrow X$  to  $\pi' : F \rightarrow Y$ .  $(\phi, \varphi)$  is an isomorphism of vector pseudo-bundles if and only if  $\phi \in \text{Diff}(E, F)$ . If so, we have that:

- $\varphi \in \text{Diff}(X, Y)$
- $(\phi^{-1}, \varphi^{-1})$  is a morphism of vector pseudo-bundles.

*Proof.* Let  $x \in X$ . Let us consider  $\phi^{-1} \circ \phi|_{\pi^{-1}(x)}$ . Since  $\phi^{-1} \circ \phi = \text{Id}_E$ , we have  $\phi^{-1} \circ \phi|_{\pi^{-1}(x)} = \text{Id}_{\pi^{-1}(x)}$  therefore  $\phi^{-1}$  is fiberwise linear.

Moreover, we have that  $X$  is isomorphic to the zero-section of  $E$  and that  $Y$  is isomorphic to the zero-section of  $F$ . Therefore, the restriction of  $\phi^{-1}$  to the zero-section of  $E$  shows existence and smoothness of  $\varphi^{-1}$ . □

We remark the following, extending the notion of homotopy of vector bundles to vector pseudo-bundles:

**Lemma 3.10.** Let  $X$  be a diffeological vector space. Then any vector pseudo-bundle  $E$  over  $X$  is homotopic to the null-vector space  $X \times \{0\}$  in the category of vector pseudo-bundles, that is, there exists a homotopy pseudo-bundle  $H$  over  $X \times [0; 1]$  such that  $H|_{X \times \{0\}}$  is isomorphic to  $E$  and  $H|_{X \times \{1\}}$  is isomorphic to  $X \times \{0\}$ .

*Proof.* We set

$$H = (E \times (X \times [0; 1])) / \sim$$

where the equivalence relation  $\sim$  identifies  $E \supset X \ni x \sim (x, 0) \in X \times [0; 1]$ . Equipped with the quotient diffeology, this is a vector pseudo-bundle which ends the proof. □

Therefore, isomorphic vector pseudo-bundles cannot be characterized by homotopies.

#### 4. ON THE GROUP OF AUTOMORPHISMS AND ON THE FRAME BUNDLE OF A VECTOR PSEUDO-BUNDLE

Let  $\pi : E \rightarrow X$  be a vector pseudo-bundle. We note by  $E_x$  the fiber  $\pi^{-1}(x)$ , for  $x \in X$ . We also note by  $GL(E)$  the gauge pseudo-bundle of endomorphisms of  $E$ , that is, the group of automorphisms of  $E$  which decompose fiberwise as an automorphism of each fiber.

##### 4.1. On the group of automorphisms.

**Definition 4.1.** We note by  $\text{Aut}(E)$  the set of automorphisms of the vector pseudo-bundle  $E$ .

**Theorem 4.2.** *Aut(E) is a diffeological group, as a diffeological subgroup of Diff(E). Moreover, there is a short exact sequence*

$$0 \rightarrow GL(E) \rightarrow Aut(E) \rightarrow G_E(X) \rightarrow 0$$

where  $G_E(X)$  is a diffeological subgroup of  $Diff(X)$ .

*Proof.* From Proposition 3.9, we already have that  $Aut(E)$  is a diffeological subgroup and that the projection  $Aut(E) \rightarrow G_E(X)$  is a morphism of diffeological groups. If  $(\phi, Id_X) \in Aut(E)$ , then  $\phi$  is fiberwise a diffeological vector space isomorphism, which ends the proof.  $\square$

Let us analyze better the group  $G_E(X)$ .

**Lemma 4.3.** *Let  $(x, y) \in X^2$  such that  $\exists(\phi, \varphi) \in Aut(E)$ ,  $y = g(x)$ . Then  $\phi|_{E_x}$  is an isomorphism from  $E_x$  to  $E_y$ .*

The proof of this easy lemma is straightforward, but has deep consequences, for which proofs are straightforward and to our opinion needless.

