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# LIOUVILLE-TYPE THEOREMS FOR THE LANE-EMDEN EQUATION IN THE HALF-SPACE AND CONES

LOUIS DUPAIGNE, ALBERTO FARINA AND TROY PETITT

## 1. INTRODUCTION

Let  $d \geq 1$  and  $p > 1$ . We consider the Lane-Emden equation

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}_+^d, \\ u = 0 & \text{on } \partial\mathbb{R}_+^d, \end{cases} \quad (1)$$

posed in the upper half-space  $\mathbb{R}_+^d := \{x = (x', x_d): x' \in \mathbb{R}^{d-1}, x_d \in (0, \infty)\}$  with homogeneous Dirichlet boundary conditions. It is conjectured that the above equation has only one nonnegative solution,  $u = 0$ .

Gidas and Spruck in [12] showed that this is indeed the case if  $1 < p \leq p_S(d)$ , where

$$p_S(d) = \frac{d+2}{(d-2)_+}$$

is Sobolev's critical exponent. In the case  $p > p_S(d)$ , only partial results are available: Dancer [4] considered bounded nonnegative solutions and proved in this case that  $u = 0$  if  $p < p_S(d-1)$ . The second named author [10] improved Dancer's result showing that the only bounded nonnegative solution is  $u = 0$  if  $p < p_{JL}(d-1)$ , where  $p_{JL}$  is the Joseph-Lundgren stability exponent given by

$$p_{JL}(d) := \frac{(d-2)^2 - 4d + 8\sqrt{d-1}}{(d-2)(d-10)_+}. \quad (2)$$

Finally Chen, Lin, and Zou [3] proved that no bounded nonnegative solution  $u \neq 0$  of (1) exists for any  $p > 1$ . In all these results, the fact that any nonnegative bounded solution of (1) is monotone in the  $x_d$ -direction, i.e.  $\partial u / \partial x_d > 0$  in  $\mathbb{R}_+^d$ , is crucially used. In fact, Sirakov, Souplet and the first named author proved in [7] that, more generally, no nontrivial monotone solution of (1) exists, whether bounded or not. Note that monotone solutions are stable, meaning they verify additionally that

$$p \int |u|^{p-1} \varphi^2 dx \leq \int |\nabla \varphi|^2 dx, \quad (3)$$

for all  $\varphi \in C_c^1(\mathbb{R}_+^d)$ . Even more generally, consider solutions which are stable only outside a compact set  $K \subset \mathbb{R}_+^d$ , i.e. such that (3) holds for  $\varphi \in C_c^1(\mathbb{R}_+^d \setminus K)$ . The second named author proved in [10, Theorem 9(b)] that there is no such solution except  $u = 0$ , provided  $1 < p < p_{JL}(d)$ . We improve this result as follows.

**Theorem 1.** *Let  $p > 1$  and let  $u \in C^2(\mathbb{R}_+^d) \cap C(\overline{\mathbb{R}_+^d})$  be a solution of (1) stable outside a compact set. Then  $u = 0$ .*

The above theorem can be partly extended to the following class of weak solutions.

**Definition 1.** *Let  $H$  denote the space of functions  $u$  such that  $u \in H^1(\mathbb{R}_+^d \cap B_R) \cap L^{p+1}(\mathbb{R}_+^d \cap B_R)$  for every  $R > 0$  and  $u = 0$  on  $\partial\mathbb{R}_+^d$  in the sense of traces. Then,  $u$  is a weak solution of (1) if  $u \in H$  and it satisfies the equation in the sense of distributions.*

**Remark 1.** *By approximation, any weak solution satisfies*

$$\int \nabla u \cdot \nabla \varphi dx = \int |u|^{p-1} u \varphi dx, \quad \text{for all } \varphi \in H. \quad (4)$$

Then,

**Corollary 1.** *Let  $p > 1$  and let  $u$  be a nonnegative weak stable solution of (1). Then  $u = 0$ .*

**Remark 2.** *Corollary 1 is sharp in the following sense. By Theorem 1.1 in [1], for  $p \in (\frac{d+1}{d-1}, p_S(d-1))$ , there exists a singular solution of the form  $u(x) = r^{-\frac{2}{p-1}} v(\theta)$ , where  $r = |x|$ ,  $\theta = x/r$  and  $v \in C^2(\overline{\mathbb{S}_+^{d-1}}) \cap H_0^1(\mathbb{S}_+^{d-1})$  is positive and radial (w.r.t the geodesic distance to the north pole). Observe that  $u \in H$  if and only if  $d \geq 3$  and  $p \in (p_S(d), p_S(d-1))$ , so that the equation is satisfied in the weak sense. However,  $u$  is always unstable, see Lemma 7 below.*

Next, we extend our study of the Lane-Emden equation to cones i.e. we consider the equation

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5)$$

where

$$\Omega = \{r\theta : r \in (0, +\infty), \theta \in A\}, \quad (6)$$

$d \geq 2$  and  $A \subset \mathbb{S}^{d-1}$  is a subset of the unit sphere of dimension  $d-1$ . Busca proved that there are no positive solutions  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  of (5) in the case where  $\Omega$  is a convex cone strictly contained in  $\mathbb{R}_+^d$ , see [2]. For such a cone, positive solutions are monotone (hence stable) and  $A = \Omega \cap \mathbb{S}^{d-1}$  is geodesically convex (see Lemma 10), hence star-shaped with respect to the north pole (up to a suitable rotation, see Lemma 12). So the following theorem extends both Busca's result and Theorem 1.

**Theorem 2.** *Let  $A$  be a  $C^{2,\alpha}$  domain contained in the (open) upper-half sphere  $\mathbb{S}_+^{d-1}$ , which is star-shaped with respect to the north pole. Let  $\Omega$  be given by (6). If  $p > 1$  and  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is a solution of (5) stable outside a compact set, then  $u = 0$ .*

**Remark 3.** *A set  $A \subseteq \mathbb{S}^{d-1}$  is said to be star-shaped with respect to a point  $\theta_0 \in A$  if for every  $\theta \in A$  any minimal geodesic path from  $\theta_0$  to  $\theta$  remains inside  $A$ . Note that when  $A \subseteq \mathbb{S}_+^{d-1}$ , there is at most one such minimal geodesic path.*

Finally, for general cones (not necessarily star-shaped, not necessarily contained in a half-space), as follows from the proof of Theorem 2, we have the following partial result

**Corollary 2.** *Let  $\Omega \subset \mathbb{R}^d$  be a cone given by (6) and  $p > 1$ . Assume that*

$$p \neq p_S(d) \quad \text{and} \quad p\mu - \frac{(d-2)^2}{4} + (p-1)\lambda_1 \geq 0,$$

where  $\mu := \frac{2}{p-1} \left( d-1 - \frac{p+1}{p-1} \right)$  and  $\lambda_1$  is the principle eigenvalue of the Laplace-Beltrami operator on  $A = \Omega \cap \mathbb{S}^{d-1}$ . If  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is a solution of (5) stable outside a compact set, then  $u = 0$ .

**Remark 4.** *In particular, the corollary applies for  $p \geq p_0$  for some (explicitly computable)  $p_0$  depending on  $d$  and  $\lambda_1$ .*

**Remark 5.** *Since  $\Omega$  is not smooth at its vertex, some care is needed when dealing with regularity/integrability properties of the solution. The reader interested only in the proof of Theorem 1 can safely skip these considerations in the proof presented below.*

## 2. PROOF OF THEOREM 2

The case  $p \leq p_S(d)$  follows from Theorem<sup>1</sup> 9 in [10]. So, we assume henceforth that  $p > p_S(d)$ . Observe that if  $u$  solves (1) and  $\lambda > 0$ , then the function  $u_\lambda$  defined for  $x \in \bar{\Omega}$  by

$$u_\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x) \quad (7)$$

is also a solution. In order to understand the asymptotic profile of  $u$  at infinity, we consider the blow-down family  $(u_\lambda)$  as  $\lambda \rightarrow +\infty$ .

