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INFINITELY MANY RADIAL SOLUTIONS OF A MEAN CURVATURE EQUATION IN LORENTZ-MINKOWSKI SPACE

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Abstract. In this paper, we show that the quasilinear equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = |u|^{\alpha-2}u, \quad \text{in } \mathbb{R}^N$$

has a positive smooth radial solution at least for any $\alpha > 2^\star = 2N/(N-2)$, $N \geq 3$. Our approach is based on the study of the optimizers for the best constant in the inequality

$$\int_{\mathbb{R}^N}(1-\sqrt{1-|\nabla u|^2}) \geq C\left(\int_{\mathbb{R}^N}|u|^\alpha\right)^{\frac{N}{\alpha+N}},$$

which holds true in the unit ball of $W^{1,\infty}(\mathbb{R}^N) \cap \mathcal{D}^{1;2}(\mathbb{R}^N)$ if and only if $\alpha \geq 2^\star$. We also prove that the best constant is not achieved for $\alpha = 2^\star$. As a byproduct, our arguments combined with Lusternik-Schnirelmann category theory allow to construct a sequence of radial solutions.

1. Introduction

It is well known [19] that the Lane-Emden equation

$$-\Delta u = |u|^{\alpha-2}u \ \text{ in } \mathbb{R}^N, \tag{1}$$

admits no nontrivial nonnegative solution for $2 < \alpha < 2^\star$, $N \geq 3$, while, for $\alpha = 2^\star$, any positive solution can be written in the form

$$u_{\delta,a}(x) = \beta_N\left(\frac{\delta}{\delta^2+|x-a|^2}\right)^{\frac{N-2}{2N}},$$

as proved by Caffarelli, Gidas and Spruck [11]. For $\alpha > 2^\star$, the set of all positive radial solutions is a one-parameter family $\{u_a(r) = au_1(a^{(\alpha-2)/2}r) : a > 0\}$, where u_1 is strictly decreasing in r (see for instance [20]). Non radial singular solutions have been constructed by Dancer, Guo and Wei [15]. We mention that it is still open whether all smooth positive solutions are radially symmetric around some point or not.

The prescribed mean curvature equation in Euclidian space

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = |u|^{\alpha-2}u \ \text{ in } \mathbb{R}^N,$$

has also been the object of many studies. It has been considered, among others, by Ni and Serrin [26] and del Pino and Guerra [17]. It is known that this problem has infinitely many radial positive solution if $\alpha \geq 2^\star$ and no smooth positive solutions if $\alpha \leq (2N-2)/(N-2)$. In contrast with the non-existence result for the Lane-Emden equation in the subcritical range, del Pino and Guerra proved the existence of many positive solutions when $\alpha = 2^\star - \epsilon$, for sufficiently small $\epsilon > 0$.

In this work, we aim to study the following prescribed mean curvature equation in the Lorentz-Minkowski space

$$Q(u) = |u|^{\alpha-2}u \ \text{ in } \mathbb{R}^N, \tag{2}$$

where

$$Q(u) = -\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right). \tag{3}$$

The quasilinear operator Q is a classical object in Riemannian geometry. The Lorentz-Minkowski space $\mathbb{L}^{N+1} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}\}$, with the flat metric $\sum_{j=1}^N (dx_j)^2 - (dt)^2$ is the natural framework of classical relativity. If M is an N-dimensional hypersurface of $\mathbb{L}^{N+1}$ that is the graph of a smooth function $u \in C^1(\Omega)$ with $\|\nabla u\|_{L^\infty} < 1$, the local mean curvature of M is given by $Q(u)$, see for instance [2, 12]. The determination of maximal or constant mean curvature hypersurfaces is an important issue in classical relativity. The volume integral $\int_\Omega \sqrt{1-|\nabla u|^2}$ gives the area integral in $\mathbb{L}^{N+1}$ and surfaces of maximal area (or simply maximal surfaces) solve the equation $Q(u) = 0$ in Ω.

For functions defined on the whole of $\mathbb{R}^N$, the operator Q is relevant in Maxwell-Born-Infeld field theory, see for instance [7, 8, 22, 23]. Basically, in this theory, which is fully relativistic, it is assumed that there is a maximal field strength. This lead Born and Infeld to consider the following Lagrangian density, expressed in Lorentz-Minkowski space,

$$\mathcal{L}_{BI} = b^2\left(1 - \sqrt{1 - \frac{|\vec{E}|^2 - |\vec{B}|^2}{b^2} - \frac{(\vec{E}.\vec{B})^2}{b^4}}\right),$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and b is the maximal admissible value of the electric field.

Up to our knowledge, the equation (2) has never been considered in the literature, at least in $\mathbb{R}^N$. We refer to [3, 4, 9, 14] for recent results on the existence of radial solutions for BVPs involving Q in the ball with either Dirichlet or Neumann conditions.

Supercritical problems are usually difficult to tackle through variational methods. For instance, concerning the Lane-Emden equation, Farina [18] has obtained a Liouville-type result for C^2 solutions of (1) with finite Morse index. Basically, if the dimension is small ($N \le 10$), the only finite Morse index solution is 0 except at the critical exponent where the above-mentioned positive solutions arise as constrained minimizers on a manifold of codimension 1.

In contrast, we show here that the quasilinear equation (2) has a smooth positive radial solution for any $\alpha > 2^\star$, $N \ge 3$ by using simple arguments from Critical Point Theory and the Calculus of Variations. In fact, when $\alpha > 2^\star$, we have enough compactness to deal with the problem in a standard way. Indeed, we minimize the volume integral

$$\int_{\mathbb{R}^N} (1 - \sqrt{1-|\nabla u|^2}), \tag{4}$$

truncated in a convenient way, constrained to the unit sphere of $L^\alpha(\mathbb{R}^N)$. Then we prove a gradient estimate which is uniform with respect to the truncation parameter.

Our first main result is the following.

Theorem 1.1. *If $\alpha > 2^\star$, equation (2) has a positive radial classical solution.*

We restrict here our attention to the existence of *radially* symmetric solutions. On the one hand, we expect that all positive smooth solutions are indeed radially symmetric, though this is an open question. On the other hand, our solution arises as a constrained minimizer and its Schwarz symmetric rearrangement yields a radially symmetric minimizer (and therefore a radially symmetric solution).

Surprisingly, our approach to establish the existence of a solution of (2) fails in the critical case $\alpha = 2^\star$. Indeed, as stated in Theorem 1.2 below, the solution of Theorem 1.1 realizes the best constant in an inequality between the volume integral (4) and the L^α-norm. This inequality still holds for $\alpha = 2^\star$ but the best constant is not achieved. We emphasize that this contrasts with the Sobolev inequality.

In the sequel, we denote by $\mathcal{X}$ the functional space

$$\mathcal{X} := \Big\{ u \in \mathcal{D}^{1;2}(\mathbb{R}^N) : \nabla u \in L^\infty(\mathbb{R}^N) \text{ and } \|\nabla u\|_{L^\infty} \leq 1 \Big\},$$

endowed with the norm

$$\|u\|_{\mathcal{D}^{1;2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}.$$

We establish the following Sobolev-type inequality.

Theorem 1.2. *There exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla u|^2}) \geq C \left(\int_{\mathbb{R}^N} |u|^\alpha \right)^{\frac{N}{\alpha+N}} \tag{5}$$

for every $u \in \mathcal{X}$ if and only if $\alpha \geq 2^\star$. Moreover, the best constant

$$\inf_{u \in \mathcal{X} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla u|^2})}{\left(\int_{\mathbb{R}^N} |u|^\alpha \right)^{\frac{N}{N+\alpha}}}$$

is achieved by a radial solution of (2) for $\alpha > 2^\star$ while it is not achieved for $\alpha = 2^\star$.

The fact that inequality (5) does not hold below the critical exponent is rather clear since the volume integral (4) is bounded from above by the Dirichlet energy. This does not mean that (2) has no non trivial nonnegative solutions for $\alpha < 2^\star$ though we conjecture that this is indeed the case. One can for instance exclude the existence of fast decaying solution but we are not able to prove a complete non-existence result for $\alpha < 2^\star$. Also the existence of a positive solution of (2) in the critical case $\alpha = 2^\star$ remains an interesting open question.

