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# Controllability of the Stefan problem by the flatness approach

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## Abstract

We show the boundary controllability to stationary states of the Stefan problem with two phases and in one dimension in the space variable. In the case of an initial condition that is a stationary state and for a time of control large enough, we also obtain the controllability to stationary states with the sign constraints associated with the problem. Our method is based on the flatness approach that consists in writing the solution and the controls through two outputs and their derivatives. We construct these outputs as Gevrey functions of order  $\sigma$  so that our solution and controls are also in a Gevrey class.

**Keywords:** Controllability, Stefan problem, Gevrey functions

**2010 Mathematics Subject Classification.** 35C10, 80A22, 93B05

## 1 Introduction and main results

In this article, we consider the controllability of the Stefan problem with two phases and in dimension 1 in space. This system can model a solid-liquid phase transition, and is illustrated on Figure 1. One of the main difficulties in the study of this system is that it contains a free boundary, that is the interface between the two phases. In dimension 1 in space, the system couples two heat equations and a differential equation for the motion of the interface. There is an important literature devoted to the study of the Stefan problem. Concerning the well-posedness, one can quote for instance in the one-dimensional case [1–3, 6, 7, 10, 14]. In [12], the author obtains the analyticity of the interface between the two phases. Let us also mention [11] where the author shows the existence and uniqueness of weak solutions for the Stefan problem in dimension  $n \geq 1$  and analyzes their properties. The regularity of the free boundary in that case is studied in [16].

Concerning the controllability of the Stefan problem, let us mention [4, 8, 9] where the authors tackle the local null-controllability of Stefan problem in dimension 1 or in dimension 2 in star-shaped domains. The method is based on a linearization of the system, a fixed point argument and Carleman estimates to handle the controllability of the corresponding parabolic system. Note that in their results, the temperature is lead to zero but there is no constraint on the interface position. Recently, in [15], the controllability to trajectories of the Stefan problem with one phase and dimension 1 in space is shown. The method is based again on a linearization of the system after a change of variables and the controllability of the linear system is obtained by using Carleman estimates. In that case, the authors control both the temperature and the position of the interface. Let us also mention [13], where the authors obtain the controllability of the linearized Stefan problem with surface tension. Finally, in the case of feedback stabilization, we can quote for instance [17–19]. The method used to stabilize the Stefan problem is based on the backstepping method.

In this article, our aim is to obtain the controllability to the steady states, and in order to do this, we are going to use the flatness method that we recall in the next section. In this approach, the state and the controls

are described by series. Such an approach was already applied for the heat equation [26], general parabolic equations [28], the Schrödinger equation [30], the linear Korteweg–de Vries equation [25] and semilinear heat equations [20]. Note in particular that the flatness approach was already applied to the Stefan problem for motion planning questions in [5].

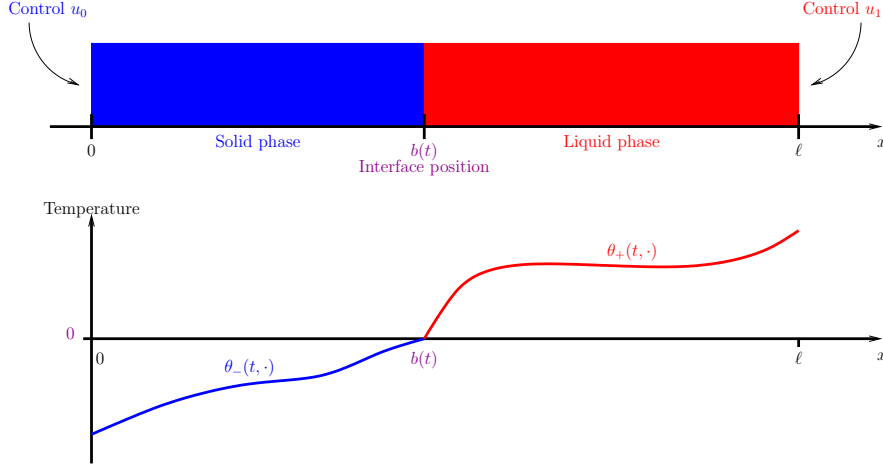


Figure 1: Schematic representation of the problem.

Let us now describe precisely our system and our results. First we consider the Stefan problem:

$$\partial_t \theta = c \partial_x^2 \theta \quad (t \geq 0, x \in (0, 1) \setminus \{b(t)\}), \quad (1.1a)$$

$$\theta(t, b(t)) = 0 \quad (t > 0), \quad (1.1b)$$

$$b'(t) = \partial_x \theta(t, b(t)^-) - \partial_x \theta(t, b(t)^+) \quad (t > 0), \quad (1.1c)$$

with initial condition:

$$b(0) = b^0, \quad \theta(0, \cdot) = \theta^0. \quad (1.2)$$

At the boundaries, we impose two controls:

$$\theta(t, 0) = u_0(t), \quad \theta(t, 1) = u_1(t) \quad (t > 0). \quad (1.3)$$

In the above system,  $\theta$  denotes the temperature of the material, and we assume that this material is liquid for  $x < b(t)$  and solid for  $x > b(t)$ . If necessary, we write

$$\theta_-(t, x) := \theta(t, x) \quad (x < b(t)) \quad \text{and} \quad \theta_+(t, x) := \theta(t, x) \quad (x > b(t)).$$

Both temperatures are equal at the interface and by a change of variables we can assume that  $\theta_-(t, b(t)) = \theta_+(t, b(t))$  which permits to use the global variable  $\theta$  as a function continuous at  $b(t)$ . The motion of the interface depends on the jump of the derivatives of  $\theta$  at  $x = b(t)$ . Here, we have assumed that the diffusion coefficient is constant on each state of the material:

$$c := \begin{cases} c_- & \text{if } x < b(t), \\ c_+ & \text{if } x > b(t), \end{cases} \quad (1.4)$$

where  $c_-, c_+ \in \mathbb{R}_+^*$ . Note that by some change of variables we can assume that all the other physical parameters are equal to 1. A natural physical condition for the above system consists in imposing

$$\theta(t, x) < 0 \quad (x \in (0, b(t))) \quad \text{and} \quad \theta(t, x) > 0 \quad (x \in (b(t), 1)) \quad (t > 0). \quad (1.5)$$

Our aim is to use the controls  $u_0$  and  $u_1$  in order to reach a stationary state  $(\bar{b}, \bar{\theta})$  of (1.1) in a finite time  $T > 0$ . Such a stationary state satisfies  $\bar{b} \in (0, 1)$  and for some constant  $\bar{v} \in \mathbb{R}$ :

$$\bar{\theta}(x) = \bar{v}(x - \bar{b}). \quad (1.6)$$

In our first result, we show that we can reach such a stationary state by using the flatness approach. Note that in this result, we do not impose the condition (1.5).