### 2.1. *A priori* estimates and convergence of the blow-down family.

We begin by proving that any classical solution is a weak solution.

**Lemma 1.** *Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of (5). Then,  $u$  is a weak solution, in the sense of Definition 1.*

*Proof.* Multiply equation (5) by  $u\zeta^2$ ,  $\zeta \in C_c^{0,1}(\mathbb{R}^d \setminus \{0\})$ , and integrate by parts to get

$$\begin{aligned} \int |\nabla u|^2 \zeta^2 &= \int |u|^{p+1} \zeta^2 - 2 \int u \zeta \nabla u \cdot \nabla \zeta \\ &\leq \int |u|^{p+1} \zeta^2 + \frac{1}{2} \int |\nabla u|^2 \zeta^2 + 2 \int u^2 |\nabla \zeta|^2 \end{aligned}$$

Hence,

$$\int |\nabla u|^2 \zeta^2 \leq 2 \|u\|_{L^\infty(\Omega_R)}^{p+1} \int \zeta^2 + 4 \|u\|_{L^\infty(\Omega_R)}^2 \int |\nabla \zeta|^2. \quad (8)$$

<sup>1</sup>Theorem 9 in [10] is stated for smooth domains, but the proof remains valid in our setting, thanks to Lemma 1, in the case  $p < p_S(d)$ . If  $p = p_S(d)$ , thanks to Lemma 11, the boundary of the cone  $\Omega$  is a graph with respect to some direction and up to rotation, we may assume that this direction is the north pole. The conservation laws Proposition I.1 and I.2 in [9] remain valid with some work.

Choosing  $\zeta = \psi\zeta_n$ , where  $\zeta_n$  is a standard cut-off function away from the origin, and passing to the limit in the latter inequality we see that (8) holds true for every  $\psi \in C_c^1(\mathbb{R}^d)$ . Hence,  $u\psi \in H_0^1(\Omega)$  and  $u$  is a weak solution of the equation.  $\square$

**Remark 6.** *By the same proof, any weak solution  $u$  is such that  $u\psi \in H_0^1(\Omega)$  for any  $\psi \in C_c^1(\mathbb{R}^d)$ .*

**Lemma 2.** *Let  $p \geq p_S(d)$ . Assume that  $u$  is a weak solution of (5) stable outside a compact set. Then, there exists a constant  $C = C(u, d, p)$  and  $R_1 = R_1(u) > 1$  such that for all  $R \geq R_1$  and  $\lambda \geq 1$ ,*

$$\int_{\Omega_R} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} \leq CR^{d-2\frac{p+1}{p-1}}, \quad (9)$$

where  $\Omega_R = B_R \cap \Omega$  and  $B_R$  is the ball of radius  $R$  centered at the origin.

This follows essentially from [10]. For completeness, we include a proof below.

*Proof.* Let  $u$  be a weak solution of (5) stable outside a compact set  $K$ . Test the equation with  $u\varphi^2$ , where  $\varphi$  is some smooth function supported on  $B_R \setminus K$ , to get

$$\int \nabla u \nabla (u\varphi^2) = \int |u|^{p+1} \varphi^2. \quad (10)$$

We estimate the left-hand side of (10) using the stability of  $u$

$$\begin{aligned} \int |u|^{p+1} \varphi^2 &= \int \nabla u \nabla (u\varphi^2) = \int |\nabla(u\varphi)|^2 - \int u^2 |\nabla\varphi|^2 \\ &\geq p \int |u|^{p+1} \varphi^2 - \int u^2 |\nabla\varphi|^2. \end{aligned} \quad (11)$$

Combining the above lines and observing that  $p > 1$ , we reach

$$(p-1) \int |u|^{p+1} \varphi^2 \leq \int u^2 |\nabla\varphi|^2. \quad (12)$$

Returning to (11), it follows that

$$\frac{p-1}{p} \int |\nabla(u\varphi)|^2 \leq \int u^2 |\nabla\varphi|^2. \quad (13)$$

Taking  $\varphi := \psi^m$ , for some smooth function  $\psi$  supported on  $B_R \setminus K$  and  $m > 1$  gives

$$(p-1) \int |u|^{p+1} \psi^{2m} \leq m^2 \int u^2 \psi^{2m-2} |\nabla\psi|^2.$$

Using Hölder's inequality,

$$(p-1) \int |u|^{p+1} \psi^{2m} \leq m^2 \left( \int |u|^{p+1} \psi^{(m-1)(p+1)} \right)^{\frac{2}{p-1}} \left( \int |\nabla\psi|^{2\frac{p+1}{p-1}} \right)^{\frac{p-1}{p-1}}.$$

We can take  $m := \frac{p+1}{p-1}$ , so the above inequality simplifies to

$$\int |u|^{p+1} \psi^{2m} \leq \left[ \frac{(p+1)^2}{(p-1)^3} \right]^{\frac{p+1}{p-1}} \int |\nabla\psi|^{2\frac{p+1}{p-1}}. \quad (14)$$

Fix  $R_0 > 0$  such that  $K \subset \Omega_{R_0}$  and so  $u$  is stable outside of  $\Omega_{R_0}$ . If  $\psi_R(x) = \psi_1(x/R)$  is a standard cutoff function, then  $\psi = (1 - \psi_{R_0})\psi_R$  is supported in  $\Omega_{2R} \setminus K$  and we may apply (14). It follows that

$$\int_{\Omega_R \setminus \Omega_{2R_0}} |u|^{p+1} \leq C_p \int_{\Omega_{2R_0}} |\nabla\psi|^{2\frac{p+1}{p-1}} + C_p \int_{\Omega_{2R} \setminus \Omega_{2R_0}} |\nabla\psi|^{2\frac{p+1}{p-1}} \leq C_{p,d} (R_0^{d-2\frac{p+1}{p-1}} + R^{d-2\frac{p+1}{p-1}})$$

Using the same test function in (13) yields similarly

$$\int_{\Omega_R \setminus \Omega_{2R_0}} |\nabla u|^2 \leq C_{p,d} (R_0^{d-2\frac{p+1}{p-1}} + R^{d-2\frac{p+1}{p-1}}). \quad (15)$$

Given  $\lambda \geq 1$ ,  $u_\lambda$  is stable outside the smaller domain  $\Omega_{R_0/\lambda}$ , so that applying the above inequalities to  $u_\lambda$  yields

$$\int_{\Omega_R \setminus \Omega_{2R_0/\lambda}} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} \leq C_{p,d,R_0} R^{d-2\frac{p+1}{p-1}},$$

where we used that  $p \geq p_S(d)$  and  $R \geq 1$ . Inside  $\Omega_{2R_0/\lambda}$ , using again that  $p \geq p_S(d)$  and  $\lambda \geq 1$ , we notice that

$$\int_{\Omega_{2R_0/\lambda}} |\nabla u_\lambda|^2 + |u_\lambda|^{p+1} = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_{2R_0}} |\nabla u|^2 + |u|^{p+1} \leq \int_{\Omega_{2R_0}} |\nabla u|^2 + |u|^{p+1} < +\infty. \quad (16)$$

The lemma follows.  $\square$

**Remark 7.** Assume  $p > p_S(d)$ . Then, the following improved version of (9) holds for some  $\gamma > 1$

$$\int_{\Omega_R} |\nabla(|u_\lambda|^{\frac{\gamma-1}{2}} u_\lambda)|^2 + |u_\lambda|^{p+\gamma} \leq CR^{d-2\frac{p+\gamma}{p-1}}. \quad (17)$$

Precisely, as in Proposition 4 in [10], testing equation (5) with  $|T_k(u)|^{\gamma-1} u \varphi^2$ , where  $T_k(u)$  is the truncation of  $u$  at level  $k$  and  $\gamma \in (1, 2p + 2\sqrt{p(p-1)} - 1)$  leads to a bound similar to (15) while choosing  $\gamma$  such that  $2\frac{p+\gamma}{p-1} - d \leq 0$  leads to a bound similar to (16).

