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Regularizing effect for unbounded flux-limited viscosity solutions of a discontinuous Hamilton-Jacobi equation on junction

N. El Khatib,¹ N . Forcadel,² M. Zaydan¹

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Abstract

In this paper, we provide regularizing effect for continuous bounded from below viscosity solutions of a discontinuous Hamilton-Jacobi equation posed on a junction. We consider different quasi-convex and coercive time-space Hamiltonians on each branch and a flux limiter condition at the junction point. We then prove that the derivative with respect to time of the solution is bounded. As consequence, we deduce that the solution of the equation is locally Lipschitz continuous.

AMS Classification: 35F21; 35D35; 35D40

Keywords: Regularizing effect, viscosity solutions, discontinuous Hamilton-Jacobi on a junction

1 Introduction

The term regularizing effect indicates that the solution of a non-linear PDE becomes more regular for $t > 0$ than it is at the initial time $t = 0$. Such effects were studied in the framework of first order continuous Hamilton-Jacobi equations in many papers like [3, 6, 12, 15]. In these works, the regularizing effect is obtained for Hamiltonian H satisfying that $(H_p \cdot p - H)(p)$ is large when $|p|$ is large or when $H(p)$ is large. This regularizing effect can be used to study the large time behavior of viscosity solutions. For example, in [6] the authors proved that the viscosity solution u satisfy $u_t \geq -\eta(t)e^u$. This result implies first more regularity in space: local Lipschitz continuity if the Hamiltonian is coercive and Holder regularity for hypo-elliptic Hamiltonian. As a second application, this result can be used in the study of the large time behavior of the viscosity solution if $\eta(+\infty) = 0$. For more details, the reader can refer to [6, 7, 4]. Let us mention also the work [15] where authors exploited the regularizing effect for a Hamilton-Jacobi equation in order to study the large time behavior of the stochastic viscosity solution.

In this paper, we want to develop regularizing results for discontinuous Hamilton-Jacobi equations. The study of discontinuous (in space) Hamilton-Jacobi equations is recent (see [1, 11]) but the basic results, like comparison principle, existence, stability, homogenization are now well understood (see [10, 13, 5, 9, 16, 2, 14, 8]) and we are now able to attack more difficult problems like the regularity or the regularizing effect.

More precisely, in this paper, we consider regularizing effect for a Hamilton-Jacobi posed on a junction, which is a metric space formed by the union of a N branches J_1, J_2, \dots, J_N glued to one

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point called *junction point*. For $i \in \{1, \dots, N\}$, the branch J_i is isometric to \mathbb{R}^+ and the junction point is $x = 0$. We denote by J the junction and it is defined by

$$J = \bigcup_{i=1, \dots, N} J_i \quad \text{with } J_i \neq J_j \text{ if } i \neq j.$$

To be more precise, the definition of elements in J is given by the following definition: if $x \in J$, then

$$x = x_i \cdot e_i, x_i > 0 \text{ if } x \in J_i^* = J_i \setminus \{0\} \quad \text{or} \quad x = (0, 0)$$

where e_i is a unit vector in \mathbb{R}^2 and $e_i \neq e_j$ if $i \neq j$. We define on the junction J the geodesic distance by

$$d(x, y) = \begin{cases} |x_i - y_i| & \text{if } x, y \in J_i^*, \\ |x_i| + |y_j| & \text{if } x \in J_i \text{ and } y \in J_j \text{ with } i \neq j. \end{cases}$$

For $T > 0$, let $u : [0, T] \times J \rightarrow \mathbb{R}$ be a real function. The space gradient of u at $(t, x) \in [0, T] \times J$ is defined by

$$u_x(t, x) = \begin{cases} \partial_i u(t, x) & \text{if } x \in J_i^*, \\ (\partial_1 u(t, 0), \dots, \partial_N u(t, 0)) & \text{if } x = 0 \end{cases}$$

where $\partial_i u(t, x)$ is the derivative of u with respect to $x \in J_i^*$. We consider a continuous bounded by below viscosity solution of the following Hamilton-Jacobi with flux limited condition at the point 0 introduced in [10] and given by

$$\begin{cases} u_t + H_i(t, x, u_x) = 0 & \text{if } (t, x) \in [0, T] \times J_i^*, \\ u_t + F_A(t, 0, u_x) = 0 & \text{if } t \in [0, T], \end{cases} \quad (1.1)$$

where for each $i \in \{1, \dots, N\}$, the Hamiltonian H_i satisfies the following assumptions:

$$\begin{cases} H_i \in C([0, T] \times J_i \times \mathbb{R}), \\ \text{for all } (t, x) \in [0, T] \times J_i, \text{ the Hamiltonian } H_i(t, x, \cdot) \text{ is quasi-convex,} \\ \text{for all } (t, x) \in [0, T] \times J_i, \text{ the Hamiltonian } H_i(t, x, \cdot) \text{ is coercive.} \end{cases}$$

For $t \in [0, T]$ and $p = (p_1, \dots, p_N) \in \mathbb{R}^N$, the flux limiter function F_A is defined by

$$F_A(t, 0, p) = \max \left(A(t), \max_{i=1, \dots, N} H_i^-(t, 0, p_i) \right) \quad (1.2)$$

where $H_i^-(t, 0, \cdot)$ is the non-increasing part of $H_i(t, 0, \cdot)$. The function A is called the "flux limiter" and we assume that $A \in C([0, T])$. Moreover, we suppose that for $t \in [0, T]$,

$$A(t) \geq \max_{i \in \{1, \dots, N\}} \min_{p \in \mathbb{R}} H_i(t, 0, p).$$

Additional assumptions (H) (see below) are required to obtain a regularizing effect. These assumptions are satisfied for example by the following Hamiltonians,

$$H_i(t, x, p) = |A_i(t, x)|^{m_i} |p|^{m_i} - f_i(t, x)$$

where $m_i > 1$, f_i is non negative (or bounded by below), A_i and f_i are space-time Lipschitz continuous functions and there exists $0 < \delta \leq 1$ such that

$$\delta \leq |A_i(t, x)| \leq \frac{1}{\delta} \text{ for } (t, x) \in [0, T] \times J_i.$$

More precisely, we will provide a lower bound for the time derivative of the viscosity solution u . Up to constant addition, we can assume that $u \geq 0$. We will prove that there exists a time $t_* > 0$ and function η defined on $(0, t_*)$ such that for all $t \in (0, t_*)$ and $x \in J$, we have

$$u_t \geq -\eta(t)e^{u(t,x)}. \quad (1.3)$$

Using inequality (1.3), we can prove that u is locally Lipschitz continuous in $J \times (0, t_*)$ (see Proposition 2.2). To obtain (1.3), we will first provide a lower gradient bound on each branch of the junction using the assumption $((H_i)_p \cdot p - H_i)(t, x, p)$ is positive and large when p is negative and small (Lemma 4.3). This bound, joint to the idea of studying the sign of the derivative of $d^2(x, y)$ used in the proof of the comparison principle in [5] will allow us to obtain our main result by exploiting the assumption $((H_i)_p \cdot p - H_i)(t, x, p)$ is positive and large when $H_i(t, x, p)$ is positive and large.

Organization of the paper In section 2, we state the assumptions imposed on the Hamiltonians and then we give the main result (Theorem 2.1) of this paper. In section 3, we derive consequences from assumptions of section 2 and then we give examples of Hamiltonians. In section 4, first, using the variable change $v = -e^{-u}$, we obtain a new Hamilton-Jacobi formulation and we derive properties of the new Hamiltonians G_i . Then, we prove the main result of this paper using this new equation (Theorem 4.2). In the last section, as application of Theorem 2.1, we show that our solution is locally Lipschitz.

2 Assumptions on the Hamiltonians and main results

2.1 Assumptions (H)

Our main goal is to process the proof in the vicinity of the junction point $x = 0$. Far from it, we work with one type of Hamiltonian and we can impose weaker assumptions (like non-coercivity) on the Hamiltonians. To be more precise, far from the junction point, we can use the assumptions on Hamiltonians provided in [6]. Therefore, all the assumptions we will see below can be imposed on a neighborhood of the junction point. For simplicity, we assume that they are satisfied for all $x \in J$. Let $u : [0, T] \times J \rightarrow \mathbb{R}$ be a continuous, non-negative viscosity solution of (1.1). We set

$$C_u = \min_{t \in [0, T]} (e^{-u(t,0)}) > 0. \quad (2.1)$$

We state now the assumptions concerning the Hamiltonians.

(H0) For all $i \in \{1, \dots, N\}$ and for all $t \in [0, T]$, $x \in J_i$, the Hamiltonian $H_i(t, x, \cdot)$ is quasi-convex and coercive on \mathbb{R} and H_i is continuous in $[0, T] \times J_i \times \mathbb{R}$.