**Theorem 1.1.** *Assume*

$$\begin{aligned} b^0 \in (0, 1), \quad \theta^0 \in C^1([0, 1] \setminus \{b^0\}) \cap C^0([0, 1]), \\ \theta^0 \leq 0 \text{ in } [0, b^0], \quad \theta^0 \geq 0 \text{ in } [b^0, 1], \quad \theta^0(0) < 0, \quad \theta^0(1) > 0. \end{aligned} \quad (1.7)$$

Then for any  $T > 0$ ,  $\sigma \in (1, 2)$ ,  $\bar{b} \in (0, 1)$  and  $\bar{v} \in \mathbb{R}$ , there exist two controls  $u_0, u_1$  Gevrey of order  $\sigma$  in  $[0, T]$  such that the solution  $(b, \theta)$  of (1.1)–(1.3) satisfies

$$b(T) = \bar{b}, \quad \theta(T, x) = \bar{v}(x - \bar{b}) \quad (x \in [0, 1]). \quad (1.8)$$

Moreover, for any  $\varepsilon > 0$ ,  $b$  is Gevrey of order  $\sigma$  in  $[\varepsilon, T]$  and there exist  $S_-$  and  $S_+$  in  $[\varepsilon, T] \times \mathbb{R}$  Gevrey of order  $\sigma$  in time and 1 in space such that

$$\begin{aligned} \theta_-(t, x) &= S_-(t, x - b(t)) & (t \in [\varepsilon, T], x \in [0, b(t)]), \\ \theta_+(t, x) &= S_+(t, x - b(t)) & (t \in [\varepsilon, T], x \in [b(t), 1]). \end{aligned}$$

In order to respect the sign condition (1.5), the stationary states, described by (1.6), need to satisfy  $\bar{v} > 0$ . The second result states that one can use the controls  $u_0$  and  $u_1$  to pass from such a stationary state to another, and to keep the sign condition (1.5) during the evolution of the system. To obtain such a result, we need a time of control large enough.

**Theorem 1.2.** *Assume*

$$b^0 \in (0, 1), \quad v^0 > 0, \quad \theta^0(x) = v^0(x - b^0) \quad (1.9)$$

and suppose that  $\bar{v} > 0$ . Then there exists  $T_1$  with the following property: if  $T \geq T_1$  one can choose the controls  $u_0$  and  $u_1$  such that the same properties of Theorem 1.1 hold with  $\varepsilon = 0$  (in particular (1.8)) and such that (1.5) holds for all  $t \in [0, T]$ .

**Remark 1.3.** *In the proof of Theorems 1.1 and 1.2, one can see that in fact our controls  $u_0$  and  $u_1$  are constructed from two Gevrey functions of order  $\sigma$ :  $\alpha_{-,0}$ ,  $\alpha_{+,0}$  and their derivatives.*

**Remark 1.4.** *We recall that  $f$  is Gevrey of order  $\sigma$  in a domain  $I \subset \mathbb{R}$  if it is  $C^\infty$ -smooth and if for any compact  $K \subset I$ , there exist  $M, R > 0$  such that*

$$\|f^{(n)}\|_{L^\infty(K)} \leq M \frac{(n!)^\sigma}{R^n} \quad (n \in \mathbb{N}).$$

If  $\sigma = 1$ , it is thus an analytic function. As explained for instance in [27], one of the interests of such a class of functions here is that for  $\sigma > 1$ , there exist step functions that are Gevrey of order  $\sigma$  (see Section 3). The notion of Gevrey function for several variables is similar: for instance for two variables as above,  $f$  is Gevrey of order  $\sigma_1$  in time and  $\sigma_2$  in space if there exist  $M, R_1, R_2 > 0$  such that

$$\|\partial_t^m \partial_x^n f\|_{L^\infty(K)} \leq M \frac{(m!)^{\sigma_1}}{R_1^m} \frac{(n!)^{\sigma_2}}{R_2^n} \quad (m, n \in \mathbb{N}).$$

**Remark 1.5.** *An important feature in our two results is that we do not linearize the system, and consequently, we do not have any smallness assumptions as this is the case for the results in [4, 8, 9]. With respect to previous systems studied with the flatness approach (see [20, 25, 26, 28, 30]), we consider here a nonlinear problem coupling PDEs and ODEs and with a free boundary.*

**Remark 1.6.** *Let us mention some results for the controllability of parabolic equations with a sign constraint on the control or on the state: [21–24], etc. It is proved in particular in [23, Theorem 3] that to control the heat equation under non negativity constraints on the state, one need a minimal time. Let us also quote a similar result [31, Theorem 1.1] in the case of the wave equation under a nonnegative control constraint.*

**Remark 1.7.** *Let us mention that in this paper, we have chosen Dirichlet boundary controls, but this is not required for the flatness approach. For instance, similar results could have been obtained with Neumann or Robin boundary controls.*

The outline of the article is as follows. In Section 2, we describe the flatness approach in the case of the Stefan problem. Then in Section 3, we present some results on the Gevrey functions. Several of these results are well-known, but we point out that a key point in the proof of the main results is Proposition 3.4. In the proof of Theorem 1.1, we need to show that the solutions of the Stefan is regular enough for any positive time. In order to do this we use [12] where the time regularity is proved and standard methods to obtain the regularity in space. Finally, in Section 5, we prove Theorems 1.1 and 1.2. The last section, Section 6, is devoted to some numerical illustrations of our method.

## 2 The flatness approach

In this section, we present the flatness approach for the Stefan problem. We decompose the solution of (1.1) as follows

$$\theta(t, x) := \sum_{i=0}^{\infty} \alpha_{-,i}(t) \frac{(x-b(t))^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty} \beta_{-,i}(t) \frac{(x-b(t))^{2i}}{(2i)!} \quad (x < b(t)), \quad (2.1a)$$

$$\theta(t, x) := \sum_{i=0}^{\infty} \alpha_{+,i}(t) \frac{(x-b(t))^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty} \beta_{+,i}(t) \frac{(x-b(t))^{2i}}{(2i)!} \quad (x > b(t)). \quad (2.1b)$$

Note that formally (1.1b) yields  $\beta_{\pm,0} = 0$ . The separation between odd and even powers is due to the case of the heat equation, see [27, 29] where only the odd or the even powers appear.