We use the preceding remark to prove convergence of  $(u_\lambda)$  to a limit  $u_\infty$  in suitable spaces.

**Lemma 3.** Let  $p > p_S(d)$ . The rescaled solutions converge along a sequence  $(u_{\lambda_n})$  to a limit  $u_\infty$  strongly in  $H^1 \cap L^{p+1}(\Omega_R)$  for every  $R > 0$ , as  $\lambda_n \rightarrow \infty$ . Furthermore,  $u_\infty$  is a weak, stable solution of (5) satisfying  $u_\infty \psi \in H_0^1(\Omega)$  for every  $\psi \in C_c^1(\mathbb{R}^d)$ .

*Proof.* By estimate (17),  $(u_\lambda)$  is bounded in  $H^1 \cap L^{p+\gamma}(\Omega_R)$  for every  $R > 0$  and for some  $\gamma > 1$ . Thus a sequence  $(u_{\lambda_n})$  converges weakly to  $u_\infty$  in  $H^1 \cap L^{p+\gamma}(\Omega_R)$  for every  $R > 0$ . In particular we get

$$\int_{\Omega} \nabla u_{\lambda_n} \nabla \varphi \rightarrow \int_{\Omega} \nabla u_\infty \nabla \varphi, \quad (18)$$

for all  $\varphi \in H$ .

We also observe that, by Rellich-Kondrachov's compactness theorem, we may and do suppose that the convergence is strong in  $L^2(\Omega_R)$ , for every  $R > 0$ .<sup>2</sup> To obtain strong convergence in  $L^{p+1}(\Omega_R)$ , we apply the following interpolation inequality

$$\|u_\lambda - u_\infty\|_{L^{p+1}(\Omega_R)} \leq \|u_\lambda - u_\infty\|_{L^2(\Omega_R)}^\theta \|u_\lambda - u_\infty\|_{L^{p+\gamma}(\Omega_R)}^{1-\theta},$$

where  $\theta = \frac{2\gamma-2}{(p+\gamma-2)(p+1)}$ , as follows from Hölder's inequality. Since  $(u_{\lambda_n})$  converges to  $u_\infty$  in  $L^2(\Omega_R)$  and  $(u_{\lambda_n})$  weakly to  $u_\infty$  in  $L^{p+\gamma}(\Omega_R)$ , it follows that  $(u_{\lambda_n})$  converges strongly in  $L^{p+1}(\Omega_R)$  to  $u_\infty$  for every  $R > 0$ . Therefore, we can find a subsequence (still denoted by  $(u_{\lambda_n})$ ) and  $g \in L^{p+1}(\Omega_R)$  for every  $R > 0$  such that

$$\begin{aligned} |u_{\lambda_n}| &\leq g && \text{a.e. in } \Omega, \\ u_{\lambda_n} &\rightarrow u_\infty && \text{a.e. in } \Omega. \end{aligned} \quad (19)$$

The latter and Lebesgue's dominated convergence theorem imply that

$$\int_{\Omega} |u_{\lambda_n}|^{p-1} u_{\lambda_n} \varphi \rightarrow \int_{\Omega} |u_\infty|^{p-1} u_\infty \varphi, \quad (20)$$

for all  $\varphi \in H$ . The latter and (18) proves that  $u_\infty$  is a weak solution of (5), i.e.,

$$\int_{\Omega} \nabla u_\infty \nabla \varphi = \int_{\Omega} |u_\infty|^{p-1} u_\infty \varphi, \quad (21)$$

for all  $\varphi \in H$ .

Next we observe that  $u_\infty \zeta \in H^1(\Omega)$  for every  $\zeta \in C_c^1(\mathbb{R}^d)$  and that  $(u_{\lambda_n} \zeta)$  converges weakly to  $u_\infty \zeta$  in  $H^1(\Omega)$ , thanks to the weak convergence of  $(u_{\lambda_n})$  in  $H^1(\Omega_R)$  for every  $R > 0$ . Hence,  $u_\infty \zeta \in H_0^1(\Omega)$  for every  $\zeta \in C_c^1(\mathbb{R}^d)$  by Remark 6 (and  $(u_{\lambda_n} \zeta)$  converges weakly to  $u_\infty \zeta$  in  $H_0^1(\Omega)$ ). In particular we have  $u_\infty \zeta \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$  for every  $\zeta \in C_c^1(\mathbb{R}^d)$ . Hence we have

$$\int_{\Omega} \nabla u_\infty \nabla (u_\infty \zeta) = \int_{\Omega} |u_\infty|^{p-1} u_\infty (u_\infty \zeta) \quad \forall \zeta \in C_c^1(\mathbb{R}^d). \quad (22)$$

Next, we claim that  $u_{\lambda_n} \varphi \rightarrow u_\infty \varphi$  in  $H_0^1(\Omega)$ . In view of (10) and (22) we can compute as in the first line of (11) and find that, for every  $\varphi \in C_c^1(\mathbb{R}^d)$

$$\int_{\Omega} |\nabla(u_{\lambda_n} \varphi)|^2 = \int_{\Omega} |u_{\lambda_n}|^2 |\nabla \varphi|^2 + \int_{\Omega} |u_{\lambda_n}|^{p+1} \varphi^2 \rightarrow \int_{\Omega} |u_\infty|^2 |\nabla \varphi|^2 + \int_{\Omega} |u_\infty|^{p+1} \varphi^2 = \int_{\Omega} |\nabla(u_\infty \varphi)|^2,$$

where in the latter we have used (19) and Lebesgue's dominated convergence theorem. It follows that  $(u_{\lambda_n} \varphi)$  converges to  $u_\infty \varphi$  in  $H_0^1(\Omega)$ , for every  $\varphi \in C_c^1(\mathbb{R}^d)$  and, in particular  $u_{\lambda_n} \rightarrow u_\infty$  in  $H^1(\Omega_R)$  for every  $R > 0$ . Stability of the weak solution  $u_\infty$  follows from the strong  $L^{p+1}(\Omega_R)$  convergence of  $(u_{\lambda_n})$  and the fact that for fixed  $n$ ,  $u_{\lambda_n}$  is stable outside  $K/\lambda_n$ .  $\square$

**Remark 8.** One may wonder if Lemma 3 still holds if  $p = p_S(d)$ . If one works in all of  $\mathbb{R}^d$ , the answer is no. Indeed, a constant multiple of the function  $u(x) = (1 + |x|^2)^{-\frac{d-2}{2}}$  solves the equation and is stable outside a compact set, thanks to Hardy's inequality. Yet, the family  $(u_\lambda)$  is clearly not compact in  $L^{p+1}(B_1)$ .