- 1) We assume that there exists an interval $[p_i^-(t, x), p_i^+(t, x)]$ such that

$$\begin{cases} H_i(t, x, \cdot) \text{ is non increasing on } (-\infty, p_i^-(t, x)), \\ H_i(t, x, \cdot) \text{ is constant on } [p_i^-(t, x), p_i^+(t, x)], \\ H_i(t, x, \cdot) \text{ is non decreasing on } (p_i^+(t, x), +\infty). \end{cases}$$

Moreover, we assume that p_i^- is bounded in $[0, T] \times J_i$ and that p_i^+ is continuous in $[0, T] \times J_i$.

- 2) We assume that for all $i \in \{1, \dots, N\}$, there exists a continuous, quasi-convex and coercive function h_i such that for all $t \in [0, T]$ and $p \in \mathbb{R}$,

$$H_i(t, 0, p) \leq h_i(p).$$

For $i \in \{1, \dots, N\}$, there exists an interval $[p_{h_i}^-, p_{h_i}^+]$ such that

$$\begin{cases} h_i \text{ is non increasing on } (-\infty, p_{h_i}^-), \\ h_i \text{ is constant on } [p_{h_i}^-, p_{h_i}^+], \\ h_i \text{ is non decreasing on } (p_{h_i}^+, +\infty). \end{cases}$$

3) For all $i \in \{1, \dots, N\}$, there exists a constant c_i such that

$$c_i \geq \max \left(\max_{t \in [0, T], x \in J_i} |p_i^-(t, x)|, |p_{h_i}^-| C_u \right)$$

such that:

- the function H_i is locally Lipschitz continuous in a neighborhood of the set $\{(t, x, p) \in [0, T] \times J_i \times \mathbb{R}; p \leq -c_i\}$.
- There exists a continuous, positive and decreasing function ϕ_i defined on $(-\infty, -c_i)$ such that for some $B_i > c_i$,

$$\int_{B_i}^{+\infty} \frac{1}{s \phi_i(-s)} ds < +\infty \quad (2.2)$$

and for almost all $t \in [0, T]$, $x \in J_i$ and $p \in \mathbb{R}$,

$$((H_i)_p \cdot p - H_i)(t, x, p) \geq \phi_i(p) \quad \text{a.e. in } \{(t, x, p) \in [0, T] \times J_i \times \mathbb{R}; p \leq -c_i\}. \quad (2.3)$$

- There exists $\kappa_i > 0$ such that

$$|(H_i)_p(t, x, p)| \leq \kappa_i \cdot ((H_i)_p \cdot p - H_i)(t, x, p) \quad \text{a.e. in } \{(t, x, p) \in [0, T] \times J_i \times \mathbb{R}; p \leq -c_i\}.$$

(H1) Let $C \geq 0$ satisfying the following property: for all $i \in \{1, \dots, N\}$,

$$h_i(p) \geq C \geq \min_{p \in \mathbb{R}} h_i(p) \Leftrightarrow p \geq \max_{t \in [0, T]} p_i^+(t, 0) \text{ or } p \leq \min_{t \in [0, T]} p_i^-(t, 0).$$

There exists a constant c satisfying

$$c \geq \max \left(C, \max_{t \in [0, T]} |A(t)|, \max_{t \in [0, T], i \in \{1, \dots, N\}} h_i(p_i^-(t, 0)), \max_{i \in \{1, \dots, N\}} h_i \left(-\frac{c_i}{C_u} \right) \right)$$

such that

- for all $i \in \{1, \dots, N\}$, the function H_i is locally Lipschitz continuous in a neighborhood of the set $\{(t, x, p); H_i(t, x, p) \geq c\}$.
- There exists a continuous, positive and increasing function ϕ defined on $(c, +\infty)$ such that for some $B > c$,

$$\int_B^{+\infty} \frac{1}{s \phi(s)} ds < +\infty \quad (2.4)$$

and for all $i \in \{1, \dots, N\}$, and for almost all $t \in [0, T]$, $x \in J_i$ and $p \in \mathbb{R}$,

$$((H_i)_p \cdot p - H_i)(t, x, p) \geq \phi(H_i(t, x, p)) \quad \text{a.e. in } \{p; H_i(t, x, p) \geq c\}. \quad (2.5)$$

- There exists $\kappa > 0$ such that for all $i \in \{1, \dots, N\}$,

$$|(H_i)_p(t, x, p)| \leq \kappa \cdot ((H_i)_p \cdot p - H_i)(t, x, p) \quad \text{a.e. in } \{p; H_i(t, x, p) \geq c\}.$$

(H2) For all $i \in \{1, \dots, N\}$, we define the following set

$$I_i = \{p \in \mathbb{R}; \quad p \leq p_{h_i}^- \text{ and } h_i(p) \geq c\}.$$

Then, we have that

$$\phi(h_i(p)) h_i(p) < \phi_i(C_u p) h_i'(p) p \quad \text{a.e. in } I_i.$$

The next two assumptions concern the dependence of the Hamiltonians on the space and the time.

(H3) For all $i \in \{1, \dots, N\}$, there exists $t_i > 0$ such that H_i is locally Lipschitz in x in a neighborhood of the set $\{(t, x, p); t \in [0, t_i], x \in J_i, p \leq -\eta_i(t)\}$ and for all $t \in [0, T], x \in J_i$ and $p \in \mathbb{R}$,

$$|(H_i)_x(t, x, p)| \leq \frac{1}{2} \eta_i(t) (p(H_i)_p - H_i)(t, x, p) \quad \text{a.e in } \{(t, x, p); t \in [0, t_i], x \in J_i, p \leq -\eta_i(t)\}$$

(H4) For all $i \in \{1, \dots, N\}$, there exists $\tau_i > 0$ such that H_i is locally Lipschitz in t in a neighborhood of the set $\{(t, x, p) \in [0, \tau_i] \times J_i \times \mathbb{R}; H_i(t, x, p) \geq \eta(t)\}$ and for all $t \in [0, T], x \in J_i$ and $p \in \mathbb{R}$,

$$|(H_i)_t(t, x, p)| \leq \frac{1}{2} \psi(\eta(t)) (p(H_i)_p - H_i)(t, x, p) \quad \text{a.e in } \{(t, x, p) \in [0, \tau_i] \times J_i \times \mathbb{R}; H_i(t, x, p) \geq \eta(t)\}$$

where ψ is defined in (3.3).

2.2 Main results

The main result of this paper is given by the following theorem.

Theorem 2.1. *Assume (H). Let u be a continuous, non-negative viscosity solution of (1.1). Then, there exists t^* and a continuous function η such that for all $t \in (0, t^*)$ and $x \in J$, we have*

$$u_t(t, x) \geq -\eta(t) e^{u(t, x)}.$$

A consequence of this result is the following local Lipschitz continuity of u :

Proposition 2.2. *Assume that (H) holds. If u is a non-negative continuous viscosity solution of (1.1), then u is locally Lipschitz continuous on $(0, t^*) \times J$.*

3 Remarks on the assumptions and examples

In this section, we first exhibit some consequences of the assumptions and we construct in particular the function η . We then provide typical examples of Hamiltonians satisfying the assumptions.

3.1 consequences of the assumptions

In this subsection, we derive consequences from assumptions (H0)-(H2).

Consequence of (H0). The function $F_i : (c_i, +\infty) \rightarrow (0, F_i(c_i))$ defined by

$$F_i(\tau) = 2 \int_{\tau}^{\infty} \frac{d\sigma}{\sigma \phi_i(-\sigma)}$$

is decreasing and invertible. We define for $s \in (0, F_i(c_i))$ the function $\eta_i(s) = F_i^{-1}(s)$. We can easily check that the following hold:

$$\begin{cases} \eta_i'(s) = -\frac{\eta_i(s) \phi_i(-\eta_i(s))}{2} \\ \eta_i \text{ is decreasing, positive and } \eta_i(0^+) = +\infty. \end{cases} \quad (3.1)$$

Consequence of (H1): The function $F : (c, +\infty) \rightarrow (0, F(c))$ defined by

$$F(\tau) = 2 \int_{\tau}^{\infty} \frac{d\sigma}{\sigma \phi(\sigma)}$$

is decreasing and invertible. We define for $s \in (0, F(c))$ the function $\eta(s) = F^{-1}(s)$. We can easily check that the following hold:

$$\begin{cases} \eta'(s) = -\frac{\eta(s)\phi(\eta(s))}{2} \\ \eta \text{ is decreasing, positive and } \eta(0^+) = +\infty. \end{cases} \quad (3.2)$$

Moreover, $x \rightarrow x + \phi(x)$ is increasing from $(c, +\infty)$ to $(c + \phi(c), +\infty)$. We consider its inverse function denoted by ψ . We have that

$$\psi(x + \phi(x)) = x \quad \text{for } x \in (c, +\infty). \quad (3.3)$$

Consequence of (H2). For $t \in (0, F(c))$ and $x \in J_i$, we define the continuous function $q_i^-(t, x)$ by

$$q_i^-(t, x) = \min\{p < p_i^-(t, x); H_i(t, x, p) = \eta(t)\}.$$

Then, for $t \in \left(0, \min\left(F(c), \min_{i \in \{1, \dots, N\}} F_i(c_i)\right)\right)$, we have for any $i \in \{1, \dots, N\}$,

$$-\eta_i(t) > q_i^-(t, 0)C_u \quad (3.4)$$

where C_u is defined in (2.1).