**Proposition 4.4.** *If  $G_E(X)$  acts transitively on  $X$ , then  $E$  has a typical fiber, i.e.  $\forall(x, y) \in X^2$ ,  $E_x$  is isomorphic to  $E_y$  as diffeological vector spaces.*

**Proposition 4.5.** *The vector pseudo-bundle  $(E, \mathcal{P}_{Aut(E)})$  has connected components with typical fiber.*

**4.2. Frame bundles for diffeological vector bundles.** We define here the geometric objects related to frame bundles of diffeological vector bundles, along the lines of the classical constructions for frames for finite or infinite dimensional vector bundles, see e.g. [33].

**Definition 4.6.** *Let  $\pi : E \rightarrow X$  be a diffeological vector space with typical fiber  $F$ . Let  $x \in X$  and let  $E_x$  be the fiber over  $x$ . We define the space*

$$Fr(E_x)$$

as the diffeological space of isomorphisms of diffeological vector spaces from  $F$  to  $E_x$ , and we define the **frame bundle** of  $E$  by:

$$Fr(E) = \coprod_{x \in X} Fr(E_x).$$

We denote by  $GL(F)$  the (diffeological) group of (diffeological) isomorphisms of the diffeological vector space  $F$ . With this notation, and considering the composition of diffeological linear maps, we remark that:

- $GL(F)$  acts on the right on  $Fr(E)$  and,  $\forall x \in X$ , on  $Fr(E_x)$ ,
- $GL(E)$  acts on the left on  $Fr(E)$ , and,  $\forall x \in X$ , through the canonical inclusion map  $GL(E_x) \hookrightarrow GL(E)$ ,  $GL(E)$  and  $GL(E_x)$  are acting on the left on  $Fr(E_x)$ ,
- The group  $Aut(E)$  is acting on the left on  $Fr(E)$ .

Then, applying Definition 3.3, we get:

**Lemma 4.7.**  *$(Fr(E), \mathcal{P}_{Aut(E)}(Fr(E)))$  is a diffeological space.*

With this diffeology, the action of  $Aut(E)$  on  $Fr(E)$  is smooth, and we moreover have:

**Theorem 4.8.** *The diffeological space  $Fr(E)$  is:*

- (1) a Souriau quantum structure, with respect to the group  $GL(F)$ ,
- (2) a diffeological fiber bundle, with typical fiber  $GL(F)$ .

*Proof.* Given two frames  $f_1$  and  $f_2$  in  $Fr(E_x)$  we remark that

- $f_2 \circ f_1^{-1} \in GL(E_x)$
- $f_1^{-1} \circ f_2 \in GL(F)$

therefore, it is straightforward to check that, on  $Fr(E_x)$ , the left action of  $GL(E_x)$  and the right action of  $GL(F)$  are free and transitive, which shows (1). Moreover, the map  $f \in Fr(E) \mapsto f_1^{-1} \circ f$  identifies  $Fr(E_x)$  with  $GL(F)$ , which shows (2).  $\square$

## 5. ON DIFFERENTIAL FORMS AND CONNECTIONS

**5.1. Differential forms with values in vector pseudo-bundles.** Let us consider the de Rham complexes  $\Omega(E, \mathbb{R})$  and  $\Omega(M, \mathbb{R})$ . As diffeological subgroups of  $Diff(E)$ ,  $Aut(E)$  and  $GL(E)$  are acting on  $\Omega(E, \mathbb{R})$  and the same way,  $G_E(X)$  is acting on  $\Omega^*(X)$ . Let us precise this action. For a plot  $p : U \rightarrow X$ , differential form  $\alpha \in \Omega^*(X)$  pulls-back, by definition, to the (classical) differential form  $p^*\alpha \in \Omega^*(U)$ . Given  $g \in Diff(X)$ , then  $q = g \circ p$  is also a plot of  $X$  and hence,  $g \in Diff(X)$  transforms  $p^*\alpha$  to  $q^*\alpha$ . The same construction holds for  $Diff(E)$ ,  $Aut(E)$ ,  $G_E(X)$  for their corresponding target spaces. In all this section, and at each time where we appeal to the notions that are developed here,  $E$  is equipped with its  $Aut(E)$ -diffeology and  $M$  with its  $G_E(X)$ -diffeology. Let  $p \in \mathcal{P}_{G_E(M)}$ . Let  $O_p$  be the domain of  $p$  and let