Formally, for both (2.1a) and (2.1b), we have

$$\begin{aligned} \partial_t \theta - c \partial_x^2 \theta &= \sum_{i=0}^{\infty} \alpha'_i(t) \frac{(x-b(t))^{2i+1}}{(2i+1)!} - \sum_{i=0}^{\infty} \alpha_i(t) b'(t) \frac{(x-b(t))^{2i}}{(2i)!} \\ &\quad + \sum_{i=0}^{\infty} \beta'_i(t) \frac{(x-b(t))^{2i}}{(2i)!} - \sum_{i=0}^{\infty} \beta_{i+1}(t) b'(t) \frac{(x-b(t))^{2i+1}}{(2i+1)!} \\ &\quad - c \sum_{i=0}^{\infty} \alpha_{i+1}(t) \frac{(x-b(t))^{2i+1}}{(2i+1)!} - c \sum_{i=0}^{\infty} \beta_{i+1}(t) \frac{(x-b(t))^{2i}}{(2i)!}, \end{aligned}$$

so that (1.1a) yields that  $\alpha_i$  and  $\beta_i$  satisfy the recurrence formula

$$\begin{cases} \beta_{-,i+1} = \frac{1}{c_-} \beta'_{-,i} - \frac{1}{c_-} b' \alpha_{-,i}, \\ \alpha_{-,i+1} = \frac{1}{c_-} \alpha'_{-,i} - \frac{1}{c_-} b' \beta_{-,i+1}, \end{cases} \quad \begin{cases} \beta_{+,i+1} = \frac{1}{c_+} \beta'_{+,i} - \frac{1}{c_+} b' \alpha_{+,i}, \\ \alpha_{+,i+1} = \frac{1}{c_+} \alpha'_{+,i} - \frac{1}{c_+} b' \beta_{+,i+1}, \end{cases} \quad (i \geq 0). \quad (2.2)$$

Finally, (1.1c) imposes,

$$b' = \alpha_{-,0} - \alpha_{+,0}. \quad (2.3)$$

We thus see that all the coefficients of  $\theta$ , that is  $(\alpha_{\pm,i})$ ,  $(\beta_{\pm,i})$ , and  $b'$  are completely given by  $\alpha_{-,0}$  and  $\alpha_{+,0}$ . The idea of the flatness method consists in constructing  $\alpha_{-,0}$  and  $\alpha_{+,0}$  so that  $\theta$  and  $b$  satisfy the initial and

final conditions and then, the controls  $u_0$  and  $u_1$  are given by

$$u_0(t) := - \sum_{i=0}^{\infty} \alpha_{-,i}(t) \frac{b(t)^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty} \beta_{-,i}(t) \frac{b(t)^{2i}}{(2i)!}, \quad (2.4a)$$

$$u_1(t) := \sum_{i=0}^{\infty} \alpha_{+,i}(t) \frac{(1-b(t))^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty} \beta_{+,i}(t) \frac{(1-b(t))^{2i}}{(2i)!}. \quad (2.4b)$$

The proof of Theorem 1.1 relies on the above formal computation that we justify by using the Gevrey functions.

### 3 Gevrey functions

We recall here some standard results on the Gevrey functions, and we study systems of the form (2.2) when the first two terms are Gevrey of order  $\sigma$ . Let us quote for instance [32–34] for some references on the Gevrey functions.

**Notation 3.1.** For  $M, R > 0$  and  $\sigma \geq 1$ , we denote by  $\mathcal{G}(M, R, \sigma)$  the set of functions  $f \in C^\infty(\mathbb{R})$  such that

$$\|f^{(n)}\|_{L^\infty(\mathbb{R})} \leq M \frac{(n!)^\sigma}{R^n} \quad (n \in \mathbb{N}).$$

We recall the following results

**Lemma 3.2.** Assume  $\sigma > 1$ . Then there exist  $\phi_\sigma : \mathbb{R} \rightarrow [0, 1]$  and  $\eta_\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$  two Gevrey functions of order  $\sigma$  such that

$$\begin{aligned} \phi_\sigma(t) &= \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t \geq 1, \end{cases} \quad \phi_\sigma(t) + \phi_\sigma(1-t) = 1 \quad (t \in \mathbb{R}), \\ \text{supp } \eta_\sigma &\subset [0, 1], \quad \eta_\sigma \geq 0, \quad \int_{\mathbb{R}} \eta_\sigma ds = 1. \end{aligned} \quad (3.1)$$

In particular, there exist  $M_0, R_0 > 0$  such that  $\phi_\sigma, \eta_\sigma \in \mathcal{G}(M_0, R_0, \sigma)$ .

*Proof.* We set  $k := (\sigma - 1)^{-1}$  and we consider the Gevrey function of order  $\sigma$

$$\phi_\sigma(t) := \begin{cases} 1 & \text{if } t \leq 0, \\ \frac{e^{-((1-t)^{-k})}}{e^{-((1-t)^{-k})} + e^{-(t^{-k})}} & \text{if } t \in (0, 1), \\ 0 & \text{if } t \geq 1. \end{cases}$$

Then, we set

$$\eta_\sigma := \frac{\phi_\sigma(t)\phi_\sigma(1-t)}{\int_{\mathbb{R}} \phi_\sigma(s)\phi_\sigma(1-s) ds} \quad (t \in \mathbb{R}).$$

One can check that  $\phi_\sigma$  and  $\eta_\sigma$  satisfy the properties of the lemma.  $\square$

**Lemma 3.3.** Assume  $\ell, i, j \in \mathbb{N}$  and  $\sigma \geq 1$ . Then

$$\sum_{m=0}^{\ell} \binom{\ell}{m} (\ell - m + i)^\sigma (m + j)^\sigma \leq \left( \frac{i!j!(i+j+\ell+1)!}{(i+j+1)!} \right)^\sigma.$$

We have the following crucial result

**Proposition 3.4.** Assume  $\sigma \geq 1$ ,  $M_f, M_\alpha, M_\beta, R, \chi \in \mathbb{R}_+^*$ . Suppose  $f \in \mathcal{G}(M_f, R, \sigma)$  and let us consider the sequences  $(\alpha_i)_{i \in \mathbb{N}}$  and  $(\beta_i)_{i \in \mathbb{N}}$  defined by

$$\alpha_0 \in \mathcal{G}(M_\alpha, R, \sigma), \quad \beta_0 \in \mathcal{G}(M_\beta, R, \sigma)$$

and for any  $i \geq 0$ ,

$$\begin{cases} \beta_{i+1} = \chi \beta'_i - \chi f \alpha_i, \\ \alpha_{i+1} = \chi \alpha'_i - \chi f \beta_{i+1}. \end{cases} \quad (3.2)$$