At last, as a natural extension of our existence result, we combine our previous approach with Lusternik-Schnirelmann category theory to obtain a sequence of solutions whose volume integral diverge. Namely we prove the following multiplicity result.

Theorem 1.3. *For any $\alpha > 2^\star$, equation (2) has a sequence of radial solutions $(u^k)_{k \in \mathbb{N}}$ such that*

$$\int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla u^k|^2}) \to +\infty \text{ as } k \to \infty.$$

Again, we first consider an auxiliary problem and conclude by a sharp uniform estimate on the gradient of our solutions. Note that we do not provide sign information on solutions though one could probably argue as in [27, 5] to obtain a sequence of sign changing solutions. We leave this, as well as the existence of infinitely many positive solutions, as open questions.

The paper is organized as follows. Section 2 contains some preliminary results on the functional spaces we will work with. In Section 3, we establish the existence of at least one classical solution of (2) (see Theorem 1.1 above). Section 4 is devoted to the proof of the inequality in Theorem 1.2 and especially to the existence of optimizers for the best constant in this inequality. Finally, in Section 5, we obtain infinitely many solutions of (2) as stated in Theorem 1.3.

With some abuse of notation, we will sometimes consider radial functions as functions of one variable, thus writing $u(|x|)$ or $u(x)$ or $u(r)$. For any set $\mathcal{A}$ of functions, $\mathcal{A}_{rad}$ is defined as the set of all radially symmetric functions of $\mathcal{A}$. Throughout the paper, C denotes a positive constant that can change from line to line.

2. Functional framework and preliminary results

Let us set $a_0(s) = (1-s)^{-1/2}$ for all $s < 1$. Equation (2) can be written as

$$-\mathrm{div}\left(a_0(|\nabla u|^2)\nabla u\right) = |u|^{\alpha-2}u \ \text{ in } \mathbb{R}^N.$$

We introduce the energy functional

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} A_0(|\nabla u|^2),$$

where $A_0(t) = \int_0^t a_0(s)\,ds$ for all $t \le 1$. This functional is well defined on $\mathcal{X} = \{u \in \mathcal{D}^{1;2}(\mathbb{R}^N) : \nabla u \in L^\infty(\mathbb{R}^N) \text{ and } \|\nabla u\|_{L^\infty} \le 1\}$, because we have

$$\frac{1}{2}|\nabla u|^2 \le 1 - \sqrt{1-|\nabla u|^2} = \frac{|\nabla u|^2}{1+\sqrt{1-|\nabla u|^2}} \le |\nabla u|^2.$$

Lemma 2.1. *Let $u \in \mathcal{X}$. Then $|\nabla u| \in L^q(\mathbb{R}^N)$ for every $q \ge 2$, and $u \in L^s(\mathbb{R}^N)$ for every $s \ge 2^\star$. Moreover, u can be assumed to be continuous and such that*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Proof. Since $|\nabla u| \le 1$ and $|\nabla u| \in L^2(\mathbb{R}^N)$, we infer that $|\nabla u| \in L^q(\mathbb{R}^N)$ for every $q \ge 2$. It then follows that $u \in L^{qN/(N-q)}(\mathbb{R}^N)$ for every $q \ge 2$, and, by interpolation, $u \in L^s(\mathbb{R}^N)$ for every $s \ge 2^\star$. Observe also that since $u \in W^{1,r}(\mathbb{R}^N)$ for some $r > N$, it can be assumed to be continuous and moreover $\lim_{|x|\to\infty} u(x) = 0$. □

Working with the functional I_0 in $\mathcal{X}$ requires some care. Since I_0 is weakly lower semi-continuous, a natural way to obtain a solution of (2) consists in minimizing I_0 constrained to the manifold

$$\mathcal{M}_0 := \left\{u \in \mathcal{X} : \int_{\mathbb{R}^N} |u|^\alpha = 1\right\}.$$

However, it is not clear that minimizers solve an associated Euler-Lagrange equation. Indeed, the functional I_0 is C^1 only at points $u \in \mathcal{X}$ with Lipschitz constant $Lip(u)$ strictly less than 1. Without this condition, minimizers solely solve a variational inequality.

To overcome this lack of differentiability on the boundary of $\mathcal{X}$, we will work with an auxiliary functional. This type of truncation argument has already been used in [13, 14] to deal with Dirichlet boundary condition in an interval or a ball. Here, one of the novelties is that an a priori L^∞ bound on minimizers cannot be derived from the solely boundedness of the gradient. Therefore, we truncate the volume integral in a different way than in [13, 14] and we deal with a different functional framework.

We now define our auxiliary functional. For $\theta \in]0,1[$, define $a_\theta : \mathbb{R} \to \mathbb{R}^+$ by

$$a_\theta(s) = a_0(s) \text{ for } 0 \le s \le 1-\theta \quad \text{and} \quad a_\theta(s) = \gamma s^p + \delta \text{ for } s > 1-\theta, \tag{6}$$

where γ and δ are chosen in such a way that a_θ is C^1. The exponent p will be chosen later according to the value of α in (2).

In the sequel, we will work with the spaces $\mathcal{D}^{1;r}_{rad}(\mathbb{R}^N)$ and $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$, defined respectively as the closure of the smooth compactly supported radially symmetric functions for the norms

$$\|u\|_{\mathcal{D}^{1;r}} := \left(\int_{\mathbb{R}^N} |\nabla u|^r\right)^{\frac{1}{r}}$$

and

$$\|u\|_{\mathcal{D}^{1;(2,q)}} := \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q\right)^{\frac{1}{q}},$$

with $1 < q, r < \infty$. Consider the manifold

$$\mathcal{M} := \left\{ u \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^\alpha = 1 \right\}.$$

We will look for critical points of I_θ constrained to $\mathcal{M}$ where

$$I_\theta : \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N) \to \mathbb{R}^+$$

is defined by

$$I_\theta(u) := \frac{1}{2}\int_{\mathbb{R}^N} A_\theta(|\nabla u|^2),$$

and $A_\theta(t) = \int_0^t a_\theta(s)\,ds$.

We next recall some elementary facts. We quote them in separate lemmas for further references in the text. We do not provide the details for Lemma 2.2 which follows from standard arguments. We refer for instance to [28, 25, 6] for Lemma 2.3, whereas Lemma 2.4 can easily be deduced from [25, Corollary II-3]. Below, $q^\star := qN/(N-q)$ for $q < N$.

Lemma 2.2. *Let $u \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$. Then $u \in \mathcal{D}^{1;r}_{rad}(\mathbb{R}^N)$ for every $r \in [2,q]$. If $q < N$ then $u \in L^s(\mathbb{R}^N)$ for every $s \in [2^\star, q^\star]$; if $q = N$ then $u \in L^s(\mathbb{R}^N)$ for every $s \in [2^\star, +\infty[$; if $q > N$ then $u \in L^s(\mathbb{R}^N)$ for every $s \in [2^\star, +\infty]$. Moreover, the embeddings are continuous.*

Lemma 2.3. *Let $r \in [2,q]$ if $q < N$, and $r \in [2, N[$ if $q \ge N$. Then there exists $C > 0$ (depending only on N and r) such that for all $u \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$, there holds*

$$|u(x)| \le C|x|^{-\frac{N-r}{r}} \|\nabla u\|_{L^r},$$

for almost all $x \in \mathbb{R}^N \setminus \{0\}$.

