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LIOUVILLE-TYPE THEOREMS FOR THE LANE-EMDEN EQUATION IN THE HALF-SPACE AND CONES

LOUIS DUPAIGNE, ALBERTO FARINA AND TROY PETITT

1. INTRODUCTION

Let $d \geq 1$ and $p > 1$. We consider the Lane-Emden equation

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}_+^d, \\ u = 0 & \text{on } \partial\mathbb{R}_+^d, \end{cases} \quad (1)$$

posed in the upper half-space $\mathbb{R}_+^d := \{x = (x', x_d) : x' \in \mathbb{R}^{d-1}, x_d \in (0, \infty)\}$ with homogeneous Dirichlet boundary conditions. It is conjectured that the above equation has only one nonnegative solution, $u = 0$.

Gidas and Spruck in [12] showed that this is indeed the case if $1 < p \leq p_S(d)$, where

$$p_S(d) = \frac{d+2}{(d-2)_+}$$

is Sobolev's critical exponent. In the case $p > p_S(d)$, only partial results are available: Dancer [4] considered bounded nonnegative solutions and proved in this case that $u = 0$ if $p < p_S(d-1)$. The second named author [10] improved Dancer's result showing that the only bounded nonnegative solution is $u = 0$ if $p < p_{JL}(d-1)$, where p_{JL} is the Joseph-Lundgren stability exponent given by

$$p_{JL}(d) := \frac{(d-2)^2 - 4d + 8\sqrt{d-1}}{(d-2)(d-10)_+}. \quad (2)$$

Finally Chen, Lin, and Zou [3] proved that no bounded nonnegative solution $u \neq 0$ of (1) exists for any $p > 1$. In all these results, the fact that any nonnegative bounded solution of (1) is monotone in the x_d -direction, i.e. $\partial u / \partial x_d > 0$ in \mathbb{R}_+^d , is crucially used. In fact, Sirakov, Souplet and the first named author proved in [7] that, more generally, no nontrivial monotone solution of (1) exists, whether bounded or not. Note that monotone solutions are stable, meaning they verify additionally that

$$p \int |u|^{p-1} \varphi^2 dx \leq \int |\nabla \varphi|^2 dx, \quad (3)$$

for all $\varphi \in C_c^1(\mathbb{R}_+^d)$. Even more generally, consider solutions which are stable only outside a compact set $K \subset \mathbb{R}_+^d$, i.e. such that (3) holds for $\varphi \in C_c^1(\mathbb{R}_+^d \setminus K)$. The second named author proved in [10, Theorem 9(b)] that there is no such solution except $u = 0$, provided $1 < p < p_{JL}(d)$. We improve this result as follows.

Theorem 1. *Let $p > 1$ and let $u \in C^2(\mathbb{R}_+^d) \cap C(\overline{\mathbb{R}_+^d})$ be a solution of (1) stable outside a compact set. Then $u = 0$.*

The above theorem can be partly extended to the following class of weak solutions.

Definition 1. *Let H denote the space of measurable functions u defined on the upper half-space \mathbb{R}_+^d such that $u \in H^1(\mathbb{R}_+^d \cap B_R) \cap L^{p+1}(\mathbb{R}_+^d \cap B_R)$ for every $R > 0$ and $u = 0$ on $\partial\mathbb{R}_+^d$ in the sense of traces. Then, u is a weak solution of (1) if $u \in H$ and it satisfies the equation $-\Delta u = |u|^{p-1}u$ in the sense of distributions.*

Then,

Corollary 1. *Let $p > 1$ and let u be a nonnegative weak stable solution of (1). Then $u = 0$.*

Remark 1. *Corollary 1 is sharp in the following sense. By Theorem 1.1 in [1], for $p \in (\frac{d+1}{d-1}, p_S(d-1))$, there exists a singular solution of the form $u(x) = r^{-\frac{2}{p-1}}v(\theta)$, where $r = |x|$, $\theta = x/r$ and $v \in C^2(\mathbb{S}_+^{d-1}) \cap H_0^1(\mathbb{S}_+^{d-1})$ is positive and radial (w.r.t the geodesic distance to the north pole). Observe that $u \in H$ if only if $d \geq 3$ and $p \in (p_S(d), p_S(d-1))$, so that the equation is satisfied in the weak sense. However, u is always unstable, see Lemma 7 and Lemma 8 below. So, the stability assumption cannot be completely removed from the statement of Corollary 1.*

Next, we extend our study of the Lane-Emden equation to cones, i.e. we consider the equation

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where

$$\Omega = \{r\theta : r \in (0, +\infty), \theta \in A\}, \quad (5)$$

$d \geq 2$ and $A \subset \mathbb{S}^{d-1}$ is a subset of the unit sphere of dimension $d - 1$.

Busca proved that there are no positive solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (4) in the case where Ω is a convex cone strictly contained in \mathbb{R}_+^d , see [2]. For such a cone, positive solutions are monotone (hence stable¹ in Ω) and $A = \Omega \cap \mathbb{S}^{d-1}$ is geodesically convex (see Lemma 11), hence star-shaped with respect to the north pole (up to a suitable rotation, see Lemma 13). So the following theorem extends both Busca's result and Theorem 1.

Theorem 2. *Let A be a $C^{2,\alpha}$ domain contained in the (open) upper-half sphere \mathbb{S}_+^{d-1} , which is star-shaped with respect to the north pole. Let Ω be given by (5). If $p > 1$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of (4) stable outside a compact set², then $u = 0$.*

Remark 2. *We recall that a set $A \subseteq \mathbb{S}^{d-1}$ is said to be star-shaped with respect to a point $\theta_0 \in A$ if for every $\theta \in A$ any minimal geodesic path from θ_0 to θ remains inside A . Note that when $A \subseteq \mathbb{S}_+^{d-1}$, there is at most one such minimal geodesic path.*

Finally, for general cones (not necessarily star-shaped, not necessarily contained in a half-space), as follows from the proof of Theorem 2, we have the following partial result

Corollary 2. *Let $\Omega \subset \mathbb{R}^d$ be a cone given by (5) and $p > 1$. Assume that*

$$p \neq p_S(d) \quad \text{and} \quad p\mu - \frac{(d-2)^2}{4} + (p-1)\lambda_1 \geq 0,$$

where $\mu = \frac{2}{p-1} \left(d - 1 - \frac{p+1}{p-1} \right)$ and λ_1 is the principal eigenvalue of the Laplace-Beltrami operator on $A = \Omega \cap \mathbb{S}^{d-1}$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of (4) stable outside a compact set, then $u = 0$.

Remark 3. *In particular, the Corollary applies to all $p \geq p_0$ for some (explicitly computable) p_0 depending on d and λ_1 .*

Remark 4. *Since Ω is not smooth at its vertex, some care is needed when dealing with regularity/integrability properties of the solution. The reader interested only in the proof of Theorem 1 can safely skip these considerations in the proof presented below.*

2. PROOF OF THEOREM 2

We begin by proving that any classical solution of (4) is a weak solution of (4) in the following sense.

Definition 2. *Set $\Omega_R := \Omega \cap B_R$ for every $R > 0$. We say that a measurable function u defined on Ω is a weak solution of (4) if $u \in H^1(\Omega_R) \cap L^{p+1}(\Omega_R)$ for every $R > 0$, u solves $-\Delta u = |u|^{p-1}u$ in the sense of distributions of Ω and if for all $\psi \in C_c^1(\mathbb{R}^d)$ we have $u\psi \in H_0^1(\Omega)$.*

Lemma 1. *Assume $p > 1$ and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of (4). Then, u is a weak solution of (4). Furthermore, every weak solution u to (4) satisfies,*

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} |u|^{p-1} u \phi, \quad (6)$$

for all $\phi \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ with bounded support.

Proof. Multiply equation (4) by $u\zeta^2$, $\zeta \in C_c^{0,1}(\mathbb{R}^d \setminus \{0\})$, and integrate by parts to get

$$\begin{aligned} \int |\nabla u|^2 \zeta^2 &= \int |u|^{p+1} \zeta^2 - 2 \int u \zeta \nabla u \cdot \nabla \zeta \\ &\leq \int |u|^{p+1} \zeta^2 + \frac{1}{2} \int |\nabla u|^2 \zeta^2 + 2 \int u^2 |\nabla \zeta|^2 \end{aligned}$$

Hence,

$$\int |\nabla u|^2 \zeta^2 \leq 2 \|u\|_{L^\infty(\Omega_R)}^{p+1} \int \zeta^2 + 4 \|u\|_{L^\infty(\Omega_R)}^2 \int |\nabla \zeta|^2. \quad (7)$$

Choosing $\zeta = \psi \zeta_n$, where ζ_n is a standard cut-off function away from the origin, and passing to the limit in the latter inequality we see that (7) remains true for every $\zeta = \psi \in C_c^1(\mathbb{R}^d)$. Hence, $u \in H^1(\Omega_R) \cap L^{p+1}(\Omega_R)$ for every $R > 0$ and u is a solution to $-\Delta u = |u|^{p-1}u$ in the sense of distributions of Ω . Also, $u\psi \in H_0^1(\Omega_R)$ for any $R \gg 1$, since ψ has compact support,³ and thus $u\psi \in H_0^1(\Omega)$. Finally, (6) follows by a standard density argument (by passing to the limit in the distributional formulation of $-\Delta u = |u|^{p-1}u$). \square

¹i.e., they satisfy (3) for all $\varphi \in C_c^1(\Omega)$.

²i.e., u satisfies (3) for all $\varphi \in C_c^1(\Omega \setminus K)$, where $K \subset \Omega$ is a fixed compact set.