Then, for any  $\rho > 0$  such that

$$\rho \leq \frac{2^{3-\sigma} R}{3} \quad \text{and} \quad \rho \leq \frac{2M_\alpha}{3\chi \left( \frac{M_\beta}{R} + M_\alpha M_f \right) M_f}, \quad (3.3)$$

we have, for any  $\ell, i \in \mathbb{N}$ ,

$$\left\| \alpha_i^{(\ell)} \right\|_{L^\infty(\mathbb{R})} \leq \frac{M_\alpha \chi^i}{R^\ell \rho^i} \frac{(\ell + 2i)!^\sigma}{i!^{2-\sigma} (2i)!^{\sigma-1}}, \quad (3.4)$$

$$\left\| \beta_{i+1}^{(\ell)} \right\|_{L^\infty(\mathbb{R})} \leq \left( \frac{M_\beta}{R} + M_\alpha M_f \right) \frac{\chi^{i+1}}{R^\ell \rho^i} \frac{(\ell + 2i + 1)!^\sigma}{i!^{2-\sigma} (2i + 1)!^{\sigma-1}}. \quad (3.5)$$

If  $\beta_0 = 0$ , then one can take  $M_\beta = 0$  in the above statements.

**Remark 3.5.** Note that in [5], the authors show an induction result similar to Proposition 3.4 (see [5, Theorem 2.1]). More precisely, they can deal with more general non-linearities as the one associated with the Stefan problem, but they also need a smallness condition that we avoid here.

*Proof of Proposition 3.4.* We consider  $\rho > 0$  satisfying (3.3). We show (3.4)–(3.5) by induction on  $i$ . For  $i = 0$ , (3.4) is a consequence of  $\alpha_0 \in \mathcal{G}(M_\alpha, R, \sigma)$ . Then, using that  $f \in \mathcal{G}(M_f, R, \sigma)$ ,  $\alpha_0 \in \mathcal{G}(M_\alpha, R, \sigma)$ , and  $\beta_0 \in \mathcal{G}(M_\beta, R, \sigma)$  combined with the Leibniz rule, we deduce,

$$\left| \beta_1^{(\ell)} \right| \leq \chi \left| \beta_0^{(\ell+1)} \right| + \chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \alpha_0^{(\ell-m)} f^{(m)} \right| \leq \chi \frac{M_\beta (\ell + 1)!^\sigma}{R^{\ell+1}} + \frac{\chi M_\alpha M_f}{R^\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} (\ell - m)!^\sigma (m!)^\sigma.$$

Using Lemma 3.3, we obtain (3.5) for  $i = 0$ .

Assume now (3.4)–(3.5) for  $i \in \mathbb{N}$ . Then (3.2) and the Leibniz rule imply

$$\left| \alpha_{i+1}^{(\ell)} \right| \leq \chi \left| \alpha_i^{(\ell+1)} \right| + \chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \beta_{i+1}^{(\ell-m)} f^{(m)} \right|. \quad (3.6)$$

Using (3.4) and that  $\sigma \geq 1$ , we have

$$\begin{aligned} \chi \left| \alpha_i^{(\ell+1)} \right| &\leq \frac{M_\alpha \chi^{i+1}}{R^\ell \rho^{i+1}} \frac{(\ell + 2 + 2i)!^\sigma}{(i + 1)!^{2-\sigma} (2i + 2)!^{\sigma-1}} \frac{\rho (i + 1)^{2-\sigma} (2i + 2)^{\sigma-1} (2i + 1)^{\sigma-1}}{(\ell + 2 + 2i)^\sigma} \\ &\leq \frac{M_\alpha \chi^{i+1}}{R^\ell \rho^{i+1}} \frac{(\ell + 2 + 2i)!^\sigma}{(i + 1)!^{2-\sigma} (2i + 2)!^{\sigma-1}} \frac{\rho 2^{\sigma-2}}{R}. \end{aligned} \quad (3.7)$$

Using (3.5) and that  $f \in \mathcal{G}(M_f, R, \sigma)$ , we have that

$$\chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \beta_{i+1}^{(\ell-m)} f^{(m)} \right| \leq \frac{\chi^{i+2}}{R^\ell \rho^i} \left( \frac{M_\beta}{R} + M_\alpha M_f \right) M_f \sum_{m=0}^{\ell} \binom{\ell}{m} (\ell - m + 2i + 1)!^\sigma (m!)^\sigma.$$

Applying Lemma 3.3 and using that  $\sigma \geq 1$ , we find

$$\chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \beta_{i+1}^{(\ell-m)} f^{(m)} \right| \leq \frac{M_{\alpha} \chi^{i+1}}{R^{\ell} \rho^{i+1}} \frac{(\ell+2+2i)^{\sigma}}{(i+1)!^{2-\sigma} (2i+2)!^{\sigma-1}} \frac{\rho \chi \left( \frac{M_{\beta}}{R} + M_{\alpha} M_f \right) M_f}{2M_{\alpha}}.$$

Using the above estimate and (3.7), in (3.6), together with the condition (3.3) on  $\rho$ , we deduce (3.4) for  $\alpha_{i+1}$ .

We proceed similarly for  $\beta_{i+1}$ : applying the Leibniz rule on (3.2), we deduce

$$\left| \beta_{i+2}^{(\ell)} \right| \leq \chi \left| \beta_{i+1}^{(\ell+1)} \right| + \chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \alpha_{i+1}^{(\ell-m)} f^{(m)} \right|. \quad (3.8)$$

From (3.5), we have

$$\chi \left| \beta_{i+1}^{(\ell+1)} \right| \leq \left( \frac{M_{\beta}}{R} + M_{\alpha} M_f \right) \frac{\chi^{i+2}}{R^{\ell} \rho^{i+1}} \frac{(\ell+2i+3)^{\sigma}}{(i+1)!^{2-\sigma} (2i+3)!^{\sigma-1}} \frac{\rho 2^{\sigma-2}}{R}. \quad (3.9)$$

Using (3.4) and that  $f \in \mathcal{G}(M_f, R, \sigma)$ , we have that

$$\chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \alpha_{i+1}^{(\ell-m)} f^{(m)} \right| \leq \frac{M_{\alpha} M_f \chi^{i+2}}{R^{\ell} \rho^{i+1}} \frac{1}{(i+1)!^{2-\sigma} (2i+2)!^{\sigma-1}} \sum_{m=0}^{\ell} \binom{\ell}{m} (\ell-m+2i+2)^{\sigma} m!^{\sigma}.$$