$$\alpha_p : \wedge^n TO \rightarrow E$$

be a smooth map such that, if  $u \in O_p$  and  $x = p(u)$ , then  $\alpha_p$  restricts to a skew-symmetric  $n$ -linear map from  $T_u^n O_p$  to  $E_x$ . With these notations, one can define a  $n$ -form  $\alpha \in \Omega^n(M, E)$ . A  $E$ -valued  $n$ -differential form  $\alpha$  on  $X$  (noted  $\alpha \in \Omega^n(X, E)$ ) is a map

$$\alpha : \{p : O_p \rightarrow X\} \in \mathcal{P} \mapsto \alpha_p$$

(where  $\alpha_p$  has the property described before) such that

- Let  $x \in X$ .  $\forall p, p' \in \mathcal{P}$  such that  $x \in Im(p) \cap Im(p')$ , the forms  $\alpha_p$  and  $\alpha_{p'}$  are of the same order  $n$ .
- Moreover, let  $y \in O_p$  and  $y' \in O_{p'}$ . If  $(X_1, \dots, X_n)$  are  $n$  germs of paths in  $Im(p) \cap Im(p')$ , if there exists two systems of  $n$ -vectors  $(Y_1, \dots, Y_n) \in (T_y O_p)^n$  and  $(Y'_1, \dots, Y'_n) \in (T_{y'} O_{p'})^n$ , if  $p_*(Y_1, \dots, Y_n) = p'_*(Y'_1, \dots, Y'_n) = (X_1, \dots, X_n)$ ,

$$\alpha_p(Y_1, \dots, Y_n) = \alpha_{p'}(Y'_1, \dots, Y'_n).$$

We note by

$$\Omega(X; E) = \bigoplus_{n \in \mathbb{N}} \Omega^n(X, E)$$

the set of  $E$ -valued differential forms. With such a definition, we feel the need to make two remarks for the reader:

- If there does not exist  $n$  linearly independent vectors  $(Y_1, \dots, Y_n)$  defined as in the last point of the definition,  $\alpha_p = 0$  at  $y$ .
- Let  $(\alpha, p, p') \in \Omega(X, V) \times \mathcal{P}^2$ . If there exists  $g \in C^\infty(D(p); D(p'))$  (in the usual sense) such that  $p' \circ g = p$ , then  $\alpha_p = g^* \alpha_{p'}$ .

**Proposition 5.1.** *The set  $\mathcal{P}(\Omega^n(X, E))$  made of maps  $q : x \mapsto \alpha(x)$  from an open subset  $O_q$  of a finite dimensional vector space to  $\Omega^n(X, E)$  such that for each  $p \in \mathcal{P}$ ,*

$$\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, E)),$$

*is a diffeology on  $\Omega^n(X, E)$ .*

Therefore, one can define the de Rham exterior differential, for  $n \in \mathbb{N}$ ,

$$d : \Omega^n(X, E) \rightarrow \Omega^{n+1}(X, E)$$

by its expression on each plots  $p$  of the diffeology.

**Remark 5.2.** *We have here to warn about the temptation of a straightforward extension of the classical wedge product of differential forms which only exists if  $E$  is fiberwise equipped with a smooth fiberwise multiplication. Therefore, the de Rham differential can only be defined as a linear map, and not as a graded differential, in this setting.*

The following proposition is straightforward from the definition of the diffeologies of  $\text{Aut}(E)$  and of  $G_E(X)$ .