³Recall that: $v \in H^1(A) \cap C^0(\overline{A})$ and $v = 0$ on ∂A implies $v \in H_0^1(A)$ for every open set $A \subset \mathbb{R}^d$.

Remark 5. If $\Omega = \mathbb{R}_+^d$, Definitions 1 and 2 coincide.

Next we recall the following

Definition 3. A weak solution of (4) is

- *stable* if u satisfies (3) for all $\varphi \in C_c^1(\Omega)$,
- *stable outside a compact set* if there is a compact set $K \subset \Omega$ such that u satisfies (3) for all $\varphi \in C_c^1(\Omega \setminus K)$.

Let us focus next on

2.1. The subcritical and critical cases $p \leq p_S(d)$.

This part of the proof is inspired by [9] but some care is needed since the domain Ω is not necessarily smooth (at the origin)⁴ and so solutions might not belong to $W^{2,q}(\Omega_R)$ with $q \in (1, +\infty)$ as in their case. The following integral estimate holds.

Lemma 2. Let $1 < p \leq p_S(d)$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of (4) stable outside a compact set K . Then, $u \in L^{p+1}(\Omega)$ and $|\nabla u| \in L^2(\Omega)$.

Proof. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of (4) stable outside a compact set K and let $\varphi \in C_c^1(\mathbb{R}^d)$ supported on $\mathbb{R}^d \setminus K$. Then, $\phi = u\varphi^2$ has bounded support and belongs to $H_0^1(\Omega) \cap L^{p+1}(\Omega)$ (thanks to Lemma 1 and Definition 2) and so, from (6), we get

$$\int \nabla u \nabla (u\varphi^2) = \int |u|^{p+1} \varphi^2. \quad (8)$$

We estimate the left-hand side of (8) using the stability of u in $\Omega \setminus K$

$$\begin{aligned} \int |u|^{p+1} \varphi^2 &= \int \nabla u \nabla (u\varphi^2) = \int |\nabla(u\varphi)|^2 - \int u^2 |\nabla \varphi|^2 \\ &\geq p \int |u|^{p+1} \varphi^2 - \int u^2 |\nabla \varphi|^2, \end{aligned} \quad (9)$$

where in the latter we have used that $\phi = u\varphi^2$ belongs to $H_0^1(\Omega \setminus K)$, and so it can be inserted into (3).

Combining the above lines and observing that $p > 1$, we reach

$$(p-1) \int |u|^{p+1} \varphi^2 \leq \int u^2 |\nabla \varphi|^2. \quad (10)$$

Returning to (9), it follows that

$$\frac{p-1}{p} \int |\nabla(u\varphi)|^2 \leq \int u^2 |\nabla \varphi|^2. \quad (11)$$

Taking $\varphi := \psi^m$, for some smooth non-negative function $\psi \in C_c^1(\mathbb{R}^d)$ supported on $\mathbb{R}^d \setminus K$ and $m > 1$ gives

$$(p-1) \int |u|^{p+1} \psi^{2m} \leq m^2 \int u^2 \psi^{2m-2} |\nabla \psi|^2.$$

Using Hölder's inequality,

$$(p-1) \int |u|^{p+1} \psi^{2m} \leq m^2 \left(\int |u|^{p+1} \psi^{(m-1)(p+1)} \right)^{\frac{2}{p+1}} \left(\int |\nabla \psi|^{2\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}}.$$

We can take $m := \frac{p+1}{p-1}$, so the above inequality simplifies to

$$\int |u|^{p+1} \psi^{2m} \leq \left[\frac{(p+1)^2}{(p-1)^3} \right]^{\frac{p+1}{p-1}} \int |\nabla \psi|^{2\frac{p+1}{p-1}}. \quad (12)$$

Fix $R_0 > 0$ such that $K \subset \Omega_{R_0}$ and let $R > 2R_0$ so u is stable outside of Ω_R . If $\psi_R(x) = \psi_1(x/R)$ is a standard cutoff function, then $\psi = (1 - \psi_{R_0})\psi_R$ is supported in $B_{2R} \setminus K$ and so we may apply (12). It follows that

$$\int_{\Omega_R \setminus \Omega_{2R_0}} |u|^{p+1} \leq C_p \int_{\Omega_{2R_0}} |\nabla \psi|^{2\frac{p+1}{p-1}} + C_p \int_{\Omega_{2R} \setminus \Omega_{2R_0}} |\nabla \psi|^{2\frac{p+1}{p-1}} \leq C_{p,d} (R_0^{d-2\frac{p+1}{p-1}} + R^{d-2\frac{p+1}{p-1}}). \quad (13)$$

Using the same test function in (11) yields similarly

$$\int_{\Omega_R \setminus \Omega_{2R_0}} |\nabla u|^2 \leq C_{p,d} (R_0^{d-2\frac{p+1}{p-1}} + R^{d-2\frac{p+1}{p-1}}). \quad (14)$$

The lemma follows by letting $R \rightarrow +\infty$, since $p \leq p_S(d)$ implies $d - 2\frac{p+1}{p-1} \leq 0$. \square

Thanks to Lemma 2, we derive the following Pohožaev identity.

⁴Take e.g. $A = \mathbb{S}^{d-1} \cap \{x \in \mathbb{R}^d : |x_d| < 1/2\}$ and observe in passing that Ω is not contained in a half-space in this case.

Proposition 1. *Under the assumptions of Lemma 2, there holds*

$$\left(1 - \frac{d}{2}\right) \int_{\Omega} |\nabla u|^2 + \frac{d}{p+1} \int_{\Omega} |u|^{p+1} = 0.$$

Proof. Let (ζ_n) denote a standard truncation sequence near the origin and (θ_n) a standard truncation sequence near infinity. Set $\psi_n = \zeta_n \theta_n$ for $n \geq 1$. Multiply equation (4) by $(x \cdot \nabla u) \psi_n$ and integrate by parts. Then,

$$- \int_{\Omega} \Delta u (x \cdot \nabla u) \psi_n = \int_{\Omega} \nabla u \cdot \nabla [(x \cdot \nabla u) \psi_n] - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) \psi_n = \int_{\Omega} |u|^{p-1} u (x \cdot \nabla u) \psi_n = \int_{\Omega} x \cdot \nabla \frac{|u|^{p+1}}{p+1} \psi_n.$$

So,

$$\begin{aligned} & \int_{\Omega} \nabla u \cdot \nabla (x \cdot \nabla u) \psi_n + \int_{\Omega} \nabla u \cdot \nabla \psi_n (x \cdot \nabla u) - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) \psi_n = \\ & - \int_{\Omega} \frac{|u|^{p+1}}{p+1} (d\psi_n + x \cdot \nabla \psi_n) + \int_{\partial\Omega \setminus \{0\}} \frac{|u|^{p+1}}{p+1} (x \cdot \nu) \psi_n = - \frac{d}{p+1} \int_{\Omega} |u|^{p+1} \psi_n - \int_{\Omega} \frac{|u|^{p+1}}{p+1} (x \cdot \nabla \psi_n). \end{aligned}$$

It follows that

$$\int_{\Omega} |\nabla u|^2 \psi_n + \int_{\Omega} \left(x \cdot \nabla \frac{|\nabla u|^2}{2} \right) \psi_n + \int_{\Omega} \nabla u \cdot \nabla \psi_n (x \cdot \nabla u) - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) \psi_n = - \frac{d}{p+1} \int_{\Omega} |u|^{p+1} \psi_n - \int_{\Omega} \frac{|u|^{p+1}}{p+1} (x \cdot \nabla \psi_n),$$

i.e.

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 \psi_n - \int_{\Omega} \frac{|\nabla u|^2}{2} (d\psi_n + x \cdot \nabla \psi_n) + \int_{\partial\Omega \setminus \{0\}} \frac{|\nabla u|^2}{2} (x \cdot \nu) \psi_n + \\ & \int_{\Omega} \nabla u \cdot \nabla \psi_n (x \cdot \nabla u) - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) \psi_n = - \frac{d}{p+1} \int_{\Omega} |u|^{p+1} \psi_n - \int_{\Omega} \frac{|u|^{p+1}}{p+1} (x \cdot \nabla \psi_n). \end{aligned}$$

On $\partial\Omega \setminus \{0\}$, $\nabla u = u_\nu \nu$ and $0 = r \partial_r u = x \cdot \nabla u$, since $u = 0$ on $\partial\Omega$ and Ω is a cone. Thus, $x \cdot \nabla u = u_\nu (x \cdot \nu)$ and $\frac{|\nabla u|^2}{2} (x \cdot \nu) = \frac{u_\nu^2}{2} (x \cdot \nu) = 0$ on $\partial\Omega \setminus \{0\}$ so both boundary integrals are equal to zero! It follows that

$$(1-d/2) \int_{\Omega} |\nabla u|^2 \psi_n + \frac{d}{p+1} \int_{\Omega} |u|^{p+1} \psi_n = \int_{\Omega} \frac{|\nabla u|^2}{2} (x \cdot \nabla \psi_n) - \int_{\Omega} \nabla u \cdot \nabla \psi_n (x \cdot \nabla u) - \int_{\Omega} \frac{|u|^{p+1}}{p+1} (x \cdot \nabla \psi_n) =: I(n).$$

For $n \geq 1$, the integrands of the right-hand side can be bounded as follows

$$\left| \frac{|\nabla u|^2}{2} (x \cdot \nabla \psi_n) - \nabla u \cdot \nabla \psi_n (x \cdot \nabla u) - \frac{|u|^{p+1}}{p+1} (x \cdot \nabla \psi_n) \right| \leq \left(\frac{3}{2} |\nabla u|^2 + \frac{|u|^{p+1}}{p+1} \right) |x| \cdot |\nabla \psi_n| =: g|x| \cdot |\nabla \psi_n| \quad \text{in } \Omega,$$

where $g \in L^1(\Omega)$ and $g \geq 0$. Taking the truncation functions of the form $\theta_n(x) = \theta(x/n)$ and $\zeta_n(x) = \zeta(nx)$ if $d \geq 3$ (and using log-type capacity truncation functions if $d = 2$), we infer that

$$\int_{\Omega} g|x| \cdot |\zeta_n \nabla \theta_n| \leq \|\nabla \theta\|_{\infty} \int_{\Omega \cap [n < |x| < 2n]} g \frac{|x|}{n} \leq 2\|\nabla \theta\|_{\infty} \int_{\Omega \cap [n < |x| < 2n]} g \rightarrow 0$$

as $n \rightarrow +\infty$ and similarly

$$\int_{\Omega} g|x| \cdot |\theta_n \nabla \zeta_n| \leq 2\|\nabla \zeta\|_{\infty} \int_{\Omega \cap [1/n < |x| < 2/n]} g \rightarrow 0.$$

Since $\nabla u \in L^2(\Omega)$ and $u \in L^{p+1}(\Omega)$, we may apply the dominated convergence theorem to pass to the limit in $(1-d/2) \int_{\Omega} |\nabla u|^2 \psi_n - \frac{d}{p+1} \int_{\Omega} |u|^{p+1} \psi_n$ and the proposition follows. \square

Next, in order to exploit Pohožaev's identity, we establish the following elementary identity.