The above estimate and Lemma 3.3 imply

$$\chi \left| \sum_{m=0}^{\ell} \binom{\ell}{m} \alpha_{i+1}^{(\ell-m)} f^{(m)} \right| \leq \frac{1}{3} \left( \frac{M_{\beta}}{R} + M_{\alpha} M_f \right) \frac{\chi^{i+2}}{R^{\ell} \rho^{i+1}} \frac{(\ell+2i+3)^{\sigma}}{(i+1)!^{2-\sigma} (2i+3)!^{\sigma-1}}.$$

Inserting the above estimate and (3.9) into (3.8), we deduce

$$\left| \beta_{i+2}^{(\ell)} \right| \leq \left( \frac{1}{3} + \frac{\rho 2^{\sigma-2}}{R} \right) \left( \frac{M_{\beta}}{R} + M_{\alpha} M_f \right) \frac{\chi^{i+2}}{R^{\ell} \rho^{i+1}} \frac{(\ell+2i+3)^{\sigma}}{(i+1)!^{2-\sigma} (2i+3)!^{\sigma-1}}.$$

Using (3.3), the above inequality implies (3.5) for  $\beta_{i+1}$ .  $\square$

**Corollary 3.6.** *With the assumptions of Proposition 3.4, if  $\sigma < 2$ , then the function defined by*

$$S(t, x) := \sum_{i=0}^{\infty} \alpha_i(t) \frac{x^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty} \beta_i(t) \frac{x^{2i}}{(2i)!} \quad (t, x \in \mathbb{R}), \quad (3.10)$$

is Gevrey of order  $\sigma$  in time and 1 in space and

$$\partial_t^{\ell} \partial_x^n S(t, x) = \sum_{2i+1 \geq n} \alpha_i^{(\ell)}(t) \frac{x^{2i+1-n}}{(2i+1-n)!} + \sum_{2i \geq n} \beta_i^{(\ell)}(t) \frac{x^{2i-n}}{(2i-n)!} \quad (t, x \in \mathbb{R}). \quad (3.11)$$

Moreover, for any  $r > 0$ , there exist  $C, \tilde{r} > 0$  such that

$$\left| \partial_t^{\ell} \partial_x^n S(t, x) \right| \leq C \frac{n!}{r^n} \frac{(\ell)^{\sigma}}{\tilde{r}^{\ell}} \quad (t, x \in \mathbb{R}, |x| \leq r). \quad (3.12)$$

*Proof.* Applying Proposition 3.4, we deduce the existence of  $C > 0$  such that for  $t, x \in \mathbb{R}$ ,

$$\left| \alpha_i^{(\ell)}(t) \frac{x^{2i+1-n}}{(2i+1-n)!} \right| \leq C \frac{\chi^i}{R^{\ell} \rho^i} |x|^{2i+1-n} \frac{(\ell+2i)^{\sigma}}{i!^{2-\sigma} (2i)!^{\sigma-1}} \frac{1}{(2i+1-n)!}. \quad (3.13)$$



$$\left| \beta_i^{(\ell)}(t) \frac{x^{2i-n}}{(2i-n)!} \right| \leq C \frac{\chi^{i+1}}{R^\ell \rho^i} |x|^{2i-n} \frac{(\ell+2i+1)!^\sigma}{i!^{2-\sigma} (2i+1)!^{\sigma-1} (2i-n)!}. \quad (3.14)$$

Using that

$$(\ell+2i)! \leq 2^{\ell+2i} \ell! (2i)!, \quad (2i)! \leq 2^{2i} (2i+1-n)! (n-1)!$$

and similar relations for  $\beta_i$ , we deduce that if  $|x| \leq r$ ,

$$\left| \alpha_i^{(\ell)}(t) \frac{x^{2i+1-n}}{(2i+1-n)!} \right| + \left| \beta_i^{(\ell)}(t) \frac{x^{2i-n}}{(2i-n)!} \right| \leq C \frac{n!}{r^n} \left( \frac{2^\sigma}{R} \right)^\ell (\ell!)^\sigma \frac{1}{i!^{2-\sigma}} \left( \frac{\chi^{4\sigma+1} r^2}{\rho} \right)^i. \quad (3.15)$$

Since  $\sigma < 2$ ,

$$\sum_{i \geq 0} \frac{1}{i!^{2-\sigma}} \left( \frac{\chi^{4\sigma+1} r^2}{\rho} \right)^i < \infty,$$

Therefore, the function  $S$  defined by (3.10) is  $C^\infty$  in time and space and satisfies for any  $t, x \in \mathbb{R}$ ,  $|x| \leq r$ ,

$$|\partial_t^\ell \partial_x^n S(t, x)| \leq C \frac{n!}{r^n} \left( \frac{2^\sigma}{R} \right)^\ell (\ell!)^\sigma.$$

□

## 4 Analyticity of the solutions of the Stefan problem

In this section, we show that the solutions of the Stefan problem are analytic in the time and space variables for any positive time and more precisely that we can write  $\theta$  under the form (2.1) provided that we choose the controls adequately. The proof is based on results of [12] that we recall here.

Let us assume that  $\theta^0$  and  $b^0$  satisfy (1.7). We consider  $u_0, u_1 : [0, T] \rightarrow \mathbb{R}$  analytic and such that  $u_0(0) = \theta^0(0)$ ,  $u_1(0) = \theta^0(1)$ ,  $u_0 < 0$  and  $u_1 > 0$ . One can for instance take  $u_0$  and  $u_1$  as constant functions. Then from [12], the system (1.1)–(1.3) admits a unique solution  $(b, \theta)$  in  $[0, T]$ . Moreover,  $b$  is analytic in time and after a change of variable, the temperature is also analytic in time. Let us precise this result.