Proof of (3.4). Let $q_{h_i}^-(t) \leq p_{h_i}^-$ such that $h_i(q_{h_i}^-(t)) = \eta(t)$. We remark that $q_{h_i}^-(t)$ exists because $\eta(t) > c \geq \min_{p \in \mathbb{R}} h_i(p)$. In addition, we have

$$q_{h_i}^-(t) \geq q_i^-(t, 0). \quad (3.5)$$

In fact, using that $\eta(t) > c \geq h_i(p_i^-(t, 0))$, we have

$$q_{h_i}^-(t) = h_i^{-1}(\eta(t)) < p_i^-(t, 0) \quad (3.6)$$

where h_i^{-1} is the inverse of the function $h_i \upharpoonright_{(-\infty, p_{h_i}^-)}$. Moreover, we have

$$H_i(t, 0, q_{h_i}^-(t)) \leq h_i(q_{h_i}^-(t)) = H_i(t, 0, q_i^-(t, 0)) = \eta(t). \quad (3.7)$$

Using (3.6) and (3.7), we deduce (3.5). We claim that

$$-\eta_i(t) > q_{h_i}^-(t)C_u. \quad (3.8)$$

Using that $\eta(t) = h_i(q_{h_i}^-(t)) > c \geq h_i(-\frac{c_i}{C_u})$ with $-\frac{c_i}{C_u} \leq p_{h_i}^-$, we get

$$q_{h_i}^-(t) \leq -\frac{c_i}{C_u}.$$

Therefore, using the inverse of η_i , (3.8) is equivalent to

$$t > F_i(-q_{h_i}^-(t)C_u). \quad (3.9)$$

We have

$$F_i(-q_{h_i}^-(t)C_u) = 2 \int_{-q_{h_i}^-(t)C_u}^{+\infty} \frac{d\sigma}{\sigma \phi_i(-\sigma)}.$$

Setting the variable change $\sigma = -q_{h_i}^-(s)C_u$, we obtain

$$F_i(-q_{h_i}^-(t)C_u) = 2 \int_t^0 \frac{(q_{h_i}^-)'(s)}{q_{h_i}^-(s) \phi_i(q_{h_i}^-(s)C_u)} ds.$$

Deriving the equality $h_i(q_{h_i}^-(s)) = \eta(s)$, we get

$$(q_{h_i}^-)'(s) = -\frac{\phi(h_i(q_{h_i}^-(s)))h_i(q_{h_i}^-(s))}{2h_i'(q_{h_i}^-(s))}.$$

Replacing the last equality in the integral, we get

$$F_i(-q_{h_i}^-(t)C_u) = \int_0^t \frac{\phi(h_i(q_{h_i}^-(s)))h_i(q_{h_i}^-(s))}{\phi_i(q_{h_i}^-(s)C_u)h_i'(q_{h_i}^-(s))q_{h_i}^-(s)} ds.$$

Using assumption (H2), we get (3.9) and (3.8). Finally, (3.5) and (3.8) implies (3.4). \square

3.2 Examples of Hamiltonians

We give now examples of Hamiltonians satisfying assumptions (H). Let us first consider the model Hamiltonians defined for $i \in \{1, \dots, N\}$ by

$$H_i(t, x, p) = |A_i(t, x)|^{m_i} |p|^{m_i} - f_i(t, x) \quad (3.10)$$

with

$$\begin{cases} m_i > 1, \\ f_i \text{ is non negative,} \\ A_i \text{ and } f_i \text{ are space-time Lipschitz functions,} \\ \text{there exists } 0 < \delta \leq 1 \text{ such that } \delta < |A_i(t, x)| < \frac{1}{\delta} \text{ for } (t, x) \in [0, T] \times J_i. \end{cases}$$

For such Hamiltonians, our assumptions are satisfied for $c_i = 0$ and $c = \max_{t \in [0, T]} |A(t)|$ where $A(t)$ is the flux limiter. We take $\phi_i(s) = (m_i - 1)\delta^{m_i}|s|^{m_i}$, $\phi(s) = (m - 1)\bar{C}s$ and $h_i(p) = \frac{|p|^{m_i}}{\delta^{m_i}}$ and $\bar{C} < (\delta^2 C_u)^{\bar{m}}$ where $m = \min_{i \in \{1, \dots, N\}} m_i$ and $\bar{m} = \max_{i \in \{1, \dots, N\}} m_i$. Moreover, the function η is defined for $s \in (0, F(c))$ by

$$\eta(s) = \frac{2}{\bar{C}(m-1)s}.$$

For $i \in \{1, \dots, N\}$, the function η_i is defined for $s > 0$ by

$$\eta_i(s) = \delta \left(\frac{2}{m_i(m_i - 1)s} \right)^{\frac{1}{m_i}}.$$

Concerning (H3) and (H4), we have the following

$$\begin{aligned}
 ((H_i)_p \cdot p - H_i)(t, x, p) &= (m_i - 1)|A_i(t, x)|^{m_i}|p|^{m_i} + f_i(t, x), \\
 |(H_i)_x(t, x, p)| &\leq m_i|(A_i)_x(t, x)||A_i(t, x)|^{m_i-1}|p|^{m_i} + |(f_i)_x(t, x)|, \\
 |(H_i)_t(t, x, p)| &\leq m_i|(A_i)_t(t, x)||A_i(t, x)|^{m_i-1}|p|^{m_i} + |(f_i)_t(t, x)|.
 \end{aligned}$$

We can easily verify that (H3) is satisfied if we take $t < t_i$ with

$$\eta_i(t_i) \geq \max \left(\frac{4K_1 m_i}{(m_i - 1)\delta^{2m_i-1}}, \left(\frac{4K_2}{(m_i - 1)\delta^{m_i}} \right)^{\frac{1}{m_i+1}} \right)$$

where K_1 and K_2 are respectively bounds of $|(A_i)_x|$ and $|(f_i)_x|$. Assumption (H4) is satisfied if we take $t < \tau_i$ with

$$\eta_i(\tau_i) \geq \max \left(\frac{4K_3 m_i ((m-1)\bar{C} + 1)}{(m_i - 1)\delta^{2m_i-1}}, \left(\frac{4K_4 ((m-1)\bar{C} + 1)}{(m_i - 1)} \right)^{\frac{1}{2}} \right)$$

where K_3 and K_4 are respectively bounds of $|(A_i)_t|$ and $|(f_i)_t|$.

Another example is the following Hamiltonians:

$$\begin{aligned}
 H_1(p) &= e^{|p|}, \\
 H_2(p) &= |p|^m, \\
 H_3(p) &= |p|^n - p, \\
 H_4(p) &= |p| \log(1 + |p|)
 \end{aligned}$$

with $m, n > 1$. In this case, $H_i = h_i$. For s negative and small enough, we define for $i = 1, 2, 3, 4$,

$$\phi_i(s) = \bar{\phi}(h_i(s))$$

with $\bar{\phi}(s) = (\log(s))^2$. We remark that for s negative and small enough, we have that

$$h_i(s) > |s| > 1.$$

Therefore,

$$\int_{\tau}^{+\infty} \frac{d\sigma}{\sigma \phi_i(-\sigma)} < \int_{\tau}^{+\infty} \frac{d\sigma}{\sigma (\log(\sigma))^2} < +\infty.$$

Moreover, we can easily verify that

$$\begin{cases} e^{|p|}(|p| - 1) - \phi_1(p) \rightarrow +\infty & \text{as } p \rightarrow -\infty \\ (m-1)|p|^m - \phi_2(p) \rightarrow +\infty & \text{as } p \rightarrow -\infty \\ (n-1)|p|^n - \phi_3(p) \rightarrow +\infty & \text{as } p \rightarrow -\infty \\ \frac{|p|^2}{1+|p|} - \phi_4(p) \rightarrow +\infty & \text{as } p \rightarrow -\infty. \end{cases}$$

Hence, we deduce that for all $i \in \{1, 2, 3, 4\}$, there exists c_i positive and big enough such that (H0) is satisfied for $\phi_i(s) = \bar{\phi}(h_i(s))$.