Proposition 2. *Under the assumptions of Lemma 2, there holds*

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{p+1}.$$

Proof. Let (θ_n) be a standard truncation sequence near infinity and apply (6) with $\psi = \theta_n$ to get

$$\int_{\Omega} \nabla u \cdot \nabla (u \theta_n) = \int_{\Omega} |u|^{p+1} \theta_n.$$

The left-hand side equals

$$\int_{\Omega} |\nabla u|^2 \theta_n + \int_{\Omega} \nabla \frac{u^2}{2} \cdot \nabla \theta_n = \int_{\Omega} |\nabla u|^2 \theta_n - \int_{\Omega} \frac{u^2}{2} \cdot \Delta \theta_n,$$

where we used that $u = 0$ on $\partial\Omega$ and $\nabla \theta_n = 0$ in $B_1 \cup \mathbb{R}^d \setminus B_{2n}$. In addition,

$$\left| \int_{\Omega} u^2 \Delta \theta_n \right| \leq \left(\int_{\Omega \cap [n < |x| < 2n]} |u|^{p+1} \right)^{\frac{2}{p+1}} \left(\int_{\Omega \cap [n < |x| < 2n]} |\Delta \theta_n|^{\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}},$$

and since $p \leq p_S(d)$,

$$\int_{\Omega \cap \{n < |x| < 2n\}} |\Delta \theta_n|^{\frac{p+1}{p-1}} \leq C n^{d-2\frac{p+1}{p-1}} \rightarrow 0,$$

as $n \rightarrow \infty$. Since $u \in L^{p+1}(\Omega)$ and $\nabla u \in L^2(\Omega)$, we may pass to the limit and obtain the desired conclusion. \square

Remark 6. *Combining Propositions 1 and 2, we immediately conclude that $u = 0$ if $p < p_S(d)$.*

To deal with the critical case $p = p_S(d)$, we prove the following

Proposition 3. *Let $p = p_S(d)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution of (4) in a cone Ω such that $\partial\Omega$ is a graph in the x_d direction. Then, under the assumptions of Lemma 2, there holds*

$$\int_{\partial\Omega \setminus \{0\}} \nu_d |\nabla u|^2 = 0,$$

where ν_d is the d -th coordinate of the outward unit normal vector ν .

Proof. We multiply (4) with the test function $\frac{\partial u}{\partial x_d} \psi_n$ (where ψ_n has been defined at the beginning of the proof of Proposition 1) and integrate by parts. The left-hand side equals

$$-\int_{\Omega} \Delta u \left(\frac{\partial u}{\partial x_d} \psi_n \right) = \int_{\Omega} \nabla u \cdot \nabla \left(\frac{\partial u}{\partial x_d} \psi_n \right) - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_d} \psi_n$$

while the right-hand side equals

$$\int_{\Omega} |u|^{p-1} u \frac{\partial u}{\partial x_d} \psi_n = \int_{\Omega} \frac{\partial}{\partial x_d} \left(\frac{|u|^{p+1}}{p+1} \right) \psi_n = - \int_{\Omega} \frac{|u|^{p+1}}{p+1} \frac{\partial \psi_n}{\partial x_d},$$

where we used the fact that $u = 0$ on $\partial\Omega \setminus \{0\}$. Hence,

$$\int_{\Omega} \nabla u \cdot \nabla \left(\frac{\partial u}{\partial x_d} \right) \psi_n + \int_{\Omega} \nabla u \cdot \nabla \psi_n \frac{\partial u}{\partial x_d} - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_d} \psi_n = - \int_{\Omega} \frac{|u|^{p+1}}{p+1} \frac{\partial \psi_n}{\partial x_d}.$$

Since $\nabla u \cdot \nabla \left(\frac{\partial u}{\partial x_d} \right) = \frac{\partial}{\partial x_d} \frac{|\nabla u|^2}{2}$, we integrate by parts the first term and find

$$-\int_{\Omega} \frac{|\nabla u|^2}{2} \frac{\partial \psi_n}{\partial x_d} + \int_{\partial\Omega \setminus \{0\}} \frac{|\nabla u|^2}{2} \psi_n \nu_d + \int_{\Omega} \nabla u \cdot \nabla \psi_n \frac{\partial u}{\partial x_d} - \int_{\partial\Omega \setminus \{0\}} \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_d} \psi_n = - \int_{\Omega} \frac{|u|^{p+1}}{p+1} \frac{\partial \psi_n}{\partial x_d}.$$

Since $\nabla u = u_\nu \nu$ on $\partial\Omega \setminus \{0\}$, $\frac{\partial u}{\partial x_d} = u_\nu \nu_d$ on $\partial\Omega \setminus \{0\}$ and so $\frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_d} = u_\nu^2 \nu_d = |\nabla u|^2 \nu_d$ on $\partial\Omega \setminus \{0\}$. We deduce that

$$I(n) := - \int_{\Omega} \frac{|\nabla u|^2}{2} \frac{\partial \psi_n}{\partial x_d} + \int_{\Omega} \nabla u \cdot \nabla \psi_n \frac{\partial u}{\partial x_d} + \int_{\Omega} \frac{|u|^{p+1}}{p+1} \frac{\partial \psi_n}{\partial x_d} = \frac{1}{2} \int_{\partial\Omega \setminus \{0\}} |\nabla u|^2 \psi_n \nu_d.$$

By the graph property of $\partial\Omega$, $-\nu_d \geq 0$. So, in order to conclude, it suffices to prove that the left-hand side in the above identity converges to 0 as $n \rightarrow +\infty$ (using Beppo Levi's theorem on $\partial\Omega \setminus \{0\}$). But

$$|I(n)| \leq \int_{\Omega} \left(\frac{3}{2} |\nabla u|^2 + \frac{|u|^{p+1}}{p+1} \right) |\nabla \psi_n| \rightarrow 0,$$

as $n \rightarrow +\infty$, as in the proof of Proposition 1. \square

Remark 7. *If $\partial\Omega$ is a graph, then there exists $p \in \partial\Omega \setminus \{0\}$ such that $\nu_d(p) \neq 0$ and so there exists $\rho \in (0, 1)$ such that $|\nabla u|^2 = 0$ in $(\partial\Omega \setminus \{0\}) \cap B(p, \rho)$. By the unique continuation principle, $u = 0$ in Ω .*

Going back to the statement of Theorem 2, our cone Ω is star-shaped with respect to the north pole, so thanks to Lemma 12 below, the boundary of the cone Ω is a graph in the x_d direction. Then we may apply Proposition 3 and by Remarks 6 and 7, we have just proved Theorems 1, 2 and Corollary 2 in the case $1 < p \leq p_S(d)$. Corollary 1 also follows for $1 < p \leq p_S(d)$ since in that case and for $\Omega = \mathbb{R}_+^d$, any weak solution is in fact a classical solution $u \in C^2(\overline{\mathbb{R}_+^d})$.

Remark 8. *Regarding weak solutions, we observe that Lemma 2 remains valid for weak solutions stable outside a compact set. Indeed, in these cases, (6) holds and so also does Lemma 2.*

We turn next to

2.2. The supercritical case $p > p_S(d)$.

Observe that if u solves (4) and $\lambda > 0$, then the function u_λ defined by

$$u_\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x), \quad x \in \Omega \tag{15}$$

is also a solution to (4). In order to understand the asymptotic profile of u at infinity, we consider the blow-down family (u_λ) as $\lambda \rightarrow +\infty$.

2.2.1. *A priori estimates and convergence of the blow-down family.*

Lemma 3. *Let $p \geq p_S(d)$. Assume that u is a weak solution of (4) stable outside a compact set. Then, there exist constants $C = C(u, d, p) > 0$ such that for all $R > 1$ and $\lambda \geq 1$,*

$$\int_{\Omega_R} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} \leq CR^{d-2\frac{p+1}{p-1}}. \quad (16)$$

This follows essentially from [10]. For completeness, we include a proof below.