<sup>2</sup>Indeed, let  $(\phi_k)$  be a sequence of standard cut-off functions of class  $C_c^1(\mathbb{R}^d)$ . Then, for every fixed  $k$ , the sequence  $(u_{\lambda_n} \phi_k)$  is bounded in  $H_0^1(\Omega_{3k})$  and by Rellich-Kondrachov's compactness theorem, it is compact in  $L^2(\Omega_{3k})$  which, in turn, implies that the sequence  $(u_{\lambda_n})$  is compact in  $L^2(\Omega_k)$ . The desired conclusion then follows by a standard diagonal argument.

**2.2. Monotonicity formula.** Define the functional

$$E(u; \lambda) = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_\lambda} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx + \lambda^{\frac{4}{p-1}+1-d} \frac{1}{p-1} \int_{\Omega \cap \partial B_\lambda} u^2 d\sigma, \quad (23)$$

where  $\lambda > 0$ ,  $\Omega_\lambda = \Omega \cap B_\lambda$  and  $u$  is a solution to (5).

**Remark 9.** Note that  $E$  satisfies the following simple scaling relation: given  $\lambda, R > 0$ ,

$$E(u; \lambda R) = E(u_\lambda; R)$$

The main result of this section is the following monotonicity formula, which extends a result due to Pacard [14].

**Lemma 4.** Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of (5) and  $\lambda > 0$ . Let  $E$  be as in (23). Then  $\lambda \mapsto E(u; \lambda)$  is a non-decreasing function of  $\lambda$ . Furthermore,  $E(u; \cdot) \in C^1(\mathbb{R}_+^*)$  and for all  $\lambda > 0$ ,

$$\frac{dE}{d\lambda}(u; \lambda) = \lambda^{\frac{4}{p-1}+2-d} \int_{\Omega \cap \partial B_\lambda} \left( \partial_r u + \frac{2}{p-1} \frac{u}{r} \right)^2 d\sigma. \quad (24)$$

*Proof.* Fix  $\varepsilon > 0$ . Let

$$E_1^\varepsilon(u; \lambda) = \lambda^{2\frac{p+1}{p-1}-d} \int_{\Omega_\lambda \setminus \Omega_{\lambda\varepsilon}} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx. \quad (25)$$

For  $x \in \bar{\Omega}$ ,  $\lambda > 0$ , let also  $u_\lambda$  be defined by (7). Then,  $u_\lambda$  satisfies the three following properties: given  $\lambda > 0$ ,  $u_\lambda$  solves (5),

$$E_1^\varepsilon(u; \lambda) = E_1^\varepsilon(u_\lambda; 1), \quad (26)$$

and

$$\lambda \partial_\lambda u_\lambda = \frac{2}{p-1} u_\lambda + r \partial_r u_\lambda \quad \text{for a.e. } (x, \lambda) \in \Omega \times \mathbb{R}_+^*. \quad (27)$$

By standard elliptic regularity,  $u_\lambda \in C^2(\bar{\Omega}_1 \setminus \bar{\Omega}_\varepsilon)$ . So  $\lambda \mapsto E_1^\varepsilon(u; \lambda)$  is  $C^1$  and differentiating the right-hand side of (26), we find

$$\frac{dE_1^\varepsilon}{d\lambda}(u; \lambda) = \int_{\Omega_1 \setminus \Omega_\varepsilon} \nabla u_\lambda \cdot \nabla \partial_\lambda u_\lambda dx - \int_{\Omega_1 \setminus \Omega_\varepsilon} |u_\lambda|^{p-1} u_\lambda \partial_\lambda u_\lambda dx.$$

Integrating by parts and using the equation, we find

$$\frac{dE_1^\varepsilon}{d\lambda}(u; \lambda) = \int_{\Omega \cap \partial B_1} \partial_r u_\lambda \partial_\lambda u_\lambda d\sigma - \int_{\Omega \cap \partial B_\varepsilon} \partial_r u_\lambda \partial_\lambda u_\lambda d\sigma.$$

Using (27),

$$\begin{aligned} \left| \int_{\Omega \cap \partial B_\varepsilon} \partial_r u_\lambda \partial_\lambda u_\lambda d\sigma \right| &= \frac{1}{\lambda} \left| \int_{\Omega \cap \partial B_\varepsilon} \frac{2}{p-1} u_\lambda \partial_r u_\lambda + \varepsilon (\partial_r u_\lambda)^2 d\sigma \right| \\ &\leq C \|u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)} \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)} + \varepsilon \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2 \\ &\leq C \varepsilon \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2, \end{aligned}$$

where we used (the sharp) Poincaré inequality in  $H_0^1(\partial B_\varepsilon \cap \Omega)$  in the last inequality. Since  $u_\lambda \in H^1(\Omega_1)$ ,  $\liminf_{\varepsilon \rightarrow 0} (\varepsilon |\ln \varepsilon|) \|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_\varepsilon)}^2 = 0$  and so there exists a sequence  $\varepsilon_n \rightarrow 0$  such that

$$\|\nabla u_\lambda\|_{L^2(\Omega \cap \partial B_{\varepsilon_n})}^2 \leq \frac{1}{\varepsilon_n |\ln \varepsilon_n|}.$$

And so

$$\begin{aligned} C |\ln \varepsilon_n|^{-1} &\geq \left| \frac{dE_1^{\varepsilon_n}}{d\lambda}(u; \lambda) - \int_{\Omega \cap \partial B_1} \partial_r u_\lambda \partial_\lambda u_\lambda d\sigma \right| \\ &\geq \left| \frac{dE_1^{\varepsilon_n}}{d\lambda}(u; \lambda) - \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 d\sigma + \frac{2}{p-1} \int_{\Omega \cap \partial B_1} u_\lambda (\partial_\lambda u_\lambda) d\sigma \right| \\ &\geq \left| \frac{d}{d\lambda} \left( E_1^{\varepsilon_n}(u; \lambda) + \frac{1}{p-1} \int_{\Omega \cap \partial B_1} u_\lambda^2 d\sigma \right) - \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 d\sigma \right|. \end{aligned}$$

For any fixed  $\lambda > 0$ , take  $0 < \lambda_1 < \lambda_2 \leq \lambda$  and integrate the above inequality between  $\lambda_1$  and  $\lambda_2$ . Then, letting  $\varepsilon_n \rightarrow 0$ , we find that

$$E(u; \lambda_2) - E(u; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \lambda \int_{\Omega \cap \partial B_1} (\partial_\lambda u_\lambda)^2 d\sigma d\lambda. \quad (28)$$

Hence,  $\lambda \mapsto E(u; \lambda) \in C^1(\mathbb{R}_+^*)$  and scaling back the above quantity, the lemma follows.  $\square$

**Remark 10.** If  $E(u; \lambda)$  is constant with respect to  $\lambda > 0$ , then by (24) it follows that  $u$  is homogeneous of degree  $-\frac{2}{p-1}$ .

**2.3. Analysis of the limiting profile  $u_\infty$ .** In this section, we will show that  $u_\infty$  is homogeneous. Given a weak solution  $u$  of (5), if  $u_\lambda = u$  for all  $\lambda > 0$  ( $u$  is homogeneous of degree  $-\frac{2}{p-1}$ ),  $u$  must have the form

$$u(r, \theta) = r^{-\frac{2}{p-1}} v(\theta), \quad (29)$$

for some function  $v: A \subset \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ . This is a simple consequence of taking  $\lambda = |x|^{-1}$  in (7). Note that by (29),  $u_\lambda = u$  automatically excludes the possibility that  $u$  is a classical solution of (1) because of the singularity near  $x = 0$  (unless  $v = 0$ ).

Now we prove an important property of the limit function  $u_\infty$  given in Lemma 3.