Lemma 2.4. *Let $(u_n)_n \subset \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ be a bounded sequence. If $q < N$ then for any $s \in]2^\star, q^\star[$, there exists a subsequence which converges weakly in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ and strongly in $L^s(\mathbb{R}^N)$. If $q > N$, the same result holds for any $s \in]2^\star, +\infty[$.*

We close this section by a uniform estimate on the regularization schema. Observe that for $\theta_1 := 1/(2p+1)$, the function a_θ defined in (6) is given by

$$a_{\theta_1}(s) = 1/\sqrt{1-s} \text{ if } 0 \le s \le 1-\theta_1 \quad \text{and} \quad a_{\theta_1}(s) = \gamma_p s^p \text{ if } s > 1-\theta_1,$$

where $\gamma_p = \sqrt{2p+1}\,((2p+1)/2p)^p$. Therefore, for all $\theta \in]0, \theta_1]$ and $s \in \mathbb{R}^+$ we have

$$\frac{\gamma_p}{p+1} s^{p+1} \le A_{\theta_1}(s) \le A_\theta(s), \tag{7}$$

and

$$\begin{array}{rcll} A_\theta(s) \ge A_{\theta_1}(s) & \ge & s, & \text{if } s \le \frac{2p}{2p+1}, \\ & \ge & \frac{\gamma_p}{p+1}\left(\frac{2p}{2p+1}\right)^p s, & \text{if } s > \frac{2p}{2p+1}. \end{array} \tag{8}$$

Inequalities (7) and (8) lead to uniform estimates (with respect to θ) in $\mathcal{D}^{1;(2,2p+2)}_{rad}(\mathbb{R}^N)$. They will be important keys in the sequel to obtain a priori bounds independent of the truncation parameter θ. As for an upper bound on I_θ, we observe that for all $u \in \mathcal{D}^{1;(2,2p+2)}(\mathbb{R}^N)$,

$$A_\theta(|\nabla u|^2) \le C\,(|\nabla u|^{2p+2} + |\nabla u|^2). \tag{9}$$

for some constant C depending on θ. The functional I_θ is then well defined in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ with $q := 2p+2$ and it is straightforward that I_θ is C^1 on $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$.

The preceding lemmas suggest to choose p in the definition of a_θ such that $q = 2p+2$ satisfies $q^\star > \alpha$. Indeed, a lower bound in $\mathcal{D}^{1;(2,2p+2)}(\mathbb{R}^N)$ will follow from (7) and (8) whereas $\mathcal{M}$ is weakly closed as soon as $q^\star > \alpha$.

3. Existence of a positive solution for supercritical exponents

In this section, we prove that equation (2) has at least one positive solution.

3.1. **The auxiliary problem.** We will first look for a solution of the modified problem

$$-\mathrm{div}\left(a_\theta(|\nabla u_\theta|^2)\nabla u_\theta\right) = \lambda_\theta \alpha |u_\theta|^{\alpha-2}u_\theta \ \text{ in } \mathbb{R}^N,$$

where a_θ is defined in (6). It will turn out that if the parameter θ is small enough, this solution also solves the original equation (2). From now on, we assume $\theta \in]0, \theta_1]$. Recall also that $q^\star > \alpha$ (which can be written as $q > N\alpha/(N+\alpha)$), $\theta_1 = 1/(2p+1)$ and $q = 2p+2$.

Proposition 3.1. *Let $\alpha > 2^\star$ and $q > \frac{N\alpha}{N+\alpha}$. Then there exists $u_\theta \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ such that*

$$c^1_\theta := \min_{v \in \mathcal{M}} I_\theta(v) = I_\theta(u_\theta) > 0. \tag{10}$$

For any minimizer u_θ of (10), there exists $\lambda_\theta \in \mathbb{R}^+$ such that u_θ is a weak solution of the equation

$$-\left(r^{N-1}a_\theta(|u_\theta'|^2)u_\theta'\right)' = \lambda_\theta \alpha r^{N-1}|u_\theta|^{\alpha-2}u_\theta, \tag{11}$$

i.e.

$$\int_0^{+\infty} r^{N-1} a_\theta(|u_\theta'|^2) u_\theta' v' = \lambda_\theta \alpha \int_0^{+\infty} r^{N-1} |u_\theta|^{\alpha-2} u_\theta v,$$

for every $v \in \mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$.

Moreover, for every $s \in [2^\star, q^\star]$, $s \in [2^\star, +\infty[$ *or* $s \in [2^\star, +\infty]$ *if* $q < N$, $q = N$ *and* $q > N$ *respectively, there exist* $C_1, M_1 > 0$ *independent of* $\theta \in]0, \theta_1]$ *such that*

$$\max\{\|u_\theta\|_{\mathcal{D}^{1;(2,q)}}, \|u_\theta\|_{L^s}\} \le C_1 \quad \text{and} \quad c_\theta^1 \le M_1. \tag{12}$$

Proof. We proceed in several steps.

Step 1: Lower bounds on c_θ^1. The inequalities (7) and (8) imply the existence of a positive constant C depending only on p such that, for all $\theta \in]0, \theta_1]$ and all $u \in \mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$,

$$I_\theta(u) \ge C \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{and} \quad I_\theta(u) \ge C \int_{\mathbb{R}^N} |\nabla u|^q. \tag{13}$$

As $\alpha > 2^\star$, we have $2 < \frac{N\alpha}{N+\alpha} < q$ and we deduce by interpolation and Sobolev inequality that for all $u \in \mathcal{M}$,

$$I_\theta(u) \ge C \int_{\mathbb{R}^N} |\nabla u|^{\frac{N\alpha}{N+\alpha}} \ge C \left(\int_{\mathbb{R}^N} |u|^\alpha \right)^{\frac{N}{N+\alpha}} = C > 0, \tag{14}$$

for some $C > 0$ which depends only on p, α and N. This implies that $\inf_{v \in \mathcal{M}} I_\theta(v) > 0$.

Step 2: Existence of a minimizer. Let $(u_n)_n \subset \mathcal{M}$ be a minimizing sequence, i.e.

$$I_\theta(u_n) \to \inf_{v \in \mathcal{M}} I_\theta(v)$$

as $n \to \infty$. Choosing $\bar{u} \in \mathcal{M}$ a smooth function such that $|\nabla \bar{u}(x)| < 1 - \theta_1$ for all $x \in \mathbb{R}^N$, we can assume w.l.g. that

$$I_\theta(u_n) \le \int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla \bar{u}|^2}) =: M_1 \tag{15}$$

for any $n \in \mathbb{N}$ and any $\theta \in]0, \theta_1]$.

It then follows from (13) and (15) that $(u_n)_n$ is bounded in $\mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$. Since $\alpha > 2^\star$ and $q > \frac{N\alpha}{N+\alpha}$, Lemma 2.4 implies that, up to a subsequence, $(u_n)_n$ converges weakly to u_θ in $\mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$ and strongly in $L^\alpha(\mathbb{R}^N)$ as $n \to \infty$. Obviously, $\int_{\mathbb{R}^N} |u_\theta|^\alpha = 1$ and $u_\theta \in \mathcal{M}$.

Moreover, I_θ being convex and continuous, I_θ is weakly lower semi-continuous and

$$I_\theta(u_\theta) \le \liminf_{n \to \infty} I_\theta(u_n) = \inf_{v \in \mathcal{M}} I_\theta(v).$$

Since $u_\theta \in \mathcal{M}$, we conclude that $I_\theta(u_\theta) = \inf_{v \in \mathcal{M}} I_\theta(v)$.

Step 3: A priori bounds on the family $\{u_\theta : \theta \in]0, \theta_1]\}$. From (15), we infer that

$$c_\theta^1 = I_\theta(u_\theta) \le M_1. \tag{16}$$

By (13) and (16), u_θ is bounded in $\mathcal{D}^{1;(2,q)}$ uniformly in θ. The a priori bound in L^s follows from Lemma 2.2 according to whether $q < N$, $q = N$ or $q > N$.

Step 4: The Euler-Lagrange equation. By the Lagrange multiplier rule, there exists $\lambda_\theta \in \mathbb{R}$ such that for all $\varphi \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$,

$$I'_\theta(u_\theta)(\varphi) = \lambda_\theta \alpha \int_{\mathbb{R}^N} |u_\theta|^{\alpha-2} u_\theta \, \varphi.$$

This means that

$$-\mathrm{div}\left(a_\theta(|\nabla u_\theta|^2)\nabla u_\theta\right) = \lambda_\theta \alpha |u_\theta|^{\alpha-2} u_\theta \ \text{ in } \mathbb{R}^N,$$

in the weak sense. As u_θ is radial, (11) follows. □

Observe that it is standard to prove that u_θ is a classical solution of (11) on $]0,+\infty[$. If $q > N$ then the solution is bounded and we can apply the regularity theory of Lieberman [24] to deduce that the weak solution u_θ is also $C^{1,\alpha}$ for some $0 < \alpha < 1$ in a neighborhood of the origin. We can deduce the regularity at the origin from even simpler arguments if $q < N$. Observe that for $\alpha > 2^\star$, we have $N - N/\alpha > N\alpha/(N+\alpha)$. In particular, Proposition 3.1 holds if $q > N - N/\alpha$.