Proof. Fix $R_0 = R_0(u) > 1$ such that u is stable outside of Ω_{R_0} . Then, given $\lambda \geq 1$, u_λ is stable outside the smaller domain $\Omega_{R_0/\lambda}$. Therefore, by proceeding as in the proof of Lemma 2, we obtain inequalities (13) and (14) for u_λ . Hence, for $R > 2R_0/\lambda$,

$$\int_{\Omega_R \setminus \Omega_{2R_0/\lambda}} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} \leq C_{p,d}((R_0/\lambda)^{d-2\frac{p+1}{p-1}} + R^{d-2\frac{p+1}{p-1}}) \leq 2C_{p,d}R^{d-2\frac{p+1}{p-1}}$$

where we have used that $p \geq p_S(d)$ implies $d - 2\frac{p+1}{p-1} \geq 0$.

On the other hand we also have

$$\int_{\Omega_{2R_0/\lambda}} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_{2R_0}} |\nabla u|^2 + |u|^{p+1} \leq \int_{\Omega_{2R_0}} |\nabla u|^2 + |u|^{p+1} = C(u, p, R_0(u)) < +\infty \quad (17)$$

since $\lambda \geq 1$. The lemma follows. \square

Remark 9. *Let $p > p_S(d)$. Assume that u is a weak solution of (4) stable outside a compact set. Then, the following improved version of (16) holds for some $\gamma > 1$ and all $R > 1$*

$$\int_{\Omega_R} |\nabla(|u_\lambda|^{\frac{\gamma-1}{2}} u_\lambda)|^2 + |u_\lambda|^{p+\gamma} \leq C(1 + R^{d-2\frac{p+\gamma}{p-1}}), \quad (18)$$

where C is constant depending only on p, d, γ and u .

Precisely, as in Proposition 4 of [10], testing equation (4) with⁵ $|T_k(u)|^{\gamma-1} T_k(u) \varphi^2$, where $T_k(u)$ is the truncation of u at level $k > 0$, $\gamma \in (1, 2p+2\sqrt{p(p-1)}-1)$ and $\varphi \in C_c^1(\mathbb{R}^d)$, and then inserting $|T_k(u)|^{\frac{\gamma-1}{2}} T_k(u) \varphi^2$ in the stability condition, we are lead to bounds (13) and (14), where the exponent $d-2\frac{p+1}{p-1}$ is replaced by $d-2\frac{p+\gamma}{p-1}$. Finally, choosing γ such that $d-2\frac{p+\gamma}{p-1} \geq 0$ (and filling the hole) leads to the desired estimate (18).

We use the preceding remark to prove convergence of (u_λ) to a limit u_∞ in suitable spaces.

Lemma 4. *Let $p > p_S(d)$. Assume that u is a weak solution of (4) stable outside a compact set. The rescaled solutions converge along a sequence (u_{λ_n}) to a limit u_∞ strongly in $H^1 \cap L^{p+1}(\Omega_R)$ for every $R > 0$, as $\lambda_n \rightarrow \infty$. Furthermore, u_∞ is a weak, stable solution of (4).*

Proof. By estimate (16) and (18) there is $\gamma > 1$ such that (u_λ) is bounded in $H^1 \cap L^{p+\gamma}(\Omega_R)$ for every $R > 0$. Thus, by a standard diagonal argument, there exists a measurable function u_∞ defined on Ω and a subsequence (u_{λ_n}) weakly converging to u_∞ in $H^1 \cap L^{p+\gamma}(\Omega_R)$ for every $R > 0$. In particular we get

$$\int_{\Omega} \nabla u_{\lambda_n} \nabla \eta \rightarrow \int_{\Omega} \nabla u_\infty \nabla \eta, \quad (19)$$

for all $\eta \in H_0^1(\Omega)$ with bounded support.

We also observe that, by Rellich-Kondrachov's compactness theorem, we may and do suppose that the convergence is strong in $L^2(\Omega_R)$, for every $R > 0$.⁶ To obtain strong convergence in $L^{p+1}(\Omega_R)$, we apply the following interpolation inequality

$$\|u_\lambda - u_\infty\|_{L^{p+1}(\Omega_R)} \leq \|u_\lambda - u_\infty\|_{L^2(\Omega_R)}^\theta \|u_\lambda - u_\infty\|_{L^{p+\gamma}(\Omega_R)}^{1-\theta},$$

where $\theta = \frac{2\gamma-2}{(p+\gamma-2)(p+1)}$, as follows from Hölder's inequality. Since (u_{λ_n}) converges to u_∞ in $L^2(\Omega_R)$ and (u_{λ_n}) is bounded in $L^{p+\gamma}(\Omega_R)$, it follows that (u_{λ_n}) converges strongly to u_∞ in $L^{p+1}(\Omega_R)$ for every $R > 0$. Therefore, we can find a subsequence (still denoted by (u_{λ_n})) and $g \in L^{p+1}(\Omega_R)$ for every $R > 0$ such that

$$\begin{aligned} |u_{\lambda_n}| &\leq g && \text{a.e. in } \Omega, \\ u_{\lambda_n} &\rightarrow u_\infty && \text{a.e. in } \Omega. \end{aligned} \quad (20)$$

⁵For any $\beta > 0$, the function $|T_k(u)|^\beta T_k(u) \varphi^2$ belongs to $H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and has bounded support. Indeed, pick $\xi \in C_c^1(\mathbb{R}^d)$ such that $\xi \equiv 1$ on a neighbourhood of the support of φ , then $u\xi \in H_0^1(\Omega)$. The latter implies $|T_k(u\xi)|^\beta T_k(u\xi) \in H_0^1(\Omega)$ and so also $|T_k(u\xi)|^\beta T_k(u\xi) \varphi^2$ belongs to $H_0^1(\Omega)$. Then observe that $|T_k(u\xi)|^\beta T_k(u\xi) \varphi^2 = |T_k(u)|^\beta T_k(u) \varphi^2$ on Ω . Since it is clear that $|T_k(u)|^\beta T_k(u) \varphi^2$ belongs to $L^{p+1}(\Omega)$ and has bounded support, the desired claim follows.

⁶Indeed, let (ϕ_k) be a sequence of standard cut-off functions of class $C_c^1(\mathbb{R}^d)$. Then, for every fixed k , the sequence $(u_{\lambda_n} \phi_k)$ is bounded in $H_0^1(\Omega_{3k})$ and by Rellich-Kondrachov's compactness theorem, it is compact in $L^2(\Omega_{3k})$ which, in turn, implies that the sequence (u_{λ_n}) is compact in $L^2(\Omega_k)$. The desired conclusion then follows by a standard diagonal argument.

The latter and Lebesgue's dominated convergence theorem imply that

$$\int_{\Omega} |u_{\lambda_n}|^{p-1} u_{\lambda_n} \eta \rightarrow \int_{\Omega} |u_{\infty}|^{p-1} u_{\infty} \eta, \quad (21)$$

for all $\eta \in L^{p+1}(\Omega)$ with bounded support.

Recalling that u_{λ_n} is a weak solution to (4), we see that the combination of (21) and (19) gives

$$\int_{\Omega} \nabla u_{\infty} \nabla \eta = \int_{\Omega} |u_{\infty}|^{p-1} u_{\infty} \eta, \quad (22)$$

for all $\eta \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ with bounded support.

Next we observe that $u_{\infty} \zeta \in H^1(\Omega)$ for every $\zeta \in C_c^1(\mathbb{R}^d)$ and that $(u_{\lambda_n} \zeta)$ converges weakly to $u_{\infty} \zeta$ in $H^1(\Omega)$, thanks to the weak convergence of (u_{λ_n}) in $H^1(\Omega_R)$ for every $R > 0$. Hence, $u_{\infty} \zeta \in H_0^1(\Omega)$ for every $\zeta \in C_c^1(\mathbb{R}^d)$, since $H_0^1(\Omega)$ is weakly closed in $H^1(\Omega)$. The latter and (22) prove that u_{∞} is a weak solution to (4) and

$$\int_{\Omega} \nabla u_{\infty} \nabla (u_{\infty} \varphi^2) = \int_{\Omega} |u_{\infty}|^{p+1} \varphi^2 \quad \forall \varphi \in C_c^1(\mathbb{R}^d) \quad (23)$$

since $u_{\infty} \varphi^2$ belongs to $H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and has bounded support.

Next, we claim that $u_{\lambda_n} \varphi \rightarrow u_{\infty} \varphi$ in $H_0^1(\Omega)$. In view of (23) and (8) we can compute as in the first line of (9) and find that, for every $\varphi \in C_c^1(\mathbb{R}^d)$

$$\int_{\Omega} |\nabla (u_{\lambda_n} \varphi)|^2 = \int_{\Omega} |u_{\lambda_n}|^2 |\nabla \varphi|^2 + \int_{\Omega} |u_{\lambda_n}|^{p+1} \varphi^2 \rightarrow \int_{\Omega} |u_{\infty}|^2 |\nabla \varphi|^2 + \int_{\Omega} |u_{\infty}|^{p+1} \varphi^2 = \int_{\Omega} |\nabla (u_{\infty} \varphi)|^2,$$

where in the latter we have used (20) and Lebesgue's dominated convergence theorem. It follows that $(u_{\lambda_n} \varphi)$ converges to $u_{\infty} \varphi$ in $H_0^1(\Omega)$, for every $\varphi \in C_c^1(\mathbb{R}^d)$ and, in particular $u_{\lambda_n} \rightarrow u_{\infty}$ in $H^1(\Omega_R)$ for every $R > 0$. Stability of the weak solution u_{∞} follows from the strong convergence of (u_{λ_n}) in $L^{p+1}(\Omega_R)$ and the fact that for fixed n , u_{λ_n} is stable outside K/λ_n (here $K \subset \Omega$ is any compact set such that u is stable outside K). \square

Remark 10. *One may wonder if Lemma 4 still holds if $p = p_S(d)$. If one works in all of \mathbb{R}^d , the answer is no. Indeed, a constant multiple of the function $u(x) = (1 + |x|^2)^{-\frac{d-2}{2}}$ solves the equation and is stable outside a compact set, thanks to Hardy's inequality. Yet, the family (u_{λ}) is clearly not compact in $L^{p+1}(B_1)$.*

2.2.2. Monotonicity formula. Define the functional

$$E(u; \lambda) = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_{\lambda}} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx + \lambda^{\frac{4}{p-1}+1-d} \frac{1}{p-1} \int_{\Omega \cap \partial B_{\lambda}} u^2 d\sigma, \quad (24)$$

where $\lambda > 0$, $\Omega_{\lambda} = \Omega \cap B_{\lambda}$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution to (4).