**Lemma 5.** Let  $p > p_S(d)$ . The blow-down limit  $u_\infty$  is homogeneous of degree  $-\frac{2}{p-1}$  and thus has the form (29) for a suitable  $v \in H_0^1 \cap L^{p+1}(A)$ .

*Proof.* After fixing two radii  $R_2 > R_1 > 0$  we set

$$a_{\lambda_n} := E(u_{\lambda_n}; R_2),$$

$$b_{\lambda_n} := E(u_{\lambda_n}; R_1),$$

and  $c_{\lambda_n} := a_{\lambda_n} - b_{\lambda_n}$ .

We use the fundamental theorem of calculus and the monotonicity formula (24) to express  $c_{\lambda_n}$  as

$$\begin{aligned} c_{\lambda_n} &= E(u_{\lambda_n}; R_2) - E(u_{\lambda_n}; R_1) = \int_{R_1}^{R_2} \frac{dE}{d\lambda}(u_{\lambda_n}; \lambda) d\lambda = \int_{R_1}^{R_2} \frac{d}{d\lambda}(E(u_{\lambda_n}; \lambda; 1)) d\lambda \\ &= \int_{R_1}^{R_2} \int_{\partial B_1 \cap \Omega} \lambda (\partial_\lambda u_{\lambda_n})^2 d\sigma d\lambda = \int_{R_1}^{R_2} \int_{\partial B_1 \cap \Omega} \frac{1}{\lambda} \left( \frac{2}{p-1} u_{\lambda_n} \lambda + |x| \partial_r u_{\lambda_n} \right)^2 d\sigma d\lambda \\ &= \int_{R_1}^{R_2} \lambda^{\frac{4}{p-1}-d} \int_{\partial B_\lambda \cap \Omega} \left( \frac{2}{p-1} u_{\lambda_n} + |x| \partial_r u_{\lambda_n} \right)^2 d\sigma d\lambda \\ &= \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left( \frac{2}{p-1} u_{\lambda_n} + |x| \partial_r u_{\lambda_n} \right)^2 dx. \end{aligned}$$

Since  $(u_{\lambda_n})$  converges strongly in  $H^1(\Omega_R)$ , we deduce that

$$\lim_{n \rightarrow +\infty} c_{\lambda_n} = \int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left( \frac{2}{p-1} u_\infty + |x| \partial_r u_\infty \right)^2 dx.$$

Next, we prove that

$$c_{\lambda_n} \rightarrow 0.$$

By Lemma 2 (and the trace inequality  $\|u\|_{L^2(\partial B_R \cap \Omega)} \leq C(R) \|u\|_{H^1(\Omega \cap B_R)}$ ), the sequences  $(a_{\lambda_n})$  and  $(b_{\lambda_n})$  are bounded. Using the monotonicity formula (24), we deduce that  $(a_{\lambda_n})$  and  $(b_{\lambda_n})$  are nondecreasing and converge to the same (finite) limit.

Hence,

$$\int_{\Omega \cap B_{R_2} \setminus B_{R_1}} |x|^{\frac{4}{p-1}-d} \left( \frac{2}{p-1} u_\infty + |x| \partial_r u_\infty \right)^2 dx = 0,$$

and  $u_\infty$  has the form (29).  $\square$

**2.4. Homogeneous and stable solutions.** In the previous section, we have shown that the blow-down limit  $u_\infty$  of the rescaled solutions  $u_\lambda$  is a weak solution of (5), and it is homogeneous and stable. Here we study such solutions with the goal of proving that they are necessarily equal to 0.

Given a homogeneous solution  $u(r, \theta) = r^{-\frac{2}{p-1}} v(\theta)$  of (1), one can make a simple formal computation to conclude that the nonradial factor  $v$  satisfies the following equation

$$\begin{cases} -\Delta' v + \mu v = |v|^{p-1} v & \text{in } A, \\ v = 0 & \text{on } \partial A, \end{cases} \quad (30)$$

where  $\mu := \frac{2}{p-1} \left( d - 1 - \frac{p+1}{p-1} \right)$  and  $-\Delta'$  is the Laplace-Beltrami operator on the  $(d-1)$ -dimensional sphere. If  $u$  is also stable then  $v$  satisfies the following inequality for suitable test functions  $\psi$ :

$$\frac{(d-2)^2}{4} \int_A \psi^2 d\sigma + \int_A |\nabla' \psi|^2 d\sigma \geq p \int_A |v|^{p-1} \psi^2 d\sigma, \quad (31)$$

where  $\nabla'$  is the Riemannian gradient on the  $(d-1)$ -dimensional sphere, see e.g. [15, pp. 5245-5246] for the proof<sup>3</sup> of (31). We work in the class of weak solutions of (1) and make these considerations precise in the following lemma.

**Lemma 6.** *Let  $u \in H^1 \cap L^{p+1}(\Omega_R)$  be a weak, homogeneous, and stable solution of (5). Then  $v \in H_0^1 \cap L^{p+1}(A)$  is a weak solution of (30) which satisfies the stability-type estimate (31) for any  $\psi \in H_0^1(A)$ .*

*Proof.* To prove (30), we begin by testing (1) with the test function  $\varphi(r, \theta) = \chi(r)\psi(\theta)$  where  $\chi: \mathbb{R}_+ \rightarrow [0, 1]$  is a smooth, positive function compactly supported away from 0, and  $\psi \in H_0^1(A) \cap L^{p+1}(A)$ . Setting  $\alpha = \frac{2}{p-1}$ ,

$$\begin{aligned} \int_{\Omega} \nabla u \nabla \varphi \, dx &= \underbrace{\int_{\Omega} \partial_r(r^{-\alpha})v(\theta)\partial_r\chi(r)\psi(\theta) \, dx}_{I_1} + \underbrace{\int_{\Omega} r^{-\alpha-2}\chi(r)\nabla'v(\theta)\nabla'\psi(\theta) \, dx}_{I_2} \\ &= \underbrace{\int_{\Omega} r^{-\frac{2p}{p-1}}\chi(r)|v(\theta)|^{p-1}v(\theta)\psi(\theta) \, dx}_{I_3}. \end{aligned}$$

We perform a separation of variables of the three integral terms.

$$\begin{aligned} I_1 &= -\alpha \int_0^{\infty} r^{-\alpha-2+d}\partial_r\chi(r)dr \int_A v(\theta)\psi(\theta)d\sigma \\ &= \alpha(d-2-\alpha) \int_0^{\infty} r^{-\alpha-3+d}\chi(r)dr \int_A v(\theta)\psi(\theta)d\sigma. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^{\infty} r^{-\alpha-3+d}\chi(r)dr \int_A \nabla'v(\theta)\nabla'\psi(\theta)d\sigma, \\ I_3 &= \int_0^{\infty} r^{-p\alpha-1+d}\chi(r)dr \int_A |v(\theta)|^{p-1}v(\theta)\psi(\theta)d\sigma. \end{aligned}$$

Since  $\alpha + 2 = p\alpha$ , we can cancel all of the integral terms involving  $r$  to conclude

$$\int_A \nabla'v\nabla'\psi \, d\sigma + \alpha(d-2-\alpha) \int_A v\psi \, d\sigma = \int_A |v|^{p-1}v\psi \, d\sigma. \quad (32)$$

That is,  $v \in H_0^1 \cap L^{p+1}(A)$  is a weak solution of (30).  $\square$

We conclude this section by proving a Liouville-type result for weak solutions of (30) which satisfy estimate (31), under the additional condition for  $p > 1$

$$p\mu - \frac{(d-2)^2}{4} + (p-1)\lambda_1 \geq 0, \quad (33)$$

where  $\lambda_1$  is the first eigenvalue of the Laplace-Beltrami operator on  $A$ . Note that the term  $(p-1)\lambda_1$  in (33) gives an improvement of the condition in [10]:  $1 < p < p_{JL}(d) \iff p\mu - \frac{(d-2)^2}{4} > 0$ . In other words, the condition  $1 < p < p_{JL}(d)$  is optimal only for stable solutions in the whole space  $\mathbb{R}^d$ .