**Proposition 5.3.** *The group  $\text{Aut}(E)$  is acting smoothly on  $\Omega(X, E)$  the following way: let  $\alpha \in \Omega(M, E)$  and let  $g = (\phi, \varphi) \in \text{Aut}(E)$ , we define  $g_*\alpha$  on each  $p \in \mathcal{P}$  by*

$$(g_*\alpha)_p = \phi \circ \alpha_{\varphi^{-1} \circ p}.$$

## 5.2. Connections and covariant derivatives on a vector pseudo-bundle.

Classically, given a principal bundle  $P$  with structure group  $G$  with Lie algebra  $\mathfrak{g}$ , connections are 1-forms  $\theta \in \Omega^1(P, \mathfrak{g})$  (see [25, 34]) that are covariant under the right action of  $G$ , that is, denoting the right action of  $g \in G$  by  $R_g$ , we have:

$$(R_g)_*\theta = \text{Ad}_{g^{-1}}\theta.$$

In this framework, the de Rham differential and wedge product can be defined by using classically the Lie bracket of  $\mathfrak{g}$ . Here, our principal bundle is  $Fr(E)$ , which may not have only one structure group since the typical fiber of  $E$  may vary, depending on the connected components of  $(E, \mathcal{P}_{\text{Aut}(E)})$ , see Lemma 8. Therefore, for this section, one can restrict the study to vector bundles, that is, vector pseudo-bundles with typical fiber  $F$ . We note by  $\mathfrak{gl}(F)$  the tangent space at identity of  $GL(F)$ , and we note by  $L(F)$  the space of endomorphisms of the diffeological vector space  $F$ .

**Remark 5.4.** *In our framework, we are not sure that  $L(E) = \mathfrak{gl}(E)$ , while this is a classical fact when  $E$  is a finite dimensional trivial vector bundle over a compact manifold. To our knowledge the best framework where one can find such properties when  $L(F)$  is an infinite dimensional algebra can be found in [18, 19]. The question of the comparison of  $L(F)$  with  $\mathfrak{gl}(F)$  remains open in the fully general framework. Moreover,  $\mathfrak{gl}(F)$  is not a priori a set of endomorphisms of the diffeological vector space  $F$  since  $F$  is not assumed to carry a complete topology which ensures an adequate convergence property. This fact will be illustrated through an example.*

Therefore, one can define, along the lines of the discussion present in e.g. [33]:

**Definition 5.5.** *We define, for  $x \in M$ ,*

$$\text{End}_L(E)_x = \{e \in C^\infty(E_x, E_x) \mid e \text{ is linear}\},$$

and let

$$End_L(E) = \prod_{x \in M} End_L(E)_x.$$

**Proposition 5.6.**

$$End_L(E) = \{e = f \circ l \circ f^{-1} \mid f \in Fr(E) \text{ and } l \in L(F)\}$$

*Proof.* Let  $(f, l) \in Fr(E) \times L(F)$ . Then  $f \circ l \circ f^{-1}$  is smooth and linear by smoothness of the composition in the functional diffeology. Conversely, let  $e \in End_L(E)_x$ , and let us fix any  $f \in Fr(E)_x$ , then  $l = f \circ e \circ f^{-1} \in L(F)$  for the same reasons.  $\square$

**Definition 5.7.** We equip  $End_L(E)$  with the push-forward diffeology  $\mathcal{P}_{L, Aut}$  of the map:

$$ReF : (f, l) \in Fr(E) \times L(F) \mapsto f \circ l \circ f^{-1}.$$

**Proposition 5.8.** For the diffeology  $\mathcal{P}_{L, Aut}$ , let

$$D_L = \prod_{x \in M} E_x \times End(E)_x$$

equipped with its subset diffeology in  $E \times End(E)$ . Then the evaluation map

$$ev : (x, e) \in D_L \mapsto ev_x(e) = e(x)$$

is smooth.

*Proof.* In order to check the smoothness of the evaluation map, it is sufficient to show that

$$(x, f, l) \in E \times Fr(E) \times L(F) \mapsto f \circ l \circ f^{-1}(x)$$

is smooth, which is true since composition is smooth.  $\square$

Therefore, one has two possible generalizations of the notion of connection to a diffeological pseudobundle:

- either connection 1-forms associated to the  $GL(F)$ -principal bundle which are  $GL(F)$ -covariant 1-forms in  $\Omega^1(Fr(E), \mathfrak{gl}(F))$ , and we note this set of connections by  $\mathcal{C}_{\mathfrak{gl}}(E)$ ,
- or connection 1-forms that lie in  $\Omega^1(M, End_L(E))$ .