Lemma 3.2. *Let $\alpha > 2^\star$ and $N - \frac{N}{\alpha} < q < N$. If u_θ is a minimizer of* (10)*, it is bounded in $\mathcal{C}^1(\mathbb{R}^N)$ and either $u_\theta > 0$ or $u_\theta < 0$ on $\mathbb{R}^N$.*

Proof. As u_θ is a solution of (11) on $]0,+\infty[$, it is standard to check that, for $r > 0$, u_θ is regular. On the other hand, one observes that $r^{N-1} a_\theta(|u'_\theta|^2)u'_\theta$ satisfies the Cauchy condition at the origin so that it has a finite limit as $r \to 0$. This limit must be zero otherwise we have

$$r^{N-1} a_\theta(|u'_\theta|^2)|u'_\theta|^2 \geq C r^{-\frac{N-1}{q-1}}$$

near 0, which is not integrable because $q < N$. This contradicts the fact that, as u_θ is a weak solution of (11), we have

$$\int_0^{+\infty} r^{N-1} a_\theta(|u'_\theta|^2)|u'_\theta|^2 = \lambda_\theta \alpha.$$

We now claim that u'_θ is bounded. Integrating the equation, we get

$$\left|a_\theta(|u'_\theta(r)|^2)u'_\theta(r)\right| = \frac{\lambda_\theta \alpha}{r^{N-1}} \int_0^r s^{N-1} |u_\theta(s)|^{\alpha-1} \, ds,$$

for all $r \in [0,\infty[$. Using the estimate from Proposition 3.1, it follows that

$$a_\theta(|u'_\theta(r)|^2)\,|u'_\theta(r)| \leq C\lambda_\theta \alpha r^{\frac{N(q^\star-\alpha+1)}{q^\star} - N + 1} \|u_\theta\|^{\alpha-1}_{L^{q^\star}},$$

with $C > 0$. Moreover, we have $N(q^\star - \alpha + 1)/q^\star - N + 1 > 0$ since we assume $N - \frac{N}{\alpha} < q$, and therefore $u'_\theta(0) = 0$ and, for $r \leq 1$, we conclude that

$$a_\theta(|u'_\theta(r)|^2)\,|u'_\theta(r)| \leq C\lambda_\theta \alpha \|u_\theta\|^{\alpha-1}_{L^{q^\star}}.$$

We next deduce from Lemma 2.3 and Proposition 3.1 that for all $r > 1$,

$$\begin{aligned}
\left|a_\theta(|u'_\theta(r)|^2)u'_\theta(r)\right| &= \frac{\lambda_\theta \alpha}{r^{N-1}} \left[\int_0^1 s^{N-1}|u_\theta(s)|^{\alpha-1}\,ds + \int_1^r s^{N-1}|u_\theta(s)|^{\alpha-1}\,ds\right] \\
&\leq C\lambda_\theta\alpha\|u_\theta\|^{\alpha-1}_{L^{q^\star}} + \frac{\lambda_\theta\alpha}{r^{N-1}}\|u'_\theta\|^{\alpha-1}_{L^2}\int_1^r s^{N-1} s^{-\frac{(N-2)(\alpha-1)}{2}} ds \\
&\leq C\left(1 + r^{1-\frac{(N-2)(\alpha-1)}{2}}\right)
\end{aligned}$$

and since $\alpha > 2^\star$, we have

$$1 - \frac{N-2}{2}(\alpha - 1) < -\frac{N}{2}$$

so that the claim follows.

As $u_\theta'(0) = 0$ on proves by standard arguments that u_θ is a classical solution.

To show that any minimizer satisfies either $u_\theta > 0$ or $u_\theta < 0$, we argue by contradiction. Indeed, if u_θ changes sign, then $|u_\theta| \in \mathcal{M}$ and $I_\theta(|u_\theta|) = I_\theta(u_\theta)$. In other words, $v = |u_\theta|$ is also a minimizer, and vanishes at some point $r_0 \in [0, \infty[$. Since v is a solution of (11) with $\min_{[0,+\infty[} v = v(r_0) = 0$ and the solutions of (11) are regular, we also have $v'(r_0) = 0$, which contradicts the local uniqueness of the solution of the Cauchy problem. This concludes the proof. □

3.2. **Back to the original equation** (2)**.** We now prove that the solution obtained in Proposition 3.1 is a solution of our original problem (2) provided the parameter θ is small enough.

In the sequel, $(u_\theta, \lambda_\theta)$ is the solution of

$$-\mathrm{div}\left(a_\theta(|\nabla u_\theta|^2)\nabla u_\theta\right) = \lambda_\theta \alpha |u_\theta|^{\alpha-2}u_\theta \ \text{ in } \mathbb{R}^N, \tag{17}$$

obtained in Proposition 3.1. We first estimate the Lagrange multiplier through an argument of the Calculus of Variations.

Lemma 3.3. *For all $\theta \in]0, \theta_1]$, we have $0 < \lambda_\theta = \frac{N}{N+\alpha} c_\theta^1$.*

Proof. Multiplying (17) by u_θ and integrating, we obtain

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 = \lambda_\theta \alpha \int_{\mathbb{R}^N} |u_\theta|^\alpha = \lambda_\theta \alpha. \tag{18}$$

Next, we prove that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 = \frac{N\alpha}{2N + 2\alpha} \int_{\mathbb{R}^N} A_\theta(|\nabla u_\theta|^2). \tag{19}$$

To this end, consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(t) := I_\theta\big(t^{\frac{N}{\alpha}} u_\theta(tx)\big)$. For all $t \in \mathbb{R}^+$, $t^{N/\alpha} u_\theta(tx) \in \mathcal{M}$, and f achieves its minimum at $t = 1$. A change of variable yields

$$f(t) = \frac{1}{2}\int_{\mathbb{R}^N} A_\theta\Big(t^{\frac{2N}{\alpha}+2}|\nabla u_\theta(tx)|^2\Big)\,dx = \frac{1}{2t^N}\int_{\mathbb{R}^N} A_\theta\Big(t^{\frac{2N}{\alpha}+2}|\nabla u_\theta(y)|^2\Big)\,dy.$$

From the last equality and Lebesgue's dominated convergence theorem, it is easy to see that f is differentiable. Hence, as $f(1)$ is a minimum, we have

$$f'(1) = \frac{1}{2}\left[\left(\frac{2N}{\alpha} + 2\right)\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2 - N\int_{\mathbb{R}^N} A_\theta(|\nabla u_\theta|^2)\right] = 0,$$

which proves (19).

Combining (19) with (18), we conclude that

$$\lambda_\theta = \frac{N}{2N + 2\alpha}\int_{\mathbb{R}^N} A_\theta(|\nabla u_\theta|^2) = \frac{N}{N+\alpha} c_\theta^1 > 0. \quad \square$$

An important consequence of this lemma is that the uniform estimate on the levels c_θ^1 from Proposition 3.1 yields a uniform estimate on the Lagrange multiplier. This estimate allows to deduce that, for θ small, our regularization leads to a solution of an unperturbed equation (with Lagrange multiplier).

Proposition 3.4. *Assume $N - N/\alpha < q < N$. For $\alpha > 2^\star$ and θ small enough, the function u_θ obtained in Proposition* 3.1 *is a radial solution of*

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda |u|^{\alpha-2}u \quad \text{in } \mathbb{R}^N, \tag{20}$$

with

$$\lambda = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} > 0.$$

Moreover either $u_\theta > 0$ or $u_\theta < 0$ on $\mathbb{R}^N$.