**Lemma 7.** *Let  $v \in H_0^1 \cap L^{p+1}(A)$  be a weak solution of (30) which satisfies the stability-type estimate (31). Then for all  $p$  satisfying (33),  $v = 0$ .*

*Proof.* First, we take the solution  $v$  (multiplied by  $p$ ) as a test function in (32) to obtain

$$p\mu \int_A v^2 \, d\sigma + p \int_A |\nabla'v|^2 \, d\sigma = p \int_A |v|^p \, d\sigma.$$

Likewise, testing (31) with  $v$  yields

$$\frac{(d-2)^2}{4} \int_A v^2 \, d\sigma + \int_A |\nabla'v|^2 \, d\sigma \geq p \int_A |v|^p \, d\sigma.$$

Combining these two lines gives

$$\left(p\mu - \frac{(d-2)^2}{4}\right) \int_A v^2 \, d\sigma + (p-1) \int_A |\nabla'v|^2 \, d\sigma \leq 0.$$

We use the trivial identity

$$\int_A |\nabla'v|^2 \, d\sigma = - \left( \lambda_1 \int_A v^2 \, d\sigma - \int_A |\nabla'v|^2 \, d\sigma \right) + \lambda_1 \int_A v^2 \, d\sigma$$

<sup>3</sup>In [15, pp. 5245-5246],  $\psi$  is supposed to be smooth but the result remains valid for  $\psi \in H_0^1(A) \cap L^{p+1}(A)$  with the same proof and can be extended to any  $\psi \in H_0^1(A)$  by Fatou's lemma.



to deduce the following:

$$\left(p\mu - \frac{(d-2)^2}{4} + \lambda_1(p-1)\right) \int_A v^2 d\sigma \leq (p-1) \int_A (\lambda_1 v^2 - |\nabla' v|^2) d\sigma \leq 0, \quad (34)$$

where the final inequality follows from Poincaré's inequality on  $A$  with optimal constant  $\lambda_1$ . By (33) and (34),  $v$  is a principle eigenfunction of the Laplace-Beltrami operator on  $A$ . Combining this with the fact that  $v$  also satisfies (30) leads trivially to the conclusion that  $v = 0$ .  $\square$

**2.5. A Pohožaev-type result on subdomains of  $\mathbb{S}_+^{d-1}$ .** The following Pohožaev-type result is proven by Bidaut-Véron, Ponce, and Véron in [1, Theorem 2.1] for smooth solutions  $v$  to (30). Since we deal with weak solutions, we construct smooth approximations  $v_\lambda$  of  $v$  and use the  $H_{loc}^1$  convergence of  $u_\lambda$  to  $u_\infty$  (along a sequence) to prove that the inequality also holds for the blow-down limit  $u_\infty$ .

**Lemma 8.** *Let  $A$  be a  $C^{2,\alpha}$  domain of  $\mathbb{S}_+^{d-1}$  which is star-shaped with respect to the north pole, and let  $u_\infty(r, \theta) = r^{-\frac{2}{p-1}} v(\theta)$  be the blow-down limit of  $(u_\lambda)$  as above. Then  $v \in H_0^1(A)$  satisfies*

$$\left(\frac{d-3}{2} - \frac{d-1}{p+1}\right) \int_A |\nabla' v|^2 \phi d\sigma - \frac{d-1}{2} \left(\frac{d-\mu(p-1)-1}{p+1}\right) \int_A v^2 \phi d\sigma \leq 0, \quad (35)$$

where  $\nabla'$  is the tangential gradient to  $\mathbb{S}_+^{d-1}$ , and  $\phi(\theta) = \theta_d$  is an eigenfunction of the Laplace-Beltrami operator  $-\Delta'$  in  $H_0^1(\mathbb{S}_+^{d-1})$  associated to the principal eigenvalue  $\lambda_1(\mathbb{S}_+^{d-1}) = d-1$ .

*Proof.* We begin with a simple geometric lemma. Denote by  $\langle \cdot, \cdot \rangle$  the inner product on the tangent space to  $\mathbb{S}^{d-1}$ ,  $\phi(\theta) = \theta_d$ , and  $\nabla'\phi$  its Riemannian gradient on  $\mathbb{S}^{d-1}$ . Then,

**Lemma 9.** *Let  $A \subset \mathbb{S}_+^{d-1}$  denote a  $C^1$  open set with outward unit normal direction  $\nu$ . If  $A$  is star-shaped with respect to the north pole, there holds  $\langle \nabla'\phi, \nu \rangle \leq 0$  on  $\partial A$ .*

*Proof.* Apply the stereographic projection  $\pi_S$  from the south pole to the set  $A$ . Then take a point  $x \in \pi_S(A) \subset B_1 \subset \mathbb{R}^{d-1}$  and consider the unique minimal geodesic  $\gamma_\theta$  contained in  $A$  connecting the north pole to  $\pi_S^{-1}(x) =: \theta$ . Its projection  $\pi_S(\gamma_\theta)$  is a straight line connecting the origin to  $x$  and is contained in  $\pi_S(A)$ . So  $\pi_S(A) \subset \mathbb{R}^{d-1}$  is star-shaped (in the usual Euclidean sense). Since the stereographic projection is conformal (it preserves angles),  $\langle \nabla'\phi, \nu \rangle$  has the same sign as  $V \cdot n$ , where  $V = d\pi_S(\nabla'\phi) = \nabla\left(\frac{1-r^2}{1+r^2}\right) = \frac{-4r}{(1+r^2)^2} \frac{x}{r}$ ,  $x \in \mathbb{R}^{d-1}$ ,  $r = |x|$ , and  $n = d\pi_S(\nu)$  is the outward unit normal direction of  $\pi_S(A)$ . In view of the Lemma in [8] p. 554, we conclude that  $\langle \nabla'\phi, \nu \rangle \leq 0$  on  $\partial A$ .  $\square$

Let us return to the proof of Lemma 8. To simplify the appearance we set  $C_1 = \left(\frac{d-3}{2} - \frac{d-1}{p+1}\right)$  and  $C_2 = \frac{d-1}{2} \left(\frac{d-\mu(p-1)-1}{p+1}\right)$ . Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of (5). Since for any  $\lambda > 0$ ,  $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x)$  is also a smooth solution of (5), we set  $v_\lambda$  to satisfy

$$u_\lambda(x) = r^{-\frac{2}{p-1}} v_\lambda(r, \theta).$$

Plugging this expression of  $u_\lambda$  into (1), it follows that  $v_\lambda$  is a classical solution of the following equation:

$$\begin{cases} -\Delta' v_\lambda + \mu v_\lambda = |v_\lambda|^{p-1} v_\lambda + e_\lambda & \text{in } \Omega, \\ v_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \quad (36)$$

where the error term  $e_\lambda$  is

$$e_\lambda = \left(d-1 - \frac{4}{p-1}\right) r \partial_r v_\lambda + r^2 \partial_r^2 v_\lambda.$$

Again for simplicity, we define  $C_3 = \left(d-1 - \frac{4}{p-1}\right)$ .