The space  $\mathcal{C}_{\mathfrak{gl}}(E)$  seems to be the most natural for a straightforward generalization of the theory of connections in a principal bundle, and we have to explain why we highlight also the second framework. For this, we have to propose a generalization of the notion of covariant derivative. But before that we need to find a way back to the initial diffeology  $\mathcal{P}$  of  $E$ , since all our constructions are performed till now in the weaker diffeology  $\mathcal{P}_{Aut(E)}$ .

## 6. RECOVERING THE INITIAL DIFFEOLOGY

The point of the diffeology  $\mathcal{P}_{Aut(E)}$  consists in making different connected components the domains where the fibers of  $E$  are not isomorphic, while the initial diffeology  $\mathcal{P}$ , which is stronger than  $\mathcal{P}_{Aut(E)}$ , may have non-isomorphic fibers in the same connected component. For this, we need to define first a diffeology  $\mathcal{P}_D$  on  $D_E$  which completes already defined. We require for this diffeology that

- for the canonical projection

$$\pi_1 : D_E = \coprod_{x \in M} E_x \times \text{End}(E)_x \rightarrow E$$

- satisfies  $(\pi_1)_*(\mathcal{P}_D) = \mathcal{P}$
- the evaluation map  $ev : (D_E, \mathcal{P}_D) \rightarrow (E, \mathcal{P})$  is smooth.

Therefore, a natural diffeology can be defined.

**Definition 6.1.** We define  $\mathcal{P}_D$  the diffeology on  $D_E$  such that

$$\mathcal{P}_D = ev^*(\mathcal{P}) \cap \pi_1^*(\mathcal{P}).$$

Then, a natural diffeology arises also for  $\text{End}(E)$  :

**Definition 6.2.** Let  $\mathcal{P}_L$  be the diffeology on  $\text{End}(E)$  defined by

$$\mathcal{P}_L = (\pi_2)_*(\mathcal{P}_D),$$

where  $\pi_2$  is the second coordinate projection

$$\pi_2 : D_E = \coprod_{x \in M} E_x \times \text{End}(E)_x \rightarrow \text{End}(E).$$

Obviously,  $\text{End}(E)$  is a vector pseudo-dundle which fibers are all diffeological algebras, but we can state a little more:

**Proposition 6.3.** Addition and multiplication are smooth maps from  $\text{End}(E)^{(2)}$  to  $\text{End}(E)$ .

*Proof.* Follows from the smoothness of addition in  $E$ . □

## 7. COVARIANT DERIVATIVES AND CONNECTIONS

**Definition 7.1.** A covariant derivative  $\nabla$  on  $E$  is a collection of covariant derivatives  $(p^*\nabla)_{p \in \mathcal{P}}$  such that

- $\forall p \in \mathcal{P}$  with domain  $O_p$ ,  $p^*\nabla$  is a covariant derivative on  $p^*E$ , that is,

$$p^*\nabla : TO_p \times \Gamma(O_p, p^*E) \rightarrow p^*E$$

is a smooth fiberwise bilinear map.

- $\forall p \in \mathcal{P}$  with domain  $O_p$ ,  $\forall f \in C_c^\infty(O_p, \mathbb{R}), \forall X, Y \in TO_p \times \Gamma(O_p, p^*E)$ ,

$$p^*\nabla_{fX}Y = fp^*\nabla_XY$$

and

$$p^*\nabla_X(fY) = D_x fY + fp^*\nabla_XY.$$

- if  $(p, p') \in \mathcal{P}^2$ , with domains  $O_p$  and  $O_{p'}$  such that there is a smooth map  $g : O_{p'} \rightarrow O_p$  that satisfies  $p' = p \circ g$  then

$$(p')^*\nabla = g^*(p^*\nabla).$$

We note by  $\mathcal{CD}(E)$  the space of covariant derivatives on  $E$ .