Concerning (H1)-(H2), we can take $\phi(s) = \bar{C}\bar{\phi}(s) = \bar{C}(\log(s))^2$ with $\bar{C} < C_u^2$. There exists c big enough such that (H1) and (H2) are satisfied. In fact, we have

$$\begin{cases} e^{|p|}(|p| - 1) - \phi(e^{|p|}) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty \\ (m-1)|p|^m - \phi(|p|^m) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty \\ (n-1)|p|^n - \phi(|p|^n - p) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty \\ \frac{|p|^2}{1+|p|} - \phi(|p| \log(1 + |p|)) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty. \end{cases}$$

We recall that the above inequalities imply that $h'_i(p)p - h_i(p) > \phi(h_i(p)) > 0$ for $|p|$ big enough. Thus, to prove (H2), it's sufficient to prove that

$$\bar{C}\bar{\phi}(h_i(p)) < \bar{\phi}(h_i(C_up)) \quad \text{in } I_i.$$

For c big enough, the last inequality is true for any $i \in \{1, 2, 3, 4\}$ since we have

$$\lim_{p \rightarrow -\infty} \frac{\bar{\phi}(h_1(C_up))}{\bar{\phi}(h_1(p))} = C_u^2 > \bar{C}$$

and for $i = 2, 3, 4$,

$$\lim_{p \rightarrow -\infty} \frac{\bar{\phi}(h_i(C_up))}{\bar{\phi}(h_i(p))} = 1 > \bar{C}.$$

4 Proof of Theorem 2.1

Adapting the same strategy in [6], we consider the function $v(t, x) = -e^{-u(t, x)}$. To prove Theorem 2.1, we will prove that for all $t \in (0, t_*)$ and $x \in J$, we have

$$v_t \geq -\eta(t).$$

This is done in Theorem 4.2. Before to do that, we give some properties of the equation satisfied by v

4.1 Viscosity solutions

The function v is a continuous viscosity solution of

$$\begin{cases} v_t + G_i(t, x, v, v_x) = 0 & \text{if } (t, x) \in [0, T] \times J_i^*, \\ v_t + G_A(t, 0, v, v_x) = 0 & \text{if } t \in [0, T] \end{cases} \quad (4.1)$$

where

$$\begin{cases} G_i(t, x, v, p) = (-v)H_i\left(t, x, -\frac{p}{v}\right), & p \in \mathbb{R}, \\ G_A(t, 0, v, p) = (-v)\max\left(A(t), \max_{i=1, \dots, N} H_i^-\left(t, 0, -\frac{p_i}{v}\right)\right), & p = (p_1, p_2, \dots, p_N) \in \mathbb{R}^N. \end{cases}$$

The definition of viscosity solutions of (4.1) can be derived from the one of (1.1). For the reader convenience, we state it: for $T > 0$, set $J_T = (0, T) \times J$. We define the class of test functions on $J \times (0, T)$ by

$$C^1(J_T) = \{\varphi \in C(J_T) \text{ such that the restriction of } \varphi \text{ to } (0, T) \times J_i \text{ is } C^1 \text{ for all } i \in \{1, \dots, N\}\}.$$

Definition 4.1. *Let $v : J_T \rightarrow \mathbb{R}$ be a continuous function. We say that v is a viscosity sub-solution (resp. super-solution) of (4.1) in J_T if for any test function $\varphi \in C^1(J_T)$ touching v from above (resp. from below) at some point $(t_0, x_0) \in J_T$, we have*

$$\begin{cases} \varphi_t(t_0, x_0) + G_i(t_0, x_0, v(t_0, x_0), \varphi_x(t_0, x_0)) \leq 0 & \text{(resp. } \geq 0) \quad \text{if } x_0 \in J_i^* \\ \varphi_t(t_0, 0) + G_A(t_0, 0, v(t_0, 0), \varphi_x(t_0, 0)) \leq 0 & \text{(resp. } \geq 0) \quad \text{if } x_0 = 0. \end{cases}$$

If v is viscosity sub-solution and viscosity super-solution, we say that v is a viscosity solution.

4.2 Properties of G_i

We derive properties of the Hamiltonian $G_i(t, x, v, p) = (-v)H_i\left(t, x, \frac{p}{-v}\right)$ from the assumptions (H). The derivative of this function are involved and we will state new conditions (G0)-(G1)-(G2) and (G3). We define first the time t_* for which assumptions (H) are satisfied. We set

$$t_* = \min\left(T, \min_{i=1, \dots, N} \eta_i^{-1}(c_i), \eta^{-1}(c), \min_{i=1, \dots, N} t_i, \min_{i=1, \dots, N} \tau_i\right). \quad (4.2)$$

By simple computations, we can check that the function G_i has the following properties.

(G0) Let $i \in \{1, \dots, N\}$. For all $t \in (0, t_*)$, if $p \leq -\eta_i(t)$, then by using (H0), we have that $(G_i)_v(t, x, v, p) \geq \phi_i(p)$ and $|(G_i)_p(t, x, v, p)| \leq \kappa_i \cdot (G_i)_v(t, x, v, p)$ and $(G_i)_p(t, x, v, p) \leq 0$.

(G1) Let $i \in \{1, \dots, N\}$. For all $t \in (0, t_*)$, if $G_i(t, x, v, p) \geq \eta(t)$, then by using (H1) we have that

$$(G_i)_v(t, x, v, p) \geq \phi(G_i(t, x, v, p)) \text{ and } |(G_i)_p(t, x, v, p)| \leq \kappa \cdot (G_i)_v(t, x, v, p).$$

(G2) Let $i \in \{1, \dots, N\}$. For all $t \in (0, t_*)$, if $p \leq -\eta_i(t)$, then by using (H3) we have that

$$|(G_i)_x(t, x, v, p)| \leq \frac{1}{2} \eta_i(t) (G_i)_v(t, x, v, p).$$

(G3) Let $i \in \{1, \dots, N\}$. For all $t \in (0, t_*)$, if $G_i(t, x, v, p) \geq \eta(t)$, then by using (H4) we have that

$$|(G_i)_t(t, x, v, p)| \leq \frac{1}{2} \psi(G_i(t, x, v, p)) (G_i)_v(t, x, v, p).$$

4.3 The main result wrote on v

Theorem 2.1 is a direct consequence of the following result on v .

Theorem 4.2. *Assume (H). Let u be a continuous, non-negative viscosity solution of (1.1). Then, for all $t \in (0, t_*)$ and $x \in J$, the function $v = -e^{-u}$ is a viscosity solution of (4.1) and it satisfies:*

$$v(t, x) - v(s, x) \geq -\eta(s)(t - s) \quad \text{if } 0 < s < t < t_* \text{ and } x \in J.$$

To prove Theorem 4.2, we need the following lemma in which we obtain a gradient lower bound on each branch.

Lemma 4.3. *Assume (H). Let u be a continuous, non-negative viscosity solution of (1.1). The function $v = -e^{-u}$ is a viscosity solution of (4.1) and it satisfies the following: for all $i \in \{1, \dots, N\}$ and $t \in (0, t_*)$, if $x, y \in J_i^*$ such that $x = x_i \cdot e_i$ and $y = y_i \cdot e_i$ with $x_i > y_i$, then*

$$v(t, x) - v(t, y) \geq -\eta_i(t)d(x, y).$$

Proof of Lemma 4.3. Let $i \in \{1, \dots, N\}$. We define

$$M = \sup_{t \in (0, t_*); x, y \in J_i^*, x_i > y_i > 0} \{v(t, y) - v(t, x) - \eta_i(t)d(x, y)\}.$$

By contradiction, we assume that $M > 0$. We introduce

$$M_{\nu, \gamma} = \sup_{t, s \in (0, t_*); x, y \in J_i^*, x_i > y_i > 0} \left\{ v(t, y) - v(s, x) - \eta_i(t)d(x, y) - \frac{(t-s)^2}{2\nu} - \frac{\gamma}{t_* - t} - \alpha d(0, x) - \frac{\gamma}{d(0, y)} \right\}.$$

Since $M > 0$, we deduce that $M_{\nu, \gamma} > 0$ for γ, α small enough. Using that $-1 \leq v < 0$, we have $0 < M_{\nu, \gamma} \leq 1 - \alpha d(0, x)$. This implies that $d(0, x) \leq \frac{1}{\alpha} < +\infty$. Moreover, using that $M_{\nu, \gamma}$ is a supremum of a continuous function, we deduce that it's reached at some point (t, s, x, y) .