Following [1, pp. 186-188], we apply the divergence theorem to the vector field

$$P = \langle \nabla'\phi, \nabla' v_\lambda \rangle \nabla' v_\lambda,$$

to eventually reach the following relation

$$\begin{aligned} C_1 \int_A |\nabla' v_\lambda|^2 \phi d\sigma + C_2 \int_A v_\lambda^2 \phi d\sigma - \frac{d-1}{p+1} \int_A e_\lambda v_\lambda \phi d\sigma + \int_A \langle \nabla'\phi, \nabla' v_\lambda \rangle e_\lambda d\sigma \\ = \frac{1}{2} \int_{\partial A} |\nabla' v_\lambda|^2 \langle \nabla'\phi, \nu \rangle d\tau \leq 0, \end{aligned} \quad (37)$$

where  $\nu$  is the unit outer normal. The right-hand side of (37) is nonpositive since  $A$  is star-shaped. Next, we multiply all terms of (37) by a standard cutoff function  $\eta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  compactly supported away from 0 and we

integrate in the radial direction (including the factor  $r^{d-1}$ ). We begin with the first error term:

$$\begin{aligned} \int_0^\infty \int_A e_\lambda v_\lambda \phi r^{d-1} \eta \, d\sigma \, dr &= C_3 \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi r^d \eta \, d\sigma \, dr + \int_0^\infty \int_A (\partial_r^2 v_\lambda) v_\lambda \phi r^{d+1} \eta \, d\sigma \, dr \\ &= C_3 \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi r^d \eta \, d\sigma \, dr - \int_0^\infty \int_A (\partial_r v_\lambda)^2 \phi r^{d+1} \eta \, d\sigma \, dr - \int_0^\infty \int_A (\partial_r v_\lambda) v_\lambda \phi \partial_r (r^{d+1} \eta) \, d\sigma \, dr, \end{aligned}$$

where we have integrated by parts in the second integral of the first line. Now, by the fact that  $v_\lambda \rightarrow v = r^{\frac{2}{p-1}} u_\infty$  in  $H^1(\Omega_R)$  (along a sequence) and that  $v$  does not depend on  $r$ , we can pass to the limit as  $\lambda \rightarrow \infty$  to conclude

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \int_A e_\lambda v_\lambda \phi r^{d-1} \eta \, d\sigma \, dr = 0.$$

For the second error term in (37), we proceed as follows:

$$\begin{aligned} \int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle e_\lambda r^{d-1} \eta \, d\sigma \, dr &= \int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle (C_3 r \partial_r v_\lambda + r^2 \partial_r^2 v_\lambda) r^{d-1} \eta \, d\sigma \, dr \\ &= C_3 \underbrace{\int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^d \eta \, d\sigma \, dr}_{I_1} \\ &\quad - \underbrace{\int_0^\infty \int_A \langle \nabla' v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda \partial_r (r^{d+1} \eta) \, d\sigma \, dr}_{I_2} \\ &\quad - \underbrace{\int_0^\infty \int_A \langle \nabla' \partial_r v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^{d-1} \eta \, d\sigma \, dr}_{I_3}, \end{aligned}$$

where we have integrated by parts in the  $r$  variable. Since  $v_\lambda \rightarrow v$  in  $H^1(\Omega_R)$  (along a sequence) and since  $v$  does not depend on  $r$ ,  $I_1$  and  $I_2$  tend to zero as  $\lambda \rightarrow \infty$ . We deal with  $I_3$  by recalling that  $\phi$  is an eigenfunction and so

$$\begin{aligned} - \int_0^\infty \int_A \langle \nabla' \partial_r v_\lambda, \nabla' \phi \rangle \partial_r v_\lambda r^{d-1} \eta \, d\sigma \, dr &= -\frac{1}{2} \int_0^\infty \int_A \langle \nabla' (\partial_r v_\lambda)^2, \nabla' \phi \rangle r^{d-1} \eta \, d\sigma \, dr \\ &= -\frac{d-1}{2} \int_\Omega (\partial_r v_\lambda)^2 \phi \eta \, dx, \end{aligned}$$

where in the latter we have used an integration by parts over  $A$  and the fact that  $\partial_r v_\lambda = 0$  on  $\partial\Omega \setminus \{0\}$ .

This term then tends to 0 as  $\lambda \rightarrow \infty$ , since  $v_\lambda$  converges to  $v$  in  $H^1(\Omega_R)$  (along a sequence).

Now we deal with the other solid integral terms arising from integrating (37) with the test function  $r^{d-1} \eta$  in the  $r$  variable. Since  $u_\lambda$  converges to  $u$  in  $H^1(\Omega_R)$  for every  $R > 0$  along a sequence, so does  $v_\lambda$  to  $v$ . Since  $v$  does not depend on  $r$ , we can apply Fubini's theorem at the limit to obtain

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \int_A (C_1 |\nabla' v_\lambda|^2 + C_2 v_\lambda^2) \phi r^{d-1} \eta \, d\sigma \, dr = \int_0^\infty r^{d-1} \eta \, dr \int_A (C_1 |\nabla' v|^2 + C_2 v^2) \phi \, d\sigma \leq 0,$$

where  $C_1$  and  $C_2$  are the fixed constants defined at the beginning of the proof. Dividing by the positive constant  $\int_0^\infty r^{d-1} \eta \, dr$  yields (35).  $\square$

## 2.6. Proof of Theorem 2.

*Proof.* By an elementary computation,  $p_S(d) < p_S(d-1) < p_{JL}(d)$ . Recall also that if  $p \leq p_{JL}(d)$ , then (33) holds. So all  $p > p_S(d)$  either satisfy (33),  $p > p_S(d-1)$ , or both. Suppose first that (33) holds. By Lemmas 5, 6, and 7, it follows that  $u_\infty = 0$ . In the remaining case  $p > p_S(d-1)$ , Lemma 8 implies that  $v = 0$  (which of course implies that  $u_\infty = 0$ ). Then, since  $(u_{\lambda_n})$  converges strongly to  $u_\infty$  in  $H^1 \cap L^{p+1}(\Omega_R)$ , it follows that  $E(u_{\lambda_n}; 1) \rightarrow 0$  as  $\lambda_n \rightarrow \infty$ . Furthermore, by the monotonicity formula (24), the convergence to 0 is monotone non-decreasing.

To conclude, we study the behavior of  $E(u_\lambda; 1) = E(u; \lambda)$  as  $\lambda \rightarrow 0^+$ . By definition (23) of the monotonicity formula, we have

$$E(u, \lambda) \geq -\frac{1}{p+1} \lambda^{\frac{2p+1}{p-1}-d} \int_{\Omega_\lambda} |u|^{p+1} \, dx = -c \lambda^{\frac{2p+1}{p-1}} \int_{\Omega_\lambda} |u|^{p+1} \, dx$$

Since  $u$  is bounded in a neighbourhood of 0, we deduce that  $\lim_{\lambda \rightarrow 0^+} E(u; \lambda) \geq 0$ . Since  $E(u, \cdot)$  is nondecreasing, we deduce that

$$E(u; \lambda) \geq 0 \quad \text{for all } \lambda > 0 \tag{38}$$

Now,  $\lim_{\lambda \rightarrow +\infty} E(u; \lambda) = E(u_\infty; 1) = 0$  and so, since  $E(u, \cdot)$  is nondecreasing and nonnegative,  $E(u; \lambda) = 0$  for every  $\lambda > 0$ . By (24),  $u$  is homogeneous, which is possible only if  $u = 0$  since  $u \in C(\bar{\Omega})$ .  $\square$