Proof. Consider the solution $(u_\theta, \lambda_\theta)$ of

$$-\mathrm{div}\left(a_\theta(|\nabla u_\theta|^2)\nabla u_\theta\right) = \lambda_\theta \alpha |u_\theta|^{\alpha-2}u_\theta \ \text{ in } \mathbb{R}^N,$$

obtained in Proposition 3.1. Let us prove the existence of a constant $E > 0$ such that for all $\theta \in]0, \theta_1]$ and all $r > 0$,

$$\left|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)\right| \le E. \tag{21}$$

We argue as in Lemma 3.2 to deduce uniform estimates. First, using the uniform estimates from Proposition 3.1 and Lemma 3.3, it follows that for all $r < 1$ and all $\theta \in]0, \theta_1]$,

$$a_\theta(|u_\theta'(r)|^2)\,|u_\theta'(r)| \le C\lambda_\theta\alpha\|u_\theta\|_{L^{q^\star}}^{\alpha-1} \le C,$$

with $C > 0$ independent of $\theta \in]0, \theta_1]$. Moreover, by Lemma 2.3, Proposition 3.1 and Lemma 3.3, we have for all $r > 1$ and $\theta \in]0, \theta_1]$,

$$\begin{aligned}\left|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)\right| &\le C\lambda_\theta\alpha\|u_\theta\|_{L^{q^\star}}^{\alpha-1} + \frac{\lambda_\theta\alpha}{r^{N-1}}\|\nabla u_\theta\|_{L^2}^{\alpha-1}\int_1^r s^{N-1}s^{-\frac{(N-2)(\alpha-1)}{2}}ds \\ &\le C\left(1 + r^{1-\frac{(N-2)(\alpha-1)}{2}}\right),\end{aligned}$$

where $C > 0$ is still independent of θ. As $\alpha > 2^\star$, we have

$$1 - \frac{N-2}{2}(\alpha-1) < -\frac{N}{2}.$$

This proves (21).

Finally, by construction of a_θ, (21) implies that $|u_\theta'(r)| \le 1 - \epsilon$ for some $\epsilon > 0$, and hence u_θ solves (20) for θ small enough. More precisely, we have for all $r \ge 0$ and all $\theta < \min\{\theta_1, 1/(1+E^2)\}$,

$$|u_\theta'(r)| \le \frac{E}{\sqrt{1+E^2}},$$

and the result follows for $\theta < \min\{\theta_1, 1/(1+E^2)\}$. The fact that $\lambda := \lambda_\theta$ is bounded away from zero follows from Lemma 3.3. □

Proof of Theorem 1.1. By Proposition 3.4, we know that for θ small enough, u_θ is a radial solution of (20) i.e. u_θ is a solution of

$$-\left(r^{N-1}\frac{v'}{\sqrt{1-|v'|^2}}\right)' = \lambda r^{N-1}|v|^{\alpha-2}v \ \text{ in }]0,+\infty[.$$

Observe that w_t defined by $w_t(r) = tu_\theta(r/t)$ solves

$$-\left(r^{N-1}\frac{w'}{\sqrt{1-|w'|^2}}\right)' = \lambda\frac{1}{t^\alpha}r^{N-1}|w|^{\alpha-2}w \ \text{ in }]0,+\infty[.$$

Then w_t is a solution of the original equation (2) if $t = \lambda^{1/\alpha}$. □

Remark 3.5. Note that, for $\theta \in]0,\theta_1]$, w_t satisfies in fact

$$I_\theta(|\nabla w_t|^2) = \min\left\{I_\theta(|\nabla v|^2) : v \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|v|^\alpha = \lambda^{\frac{\alpha+N}{\alpha}}\right\},$$

where $\lambda = c^1_\theta N/(N+\alpha) > 0$.

4. Optimizers in the inequality involving the volume integral

This section deals with the proof of Theorem 1.2 stated in the introduction. This theorem will follow from Proposition 4.1, Proposition 4.2, Proposition 4.4 and Proposition 4.5 below.

Proposition 4.1. *Assume $\alpha \geq 2^\star$. Then there exists a constant $C > 0$, depending only on α and N, such that for all $u \in \mathcal{X}$,*

$$\int_{\mathbb{R}^N}(1-\sqrt{1-|\nabla u|^2}) \geq C\left(\int_{\mathbb{R}^N}|u|^\alpha\right)^{\frac{N}{\alpha+N}}. \tag{22}$$

Proof. If $\alpha \geq 2^\star$ then $2 \leq N\alpha/(N+\alpha)$. Hence, using the fact that $\|\nabla u\|_{L^\infty} \leq 1$ and Sobolev inequality, we have for all $u \in \mathcal{X}$,

$$\int_{\mathbb{R}^N}(1-\sqrt{1-|\nabla u|^2}) \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^{\frac{N\alpha}{N+\alpha}} \geq C\left(\int_{\mathbb{R}^N}|u|^\alpha\right)^{\frac{N}{N+\alpha}},$$

where $C > 0$ depends only on α and N. □

Observe that the exponent $N\alpha/(\alpha+N)$ in the L^α-norm naturally arises in the proof when using Sobolev inequality. The presence of this exponent can also be explained from the invariance of the inequality (22) under the homeomorphisms $\phi_t : u(\cdot) \mapsto tu(\cdot/t)$ for $t > 0$.

We next show that the inequality (22) does not hold whatever $C > 0$ when $\alpha < 2^\star$.

Proposition 4.2. *If $\alpha < 2^\star$ then*

$$\inf_{u\in\mathcal{X}\setminus\{0\}}\frac{\int_{\mathbb{R}^N}(1-\sqrt{1-|\nabla u|^2})}{\left(\int_{\mathbb{R}^N}|u|^\alpha\right)^{\frac{N}{N+\alpha}}} = 0.$$

Proof. It is straightforward to construct a sequence $(u_n)_n \subset \mathcal{D}^{1;2}(\mathbb{R}^N)$ such that $\|\nabla u_n\|_{L^\infty} \leq 1$, $\|u_n\|_{L^\alpha} = 1$, and $\int_{\mathbb{R}^N} |\nabla u_n|^2 \to 0$ as $n \to \infty$. Then we have for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla u_n|^2}) = \int_{\mathbb{R}^N} \frac{|\nabla u_n|^2}{1 + \sqrt{1 - |\nabla u_n|^2}} \leq \int_{\mathbb{R}^N} |\nabla u_n|^2,$$

and the conclusion follows. □

We now focus on the best constant for which (22) holds when $\alpha > 2^\star$. We will use the following lemma. For the definition and basic properties of the symmetric rearrangement, the reader is referred to [21, 29] (among many others). Since we adapt a rather classical lemma from [29], we keep the notations therein. In particular, the *symmetric rearrangement* $u^\bigstar$ *of* u is the function whose graph is the Schwarz symmetrization of $|u|$, see for instance [29, Definition 1.C].

Lemma 4.3. *For all $u \in \mathcal{X}$, we have the inequality*

$$\int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right) \geq \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u^\bigstar|^2}\right), \tag{23}$$

where $u^\bigstar : \mathbb{R} \to \mathbb{R}$ denotes the symmetric rearrangement of u.

Proof. First we observe that $u^\bigstar$ is well defined if $u \in \mathcal{X}$ because u is Lipschitz continuous and all the level sets $\{x \in \mathbb{R}^N : u(x) > t\}$ $(t \in \mathbb{R})$ have finite measure. In addition, by the Pólya-Szegö inequality (see for instance [10, Theorem 4.7]), we have

$$\|\nabla u^\bigstar\|_{L^\infty} \leq \|\nabla u\|_{L^\infty} \leq 1$$

and

$$\int_{\mathbb{R}^N} |\nabla u^\bigstar|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2.$$

Therefore the right-hand side of (23) makes sense and both sides of the inequality are finite because ∇u is square integrable for $u \in \mathcal{X}$.

It is proven in [29, Theorem 1.C] (see also [21]) that the inequality

$$\int_{\mathbb{R}^N} \Phi(|\nabla u|) \geq \int_{\mathbb{R}^N} \Phi(|\nabla u^\bigstar|)$$

holds for any Lipschitz-continuous u which decays at infinity and any convex, increasing function $\Phi : [0, \infty[\to [0, \infty[$ satisfying $\Phi(0) = 0$.