Step 1: $t, s \neq 0$ and $t, s \neq t_*$: Using that $\eta_i(t) \rightarrow +\infty$ as $t \rightarrow 0$ and $\frac{\gamma}{t_* - t} \rightarrow +\infty$ if $t \rightarrow t_*$, we have that $t \neq 0$ and $t \neq t_*$ for all $\nu > 0$. Moreover, we have

$$\frac{(t-s)^2}{2\nu} \leq 1. \quad (4.3)$$

Inequality (4.3) implies that $|t-s| \rightarrow 0$ as $\nu \rightarrow 0$. Thus, for ν small enough, we have that $s \neq 0$ and $s \neq t_*$.

Step 2: $y_i \neq 0$ and $x_i > y_i$ for ν small enough. Using that $\frac{\gamma}{d(0,y)} \rightarrow +\infty$ if $d(0,y) \rightarrow 0$, we get that $d(0,y) > 0$ and thus $y_i > 0$. Assume that there exists a $\nu \rightarrow 0$ such that we have $x_i = y_i$. This implies that

$$v(t, x) - v(s, x) > \frac{\gamma}{t_*}. \quad (4.4)$$

Using that $d(0, x) = x_i \leq \frac{1}{\alpha}$, we deduce that x converges (up to a sub-sequence) as $\nu \rightarrow 0$. Denoting by \bar{t} (resp. \bar{x}) the common limit of t and s (resp. x and y), and taking ν to zero in (4.4), we obtain

$$v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}) \geq \frac{\gamma}{t_*} > 0$$

which gives a contradiction. Therefore, $x_i > y_i$ for ν small enough.

Step 3: Use of the viscosity inequalities. Using the viscosity inequalities, we obtain

$$\begin{cases} \eta'_i(t)(x_i - y_i) + \frac{t-s}{\nu} + \frac{\gamma}{(t_* - t)^2} + G_i\left(t, y, v(t, y), -\eta_i(t) - \frac{\gamma}{(y_i)^2}\right) \leq 0 \\ \frac{t-s}{\nu} + G_i(s, x, v(s, x), -\eta_i(t) - \alpha) \geq 0. \end{cases}$$

This implies

$$\eta'_i(t)(x_i - y_i) + \frac{\gamma}{(t_* - t)^2} + G_i\left(t, y, v(t, y), -\eta_i(t) - \frac{\gamma}{(y_i)^2}\right) - G_i(s, x, v(s, x), -\eta_i(t) - \alpha) \leq 0.$$

Denoting by \bar{t} the common limit of t and s and by \bar{x} (resp. \bar{y}) the limit of x (resp. y) as ν goes to zero and using the continuity of G_i and v , we obtain after taking ν to zero,

$$\eta'_i(\bar{t})(\bar{x}_i - \bar{y}_i) + \frac{\gamma}{(t_* - \bar{t})^2} + G_i\left(\bar{t}, \bar{y}, v_1, -\eta_i(\bar{t}) - \frac{\gamma}{(\bar{y}_i)^2}\right) - G_i(\bar{t}, \bar{x}, v_2, -\eta_i(\bar{t}) - \alpha) \leq 0 \quad (4.5)$$

where $v_1 = v(\bar{t}, \bar{y})$ and $v_2 = v(\bar{t}, \bar{x})$.

For $\sigma \in [0, 1]$, we define

$$\xi_\sigma = (\bar{t}, \bar{x}, v_2, -\eta_i(\bar{t}) - \alpha) + \sigma \left(0, \bar{y} - \bar{x}, v_1 - v_2, -\frac{\gamma}{\bar{y}_i^2} + \alpha\right)$$

We argue as if G_i is C^1 (otherwise a standard mollification argument allows to reduce to this case). Using (4.5), we have

$$\eta'_i(\bar{t})(\bar{x}_i - \bar{y}_i) + \frac{\gamma}{(t_* - \bar{t})^2} + \int_0^1 \frac{d}{d\sigma} G_i(\xi_\sigma) d\sigma \leq 0. \quad (4.6)$$

Step 4: Getting the contradiction. We have that

$$\begin{aligned} \frac{d}{d\sigma} G_i(\xi_\sigma) &= (G_i)_x(\xi_\sigma)(\bar{y} - \bar{x}) + (G_i)_v(\xi_\sigma)(v_1 - v_2) + (G_i)_p(\xi_\sigma) \left(-\frac{\gamma}{\bar{y}_i^2} + \alpha \right) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

with

$$\begin{cases} I_1 = (G_i)_x(\xi_\sigma)(\bar{y} - \bar{x}) + \frac{1}{2}(G_i)_v(\xi_\sigma)(v_1 - v_2), \\ I_2 = \frac{1}{2}(G_i)_v(\xi_\sigma)(v_1 - v_2) + (G_i)_p(\xi_\sigma)\alpha, \\ I_3 = -(G_i)_p(\xi_\sigma)\frac{\gamma}{\bar{y}_i^2}. \end{cases}$$

Bound for I_1 : We recall that

$$\xi_\sigma = (\bar{t}, \bar{x} + \sigma(\bar{y} - \bar{x}), v_2 + \sigma(v_1 - v_2), p)$$

with

$$p = -\eta_i(\bar{t}) - \alpha + \sigma \left(-\frac{\gamma}{\bar{y}_i^2} + \alpha \right).$$

Using that $p \leq -\eta_i(\bar{t})$, we can use (G0) and (G2) and we have

$$\begin{cases} (G_i)_v(\xi_\sigma) \geq \phi_i(p) \geq \phi_i(-\eta_i(\bar{t})), \\ |(G_i)_x(\xi_\sigma)| \leq \frac{1}{2}(G_i)_v(\xi_\sigma)\eta_i(\bar{t}) \end{cases}$$

where we use that ϕ_i is decreasing. Using that

$$v_1 - v_2 > \eta_i(\bar{t})(\bar{x}_i - \bar{y}_i) + \frac{\gamma}{t_* - \bar{t}} + \alpha\bar{x}_i$$

we obtain that

$$I_1 \geq 0.$$

Bound for I_2 : Using again (G0), we have

$$(G_i)_p(\xi_\sigma) \geq -\kappa_i(G_i)_v(\xi_\sigma).$$

We obtain that

$$\begin{aligned} I_2 &\geq \frac{1}{2}(G_i)_v(\xi_\sigma)(v_1 - v_2) - \kappa_i(G_i)_v(\xi_\sigma)\alpha \\ &\geq \frac{1}{2}\phi_i(-\eta_i(\bar{t}))\eta_i(\bar{t})(\bar{x}_i - \bar{y}_i) + (G_i)_v(\xi_\sigma) \left(\frac{1}{2}\alpha\bar{x}_i + \frac{\gamma}{2t_*} - \kappa\alpha \right) \\ &> \frac{1}{2}\phi_i(-\eta_i(\bar{t}))\eta_i(\bar{t})(\bar{x}_i - \bar{y}_i) \end{aligned}$$

where we use that $(G_i)_v(\xi_\sigma) > 0$ and $\frac{\gamma}{2t_*} - \kappa_i\alpha > 0$ for α small enough.

Bound for I_3 : Using (G0), we have $(G_i)_p(\xi_\sigma) \leq 0$ and thus $I_3 \geq 0$. Using bounds of I_1, I_2 and I_3 , we obtain in (4.6)

$$\frac{\gamma}{(t_* - \bar{t})^2} + (\bar{x}_i - \bar{y}_i) \left(\eta'_i(\bar{t}) + \frac{1}{2} \phi_i(-\eta_i(\bar{t})) \eta_i(\bar{t}) \right) \leq 0$$

which gives a contradiction. □

We turn now to the proof of Theorem 4.2.

Proof of Theorem 4.2. We introduce

$$M = \sup_{0 < s < t < t_*, x \in J} \{v(s, x) - v(t, x) - \eta(s)(t - s)\}.$$

By contradiction, assume that $M > 0$. We define

$$M_\alpha = \sup_{0 < s < t < t_*, x \in J} \left\{ v(s, x) - v(t, x) - \eta(s)(t - s) - \beta \eta(s) - \frac{\delta}{t_* - t} - \alpha d(0, x)^2 \right\}.$$