## 3. PROOFS OF COROLLARY 1 AND COROLLARY 2

**3.1. Proof of Corollary 1.** We just need to inspect the proof of Theorem 1 and adapt it as follows. Let  $u$  be a nonnegative weak stable solution. By (the proof of) Proposition 13 in [6], for every  $R > 0$ , there exists a sequence of solutions  $u_n^R \in C^2(\overline{B_R^+})$  such that  $u_n^R = 0$  on  $\partial\mathbb{R}_+^d \cap B_R$ ,  $u_n^R \nearrow u$  a.e. in  $\Omega_R$ , and  $u_n^R \rightarrow u$  in  $H^1(\Omega_R)$ , as  $n \rightarrow +\infty$ . So choosing  $R > \lambda_2$ , we may apply (28) to  $u_n^R$ , pass to the limit as  $n \rightarrow +\infty$  and conclude that the monotonicity formula  $E(u; \lambda)$  is monotone, absolutely continuous and its derivative is given by (24) for a.e.  $\lambda > 0$ . Regarding Pohozaev's identity Lemma 8, we use again [6] to deduce that the blow-down limit  $u_\infty$  is also the limit (in  $H^1(B_R^+)$ ) of classical solutions (with zero value on  $\partial\mathbb{R}_+^d \cap B_R$ ) and so the same proof applies. Finally, we adapt Section 2.6 by observing that using the same approximating procedure, (38) remains true i.e.  $E(u; \lambda)$  is nonnegative. As before, this and the blow-down analysis force  $u$  to be homogeneous, hence  $u = 0$  by our classification of homogeneous weak solutions.  $\square$

**3.2. Proof of Corollary 2.** In the subcritical case, the proof of Theorem 9 in [10], stated for smooth domains, remains valid in our setting, thanks to Lemma 1. In the supercritical case  $p > p_S(d)$ , the proof is identical to that of Theorem 2. We only need to observe that in the range (33) of exponents  $p$ , no geometrical condition on the cone  $\Omega$  is needed.  $\square$

**3.3. Positive solutions in convex cones.** In this last section, we explain more carefully how Busca's result [2] for positive classical solutions defined on convex cones can be recovered thanks to Theorem 2. This is the content of Lemma 12 below. First, we extend the following natural result proven in [11, Proposition 2] in the case where  $\Omega$  is convex.

**Lemma 10.** *Let  $\Omega$  be defined as in (6) for some  $A \subset \mathbb{S}_+^{d-1}$ . Then  $\Omega$  is convex (resp. star-shaped) if and only if  $A$  is geodesically convex (resp. geodesically star-shaped).*

*Proof.* Let  $\Omega$  be convex. Now take two points  $\varphi_1, \varphi_2 \in A$ . (By the hypothesis that  $A \subset \mathbb{S}_+^{d-1}$ ,  $\varphi_1$  and  $\varphi_2$  may not be antipodal points.) By convexity,  $t\varphi_2 + (1-t)\varphi_1 \in \Omega$ , so

$$S_t = \frac{t\varphi_2 + (1-t)\varphi_1}{|t\varphi_2 + (1-t)\varphi_1|} \in A$$

for all  $t \in [0, 1]$  by the cone property of  $\Omega$ . It can be directly verified, for example using a stereographic projection from  $-\varphi_2$ , that  $\{S_t\}_{0 \leq t \leq 1}$  coincides with the unique geodesic curve from  $\varphi_1$  to  $\varphi_2$ . Finally, since  $\{S_t\}_{t > 0} \subset A$ , we have that  $A$  is geodesically convex.

Conversely, let us choose two general points  $x_1, x_2 \in \Omega$ . In the same way as above, we can show that

$$\left\{ \frac{tx_2 + (1-t)x_1}{|tx_2 + (1-t)x_1|} \right\}_{0 \leq t \leq 1} \subset A$$

is a geodesic curve on  $A$ , whereby we find that  $tx_2 + (1-t)x_1 \in \Omega$  due to the cone property of  $\Omega$ .

In the case of star-shaped domains, we repeat the above method with  $\varphi_2$  and  $x_2$  fixed taken to be the points with respect to which  $A$  and  $\Omega$ , respectively, are star-shaped.  $\square$

**Lemma 11.** *If  $A \subset \mathbb{S}_+^{d-1}$  is star-shaped with respect to some direction  $\varphi_0 \in A$ , then  $\Omega$  is convex in that direction in the sense that for all  $x \in \Omega$  and  $t > 0$ ,  $x + t\varphi_0 \in \Omega$ .*

*Proof.* First, by virtue of the radial scaling property of cones, it is enough to prove the result for all  $x = \varphi_1 \in A$ . It can then be verified that the curve

$$S = \{S_t\}_{t > 0} = \left\{ \frac{\varphi_1 + t\varphi_0}{|\varphi_1 + t\varphi_0|} \right\}_{t > 0} \subset \mathbb{S}_+^{d-1},$$

coincides with the unique minimal geodesic connecting  $\varphi_1$  to  $\varphi_0$  (for example by taking the stereographic projection from  $-\varphi_0$ ). Then, since  $A$  is star-shaped with respect to  $\varphi_0$ ,  $S_t \in A$  for all  $t$ , whereby  $x + t\varphi_0 \in \Omega$  due to the cone property of  $\Omega$ .  $\square$

**Lemma 12.** *A convex set  $A \subset \mathbb{S}_+^{d-1}$  is star-shaped with respect to the north pole and still contained in  $\mathbb{S}_+^{d-1}$  after taking a suitable rotation.*

*Proof.* If the north pole  $\vec{n}$  belongs to  $A$  the statement is trivially proven because convex sets are in particular star-shaped. Therefore, let us assume that  $\vec{n}$  does not belong to  $A$ . Then  $\Omega \cap \langle \vec{n} \rangle = \emptyset$  by the cone property of  $\Omega$ , and  $\Omega$  is convex by Lemma 10. So by the geometric form of the Hahn-Banach theorem, there exists a hyperplane  $H$  normal to some  $\nu \in \mathbb{S}^{d-1}$  (w.l.o.g. take  $\nu = (-1, 0, \dots, 0)$ ) which includes the  $x_d$  axis and which satisfies  $H \cap \Omega = \emptyset$ . By the convexity assumption,  $\Omega$  (resp.  $A$ ) must lie on one side of  $H$  (resp.  $H \cap \mathbb{S}_+^{d-1}$ ). That is, w.l.o.g.  $A \subset \mathbb{S}_+^{d-1} \cap \{x_1 > 0\}$ .

Let us consider the great circle  $G = \{(x_1, 0, \dots, 0, x_d) : x_i \in [-1, 1], |x| = 1\}$ . Up to a suitable modification of the choice of  $\nu$ , we can assume (by convexity of  $A$ ) that there is a point  $a = (\cos \theta_a, 0, \dots, 0, \sin \theta_a) \in G \cap A$

for its corresponding  $\theta_a \in (0, \pi/2)$ . Then we take the rotation that sends  $a$  to  $\vec{n}$ . This rotation has a specific form: for a general  $p = (\alpha_p \cos \theta, x_2, \dots, x_{d-1}, \alpha_p \sin \theta) \in A$  for its corresponding  $\alpha_p \in [0, 1]$ ,  $\theta \in (0, \pi/2)$ , we have

$$p \mapsto \left( \alpha_p \cos \left( \frac{\pi}{2} - \theta_a + \theta \right), x_2, \dots, x_{d-1}, \alpha_p \sin \left( \frac{\pi}{2} - \theta_a + \theta \right) \right).$$

Since  $\theta, \theta_a \in (0, \pi/2)$ , it can be verified that the image of  $A$  under this rotation is still contained in  $\mathbb{S}_+^{d-1}$ .  $\square$

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