For all $n \in \mathbb{N}$, let us consider the functions $H_n, G_n : [0, \infty[\to [0, \infty[$ defined by

$$\begin{aligned} H_n(s) &= 1 - (1 - s)^{1/2}, && \text{for } s < 1 - 1/n^2, \\ &= 1 - \frac{1}{n} + \frac{n}{2}(s - 1 + \frac{1}{n^2}), && \text{for } s \geq 1 - 1/n^2, \end{aligned}$$

and $G_n(s) = H_n(s^2)$. Observe that G_n is convex, increasing and satisfies $G_n(0) = 0$. Hence, by [29, Theorem 1.C], we know that

$$\int_{\mathbb{R}^N} G_n(|\nabla u|) \geq \int_{\mathbb{R}^N} G_n(|\nabla u^\bigstar|). \tag{24}$$

As $u \in \mathcal{D}^{1;2}(\mathbb{R}^N)$, the measure of the set $A := \{x \in \mathbb{R}^N : |\nabla u| \geq 1/2\}$ is finite and the fact that $u \in \mathcal{X}$ implies that, for all $n \geq 2$, $|G_n(|\nabla u(x)|^2)| \leq h(x)$ with $h \in L^1(\mathbb{R}^N)$ defined by

$$\begin{aligned} h(x) &= 1, && \text{for } x \in A, \\ &= |\nabla u|^2, && \text{for } x \notin A. \end{aligned}$$

Hence, we can apply Lebesgue's dominated convergence theorem to prove that

$$\int_{\mathbb{R}^N} G_n(|\nabla u|) \to \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right). \tag{25}$$

as n goes to infinity. We can argue in the same way to prove that

$$\int_{\mathbb{R}^N} G_n(|\nabla u^\bigstar|) \to \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u^\bigstar|^2}\right). \tag{26}$$

We then conclude by (24), (25) and (26). □

With this lemma at hand we can prove the following proposition.

Proposition 4.4. *If $\alpha > 2^\star$, the infimum*

$$C(\alpha) := \inf_{u \in \mathcal{X} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (1 - \sqrt{1 - |\nabla u|^2})}{\left(\int_{\mathbb{R}^N} |u|^\alpha\right)^{\frac{N}{N+\alpha}}}$$

is achieved by a radial solution of (2).

Proof. By Lemma 4.3 and the L^α-norm-preserving property of the symmetric rearrangement, we may restrict our attention to a minimizing sequence $(u_n)_n \subset \mathcal{X}$ of radial functions. Since the quotient is invariant under the family of homeomorphisms $\phi_t : v(\cdot) \mapsto tv(\cdot/t)$ $(t > 0)$, we may assume that $\int_{\mathbb{R}^N} |u_n|^\alpha = 1$. It is easily seen that $(u_n)_n$ is a priori bounded in $\mathcal{X}$. Lemma 2.1 then provides a bound in $\mathcal{D}^{1;q}(\mathbb{R}^N)$ for every $q \geq 2$. From Lemma 2.4, we deduce the required compactness to conclude that $(u_n)_n$ weakly converges in $\mathcal{D}^{1;2}(\mathbb{R}^N)$ to a function $u \in \mathcal{X}$ with $\int_{\mathbb{R}^N} |u|^\alpha = 1$. The fact that u realizes the infimum $C(\alpha)$ follows from the weak lower semi-continuity (with respect to the weak convergence in $\mathcal{D}^{1;2}(\mathbb{R}^N)$) of the volume integral.

To show that $w_t = t\, u(\cdot/t)$ solves (2) for some $t > 0$, we first prove that $|\nabla u|$ is bounded away from 1. Denoting by u_θ a minimizer of I_θ over $\mathcal{M}$ (see Proposition 3.1), we have for all $\theta > 0$,

$$I_\theta(u_\theta) \leq I_\theta(u) \leq I_0(u), \tag{27}$$

where the second inequality follows from the ordering property of the family I_θ. Moreover, we have established in Section 3 that $I_\theta(u_\theta) = I_0(u_\theta)$ for θ small enough. As u is a minimizer of I_0 this implies that the inequalities in (27) are in fact equalities. In particular, $I_\theta(u) = I_\theta(u_\theta)$, and u is a minimizer of I_θ over $\mathcal{M}$ too. The arguments of Proposition 3.4 now apply so that

$$|u'(r)| \leq \frac{E}{\sqrt{1 + E^2}} < 1,$$

for some $E > 0$ and we conclude as in the proof of Theorem 1.1. □

We now turn to the case of the critical exponent $\alpha = 2^\star$.

Proposition 4.5. *The infimum*

$$C(2^\star) = \inf_{u\in\mathcal{X}\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(1-\sqrt{1-|\nabla u|^2})}{\left(\int_{\mathbb{R}^N}|u|^{2^\star}\right)^{\frac{N}{N+2^\star}}}$$

is not achieved.

Proof. Assume by contradiction that $C(2^\star)$ is achieved by some $u\in\mathcal{X}$. As above, we may suppose that u is radial. Let us prove that

$$\int_0^\infty r^{N-1}\left[\left(1+\frac{N}{\alpha}\right)\frac{u'^2}{\sqrt{1-|u'|^2}} - N(1-\sqrt{1-|u'|^2})\right] \le 0. \tag{28}$$

Define for all $t\in[0,1]$,

$$f(t) := \frac{1}{2}\int_{\mathbb{R}^N} A_0\Big(t^{\frac{2N}{\alpha}+2}|\nabla u(tx)|^2\Big)\,dx = \frac{1}{2t^N}\int_{\mathbb{R}^N} A_0\Big(t^{\frac{2N}{\alpha}+2}|\nabla u(y)|^2\Big)\,dy.$$

Let $t\in(0,1)$ be fixed. As 1 is a minimum of f, the mean value theorem yields the existence of $\tilde t\in(t,1)$ such that

$$f'(\tilde t)\le 0. \tag{29}$$

(Note that we cannot conclude as in Lemma 3.3 that $f'(1)=0$ because f may not be well defined for $t>1$.) Here, the mean value theorem applies because f is continuous on $[t,1]$ and differentiable on $(t,1)$. In order to prove the differentiability of f in $\tilde s\in(t,1)$, observe first that from the strict inequality $\tilde s^{(2N/\alpha)+2}\,|\nabla u|^2<1$ a.e. in $\mathbb{R}^N$, we deduce the differentiability of the integrand. Moreover the derivative of the integrand satisfies the uniform estimate

$$\left|\left(\frac{2N}{\alpha}+2\right)\frac{\tilde s^{\frac{2N}{\alpha}+1}\,|\nabla u|^2}{\sqrt{1-\tilde s^{\frac{2N}{\alpha}+2}\,|\nabla u|^2}}\right| \le C\,|\nabla u|^2,$$

which holds for all s close to $\tilde s$ and almost every $x\in\mathbb{R}^N$. Lebesgue's dominated convergence theorem implies then that f is differentiable on $(t,1)$, and the inequality (29) is equivalent to

$$\begin{aligned} -N\tilde t^{-N-1}\int_{\mathbb{R}^N}\Big(1-\sqrt{1-\tilde t^{\frac{2N}{\alpha}+2}\,|\nabla u|^2}\Big) \\ +\tilde t^{-N}\int_{\mathbb{R}^N}\frac{1}{2}\left(\frac{2N}{\alpha}+2\right)\frac{\tilde t^{\frac{2N}{\alpha}+1}\,|\nabla u|^2}{\sqrt{1-\tilde t^{\frac{2N}{\alpha}+2}\,|\nabla u|^2}} \le 0. \end{aligned} \tag{30}$$

Next, we consider $(t_k)\subset(0,1)$ such that $t_k\to 1$ as $k\to\infty$. From what precedes, we infer the existence of a sequence $(\tilde t_k)\subset(0,1)$ still converging to 1 as k goes to

∞, and satisfying (30) with $\tilde{t} = \tilde{t}_k$ for all $k \in \mathbb{N}$. This implies that, for every $k \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left(\frac{N}{\alpha}+1\right) \frac{\tilde{t}_k^{\frac{2N}{\alpha}+1} |\nabla u|^2}{\sqrt{1-\tilde{t}_k^{\frac{2N}{\alpha}+2} |\nabla u|^2}} \\ &\leq \frac{N}{\tilde{t}_k} \int_{\mathbb{R}^N} \left(1-\sqrt{1-\tilde{t}_k^{\frac{2N}{\alpha}+2} |\nabla u|^2}\right) \\ &\leq N \tilde{t}_k^{\frac{2N}{\alpha}+1} \int_{\mathbb{R}^N} |\nabla u|^2 \\ &\leq N \int_{\mathbb{R}^N} |\nabla u|^2. \end{aligned}$$

Hence, it follows from Fatou's Lemma, Lebesgue's dominated convergence theorem and (30) that $\frac{|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} \in L^1(\mathbb{R}^N)$ and

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} \left(\frac{N}{\alpha}+1\right) \frac{\tilde{t}_k^{\frac{2N}{\alpha}+1} |\nabla u|^2}{\sqrt{1-\tilde{t}_k^{\frac{2N}{\alpha}+2} |\nabla u|^2}} \\ &\leq \liminf_{k\to\infty} \frac{N}{\tilde{t}_k} \int_{\mathbb{R}^N} \left(1-\sqrt{1-\tilde{t}_k^{\frac{2N}{\alpha}+2} |\nabla u|^2}\right) \\ &= N \int_{\mathbb{R}^N} \left(1-\sqrt{1-|\nabla u|^2}\right). \end{aligned}$$

This implies that (28) holds.