We have that $M_\alpha \geq \frac{M}{2} > 0$ for β, δ and α small enough. We classically have that M_α is reached at some point $(s_\alpha, t_\alpha, x_\alpha)$. Using that $\eta(s) \rightarrow +\infty$ as $s \rightarrow 0$ and $\frac{\delta}{t_* - t} \rightarrow +\infty$ as $t \rightarrow t_*$, we have $s_\alpha \neq 0$ and $t_\alpha \neq t_*$. If $t_\alpha = s_\alpha$, we get a contradiction using that $M_\alpha > 0$. Till the end of this proof, we will denote

$$v(s_\alpha, x_\alpha) = a \quad \text{and} \quad v(t_\alpha, x_\alpha) = b.$$

Case 1: $x_\alpha = 0$. For $i \in \{1, \dots, N\}$, we define the constants λ_i and γ_i that satisfy the following:

$$\begin{cases} \lambda_i \geq \gamma_i \quad \text{and} \quad \frac{\gamma_i}{-b} \geq p_i^+(t_\alpha, 0), \\ G_i^+(s_\alpha, 0, a, \lambda_i) > \eta(s_\alpha) - \eta'(s_\alpha)(t_\alpha - s_\alpha + \beta) \quad \text{and} \quad G_i^+(t_\alpha, 0, b, \gamma_i) < \eta(t_\alpha) \end{cases} \quad (4.7)$$

where

$$G_i^+(t, 0, v, p) = (-v)H_i^+\left(t, 0, \frac{p}{-v}\right)$$

with $-1 \leq v < 0$ and $H_i^+(t, 0, \cdot)$ is non-decreasing part of $H_i(t, 0, \cdot)$. For $\varepsilon > 0$ small, we define

$$M_{\alpha, \varepsilon} = \sup_{0 < s < t < t_*, x, y \in \bar{B}_R(0)} \left\{ v(s, x) - v(t, y) - \eta(s)(t - s) - \beta \eta(s) - \frac{\delta}{t_* - t} - \alpha d(0, x)^2 - \frac{(d(x, y))^2}{2\varepsilon} - L(s, t, x) - \varphi(x) + \Psi(y) \right\}$$

where

$$\begin{cases} L(s, t, x) = (s - s_\alpha)^2 + (t - t_\alpha)^2 + d(x_\alpha, x)^2, \\ \varphi(x) = \lambda_i x_i \quad \text{if} \quad x = x_i e_i \in J_i, \\ \Psi(x) = \gamma_i x_i \quad \text{if} \quad x = x_i e_i \in J_i. \end{cases}$$

We have $M_{\alpha, \varepsilon} \geq M_\alpha > 0$ and $M_{\alpha, \varepsilon}$ is reached at some point (s, t, x, y) . Classically, we have

$$d(x, y) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Denoting by \bar{s} (resp. \bar{t}, \bar{x}) the limit of s (resp. t, x and y) as $\varepsilon \rightarrow 0$, we have

$$M_\alpha \leq M_{\alpha, \varepsilon} \leq M_\alpha - L(\bar{s}, \bar{t}, \bar{x}) - \varphi(\bar{x}) + \psi(\bar{x}).$$

Using that $\psi(x) - \varphi(x) \leq 0$, we obtain that $L(\bar{s}, \bar{t}, \bar{x}) = 0$ and hence $(\bar{s}, \bar{t}, \bar{x}) = (s_\alpha, t_\alpha, 0)$.

Case 1.1: $x = x_i \cdot e_i \in J_i^*, y = y_j \cdot e_j \in J_j$ with $i \neq j$. Writing the viscosity sub-solution's inequality, we obtain

$$\eta'(s)(t - s + \beta) - \eta(s) + 2(s - s_\alpha) + G_i(s, x, v(s, x), p_\varepsilon + 2\alpha x_i + 2x_i + \lambda_i) \leq 0$$

with $p_\varepsilon = \frac{x_i + y_j}{\varepsilon}$. Using that

$$\begin{aligned} G_i(s, x, v, p) &\geq G_i^+(s, x, v, p), \\ p_\varepsilon + 2\alpha x_i + 2x_i + \lambda_i &> \lambda_i \end{aligned}$$

we get

$$\eta'(s)(t - s + \beta) - \eta(s) + 2(s - s_\alpha) + G_i^+(s, x, v(s, x), \lambda_i) \leq 0.$$

Taking ε to zero, we obtain a contradiction using (4.7).

Case 1.2: $x = 0$ and $\exists j$ such that $y \in J_j^*$. Writing the viscosity super-solution's inequality, we obtain

$$G_j(t, y, v(t, y), p_\varepsilon + \gamma_j) \geq \eta(s) + \frac{\delta}{(t_* - t)^2} + 2(t - t_\alpha)$$

with $p_\varepsilon = -\frac{y_j}{\varepsilon}$. This implies that for ε small enough,

$$H_j\left(t, y, \frac{p_\varepsilon + \gamma_j}{-v(t, y)}\right) > \eta(s) > \eta(t) \quad (4.8)$$

where we use that η is decreasing. We deduce that

$$\begin{cases} \frac{p_\varepsilon + \gamma_j}{-v(t, y)} > q_j^+(t, y) \quad \text{or} \\ \frac{p_\varepsilon + \gamma_j}{-v(t, y)} < q_j^-(t, y) \end{cases} \quad (4.9)$$

where $q_j^-(t, y)$ and $q_j^+(t, y)$ are defined by

$$\begin{aligned} q_j^+(t, y) &= \max\{p > p_j^+(t, y); H_j(t, y, p) = \eta(t)\} \\ q_j^-(t, y) &= \min\{p < p_j^-(t, y); H_j(t, y, p) = \eta(t)\}. \end{aligned}$$

Using successively Lemma 4.3 and (3.4), we have

$$p_\varepsilon + \gamma_j \geq -\eta_j(t) > q_j^-(t, 0)C_u$$

which implies for ε small enough

$$p_\varepsilon + \gamma_j > q_j^-(t, y)(-v(t, y)).$$

Therefore, we obtain $\frac{p_\varepsilon + \gamma_j}{-v(t, y)} > q_j^-(t, y)$. Using (4.8) and (4.9), we deduce that

$$\frac{p_\varepsilon + \gamma_j}{-v(t, y)} > q_j^+(t, y) \quad (4.10)$$

and

$$G_j^+(t, y, v(t, y), p_\varepsilon + \gamma_j) > \eta(t).$$

But using that $p_\varepsilon < 0$, we have for ε small enough

$$G_j^+(t, y, v(t, y), p_\varepsilon + \gamma_j) \leq G_j^+(t, y, v(t, y), \gamma_j) < \eta(t)$$

where we use (4.7). This gives a contradiction.

Case 1.3: $x = 0$ and $y = 0$. Writing the viscosity super-solution's inequality, we obtain

$$G_A(t, 0, v(t, 0), p) \geq \eta(s) + \frac{\delta}{t_*^2} + 2(t - t_\alpha) \geq \max_{[0, t_*]} A(t) + \frac{\delta}{t_*^2} + 2(t - t_\alpha)$$

with $p = (\gamma_1, \gamma_2, \dots, \gamma_N)$. This implies

$$\max \left(A(t), \max_{i=1, \dots, N} H_i^- \left(t, 0, \frac{\gamma_i}{-v(t, 0)} \right) \right) \geq \max_{[0, t_*]} A(t) + \frac{\delta}{t_*^2} + 2(t - t_\alpha). \quad (4.11)$$

Taking ε to zero, we obtain a contradiction using (4.7). In fact, using that $\frac{\gamma_i}{-b} \geq p_i^+(t_\alpha, 0)$, we have

$$H_i^- \left(t_\alpha, 0, \frac{\gamma_i}{-b} \right) = \min_{p \in \mathbb{R}} H_i(t_\alpha, 0, p) \leq A(t_\alpha).$$

Therefore (4.11) implies (after taking ε to zero) that

$$A(t_\alpha) \geq A(t_\alpha) + \frac{\delta}{t_*^2}.$$

Case 1.4: $\exists i$ such that $x, y \in J_i^*$. Writing the viscosity sub and super-solution's inequalities, we have

$$\eta'(s)(t - s + \beta) - \eta(s) + 2(s - s_\alpha) + G_i(s, x, v(s, x), p_\varepsilon + 2\alpha x_i + 2x_i + \lambda_i) \leq 0 \quad (4.12)$$

and

$$-\eta(s) - \frac{\delta}{(t_* - t)^2} - 2(t - t_\alpha) + G_i(t, y, v(t, y), p_\varepsilon + \gamma_i) \geq 0 \quad (4.13)$$

where $p_\varepsilon = \frac{x_i - y_i}{\varepsilon}$. Using the super-solution inequality, we have

$$H_i \left(t, y, \frac{p_\varepsilon + \gamma_i}{-v(t, y)} \right) > \eta(s) > \eta(t). \quad (4.14)$$

Arguing as in case 1.2 (see (4.10)), we can prove that for ε small enough

$$\frac{p_\varepsilon + \gamma_i}{-v(t, y)} > p_i^+(t, 0). \quad (4.15)$$

We claim that

$$\frac{p_\varepsilon + \gamma_i}{-v(t, y)} \geq \max_{\tau \in [0, t_*]} p_i^+(\tau, 0). \quad (4.16)$$

Using the fact that $h_i \geq H_i$, inequality (4.14) and $\eta(t) \geq C$, we have for ε small enough,

$$h_i \left(\frac{p_\varepsilon + \gamma_i}{-v(t, y)} \right) > C.$$

Using the definition of C in (H1), we obtain that

$$\frac{p_\varepsilon + \gamma_i}{-v(t, y)} \geq \max_{\tau \in [0, t_*]} p_i^+(\tau, 0) \text{ or } \frac{p_\varepsilon + \gamma_i}{-v(t, y)} \leq \min_{\tau \in [0, t_*]} p_i^-(\tau, 0).$$