First, we consider the map

$$\mathbb{T}(t) := \int_0^t \frac{1}{b(s)^2} ds.$$

We have that  $\mathbb{T}$  is a bijection from  $[0, T]$  onto  $[0, T_b]$ , where  $T_b := \int_0^T \frac{1}{b^2} ds$ , and we can define

$$f(\tau) := u_0(\mathbb{T}^{-1}(\tau)) \quad (\tau \in [0, T_b]),$$

$$v(\tau, y) := \theta(\mathbb{T}^{-1}(\tau), y b(\mathbb{T}^{-1}(\tau))) - (1-y)f(\tau) \quad (\tau \in [0, T_b], y \in [0, 1]).$$

It is proven in [12] that for any closed interval  $\mathcal{I}$  in  $(0, T_b]$ , there exist  $M > 0$ ,  $K_\tau > 0$  and  $K_y > 0$  such that for all  $\tau \in \mathcal{I}$  and  $y \in [0, 1]$ ,

$$|\partial_\tau^n \partial_y^q v(\tau, y)| \leq M K_\tau^n K_y^q (n+q)! \quad (n \geq 0, q \in \{0, 1, 2\}). \quad (4.1)$$

The above estimates yield in a standard way the analyticity of  $v$  in time and space. For sake of completeness, let us give a sketch of proof of this property. First, we deduce from (1.1a) that for  $\tau \in [0, T_b]$ , and  $y \in [0, 1]$ ,

$$\partial_\tau v - c \partial_{yy} v = b(\mathbb{T}^{-1}(\tau)) b'(\mathbb{T}^{-1}(\tau)) y (\partial_y v - f(\tau)) - (1-y) f'(\tau)$$

and thus

$$\partial_{yy} v = \frac{1}{c} \partial_\tau v + y d(\tau) \partial_y v + f_0(\tau) + y f_1(\tau) \quad (4.2)$$

with  $d, f_0, f_1$  analytic in  $(0, T]$ . By taking possibly  $M$  and  $K_\tau$  larger, we can also assume that for all  $\tau \in \mathcal{I}$ ,

$$\left|d^{(n)}(\tau)\right| + \left|f_1^{(n)}(\tau)\right| \leq MK_\tau^n n! \quad (n \geq 0). \quad (4.3)$$

We can also suppose that

$$K_y \geq \frac{1}{c}K_\tau + 2M + 1. \quad (4.4)$$

Then we can show by induction on  $q$  that for any  $\tau \in \mathcal{I}$ ,  $y \in [0, 1]$ , and  $n \geq 0$ ,

$$\left|\partial_\tau^n \partial_y^q v(\tau, y)\right| \leq MK_\tau^n K_y^q (n+q)!. \quad (4.5)$$

From (4.2), we deduce

$$\partial_\tau^n \partial_y^3 v = \frac{1}{c} \partial_\tau^{n+1} \partial_y v + \sum_{k=0}^n \binom{n}{k} d^{(n-k)} \partial_\tau^k \partial_y v + y \sum_{k=0}^n \binom{n}{k} d^{(n-k)} \partial_\tau^k \partial_y^2 v + f_1^{(n)}(\tau) \quad (4.6)$$

From (4.1) and (4.3), the above relation yields

$$\begin{aligned} \left|\partial_\tau^n \partial_y^3 v(\tau, y)\right| &\leq \frac{1}{c} MK_\tau^{n+1} K_y (n+2)! + M^2 K_\tau^n K_y n! (n+1)^2 + M^2 K_\tau^n K_y^2 n! (n+2)(n+1)^2 + MK_\tau^n n! \\ &= MK_\tau^n K_y^2 (n+3)! \left( \frac{1}{c} \frac{K_\tau}{K_y} \frac{1}{n+3} + \frac{M}{K_y} \frac{n+1}{(n+2)(n+3)} + M \frac{n+1}{n+3} + \frac{1}{K_y^2} \frac{1}{(n+1)(n+2)(n+3)} \right). \end{aligned}$$

Combining this relation with (4.4), we obtain (4.5) for  $q = 3$ . The induction for  $q \geq 4$  is completely similar.

We deduce from (4.5) the existence of  $\varepsilon > 0$  such that for  $y \in [1 - \varepsilon, 1]$  and  $\tau \in \mathcal{I}$ ,

$$v(\tau, y) = \sum_{q=1}^{\infty} v_q(\tau) \frac{(y-1)^q}{q!}.$$

Thus, for any interval  $\mathcal{J}$  of  $(0, T]$ , we can decompose  $\theta$  as (2.1a) for  $t \in \mathcal{J}$  and for  $x \in [(1 - \varepsilon)b(t), b(t)]$ . We can proceed similarly for  $x \in [b(t), (1 + \varepsilon)b(t)]$  and we obtain (2.1b). Due to (4.5), the formal computation done in Section 2 is justified, and we obtain (2.2) and (2.3). We can thus apply Proposition 3.4 (with  $\sigma = 1$ ) and Corollary 3.6. This shows that the decompositions (2.1a) and (2.1b) hold for  $x \in [0, b(t)]$  and  $x \in [b(t), 1]$ .

## 5 Proof of the main results

We are now in position to prove Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* We set

$$\tilde{u}_0(t) := \theta^0(0) < 0 \quad \text{and} \quad \tilde{u}_1(t) := \theta^0(1) > 0.$$

From Section 4, the corresponding solution  $(\tilde{b}, \tilde{\theta})$  of (1.1)–(1.3) with the above controls satisfies (2.1) for  $t \in (0, T]$ . The corresponding coefficients  $\tilde{\alpha}_{\pm, i}$  and  $\tilde{\beta}_{\pm, i}$  verify  $\tilde{\beta}_{\pm, 0} = 0$ , (2.2) and (2.3). Since  $\tilde{b} \in (0, 1)$  in  $(0, T]$ , there exists  $\varepsilon \in (0, 1)$  such that

$$\frac{\varepsilon T}{3} \|\tilde{b}'\|_{L^\infty(0, T)} < \tilde{b} \left( \frac{T}{3} \right) < 1 - \frac{\varepsilon T}{3} \|\tilde{b}'\|_{L^\infty(0, T)}.$$

We consider now  $\phi_\sigma$  and  $\eta_\sigma$  satisfying the property of Lemma 3.2. The function defined by

$$\hat{b}(t) := \tilde{b} \left( \frac{T}{3} \right) + \int_{\frac{T}{3}}^t \tilde{b}'(s) \phi_\sigma \left( \frac{3s - T}{\varepsilon T} \right) ds \quad (5.1)$$

satisfies  $\widehat{b}(t) \in (0, 1)$  for  $t \geq T/3$  and is constant after  $t \geq (1 + \varepsilon)T/3$ . We denote by  $\widehat{b}^0$  the value of  $\widehat{b}(t)$  after  $t \geq (1 + \varepsilon)T/3$ , and we introduce,

$$\gamma(t) := \frac{3(\widehat{b}^0 - \bar{b})}{T(2 - \varepsilon)} \eta_\sigma \left( \frac{3t - T(1 + \varepsilon)}{T(2 - \varepsilon)} \right) \quad (t \in \mathbb{R})$$

and for  $\tau \in (0, T)$ ,

$$\alpha_{-,0}(t) := \widetilde{\alpha}_{-,0}(t) \phi_\sigma \left( \frac{3t - T}{\varepsilon T} \right) + \bar{v} \phi_\sigma \left( \frac{T - t}{T - \tau} \right) \quad (t \in (0, T]), \quad (5.2)$$

$$\alpha_{+,0}(t) := \widetilde{\alpha}_{+,0}(t) \phi_\sigma \left( \frac{3t - T}{\varepsilon T} \right) + \bar{v} \phi_\sigma \left( \frac{T - t}{T - \tau} \right) + \gamma(t) \quad (t \in (0, T]). \quad (5.3)$$