Remark 11. *Note that E satisfies the following simple scaling relation: given $\lambda, R > 0$,*

$$E(u; \lambda R) = E(u_{\lambda}; R)$$

The main result of this section is the following monotonicity formula, which extends a result due to Pacard [14].

Lemma 5. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of (4) and $\lambda > 0$. Let E be as in (24). Then $\lambda \mapsto E(u; \lambda)$ is a non-decreasing function. Furthermore, $E(u; \cdot) \in C^1(\mathbb{R}_+^*)$ and for all $\lambda > 0$,*

$$\frac{dE}{d\lambda}(u; \lambda) = \lambda^{\frac{4}{p-1}+2-d} \int_{\Omega \cap \partial B_{\lambda}} \left(\partial_r u + \frac{2}{p-1} \frac{u}{r} \right)^2 d\sigma. \quad (25)$$

Proof. Fix $\varepsilon \in (0, 1)$. Let

$$E_1^{\varepsilon}(u; \lambda) = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_{\lambda} \setminus \Omega_{\lambda\varepsilon}} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx. \quad (26)$$

For $\lambda > 0$, let also u_{λ} be defined by (15). Then, u_{λ} satisfies the three following properties: given $\lambda > 0$, u_{λ} solves (4),

$$E_1^{\varepsilon}(u; \lambda) = E_1^{\varepsilon}(u_{\lambda}; 1), \quad (27)$$

and

$$\lambda \partial_{\lambda} u_{\lambda} = \frac{2}{p-1} u_{\lambda} + r \partial_r u_{\lambda} \quad \text{for every } (x, \lambda) \in \Omega \times \mathbb{R}_+^*. \quad (28)$$

By standard elliptic regularity, $u_{\lambda} \in C^2(\overline{\Omega_1} \setminus \overline{\Omega_{\varepsilon}})$. So $\lambda \mapsto E_1^{\varepsilon}(u; \lambda)$ is C^1 and differentiating the right-hand side of (27), we find

$$\frac{dE_1^{\varepsilon}}{d\lambda}(u; \lambda) = \int_{\Omega_1 \setminus \Omega_{\varepsilon}} \nabla u_{\lambda} \cdot \nabla \partial_{\lambda} u_{\lambda} dx - \int_{\Omega_1 \setminus \Omega_{\varepsilon}} |u_{\lambda}|^{p-1} u_{\lambda} \partial_{\lambda} u_{\lambda} dx.$$

Integrating by parts, using the equation, the boundary condition and the fact that Ω is a cone, we find

$$\frac{dE_1^\varepsilon}{d\lambda}(u; \lambda) = \int_{\Omega \cap \partial B_1} \partial_r u_\lambda \partial_\lambda u_\lambda \, d\sigma - \int_{\Omega \cap \partial B_\varepsilon} \partial_r u_\lambda \partial_\lambda u_\lambda \, d\sigma.$$

Using (28),

$$\begin{aligned} \left| \int_{\Omega \cap \partial B_\varepsilon} \partial_r u_\lambda \partial_\lambda u_\lambda \, d\sigma \right| &= \frac{1}{\lambda} \left| \int_{\Omega \cap \partial B_\varepsilon} \frac{2}{p-1} u_\lambda \partial_r u_\lambda + \varepsilon (\partial_r u_\lambda)^2 \, d\sigma \right| \\ &\leq C \|u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)} \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)} + \varepsilon \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2 \\ &\leq C\varepsilon \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2, \end{aligned}$$

where we used (the sharp) Poincaré inequality in $H_0^1(\partial B_\varepsilon \cap \Omega)$ in the last inequality. Since $u_\lambda \in H^1(\Omega_1)$, $\liminf_{\varepsilon \rightarrow 0} (\varepsilon |\ln \varepsilon|) \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2 = 0$ and so there exists a sequence $\varepsilon_n \rightarrow 0$ such that

$$\|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_{\varepsilon_n})}^2 \leq \frac{1}{\varepsilon_n |\ln \varepsilon_n|}.$$

And so

$$\begin{aligned} C |\ln \varepsilon_n|^{-1} &\geq \left| \frac{dE_1^{\varepsilon_n}}{d\lambda}(u; \lambda) - \int_{\Omega \cap \partial B_1} \partial_r u_\lambda \partial_\lambda u_\lambda \, d\sigma \right| \\ &\geq \left| \frac{dE_1^{\varepsilon_n}}{d\lambda}(u; \lambda) - \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 \, d\sigma + \frac{2}{p-1} \int_{\Omega \cap \partial B_1} u_\lambda (\partial_\lambda u_\lambda) \, d\sigma \right| \\ &\geq \left| \frac{d}{d\lambda} \left(E_1^{\varepsilon_n}(u; \lambda) + \frac{1}{p-1} \int_{\Omega \cap \partial B_1} u_\lambda^2 \, d\sigma \right) - \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 \, d\sigma \right|. \end{aligned}$$

For any fixed $\lambda > 0$, take $0 < \lambda_1 < \lambda_2 \leq \lambda$ and integrate the above inequality between λ_1 and λ_2 . Then, letting $\varepsilon_n \rightarrow 0$, we find that

$$E(u; \lambda_2) - E(u; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 \, d\sigma \, d\lambda. \quad (29)$$

Hence, $\lambda \mapsto E(u; \lambda) \in C^1(\mathbb{R}_+^*)$ and scaling back the above quantity, the lemma follows. \square

Remark 12. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of (4). If the function $\lambda \mapsto E(u; \lambda)$ is constant on $(0, +\infty)$, then by (25) it follows that u is homogeneous of degree $-\frac{2}{p-1}$.

Remark 13. The proof of Lemma 5 immediately extends to solutions defined on Ω_R , $R > 0$ i.e. let $R > 0$ and $u \in C^2(\Omega_R) \cap C(\bar{\Omega}_R)$ be a solution of

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega \cap B_R. \end{cases} \quad (30)$$

For $\lambda \in (0, R)$ let E be as in (24). Then $\lambda \mapsto E(u; \lambda)$ is a non-decreasing function. Moreover, $E(u; \cdot) \in C^1(0, R)$ and, for all $\lambda \in (0, R)$,

$$\frac{dE}{d\lambda}(u; \lambda) = \lambda^{\frac{4}{p-1}+2-d} \int_{\Omega \cap \partial B_\lambda} \left(\partial_r u + \frac{2}{p-1} \frac{u}{r} \right)^2 \, d\sigma. \quad (31)$$

and for any $0 < \lambda_1 < \lambda_2 < R$ we find that

$$E(u; \lambda_2) - E(u; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 \, d\sigma \, d\lambda. \quad (32)$$

Remark 14. The monotonicity formula remains valid for weak stable solutions in the sense that the function defined on \mathbb{R}_+^* by $\lambda \mapsto E(u; \lambda)$ is monotone, absolutely continuous and its derivative is given by (25) for a.e. $\lambda \in (0, +\infty)$.

Indeed, if u is weak stable solution of (4), then, by Proposition 13 in [6], for every $R > 0$, there exists a sequence of non-negative solutions $u_n^R \in C^2(\bar{B}_R^+)$ such that $u_n^R = 0$ on $\partial\mathbb{R}_+^d \cap B_R$, $u_n^R \rightarrow u$ a.e. in $\Omega_R = B_R^+$, and $u_n^R \rightarrow u$ in $H^1(\Omega_R)$, as $n \rightarrow +\infty$. By Remark 13, $\lambda \mapsto E(u_n^R; \lambda)$ is a non-decreasing function of $\lambda \in (0, R)$, $E(u_n^R; \cdot) \in C^1(0, R)$ and for any $0 < \lambda_1 < \lambda_2 < R$ we find that

$$E(u_n^R; \lambda_2) - E(u_n^R; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda (u_n^R)_\lambda)^2 \, d\sigma \, d\lambda. \quad (33)$$

Passing to the limit in (33) as $n \rightarrow +\infty$, our claim follows.

2.2.3. *Analysis of the limiting profile u_∞ .* In this section, we will show that u_∞ (given in Lemma 4) is a homogeneous function. To do so, first note that given a weak solution u of (4), if $u_\lambda = u$ for all $\lambda > 0$, then u is homogeneous of degree $-\frac{2}{p-1}$, and so it must have the form

$$u(r, \theta) = r^{-\frac{2}{p-1}} v(\theta), \quad (34)$$

for some measurable function $v: A \subset \mathbb{S}^{d-1} \rightarrow \mathbb{R}$. This is a simple consequence of taking $\lambda = |x|^{-1}$ in (15). Note that by (34), $u_\lambda = u$ automatically excludes the possibility that u is a classical solution of (4) because of the singularity near $x = 0$ (unless $v = 0$).

Now we prove an important property of the limit function u_∞ given in Lemma 4.