But, from (4.15), we know that $\frac{p_\varepsilon + \gamma_i}{-v(t, y)} > p_i^+(t, 0)$. Therefore, we deduce that

$$\frac{p_\varepsilon + \gamma_i}{-v(t, y)} \geq \max_{\tau \in [0, t_*]} p_i^+(\tau, 0).$$

The coercivity of H_i and inequality (4.12) implies that

$$|p_\varepsilon + 2\alpha x_i + 2x_i + \lambda_i| \leq C_{t_\alpha, s_\alpha, o(\alpha)}$$

and in particular

$$|p_\varepsilon| \leq C_{t_\alpha, s_\alpha, o(\alpha)} + o(\alpha) + 2r + \lambda_i.$$

We deduce that up to a sub-sequence, $\lim_{\varepsilon \rightarrow 0} p_\varepsilon$ exists and we denote by $\bar{p} = \lim_{\varepsilon \rightarrow 0} p_\varepsilon$. Inequalities (4.12) and (4.13) gives after taking ε to zero

$$\begin{aligned} & \frac{\delta}{(t_* - t_\alpha)^2} + \eta'(s_\alpha)(t_\alpha - s_\alpha + \beta) + G_i(s_\alpha, 0, a, \bar{p} + \lambda_i) - G_i(t_\alpha, 0, b, \bar{p} + \gamma_i) = \\ & \frac{\delta}{(t_* - t_\alpha)^2} + \eta'(s_\alpha)(t_\alpha - s_\alpha + \beta) + \int_0^1 \frac{d}{d\sigma} G_i(\xi_\sigma) d\sigma \leq 0 \end{aligned} \quad (4.17)$$

where $\xi_\sigma = (t_\alpha + \sigma(s_\alpha - t_\alpha), 0, b + \sigma(a - b), \bar{p} + \gamma_i + \sigma(\lambda_i - \gamma_i))$ and $\sigma \in [0, 1]$. We have that

$$\begin{aligned} \frac{d}{d\sigma} G_i(\xi_\sigma) &= (s_\alpha - t_\alpha)(G_i)_t(\xi_\sigma) + (a - b)(G_i)_v(\xi_\sigma) + (\lambda_i - \gamma_i)(G_i)_p(\xi_\sigma) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

with

$$\begin{cases} I_1 = (s_\alpha - t_\alpha)(G_i)_t(\xi_\sigma) + \frac{1}{2}(a - b)(G_i)_v(\xi_\sigma), \\ I_2 = \frac{1}{2}(a - b)(G_i)_v(\xi_\sigma), \\ I_3 = (\lambda_i - \gamma_i)(G_i)_p(\xi_\sigma). \end{cases}$$

Our goal is to prove that

$$I_1 + I_2 + I_3 \geq \frac{\phi(\eta(s_\alpha)\eta(s_\alpha))}{2}(t_\alpha - s_\alpha + \beta). \quad (4.18)$$

To obtain our result, we need the following lemma whose proof is postponed.

Lemma 4.4. *For all $\sigma \in [0, 1]$, we have*

$$\eta(s_\alpha) \leq G_i(\xi_\sigma) \leq \eta(s_\alpha) + \phi(\eta(s_\alpha)). \quad (4.19)$$

We prove now (4.18). We denote by $t_\sigma = t_\alpha + \sigma(s_\alpha - t_\alpha)$. Using (4.19) and (G3), we have

$$\begin{aligned} I_1 &\geq \frac{1}{2}(s_\alpha - t_\alpha)(G_i)_v(\xi_\sigma)\psi(\eta(t_\sigma)) + \frac{1}{2}(a - b)(G_i)_v(\xi_\sigma) \\ &\geq \frac{1}{2}(G_i)_v(\xi_\sigma)(\psi(\eta(t_\sigma))(s_\alpha - t_\alpha) + (a - b)) \\ &\geq \frac{1}{2}(G_i)_v(\xi_\sigma)(t_\alpha - s_\alpha)(\eta(s_\alpha) - \psi(\eta(t_\sigma))) \end{aligned} \quad (4.20)$$

where we use that $a - b \geq \eta(s_\alpha)(t_\alpha - s_\alpha)$. Using (4.19), we have

$$\psi(\eta(t_\sigma)) < \psi(\eta(s_\alpha)) \leq \psi(G_i(\xi_\sigma)) \leq \psi(\eta(s_\alpha) + \phi(\eta(s_\alpha))) = \eta(s_\alpha)$$

where we use that η is decreasing, ψ is increasing and $\psi(x + \phi(x)) = x$. Injecting this inequality in (4.20), we get that

$$I_1 \geq 0.$$

Concerning I_2 , we use (G1) to obtain

$$I_2 \geq \frac{1}{2}\eta(s_\alpha)(t_\alpha - s_\alpha + \beta)\phi(\eta(s_\alpha)).$$

We will show now that $I_3 \geq 0$. Using (4.16), we have that

$$\frac{\bar{p} + \gamma_i}{-b} \geq p_i^+(t_\alpha + \sigma(s_\alpha - t_\alpha), 0).$$

This implies that

$$\frac{\bar{p} + \gamma_i + \sigma(\lambda_i - \gamma_i)}{-b - \sigma(a - b)} \geq p_i^+(t_\alpha + \sigma(s_\alpha - t_\alpha), 0)$$

and therefore $(G_i)_p(\xi_\sigma) \geq 0$. Hence, $I_3 \geq 0$. Finally, injecting (4.18) in (4.17), we obtain

$$\frac{\delta}{(t_* - t_\alpha)^2} + (t_\alpha - s_\alpha + \beta) \left(\eta'(s_\alpha) + \frac{\eta(s_\alpha)\phi(\eta(s_\alpha))}{2} \right) \leq 0$$

which gives a contradiction and the proof is now complete.

Case 2: $x_\alpha \neq 0$. In this case, there exists $i \in \{1, \dots, N\}$ such that $x_\alpha = x_{\alpha,i}e_i \in J_i^*$. We then define $M_{\alpha,\varepsilon}$ without the function $\Psi(y) - \varphi(x)$. The supremum $M_{\alpha,\varepsilon}$ is reached at some point (s, t, x, y) with $x, y \in J_i^*$ for ε small. Writing the viscosity inequalities, we get equation (not necessarily the same i) (4.17) with $\xi_\sigma = (t_\alpha + \sigma(s_\alpha - t_\alpha), x_\alpha, b + \sigma(a - b), \bar{p} + \sigma 2\alpha x_{\alpha,i})$ and $\sigma \in [0, 1]$. We get that

$$\begin{aligned} \frac{d}{d\sigma} G_i(\xi_\sigma) &= (s_\alpha - t_\alpha)(G_i)_t(\xi_\sigma) + (a - b)(G_i)_v(\xi_\sigma) + (2\alpha x_{\alpha,i})(G_i)_p(\xi_\sigma) \\ &= I_1 + I_2 \end{aligned}$$

with

$$\begin{cases} I_1 = (s_\alpha - t_\alpha)(G_i)_t(\xi_\sigma) + \frac{1}{2}(a - b)(G_i)_v(\xi_\sigma), \\ I_2 = \frac{1}{2}(a - b)(G_i)_v(\xi_\sigma) + (2\alpha x_{\alpha,i})(G_i)_p(\xi_\sigma). \end{cases}$$

We proceed as in case 1.4. The difference comes from the term $(2\alpha x_{\alpha,i})(G_i)_p(\xi_\sigma)$. Using Lemma 4.4, we can use (G1) and we get

$$I_2 \geq \frac{\eta(s_\alpha)\phi(\eta(s_\alpha))(t_\alpha - s_\alpha + \beta)}{2} + (G_i)_v(\xi_\sigma) \left(\frac{\delta}{2t_*} - 2\alpha x_{\alpha,i} \right).$$

Finally, using that $\alpha x_{\alpha,i} \rightarrow 0$ as $\alpha \rightarrow 0$, we have $\frac{\delta}{2t_*} - 2\alpha x_{\alpha,i} \geq 0$ for α small enough.

We turn now to the proof of Lemma 4.4.