We define  $b$  by  $b = \widetilde{b}$  in  $[0, \min(\tau, T/3)]$  and satisfying (2.3) in  $[\min(\tau, T/3), T]$ . Taking the difference between (5.2) and (5.3), we can check that

$$b(t) := \widehat{b}(t) + (\bar{b} - \widehat{b}^0) \int_{-(1+\varepsilon)/(2-\varepsilon)}^{(3t-T(1+\varepsilon))/(T(2-\varepsilon))} \eta_\sigma(s) ds \quad (t \in [T/4, T]).$$

In particular, we have  $b(t) = \widehat{b}(t)$  for  $t < T(1 + \varepsilon)/3$ .

We also define  $\alpha_{\pm,i}$  and  $\beta_{\pm,i}$  with (2.2) and with  $\beta_{\pm,0} = 0$ . From the properties of  $\phi_\sigma$  and  $\eta_\sigma$  (see Lemma 3.2), we deduce that

$$\begin{aligned} \alpha_{\pm,i} &= \widetilde{\alpha}_{\pm,i}, \quad \beta_{\pm,i} = \widetilde{\beta}_{\pm,i} \quad (i \geq 0, t \in (0, \min(\tau, T/3)]), \\ \alpha_{-,0}(T) &= \bar{v}, \quad \alpha_{+,0}(T) = \bar{v}, \quad \alpha_{\pm,i}(T) = 0 \quad (i \geq 1), \quad \beta_{\pm,i}(T) = 0 \quad (i \geq 0), \quad b(T) = \bar{b}. \end{aligned}$$

Moreover, there exist  $M, R > 0$  such that  $\alpha_{-,0}, \alpha_{+,0}, b' \in \mathcal{G}(M, R, \sigma)$ . We can apply Proposition 3.4 and Corollary 3.6 and define  $\theta$  by (2.1). Following the computation of Section 2, we deduce that  $\theta$  and  $b$  satisfy (1.1) with

$$(b, \theta) = (\widetilde{b}, \widetilde{\theta}) \quad \text{in } (0, \min(\tau, T/3)] \quad \text{and} \quad b(T) = \bar{b}, \quad \theta(T, x) = \bar{v}(x - \bar{b}) \quad (x \in [0, 1]).$$

□

*Proof of Theorem 1.2.* We define

$$\alpha(s) := v^0 \phi_\sigma(s) + \bar{v} \phi_\sigma(1 - s) \quad (s \in \mathbb{R}).$$

Then, we set

$$\text{if } b^0 \geq \bar{b}, \quad \alpha_{-,0} := \alpha \left( \frac{t}{T} \right), \quad \alpha_{+,0} := \alpha \left( \frac{t}{T} \right) + \frac{b^0 - \bar{b}}{T} \eta_\sigma \left( \frac{t}{T} \right) \quad (t \in \mathbb{R}),$$

and

$$\text{if } b^0 < \bar{b}, \quad \alpha_{-,0} := \alpha \left( \frac{t}{T} \right) + \frac{\bar{b} - b^0}{T} \eta_\sigma \left( \frac{t}{T} \right), \quad \alpha_{+,0} := \alpha \left( \frac{t}{T} \right) \quad (t \in \mathbb{R}).$$

We also set

$$b(t) := b^0 + (\bar{b} - b^0) \int_0^{t/T} \eta_\sigma(s) ds$$

and we consider  $\alpha_{\pm,i}$  and  $\beta_{\pm,i}$  for  $i \geq 1$  given by (2.2) with  $\beta_{\pm,0} = 0$ . We can check that (2.3) holds true.

We assume  $T \geq 1$ , and we thus deduce from the above construction that there exist  $M > 0$  and  $R > 0$  such that  $\alpha_{-,0}, \alpha_{+,0} \in \mathcal{G}(M, RT, \sigma)$  and  $b' \in \mathcal{G}(\frac{M}{T}, RT, \sigma)$ . We apply Proposition 3.4: the condition (3.3) writes

$$\rho \leq \frac{2^{3-\sigma} RT}{3} \quad \text{and} \quad \rho \leq \frac{2T^2}{3\chi M^2}. \quad (5.4)$$

In particular, for  $T$  large enough, one can take

$$\rho := \rho_0 T, \quad \text{with } \rho_0 := \frac{2^{3-\sigma} R}{3}.$$

With this choice, we can apply Corollary 3.6 and define  $\theta$  by (2.1). Following the computation of Section 2, we deduce that  $\theta$  and  $b$  satisfy (1.1) with

$$(b(0), \theta(0, \cdot)) = (b^0, v^0(\cdot - b^0)) \quad \text{and} \quad (b(T), \theta(0, \cdot)) = (\bar{b}, \bar{v}(\cdot - \bar{b})).$$

From the property of  $\eta_\sigma$ , we have that  $b$  is monotone and thus  $b(t) \in (0, 1)$  for any  $t \in [0, T]$ . We also deduce that

$$\alpha_{\pm, 0}(t) \geq \min(v^0, \bar{v}) > 0. \quad (5.5)$$

Moreover, we have the following estimates, for any  $i \in \mathbb{N}$ :

$$\|\alpha_{\pm, i}\|_{L^\infty(\mathbb{R})} \leq \frac{M \chi^i (2i)!}{\rho_0^i T^i i!^{2-\sigma}}, \quad \|\beta_{\pm, i+1}\|_{L^\infty(\mathbb{R})} \leq \frac{M^2 \chi^{i+1} (2i+1)!}{T \rho_0^i T^i i!^{2-\sigma}}. \quad (5.6)$$