Lemma 6. *Let $p > p_S(d)$ and assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of (4) stable outside a compact set. The blow-down limit u_∞ is homogeneous of degree $-\frac{2}{p-1}$ and thus has the form (34) for a suitable $v \in H_0^1 \cap L^{p+1}(A)$.*

Proof. After fixing two radii $R_2 > R_1 > 0$ we set

$$a_{\lambda_n} := E(u_{\lambda_n}; R_2),$$

$$b_{\lambda_n} := E(u_{\lambda_n}; R_1),$$

and $c_{\lambda_n} := a_{\lambda_n} - b_{\lambda_n}$.

We use the fundamental theorem of calculus and the monotonicity formula (25) to express c_{λ_n} as

$$\begin{aligned} c_{\lambda_n} &= E(u_{\lambda_n}; R_2) - E(u_{\lambda_n}; R_1) = \int_{R_1}^{R_2} \frac{dE}{d\lambda}(u_{\lambda_n}; \lambda) d\lambda = \int_{R_1}^{R_2} \frac{d}{d\lambda}(E(u_{\lambda_n}; \lambda; 1)) d\lambda \\ &= \int_{R_1}^{R_2} \int_{\partial B_1 \cap \Omega} \lambda (\partial_\lambda u_{\lambda_n})^2 d\sigma d\lambda = \int_{R_1}^{R_2} \int_{\partial B_1 \cap \Omega} \frac{1}{\lambda} \left(\frac{2}{p-1} u_{\lambda_n} \lambda + |x| \partial_r u_{\lambda_n} \right)^2 d\sigma d\lambda \\ &= \int_{R_1}^{R_2} \lambda^{\frac{4}{p-1}-d} \int_{\partial B_\lambda \cap \Omega} \left(\frac{2}{p-1} u_{\lambda_n} + |x| \partial_r u_{\lambda_n} \right)^2 d\sigma d\lambda \\ &= \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left(\frac{2}{p-1} u_{\lambda_n} + |x| \partial_r u_{\lambda_n} \right)^2 dx. \end{aligned}$$

Since (u_{λ_n}) converges strongly in $H^1(\Omega_R)$, we deduce that

$$\lim_{n \rightarrow +\infty} c_{\lambda_n} = \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left(\frac{2}{p-1} u_\infty + |x| \partial_r u_\infty \right)^2 dx.$$

Next, we prove that

$$c_{\lambda_n} \rightarrow 0.$$

By Lemma 3 and the trace inequality $\int_{\partial B_R \cap \Omega} u^2 d\sigma \leq C(R \int_{B_R \cap \Omega} |\nabla u|^2 dx + R^{-1} \int_{B_R \cap \Omega} u^2 dx)$, the sequences (a_{λ_n}) and (b_{λ_n}) are bounded. Using the monotonicity formula (25), we deduce that (a_{λ_n}) and (b_{λ_n}) are nondecreasing and converge to the same finite limit.

Hence,

$$0 = \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left(\frac{2}{p-1} u_\infty + |x| \partial_r u_\infty \right)^2 dx = \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{2-d} \left(\partial_r (|x|^{\frac{2}{p-1}} u_\infty) \right)^2 dx$$

so that $w := |x|^{\frac{2}{p-1}} u_\infty$ is homogeneous of degree zero. Therefore we can find a measurable function $v: A \subset \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ such that $w(x) = v(\frac{x}{|x|})$ and thus u_∞ has the form (34). Note that $w \in H^1 \cap L^{p+1}(\Omega_2 \setminus \Omega_{\frac{1}{2}})$, since $u_\infty \in H^1 \cap L^{p+1}(\Omega_R)$ for every $R > 0$, and so $v \in H^1 \cap L^{p+1}(A)$ by Fubini's theorem. To conclude we need to prove that v belongs to $H_0^1(A)$. To this end we set $w_n = |x|^{\frac{2}{p-1}} u_{\lambda_n}$ and observe that the sequence (w_n) converges to w in $H^1(\Omega_2 \setminus \Omega_{\frac{1}{2}})$, that is, $\int_{\frac{1}{2}}^2 \int_A (|w_n - w|^2 + |\nabla(w_n - w)|^2)(r\theta) r^{d-1} d\sigma dr \rightarrow 0$, as $n \rightarrow \infty$. Therefore, up to a subsequence, for almost every $r \in (\frac{1}{2}, 2)$ we get that $\int_A (|w_n - w|^2 + |\nabla(w_n - w)|^2)(r \cdot) d\sigma \rightarrow 0$ and so we can find $\bar{r} \in (\frac{1}{2}, 2)$ such that

$$\int_A (|w_n - w|^2 + |\nabla'(w_n - w)|^2)(\bar{r} \cdot) d\sigma \rightarrow 0,$$

where ∇' is the Riemannian gradient on the $(d-1)$ -dimensional sphere. The latter and the fact that $w|_A = v$ imply that the sequence of $C^2(\bar{A})$ functions $\theta \mapsto w_n(\bar{r}\theta)$ converges to v in $H^1(A)$. It follows that $v \in H_0^1(A)$ since $w_n(\bar{r}\cdot) = 0$ on ∂A . \square

2.2.4. *Homogeneous and stable solutions.* In the previous section, we have shown that the blow-down limit u_∞ of the rescaled solutions u_λ is a weak solution of (4), and it is homogeneous and stable. Here we study such solutions with the goal of proving that they are necessarily equal to 0.

Given a homogeneous solution $u(r, \theta) = r^{-\frac{2}{p-1}}v(\theta)$ of (4), one can make a simple formal computation to conclude that the nonradial factor v satisfies the following equation

$$\begin{cases} -\Delta'v + \mu v = |v|^{p-1}v & \text{in } A, \\ v = 0 & \text{on } \partial A, \end{cases} \quad (35)$$

where $\mu := \frac{2}{p-1} \left(d - 1 - \frac{p+1}{p-1} \right)$ and $-\Delta'$ is the Laplace-Beltrami operator on the $(d-1)$ -dimensional sphere. If u is also stable then v satisfies the following inequality for suitable test functions ψ :

$$\frac{(d-2)^2}{4} \int_A \psi^2 d\sigma + \int_A |\nabla' \psi|^2 d\sigma \geq p \int_A |v|^{p-1} \psi^2 d\sigma, \quad (36)$$

where ∇' is the Riemannian gradient on the $(d-1)$ -dimensional sphere, see e.g. [15, pp. 5245-5246] for the proof⁷ of (36). We work in the class of weak solutions of (4) and make these considerations precise in the following lemma.

Lemma 7. *Let u be a weak, homogeneous, and stable solution of (4). Then $v \in H_0^1 \cap L^{p+1}(A)$ is a weak solution of (35) which satisfies the stability-type estimate (36) for any $\psi \in H_0^1(A)$.*

Proof. To prove (35), we begin by testing (4) with the test function $\varphi(r, \theta) = \chi(r)\psi(\theta)$ where $\chi: \mathbb{R}_+ \rightarrow [0, 1]$ is a smooth, positive function compactly supported away from 0, and $\psi \in H_0^1(A) \cap L^{p+1}(A)$. Setting $\alpha = \frac{2}{p-1}$,

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi \, dx &= \underbrace{\int_\Omega \partial_r(r^{-\alpha}v(\theta)) \partial_r \chi(r) \psi(\theta) \, dx}_{I_1} + \underbrace{\int_\Omega r^{-\alpha-2} \chi(r) \nabla' v(\theta) \nabla' \psi(\theta) \, dx}_{I_2} \\ &= \underbrace{\int_\Omega r^{-\frac{2p}{p-1}} \chi(r) |v(\theta)|^{p-1} v(\theta) \psi(\theta) \, dx}_{I_3}. \end{aligned}$$

We perform a separation of variables of the three integral terms.

$$\begin{aligned} I_1 &= -\alpha \int_0^\infty r^{-\alpha-2+d} \partial_r \chi(r) dr \int_A v(\theta) \psi(\theta) d\sigma \\ &= \alpha(d-2-\alpha) \int_0^\infty r^{-\alpha-3+d} \chi(r) dr \int_A v(\theta) \psi(\theta) d\sigma. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^\infty r^{-\alpha-3+d} \chi(r) dr \int_A \nabla' v(\theta) \nabla' \psi(\theta) d\sigma, \\ I_3 &= \int_0^\infty r^{-p\alpha-1+d} \chi(r) dr \int_A |v(\theta)|^{p-1} v(\theta) \psi(\theta) d\sigma. \end{aligned}$$

Since $\alpha + 2 = p\alpha$, we can cancel all of the integral terms involving r to conclude

$$\int_A \nabla' v \nabla' \psi \, d\sigma + \alpha(d-2-\alpha) \int_A v \psi \, d\sigma = \int_A |v|^{p-1} v \psi \, d\sigma. \quad (37)$$

That is, $v \in H_0^1 \cap L^{p+1}(A)$ is a weak solution of (35). \square

We conclude this section by proving a Liouville-type result for weak solutions of (35) which satisfy estimate (36), under the additional condition for $p > 1$

$$p\mu - \frac{(d-2)^2}{4} + (p-1)\lambda_1 \geq 0, \quad (38)$$

where λ_1 is the first eigenvalue of the Laplace-Beltrami operator on A . Note that the term $(p-1)\lambda_1$ in (38) gives an improvement of the condition in [10]: $1 < p < p_{JL}(d) \iff p\mu - \frac{(d-2)^2}{4} > 0$. In other words, the condition $1 < p < p_{JL}(d)$ is optimal only for stable solutions in the whole space \mathbb{R}^d .

Lemma 8. *Let $v \in H_0^1 \cap L^{p+1}(A)$ be a weak solution of (35) which satisfies the stability-type estimate (36). Then for all p satisfying (38), $v = 0$.*

⁷In [15, pp. 5245-5246], ψ is supposed to be smooth but the result remains valid for $\psi \in H_0^1(A) \cap L^{p+1}(A)$ with the same proof and can be extended to any $\psi \in H_0^1(A)$ by Fatou's lemma.