Proof. Firstly, we will show that for $\sigma \in [0, 1]$,

$$G_i(\xi_\sigma) \leq \eta(s_\alpha) + \phi(\eta(s_\alpha)).$$

We have

$$\begin{aligned} G_i(\xi_1) &\leq \eta(s_\alpha) - \eta'(s_\alpha)(t_\alpha - s_\alpha + \beta) \\ &\leq \eta(s_\alpha) + \frac{\phi(\eta(s_\alpha))\eta(s_\alpha)}{2}(t_\alpha - s_\alpha + \beta) \\ &\leq \eta(s_\alpha) + \frac{\phi(\eta(s_\alpha))\eta(s_\alpha)}{2} \left(1 - \frac{\delta}{t_*}\right) \\ &< \eta(s_\alpha) + \phi(\eta(s_\alpha)) \end{aligned} \tag{4.21}$$

where we use that $1 \geq a - b \geq \eta(s_\alpha)(t_\alpha - s_\alpha) + \frac{\delta}{t_*}$. By contradiction, assume that there exists $\sigma \in [0, 1)$ such that $G_i(\xi_\sigma) > \eta(s_\alpha) + \phi(\eta(s_\alpha))$. We define

$$\bar{\sigma} = \sup\{\sigma \in [0, 1] \text{ such that } G_i(\xi_\sigma) > \eta(s_\alpha) + \phi(\eta(s_\alpha))\}.$$

By (4.21), we have that $\bar{\sigma} < 1$. By definition of $\bar{\sigma}$, we also have

$$G_i(\xi_\sigma) = \eta(s_\alpha) + \phi(\eta(s_\alpha)).$$

Using the continuity of G_i , we deduce that there exists $\bar{\bar{\sigma}} \in (\bar{\sigma}, 1]$ such that for $\sigma \in [\bar{\sigma}, \bar{\bar{\sigma}}]$, we have

$$G_i(\xi_\sigma) \in [\eta(s_\alpha), \eta(s_\alpha) + \phi(\eta(s_\alpha))] \text{ and } G_i(\bar{\bar{\sigma}}) < G_i(\bar{\sigma}).$$

However, arguing as above, we can prove that $\frac{d}{d\sigma} G_i(\xi_\sigma) \geq 0$ for $\sigma \in [\bar{\sigma}, \bar{\bar{\sigma}}]$. This gives a contradiction and therefore $G_i(\xi_\sigma) \leq \eta(s_\alpha) + \phi(\eta(s_\alpha))$. Let us now prove that

$$G_i(\xi_\sigma) \geq \eta(s_\alpha).$$

The proof is similar to the above case. We have $G_i(\xi_0) > \eta(s_\alpha)$. We define

$$\bar{\sigma} = \inf\{\sigma > 0 \text{ such that } G_i(\xi_\sigma) < \eta(s_\alpha)\}.$$

We have $\bar{\sigma} > 0$, $G_i(\xi_{\bar{\sigma}}) = \eta(s_\alpha)$ and $G_i(\xi_\sigma) \in [\eta(s_\alpha), \eta(s_\alpha) + \phi(\eta(s_\alpha))]$ for $\sigma \in [0, \bar{\sigma}]$. The inequality $G_i(\xi_0) > G_i(\xi_{\bar{\sigma}})$ gives a contradiction. □

□

Remark 4.5. We remark that the function η depends on the time T . The main reasons of this dependence are two: firstly, we need that $\eta(t) > c > \max_{t \in [0, T]} A(t)$. Secondly, we need to satisfy assumption (H2) in which the bounds of the solution u on $[0, T]$ are involved. For example, if we consider the model Hamiltonians (3.10) with $A_i = 1, f_i = 0$ and $A(t) = A > 0$, we obtain $H_i(p) = |p|^{m_i}$. Although that the Hamiltonians are not space-time depending, we still need

$$c \geq A \text{ and } \eta(s) = \frac{2}{\bar{C}(m-1)s}$$

with $\bar{C} < (C_u)^{\bar{m}}$. The constant \bar{C} is needed to satisfy assumption (H2). However, if u is bounded in $\mathbb{R}^+ \times \{0\}$, we can construct η without T -dependence (since C_u does not depend on T). Let us

mention that if $u(\cdot, 0)$ is bounded for $t > 0$, then we can obtain our main result (Theorem 4.2) for $t > 0$ and

$$\eta(+\infty) = 0. \quad (4.22)$$

In fact, for the Hamiltonians $H_i(p) = |p|^{m_i}$, taking $A = 0$, we can take $c = 0$ and in this case η is defined on $(0, +\infty)$ and $\eta(+\infty) = 0$. It may be useful to exploit (4.22) in any future study of the large time behavior of viscosity solutions of (1.1). In fact, having (4.22) in the case of continuous Hamilton-Jacobi equation was a key idea to obtain the large time behavior of the viscosity solutions, see Corollary 6 in [6] and Theorem 10.6 in [4]. However, we believe that we can not easily extend the aforementioned results. In fact, a crucial idea in [7] was the use of the periodicity assumption to construct the ergodic constant. This property is missing even in the case of Hamiltonian depending only on the gradient variable.

5 Local Lipschitz bound

The proof of Proposition 2.2 is a consequence of the following lemma.

Lemma 5.1. *Let $z = z_i e_i \in J_i^*$ and $s \in (0, t_*)$. Let $r > 0$ be such that $z_i - r > 0$ and the ball $B_r(z) \subset (0, t_*)$. There exists a constant K_i depending on z_i, t_* and r such that for all $x, y \in B_r(z)$ and for all $t \in B_r(s)$, we have*

$$|u(t, x) - u(t, y)| \leq K_i d(x, y). \quad (5.1)$$

Proof of Lemma 5.1. From Lemma 4.3, we have for $t \in (0, t_*)$, $x = x_i e_i$, $y = y_i e_i$ and $x_i > y_i$,

$$u(t, x) - u(t, y) \geq -\eta_i(t) K d(x, y) \geq -\eta_i(s - r) K d(x, y)$$

where $K = \max_{(\tau, l) \in B_r(s) \times B_r(z)} e^{u(\tau, l)}$.

It remains to prove the upper bound. Let $P_i > \max_{(t, x) \in B_r(s) \times B_r(z)} p_i^+(t, x)$ such that for all $(t, x) \in B_r(s) \times B_r(z)$,

$$H_i(t, x, P_i) > \eta(s - r) K.$$

Let $(t_0, y) \in B_r(s) \times B_r(z)$. Let $\nu > 0$ and define

$$M = \sup_{t \in B_r(s), x = x_i e_i, x \in B_r(z), y_i < x_i < z_i + r} \left\{ u(t, x) - u(t_0, y) - P_i(x_i - y_i) - \frac{(t - t_0)^2}{2\nu} \right\}.$$

We claim that $M \leq 0$. Assume by contradiction that $M > 0$. For $\delta > 0$ small enough, we define

$$M_\delta = \sup_{t \in B_r(s), x = x_i e_i, x \in B_r(z), y_i < x_i < z_i + r} \left\{ u(t, x) - u(t_0, y) - P_i(x_i - y_i) - \frac{(t - t_0)^2}{2\nu} - \frac{\delta}{z_i + r - x_i} \right\}.$$

Classically, $M_\delta > 0$ for δ small enough and is reached at some point (t, x) . If $x_i = y_i$, we get $\frac{\delta}{z_i + r} < 0$. If $x_i = z_i + r$, we get a contradiction using that $\frac{\delta}{z_i + r - x_i} \rightarrow +\infty$. We deduce that $y_i < x_i < z_i + r$. On the other hand, using the continuity of u , there exists a constant $C_{r, s, z}$ such that

$$0 < M_\delta \leq C_{r, s, z} - \frac{(t - t_0)^2}{2\nu}.$$

We deduce that $t \in B_r(s)$ for ν small enough. Writing the viscosity inequality, we get

$$H_i \left(t, x, P_i + \frac{\delta}{(z_i + r - x_i)^2} \right) \leq \frac{t_0 - t}{\nu}.$$

Using Theorem 2.1, we have $\frac{t_0 - t}{\nu} \leq \eta(t)K$. Therefore, we obtain

$$H_i(t, x, P_i) \leq H_i\left(t, x, P_i + \frac{\delta}{(z_i + r - x_i)^2}\right) \leq \eta(t)K < \eta(s - r)K$$

where we use that $H_i(t, x, q) \geq H_i(t, x, p)$ if $q \geq p \geq p_i^+(t, x)$. We get a contradiction using the definition of P_i . Therefore, $M \leq 0$, and taking $t = t_0$, we get the desired result. Finally, we can take $K_i = \max(P_i, \eta_i(s - r)K)$. □

Proof of Proposition 2.2. The proof of this proposition can be easily derived from the previous lemma. In fact, let us consider the ball $B_r(0)$ and take $x, y \in B_r(0)$ such that $x \in J_i$ and $y \in J_j$ with $i \neq j$. Using (5.1), we have

$$|u(t, x) - u(t, y)| \leq K_i d(0, x) + K_j d(0, y) \leq \max(K_i, K_j)(|x_i| + |y_j|) = \max(K_i, K_j)d(x, y).$$
□

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