**Theorem 7.2.** If  $\mathcal{CD}(E)$  is non empty, then  $\mathcal{CD}(E)$  is a diffeological affine space modelled on  $\Omega^1(M, E)$ .

*Proof.* The proof follows the standard proof for covariant derivatives in the category of (finite dimensional) vector bundles. We evaluate the conariant derivatives on each plot  $p \in \mathcal{P}$ . Let  $\nabla^1$  and  $\nabla^2$  be two covariant derivatives. Then,  $\forall p \in \mathcal{P}$  with domain  $O_p$ ,  $\forall f \in C_c^\infty(O_p, \mathbb{R})$ ,  $\forall X, Y \in TO_p \times \Gamma(O_p, p^*E)$ ,

$$p^*\nabla_X^{(1)}(fY) - p^*\nabla_X^{(2)}(fY) = fp^*\nabla_X Y,$$

and since  $O_p$  is an open subset of an Euclidian space, it is sufficient to show that  $p^*\nabla_X^{(1)} - p^*\nabla_X^{(2)}$  is a fiberwise multiplication operator. Therefore,  $p^*\nabla^{(1)} - p^*\nabla^{(2)} \in \Omega^1(M, E)$ .

Conversely, let  $\alpha \in \Omega^1(M, E)$ , let  $\nabla \in \mathcal{CD}(E)$ , then we have analyze  $\nabla' = \nabla + \alpha$ . A straightforward computation on each  $p \in \mathcal{P}$  shows that the desired relations are fulfilled.  $\square$

## 8. OUTLOOK

The structures that we highlight here seem to be present but forgotten in more standard settings. Indeed,

- the existence of the group  $G_E(X)$ , as a subgroup of  $Diff(X)$ , is already known for classical finite dimensional vector bundles see e.g. [45], but it seems to be defined and not described in the common litterature. In the more general setting of vector pseudo-bundles, we can show here, through Lemma , that  $G_E(X) \neq Diff(X)$ . Moreover, through the existence of the diffeology  $\mathcal{P}_{Aut(E)}$ , one can guess that the following open question needs an answer:

**Open question:** in which case and for which diffeologies do we have  $T_{Id_M}G_E(X) = T_{Id_M}Diff(X)$  and  $T^*_{Id_M}G_E(X) = T^*_{Id_M}Diff(X)$ ?

- the classical frame bundle  $Fr(E)$ , that we have constructed along the lines of the theory of principal bundles associated to finite dimensional fiber or vector bundles can be extended in the diffeological vector pseudo-bundles setting, seemingly, only with respect to the “restricted” diffeology  $\mathcal{P}_{Aut(E)}$  and not with respect to the diffeology  $\mathcal{P}$  of  $E$ .
- the notion of covariant derivative then seems somewhat disconnected to the notion of connection 1-form for diffeological vector pseudo-bundles because we showed how one can naturally extend the classical notion of a covariant derivative taking into account the diffeology  $\mathcal{P}$  of  $E$ .
- Since our constructions do not assume at any step that the objects are finite dimensional, the constructions remain valid for infinite dimensional vector bundles modelled on complete locally convex vector spaces  $F$ . In this setting, if one does not assume a restricted class of objects (typically, ILB objects along the lines of [46, 13]), composition of endomorphisms is hypocontinuous and leads to technical difficulties pointed our principally by Glöckner (see e.g. the recent review in [20]) while assuming only diffeological smoothness allows to state that composition is smooth in  $L(F)$  and hence that  $GL(F)$  is a smooth diffeological subgroup of  $Diff(F)$ . This idea was already present, at least intuitively, in the ILB setting, when [13] use constructions via tame maps in order to circumvent the statement of smoothness of the composition and of the inverse map in a structure group for ILB vector bundles, higlighted in [34] as a Frölicher Lie group. In this direction, diffeologies seem to simplify topological issues.

Therefore, the extension of the classical geometry of covariant derivatives along the diffeological framework here described seems to be possible. Future works will help us to decide how far.

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