Proof. First, we take the solution v (multiplied by p) as a test function in (37) to obtain

$$p\mu \int_A v^2 d\sigma + p \int_A |\nabla' v|^2 d\sigma = p \int_A |v|^p v d\sigma.$$

Likewise, testing (36) with v yields

$$\frac{(d-2)^2}{4} \int_A v^2 d\sigma + \int_A |\nabla' v|^2 d\sigma \geq p \int_A |v|^p v d\sigma.$$

Combining these two lines gives

$$\left(p\mu - \frac{(d-2)^2}{4}\right) \int_A v^2 d\sigma + (p-1) \int_A |\nabla' v|^2 d\sigma \leq 0.$$

We use the trivial identity

$$\int_A |\nabla' v|^2 d\sigma = - \left(\lambda_1 \int_A v^2 d\sigma - \int_A |\nabla' v|^2 d\sigma \right) + \lambda_1 \int_A v^2 d\sigma$$

to deduce the following:

$$\left(p\mu - \frac{(d-2)^2}{4} + \lambda_1(p-1)\right) \int_A v^2 d\sigma \leq (p-1) \int_A (\lambda_1 v^2 - |\nabla' v|^2) d\sigma \leq 0, \quad (39)$$

where the final inequality follows from Poincaré's inequality on A with optimal constant λ_1 . By (38) and (39), v is a principle eigenfunction of the Laplace-Beltrami operator on A . Combining this with the fact that v also satisfies (35) leads trivially to the conclusion that $v = 0$. \square

2.2.5. A Pohožaev-type result on subdomains of \mathbb{S}_+^{d-1} . The following Pohožaev-type result is proven by Bidaut-Véron, Ponce, and Véron in [1, Theorem 2.1] for smooth solutions v to (35). Since we deal with weak solutions, we construct smooth approximations v_λ of v and use the H_{loc}^1 convergence of u_λ to u_∞ (along a sequence) to prove that the inequality also holds for the blow-down limit u_∞ .

Lemma 9. *Let A be a $C^{2,\alpha}$ domain of \mathbb{S}_+^{d-1} which is star-shaped with respect to the north pole, and let $u_\infty(r, \theta) = r^{-\frac{2}{p-1}} v(\theta)$ be the blow-down limit of (u_λ) as above. Then $v \in H_0^1(A)$ satisfies*

$$\left(\frac{d-3}{2} - \frac{d-1}{p+1}\right) \int_A |\nabla' v|^2 \phi d\sigma - \frac{d-1}{2} \left(\frac{d-\mu(p-1)-1}{p+1}\right) \int_A v^2 \phi d\sigma \leq 0, \quad (40)$$

where ∇' is the tangential gradient to \mathbb{S}_+^{d-1} , and $\phi(\theta) = \theta_d$ is an eigenfunction of the Laplace-Beltrami operator $-\Delta'$ in $H_0^1(\mathbb{S}_+^{d-1})$ associated to the principal eigenvalue $\lambda_1(\mathbb{S}_+^{d-1}) = d-1$.

Proof. We begin with a simple geometric lemma. Denote by $\langle \cdot, \cdot \rangle$ the inner product on the tangent space to \mathbb{S}^{d-1} , $\phi(\theta) = \theta_d$, and $\nabla' \phi$ its Riemannian gradient on \mathbb{S}^{d-1} . Then,

Lemma 10. *Let $A \subset \mathbb{S}_+^{d-1}$ denote a C^1 open set with outward unit normal direction ν . If A is star-shaped with respect to the north pole, there holds $\langle \nabla' \phi, \nu \rangle \leq 0$ on ∂A .*

Proof. Apply the stereographic projection π_S from the south pole to the set A . Then take a point $x \in \pi_S(A) \subset B_1 \subset \mathbb{R}^{d-1}$ and consider the unique minimal geodesic γ_θ contained in A connecting the north pole to $\pi_S^{-1}(x) =: \theta$. Its projection $\pi_S(\gamma_\theta)$ is a straight line connecting the origin to x and is contained in $\pi_S(A)$. So $\pi_S(A) \subset \mathbb{R}^{d-1}$ is star-shaped (in the usual Euclidean sense). Since the stereographic projection is conformal (it preserves angles), $\langle \nabla' \phi, \nu \rangle$ has the same sign as $V \cdot n$, where $V = d\pi_S(\nabla' \phi) = \nabla \left(\frac{1-r^2}{1+r^2} \right) = \frac{-4r}{(1+r^2)^2} \frac{x}{r}$, $x \in \mathbb{R}^{d-1}$, $r = |x|$, and $n = d\pi_S(\nu)$ is the outward unit normal direction of $\pi_S(A)$. In view of the Lemma in [8] p. 554, we conclude that $\langle \nabla' \phi, \nu \rangle \leq 0$ on ∂A . \square

Let us return to the proof of Lemma 9. To simplify the appearance we set $C_1 = \left(\frac{d-3}{2} - \frac{d-1}{p+1}\right)$ and $C_2 = \frac{d-1}{2} \left(\frac{d-\mu(p-1)-1}{p+1}\right)$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of (4). Since for any $\lambda > 0$, $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x)$ is also a smooth solution of (4), we set v_λ to satisfy

$$u_\lambda(x) = r^{-\frac{2}{p-1}} v_\lambda(r, \theta).$$

Plugging this expression of u_λ into (4), it follows that v_λ is a classical solution of the following equation:

$$\begin{cases} -\Delta' v_\lambda + \mu v_\lambda = |v_\lambda|^{p-1} v_\lambda + e_\lambda & \text{in } \Omega, \\ v_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \quad (41)$$

where the error term e_λ is

$$e_\lambda = \left(d-1 - \frac{4}{p-1}\right) r \partial_r v_\lambda + r^2 \partial_r^2 v_\lambda.$$

Again for simplicity, we define $C_3 = \left(d - 1 - \frac{4}{p-1}\right)$.

Following [1, pp. 186-188], we apply the divergence theorem to the vector field

$$P = \langle \nabla' \phi, \nabla' v_\lambda \rangle \nabla' v_\lambda,$$

to eventually reach the following relation

$$\begin{aligned} C_1 \int_A |\nabla' v_\lambda|^2 \phi \, d\sigma + C_2 \int_A v_\lambda^2 \phi \, d\sigma - \frac{d-1}{p+1} \int_A e_\lambda v_\lambda \phi \, d\sigma + \int_A \langle \nabla' \phi, \nabla' v_\lambda \rangle e_\lambda \, d\sigma \\ = \frac{1}{2} \int_{\partial A} |\nabla' v_\lambda|^2 \langle \nabla' \phi, \nu \rangle \, d\tau \leq 0, \end{aligned} \quad (42)$$

where ν is the unit outer normal. The right-hand side of (42) is nonpositive since A is star-shaped. Next, we multiply all terms of (42) by a standard cutoff function $\eta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ compactly supported away from 0 and we integrate in the radial direction (including the factor r^{d-1}). We begin with the first error term:

$$\begin{aligned} \int_0^\infty \int_A e_\lambda v_\lambda \phi r^{d-1} \eta \, d\sigma \, dr &= C_3 \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi r^d \eta \, d\sigma \, dr + \int_0^\infty \int_A (\partial_r^2 v_\lambda) v_\lambda \phi r^{d+1} \eta \, d\sigma \, dr \\ &= C_3 \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi r^d \eta \, d\sigma \, dr - \int_0^\infty \int_A (\partial_r v_\lambda)^2 \phi r^{d+1} \eta \, d\sigma \, dr - \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi \partial_r (r^{d+1} \eta) \, d\sigma \, dr, \end{aligned}$$

where we have integrated by parts in the second integral of the first line. Now, by the fact that $v_\lambda \rightarrow v = r^{\frac{2}{p-1}} u_\infty$ in $H^1(\Omega_R)$ (along a sequence) and that v does not depend on r , we can pass to the limit as $\lambda \rightarrow \infty$ to conclude

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \int_A e_\lambda v_\lambda \phi r^{d-1} \eta \, d\sigma \, dr = 0.$$

For the second error term in (42), we proceed as follows:

$$\begin{aligned} \int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle e_\lambda r^{d-1} \eta \, d\sigma \, dr &= \int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle (C_3 r \partial_r v_\lambda + r^2 \partial_r^2 v_\lambda) r^{d-1} \eta \, d\sigma \, dr \\ &= C_3 \underbrace{\int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^d \eta \, d\sigma \, dr}_{I_1} \\ &\quad - \underbrace{\int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda \partial_r (r^{d+1} \eta) \, d\sigma \, dr}_{I_2} \\ &\quad - \underbrace{\int_0^\infty \int_A \langle \nabla' \partial_r v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^{d-1} \eta \, d\sigma \, dr}_{I_3}, \end{aligned}$$

where we have integrated by parts in the r variable. Since $v_\lambda \rightarrow v$ in $H^1(\Omega_R)$ (along a sequence) and since v does not depend on r , I_1 and I_2 tend to zero as $\lambda \rightarrow \infty$. We deal with I_3 by recalling that ϕ is an eigenfunction and so

$$\begin{aligned} - \int_0^\infty \int_A \langle \nabla' \partial_r v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^{d-1} \eta \, d\sigma \, dr &= -\frac{1}{2} \int_0^\infty \int_A \langle \nabla' (\partial_r v_\lambda)^2, \nabla' \phi \rangle r^{d-1} \eta \, d\sigma \, dr \\ &= -\frac{d-1}{2} \int_\Omega (\partial_r v_\lambda)^2 \phi \eta \, dx, \end{aligned}$$

where in the latter we have used an integration by parts over A and the fact that $\partial_r v_\lambda = 0$ on $\partial\Omega \setminus \{0\}$.