To conclude, we define the function $g : [0,1[\to \mathbb{R}$ by $g(s) := (1+\frac{N}{\alpha}-N)s - N\sqrt{1-s}+N$ and we compute $g(0)=0$, $g'(0) = 1+\frac{N}{\alpha}-\frac{N}{2}$ and $g''(s) = \frac{N}{4(1-s)^{3/2}}$. Therefore we have $g(s) > 0$ for $s \in]0,1[$ if and only if $1+\frac{N}{\alpha}-\frac{N}{2} \geq 0$, which is true if and only if $\alpha \leq 2^\star$. Hence, we infer that

$$\begin{aligned} 0 < \int_0^\infty r^{N-1} &\frac{g(u'^2)}{\sqrt{1-|u'|^2}} \\ &= \int_0^\infty r^{N-1} \left[\left(1+\frac{N}{\alpha}\right) \frac{u'^2}{\sqrt{1-|u'|^2}} - N(1-\sqrt{1-|u'|^2})\right], \end{aligned}$$

which contradicts (28). □

5. A multiplicity result

In this section, we use again the auxiliary functional I_θ defined in Section 2.

Since the manifold $\mathcal{M}$ is symmetric and I_θ is an even functional, Lusternik-Schnirelmann category theory provides a sequence of critical values for I_θ constrained to $\mathcal{M}$. More precisely, let $\mathcal{A}$ denote the set of closed and symmetric (with respect to the origin) subsets of $\mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$. We define the usual min-max values

$$c_\theta^k := \inf_{A\in\Gamma_k} \max_{u\in A} I_\theta(u),$$

where

$$\Gamma_k := \{A \subset \mathcal{M} : A \in \mathcal{A}, A \text{ is compact and } \gamma(A) \geq k\},$$

and $\gamma(A)$ is the genus of the set A. We refer e.g. to [1] for the definition of the genus and for more details on Lusternik-Schnirelmann theory.

We first show that these levels are indeed critical levels of I_θ. It is clear that $\mathcal{M} \subset \mathcal{A}$ and $\gamma(\mathcal{M}) = +\infty$. Next we show that I_θ satisfies the Palais-Smale condition on $\mathcal{M}$ by which we mean that every sequence $(u_n)_n \subset \mathcal{M}$ such that $I_\theta(u_n)$ is bounded and

$$I'_{\theta|\mathcal{M}}(u_n) \to 0$$

admits a converging subsequence. Here $I'_{\theta|\mathcal{M}}$ denotes the derivative of I_θ constrained to $\mathcal{M}$. Denoting by

$$T_u\mathcal{M} := \{v \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{\alpha-2}uv = 0\}$$

the tangent space to $\mathcal{M}$ at u, the projection $P_u : \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N) \to T_u\mathcal{M}$ is given by

$$P_u(w) = w - u\int_{\mathbb{R}^N} |u|^{\alpha-2}uw.$$

Then, for any $w \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ we have $v = P_u(w) \in T_u\mathcal{M}$ and

$$I'_{\theta|\mathcal{M}}(u)(v) = I'_{\theta|\mathcal{M}}(u)(P_u(w)) = I'_\theta(u)(w) - \lambda I'_\theta(u)(u),$$

where $\lambda = \int_{\mathbb{R}^N} |u|^{\alpha-2}uw$.

To prove the Palais-Smale condition, we will use the following convexity inequalities.

Lemma 5.1. *There exist $\gamma_2, \gamma_q > 0$ such that for every $u, v \in \mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$,*

$$I_\theta(\frac{u+v}{2}) \le \frac{1}{2}I_\theta(u) + \frac{1}{2}I_\theta(v) - \gamma_2\int_{\mathbb{R}^N} |\nabla u - \nabla v|^2 \tag{31}$$

and

$$I_\theta(\frac{u+v}{2}) \le \frac{1}{2}I_\theta(u) + \frac{1}{2}I_\theta(v) - \gamma_q\int_{\mathbb{R}^N} |\nabla u - \nabla v|^q. \tag{32}$$

Proof. Since I_θ has a uniformly positive definite second derivative, we can apply [16, Lemma 2.3] to deduce (31). In order to prove (32), we first observe that [16, Lemma 2.1] allows to show that $s \to A_\theta(s^2)$ is strongly q-monotone. This yields, for some $\gamma_q > 0$, the inequality

$$A_\theta\left(\left[\frac{u'(r)+v'(r)}{2}\right]^2\right) \le \frac{1}{2}A_\theta(u'(r)^2) + \frac{1}{2}A_\theta(v'(r)^2) - 2\gamma_q|u'(r) - v'(r)|^q$$

where u, v are given functions in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ and $r > 0$. Mutliplying by r^{N-1} and integrating from 0 to $+\infty$, we deduce (32). □

We now turn to the verification of the Palais-Smale condition.

Lemma 5.2. *For $\alpha > 2^\star$, the functional I_θ satisfies the Palais-Smale condition on $\mathcal{M}$.*

Proof. Let $(u_n)_n \subset \mathcal{M}$ be a Palais-Smale sequence, i.e. $I_\theta(u_n)$ is bounded and

$$I'_{\theta|\mathcal{M}}(u_n) \to 0.$$

Since I_θ is coercive, it is clear that $(u_n)_n$ is bounded in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ and therefore, by Lemma 2.4, up to a subsequence, there exists $u \in \mathcal{M}$ such that u_n converges

weakly to u in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ and strongly in $L^\alpha(\mathbb{R}^N)$ as $n\to\infty$. Since $(u_n)_n$ is a Palais-Smale sequence, we have, as $n\to\infty$,

$$I'_\theta(u_n)(P_{u_n}(u_n-u)) = I'_\theta(u_n)(u_n-u) - \lambda_n I'_\theta(u_n)(u_n)\to 0,$$

where we have written $\lambda_n = \int_{\mathbb{R}^N}|u_n|^{\alpha-2}u_n(u_n-u)$. Now, using the fact that $(u_n)_n$ is bounded in $\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ and $u_n\to u$ in $L^\alpha(\mathbb{R}^N)$, we infer $\lambda_n\to 0$ and $I'_\theta(u_n)(u_n)$ is bounded. Hence, we deduce that

$$\limsup_{n\to\infty} I'_\theta(u_n)(u_n-u)\le 0. \tag{33}$$

To complete the proof, it remains to show that $(u_n)_n$ converges strongly to u, which amounts to prove that

$$\|u_n-u\| = \left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^2\right)^{\frac12} + \left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^q\right)^{\frac1q}\to 0,$$

as $n\to\infty$. Since I_θ is locally bounded, we may assume that $I_\theta(u_n)$ converges. By weak lower semi-continuity, we infer

$$I_\theta(u)\le \liminf_{n\to\infty} I_\theta(u_n),$$

whereas the convexity of I_θ and (33) implies

$$\limsup_{n\to\infty} I_\theta(u_n)\le I_\theta(u)+\limsup_{n\to\infty} I'_\theta(u_n)(u_n-u)\le I_\theta(u).$$

Hence $I_\theta(u_n)$ converges to $I_\theta(u)$. Using again the lower semi-continuity of I_θ, (31) and (32), we conclude that

$$I_\theta(u)\le \liminf_{n\to\infty} I_\theta(\frac{u_n+u}{2})\le I_\theta(u)-\gamma_2\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^2$$

and

$$I_\theta(u)\le \liminf_{n\to\infty} I_\theta(\frac{u_n+u}{2})\le I_\theta(u)-\gamma_q\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^q.$$

This concludes the proof. □

Classical arguments now show that the level c^k_θ are critical values. We keep the notation $\theta_1 = 1/(2p+1)$.