This term then tends to 0 as $\lambda \rightarrow \infty$, since v_λ converges to v in $H^1(\Omega_R)$ (along a sequence).

Now we deal with the other solid integral terms arising from integrating (42) with the test function $r^{d-1} \eta$ in the r variable. Since u_λ converges to u in $H^1(\Omega_R)$ for every $R > 0$ along a sequence, so does v_λ to v . Since v does not depend on r , we can apply Fubini's theorem at the limit to obtain

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \int_A (C_1 |\nabla' v_\lambda|^2 + C_2 v_\lambda^2) \phi r^{d-1} \eta \, d\sigma \, dr = \int_0^\infty r^{d-1} \eta \, dr \int_A (C_1 |\nabla' v|^2 + C_2 v^2) \phi \, d\sigma \leq 0,$$

where C_1 and C_2 are the fixed constants defined at the beginning of the proof. Dividing by the positive constant $\int_0^\infty r^{d-1} \eta \, dr$ yields (40). \square

2.2.6. Proof of Theorem 2.

Proof. By an elementary computation, $p_S(d) < p_S(d-1) < p_{JL}(d)$. Recall also that if $p \leq p_{JL}(d)$, then (38) holds. So all $p > p_S(d)$ either satisfy (38), $p > p_S(d-1)$, or both. Suppose first that (38) holds. By Lemmas 6, 7, and 8, it follows that $u_\infty = 0$. In the remaining case $p > p_S(d-1)$, Lemma 9 implies that $v = 0$ (which of course implies that $u_\infty = 0$). Then, since (u_{λ_n}) converges strongly to u_∞ in $H^1 \cap L^{p+1}(\Omega_R)$, it follows that $E(u_{\lambda_n}; 1) \rightarrow 0$ as $\lambda_n \rightarrow \infty$. Furthermore, by the monotonicity formula (25), the convergence to 0 is monotone non-decreasing.

To conclude, we study the behavior of $E(u_\lambda; 1) = E(u; \lambda)$ as $\lambda \rightarrow 0^+$. By definition (24) of the monotonicity formula, we have

$$E(u, \lambda) \geq -\frac{1}{p+1} \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_\lambda} |u|^{p+1} dx = -c \lambda^{2\frac{p+1}{p-1}} \int_{\Omega_\lambda} |u|^{p+1} dx$$

Since u is bounded in a neighbourhood of 0, we deduce that $\lim_{\lambda \rightarrow 0^+} E(u; \lambda) \geq 0$. Since $E(u, \cdot)$ is nondecreasing, we deduce that

$$E(u; \lambda) \geq 0 \quad \text{for all } \lambda > 0 \quad (43)$$

Now, $\lim_{\lambda \rightarrow +\infty} E(u; \lambda) = E(u_\infty; 1) = 0$ and so, since $E(u, \cdot)$ is nondecreasing and nonnegative, $E(u; \lambda) = 0$ for every $\lambda > 0$. By (25), u is homogeneous, which is possible only if $u = 0$ since $u \in C(\bar{\Omega})$. \square

3. PROOFS OF COROLLARY 1 AND COROLLARY 2 IN THE SUPERCRITICAL CASE

3.1. Proof of Corollary 1. We just need to inspect the proof of Theorem 1 and adapt it as follows. Let u be a nonnegative weak stable solution. By Remark 14, the monotonicity formula remains valid and so⁸ does Lemma 6. So, the blow-down limit u_∞ is homogeneous. Regarding Pohozaev's identity Lemma 9, we use Proposition 13 in [6] to deduce that u_∞ is also the limit (in $H^1(B_R^+)$) of classical solutions u_n^R (with zero value on $\partial\mathbb{R}_+^d \cap B_R$) and so the same proof applies. Finally, we adapt Section 2.2.6 by observing that by the same proof $E(u_n^R; \lambda) \geq 0$ for all $\lambda \in (0, R)$ and so $E(u; \lambda)$ is nonnegative for all $\lambda > 0$. As before, this forces u to be homogeneous, hence $u = 0$ by our classification of homogeneous weak solutions. \square

3.2. Proof of Corollary 2. In the supercritical case $p > p_S(d)$, the proof is identical to that of Theorem 2. We only need to observe that in the range (38) of exponents p , no geometrical condition on the cone Ω is needed. \square

3.3. Positive solutions in convex cones. In this last section, we explain more carefully how Busca's result [2] for positive classical solutions defined on convex cones can be recovered thanks to Theorem 2. This is the content of Lemma 13 below. First, we extend the following natural result proven in [11, Proposition 2] in the case where Ω is convex.

Lemma 11. *Let Ω be defined as in (5) for some $A \subset \mathbb{S}_+^{d-1}$. Then Ω is convex (resp. star-shaped) if and only if A is geodesically convex (resp. geodesically star-shaped).*

Proof. Let Ω be convex. Now take two points $\varphi_1, \varphi_2 \in A$. (By the hypothesis that $A \subset \mathbb{S}_+^{d-1}$, φ_1 and φ_2 may not be antipodal points.) By convexity, $t\varphi_2 + (1-t)\varphi_1 \in \Omega$, so

$$S_t = \frac{t\varphi_2 + (1-t)\varphi_1}{|t\varphi_2 + (1-t)\varphi_1|} \in A$$

for all $t \in [0, 1]$ by the cone property of Ω . It can be directly verified, for example using a stereographic projection from $-\varphi_2$, that $\{S_t\}_{0 \leq t \leq 1}$ coincides with the unique geodesic curve from φ_1 to φ_2 . Finally, since $\{S_t\}_{t > 0} \subset A$, we have that A is geodesically convex.

Conversely, let us choose two general points $x_1, x_2 \in \Omega$. In the same way as above, we can show that

$$\left\{ \frac{tx_2 + (1-t)x_1}{|tx_2 + (1-t)x_1|} \right\}_{0 \leq t \leq 1} \subset A$$

is a geodesic curve on A , whereby we find that $tx_2 + (1-t)x_1 \in \Omega$ due to the cone property of Ω .

In the case of star-shaped domains, we repeat the above method with φ_2 and x_2 fixed taken to be the points with respect to which A and Ω , respectively, are star-shaped. \square

Lemma 12. *If $A \subset \mathbb{S}_+^{d-1}$ is star-shaped with respect to some direction $\varphi_0 \in A$, then Ω is convex in that direction in the sense that for all $x \in \Omega$ and $t > 0$, $x + t\varphi_0 \in \Omega$. In particular, the boundary of the cone Ω is a graph with respect to the direction $\varphi_0 \in A$.*

⁸In the second part of the proof of Lemma 6, in order to prove that $v \in H_0^1(A)$, one just needs to replace the sequence (u_{λ_n}) by the approximating sequence (u_n^R) , $R = 2$, from Remark 14.

Proof. First, by virtue of the radial scaling property of cones, it is enough to prove the result for all $x = \varphi_1 \in A$. It can then be verified that the curve

$$S = \{S_t\}_{t>0} = \left\{ \frac{\varphi_1 + t\varphi_0}{|\varphi_1 + t\varphi_0|} \right\}_{t>0} \subset \mathbb{S}_+^{d-1},$$

coincides with the unique minimal geodesic connecting φ_1 to φ_0 (for example by taking the stereographic projection from $-\varphi_0$). Then, since A is star-shaped with respect to φ_0 , $S_t \in A$ for all t , whereby $x + t\varphi_0 \in \Omega$ due to the cone property of Ω . \square

Lemma 13. *Assume that $A \subset \mathbb{S}_+^{d-1}$ is geodesically convex. Then, a suitable rotation of A is star-shaped with respect to the north pole and still contained in \mathbb{S}_+^{d-1} .*

Proof. If the north pole \vec{n} belongs to A the statement is trivially proven because convex sets are in particular star-shaped. Therefore, let us assume that \vec{n} does not belong to A . Then $\Omega \cap \langle \vec{n} \rangle = \emptyset$ by the cone property of Ω , and Ω is convex by Lemma 11. So by the geometric form of the Hahn-Banach theorem, there exists a hyperplane H normal to some $\nu \in \mathbb{S}^{d-1}$ (w.l.o.g. take $\nu = (-1, 0, \dots, 0)$) which includes the x_d axis and which satisfies $H \cap \Omega = \emptyset$. By the convexity assumption, Ω (resp. A) must lie on one side of H (resp. $H \cap \mathbb{S}_+^{d-1}$). That is, w.l.o.g, $A \subset \mathbb{S}_+^{d-1} \cap \{x_1 > 0\}$.

Let us consider the great circle $G = \{(x_1, 0, \dots, 0, x_d) : x_i \in [-1, 1], |x| = 1\}$. Up to a suitable modification of the choice of ν , we can assume (by convexity of A) that there is a point $a = (\cos \theta_a, 0, \dots, 0, \sin \theta_a) \in G \cap A$ for its corresponding $\theta_a \in (0, \pi/2)$. Then we take the rotation that sends a to \vec{n} . This rotation has a specific form: for a general $p = (\alpha_p \cos \theta, x_2, \dots, x_{d-1}, \alpha_p \sin \theta) \in A$ for its corresponding $\alpha_p \in [0, 1]$, $\theta \in (0, \pi/2)$, we have

$$p \mapsto \left(\alpha_p \cos \left(\frac{\pi}{2} - \theta_a + \theta \right), x_2, \dots, x_{d-1}, \alpha_p \sin \left(\frac{\pi}{2} - \theta_a + \theta \right) \right).$$

Since $\theta, \theta_a \in (0, \pi/2)$, it can be verified that the image of A under this rotation is still contained in \mathbb{S}_+^{d-1} . \square

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