Proposition 5.3. *Assume $\alpha>2^\star$ and $N-\frac{N}{\alpha}<q<N$. For every $k\ge 1$, there exists $\mu^k_\theta\in\mathbb{R}^+$ and $u^k_\theta\in\mathcal{D}^{1;(2,q)}_{rad}(\mathbb{R}^N)$ such that u^k_θ is a weak solution of*

$$-\mathrm{div}\left(a_\theta(|\nabla u^k_\theta|^2)\nabla u^k_\theta\right) = \mu^k_\theta\alpha|u^k_\theta|^{\alpha-2}u^k_\theta \quad \text{in } \mathbb{R}^N, \tag{34}$$

and $I_\theta(u^k_\theta)=c^k_\theta\to+\infty$ as $k\to\infty$. Moreover, u^k_θ is bounded in $\mathcal{C}^1(\mathbb{R}^N)$ and there exists $C_k>0$, $M_k>0$ such that, for all $\theta\in]0,\theta_1]$,

$$\max\{\|u^k_\theta\|_{\mathcal{D}^{1;(2,q)}},\|u^k_\theta\|_{L^{q^\star}}\}\le C_k \quad \text{and} \quad c^k_\theta\le M_k.$$

Proof. The proof follows easily from [1, Theorem 10.9 and Theorem 10.10] observing also that one can bound the min-max levels taking smooth functions such that $|\nabla u(x)|<1-\theta_1$ as competitors in the definition of c^k_θ. Then it is enough to follow the lines of the proof of Proposition 3.1 and Lemma 3.2. □

The next step towards the proof of Theorem 1.3 consists in finding a priori bounds for the Lagrange multiplier μ_θ^k with respect to θ. The argument in Lemma 3.3 cannot be used here (except for u_θ^1 which is a global minimizer). We then go back to the equation to derive the identity (19) for the solutions u_θ^k. In fact, we just need an inequality.

Lemma 5.4. *For all $\theta \in]0,\theta_1]$, $0 < \mu_\theta^k \leq \frac{N}{N+\alpha} c_\theta^k$.*

Proof. Multiplying (34) by u_θ^k and integrating, remembering also that $u_\theta^k \in \mathcal{M}$, we obtain

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta^k|^2)|\nabla u_\theta^k|^2 = \mu_\theta^k \alpha \int_{\mathbb{R}^N} |u_\theta^k|^\alpha = \mu_\theta^k \alpha.$$

This shows $\mu_\theta^k > 0$.

Let us prove that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta^k|^2)|\nabla u_\theta^k|^2 \leq \frac{N\alpha}{2N+2\alpha} \int_{\mathbb{R}^N} A_\theta(|\nabla u_\theta^k|^2). \tag{35}$$

This implies that

$$\mu_\theta^k \leq \frac{N}{2N+2\alpha} \int_{\mathbb{R}^N} A_\theta(|\nabla u_\theta^k|^2) = \frac{N}{N+\alpha} c_\theta^k.$$

We know that u_θ^k is a solution of

$$-\left(r^{N-1} a_\theta(v'^2) v'\right)' = \mu_\theta^k \alpha r^{N-1} |v|^{\alpha-2} v,$$

bounded in $\mathcal{C}^1(\mathbb{R}^N)$ and satisfying $\int_{\mathbb{R}^N} |\nabla u_\theta^k(x)|^2 < \infty$. Let us define the function

$$\begin{aligned} F(r) &= r^N a_\theta(|v'|^2)|v'|^2 - \frac{1}{2} r^N A_\theta(|v'|^2) + \mu r^N |v|^\alpha + \frac{N}{\alpha} r^{N-1} v' v a(|v'|^2) \\ &= r\left(r^{N-1} v' a_\theta(|v'|^2)\right) v' - \frac{1}{2} r^N A_\theta(|v'|^2) + \mu r^N |v|^\alpha \\ &\qquad + \frac{N}{\alpha} v\left(r^{N-1} a_\theta(|v'|^2) v'\right), \end{aligned}$$

where for short we have written $\mu = \mu_\theta^k$ and $v = u_\theta^k$. Then, using the equation, we compute

$$\begin{aligned} F'(r) &= r^{N-1} a_\theta(|v'|^2)|v'|^2 + r[\left(r^{N-1} a_\theta(|v'|^2) v'\right)' v' + r^{N-1} a_\theta(|v'|^2) v' v''] \\ &\quad - r^N v' v'' a_\theta(|v'|^2) - \frac{N}{2} r^{N-1} A_\theta(|v'|^2) + \mu N r^{N-1} |v|^\alpha \\ &\quad + \mu\alpha r^N |v|^{\alpha-2} v v' + \frac{N}{\alpha} v (r^{N-1} a_\theta(|v'|^2) v')' + \frac{N}{\alpha} r^{N-1} a_\theta(|v'|^2)|v'|^2 \\ &= r^{N-1}\left[(1+\frac{N}{\alpha}) a_\theta(|v'|^2)|v'|^2 - \frac{N}{2} A_\theta(|v'|^2)\right]. \end{aligned}$$

As v' and v are bounded, we have $F(0) = 0$. To estimate F at $+\infty$, we integrate the equation and we obtain

$$r^{N-1} a_\theta(|v'|^2) v' = -\mu\alpha \int_0^r (s^{N-1} |v|^{\alpha-2}(s) v(s))\, ds.$$

Using the decay estimate of Lemma 2.3, the a priori bound of Proposition 5.3 and the arguments of Lemma 3.2, we deduce that

$$\begin{aligned} r^{N-1}a_\theta(|v'|^2)|v'| &\le \mu\alpha\int_0^r (s^{N-1}|v|^{\alpha-1}(s))\,ds. \\ &\le \mu\alpha\int_0^1 (s^{N-1}|v|^{\alpha-1}(s))\,ds + C\int_1^r (s^{N-1}s^{-\frac{N-2}{2}(\alpha-1)})\,ds \\ &\le C(1+\int_1^r (s^{\frac{3N-4}{2}-\frac{N-2}{2}\alpha})\,ds) \\ &\le C(1+r^{\frac{3N-2}{2}-\frac{N-2}{2}\alpha}). \end{aligned}$$

Hence, we deduce that

$$a_\theta(|v'|^2)|v'| \quad \le \quad C(r^{1-N}+r^{\frac{N}{2}-\frac{N-2}{2}\alpha}),$$

and since $a_\theta(|v'|^2)\ge 1$, the same estimate holds for $|v'|$. This implies, again by Lemma 2.3 and Proposition 5.3, that

$$\begin{aligned} F(r) &\le r^N a_\theta(|v'|^2)|v'|^2 + \mu r^N|v|^\alpha + \frac{N}{\alpha}r^{N-1}v'va_\theta(|v'|^2) \\ &\le C(r^{2-N}+r^{\frac{2+N-(N-2)\alpha}{2}}+r^{2N-(N-2)\alpha}+r^{N-\frac{N-2}{2}\alpha}+r^{-\frac{N-2}{2}}). \end{aligned}$$

Since $\alpha > 2^\star$, we infer that

$$\limsup_{r\to\infty} F(r) \le 0$$

and therefore

$$\int_0^\infty F'(r) \le \limsup_{r\to\infty} F(r) - \lim_{r\to 0} F(r) = 0. \tag{36}$$

This completes the proof of (35). □

We are now able to complete the proof of Theorem 1.3.

Proposition 5.5. *For $\alpha > 2^\star$ and θ small enough, the function $t_k u_\theta^k(r/t_k)$ where u_θ^k is given by Proposition 5.3, and*

$$t_k = \left(\int_{\mathbb{R}^N} \frac{|\nabla u_\theta^k|^2}{\sqrt{1-|\nabla u_\theta^k|^2}}\right)^{1/\alpha},$$

is a solution of (2).

Proof. Fix $k \ge 1$. The proof follows from arguments that were used in Section 3. Indeed, since we have an estimate of the Lagrange multiplier μ_θ^k and on u_θ^k in $\mathcal{D}_{rad}^{1;(2,q)}(\mathbb{R}^N)$ as well as in $L^{q^\star}(\mathbb{R}^N)$ which are independent of θ, we infer, as in the proof of Proposition 3.4, that

$$|\nabla u_\theta^k| \le \frac{E}{\sqrt{1+E^2}},$$

for some $E > 0$. The result then follows for $\theta < \min\{\theta_1, 1/(1+E^2)\}$ as in the proof of Theorem 1.1. □

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