Thus

$$\delta(t, x) := \sum_{i=1}^{\infty} \alpha_{-, i}(t) \frac{(x - b(t))^{2i}}{(2i+1)!} + \sum_{i=1}^{\infty} \beta_{-, i}(t) \frac{(x - b(t))^{2i-1}}{(2i)!}$$

satisfies for  $0 < x < b(t)$ ,

$$|\delta(t, x)| \leq \frac{C_\delta}{T} \quad \text{where} \quad C_\delta := \sum_{i=1}^{\infty} \frac{M \chi^i}{\rho_0^i} \frac{1}{i!^{2-\sigma}} + \sum_{i=0}^{\infty} \frac{M^2 \chi^{i+1}}{\rho_0^i} \frac{1}{i!^{2-\sigma}} < \infty.$$

We thus deduce from (2.1a), the above relation and (5.5) that for  $T$  large enough

$$\frac{\theta(t, x)}{x - b(t)} = \alpha_{-, 0}(t) + \delta(t, x) \geq \frac{1}{2} \min(v^0, \bar{v}) > 0 \quad (x < b(t))$$

and with a similar argument, one can show that for  $T$  large enough,

$$\frac{\theta(t, x)}{x - b(t)} \geq \frac{1}{2} \min(v^0, \bar{v}) > 0 \quad (x > b(t)).$$

□

## 6 Numerical illustrations

In this section, we present some numerical tests based on our theoretical results. First, we estimate the error associated with the truncation of the series defining the solutions, see (2.1).

**Corollary 6.1.** *Assume the same hypotheses of Proposition 3.4 and that  $\sigma < \sigma' < 2$ . Then the series defined by (3.10) satisfies the following property:*

$$\left\| \sum_{i=N}^{\infty} \alpha_i(t) \frac{x^{2i+1}}{(2i+1)!} + \sum_{i=N}^{\infty} \beta_i(t) \frac{x^{2i}}{(2i)!} \right\|_{L^\infty((0, T) \times (0, 1))} = o\left(\frac{1}{(\ln N)(N!)^{2-\sigma'}}\right). \quad (6.1)$$

*Proof.* Using (3.4) and (3.5),

$$\left\| \sum_{i=N}^{\infty} \alpha_i(t) \frac{x^{2i+1}}{(2i+1)!} + \sum_{i=N}^{\infty} \beta_i(t) \frac{x^{2i}}{(2i)!} \right\|_{L^\infty((0, T) \times (0, 1))} = o\left(\sum_{i=N}^{\infty} \frac{1}{i!^{2-\sigma'}}\right). \quad (6.2)$$

Now, using Stirling formula,

$$\sum_{i=N}^{\infty} \frac{1}{i!^{2-\sigma'}} \leq \sum_{i=N}^{\infty} \exp\left(- (2 - \sigma') \left(i + \frac{1}{2}\right) \ln i - i\right).$$

Since the function  $g(x) := (x + \frac{1}{2}) \ln x - x$  is increasing and convex in  $[1, \infty)$ , we deduce from the above estimate that

$$\sum_{i=N}^{\infty} \frac{1}{i!^{2-\sigma'}} \leq \int_N^{\infty} \exp(-(2 - \sigma')g(x)) dx \leq \frac{\exp(-(2 - \sigma')g(N))}{(2 - \sigma') \ln N}.$$

Using again the Stirling formula, we deduce the result.  $\square$

Now to illustrate the construction of our control, we follow for instance the proof of Theorem 1.2 to pass from a stationary state to another. We only need some functions  $\phi_\sigma$  and  $\eta_\sigma$ , and we can use the explicit formula given in Lemma 3.2. This allows us to build  $\alpha_{\pm,0}$  and  $b$ . Then we use (2.2) for  $i = 1, \dots, N$  and build the approximations of the solutions with the formula (2.1) where the series are truncated at  $i = N$ . Due to Corollary 6.1, we know that the convergence of these series is quite quick which allows us to consider small values of  $N$ . In Figure 4, we illustrate the convergence of the controls  $u_0$  and  $u_1$  with respect to  $N$ . In this test, we take  $c_- = c_+ = 1$ ,  $T = 1$ ,  $b^0 = 1/2$ ,  $v^0 = 1$ ,  $\bar{b} = 3/4$  and  $\bar{v} = 4$ .

Let us mention that in order to compute the coefficients  $\alpha_{\pm,i}$  and  $\beta_{\pm,i}$  defined by induction in (2.2), we use the symbolic calculus in `Matlab` to differentiate the formulas.

With the same parameters as above, Figure 2 corresponds to the numerical test for  $N = 7$ . We represent the two controls  $u_0$  and  $u_1$ , and the evolution of  $b$  and of  $\theta$  with respect to time.

Finally, in Figure 3, we consider the dependence of the controls  $u_0$  and  $u_1$  with respect to  $T$ . We see in particular that for  $T = 0.5$ , the control  $u_0$  have positive values whereas for  $T = 0.75$  and for  $T = 1$ , this control remains negative.

We can observe the convergence of the controls when  $N$  is growing on Figure 4 where we observe a “smoothing effect”.

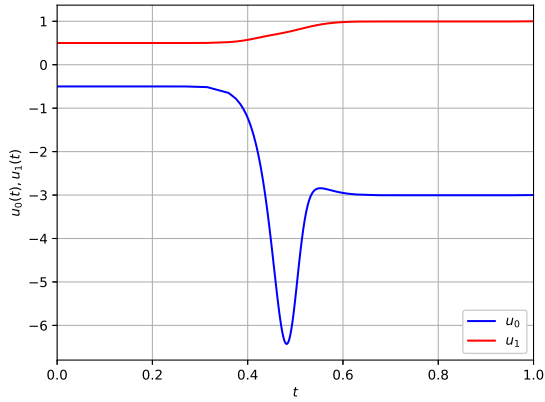
On Figures 3 and 4, the controls are almost constant for abscisses less than 0.25 and larger than 0.75 and thus those areas are excluded from these figure.

## Acknowledgments

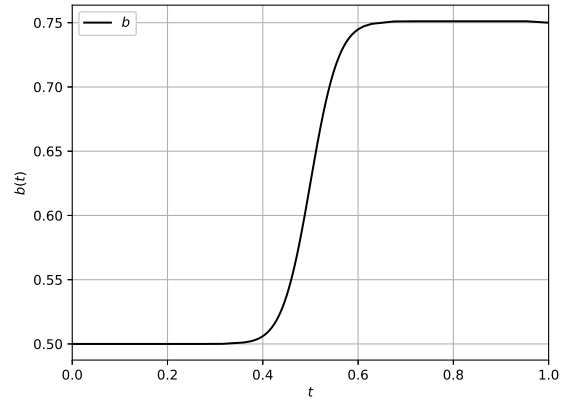
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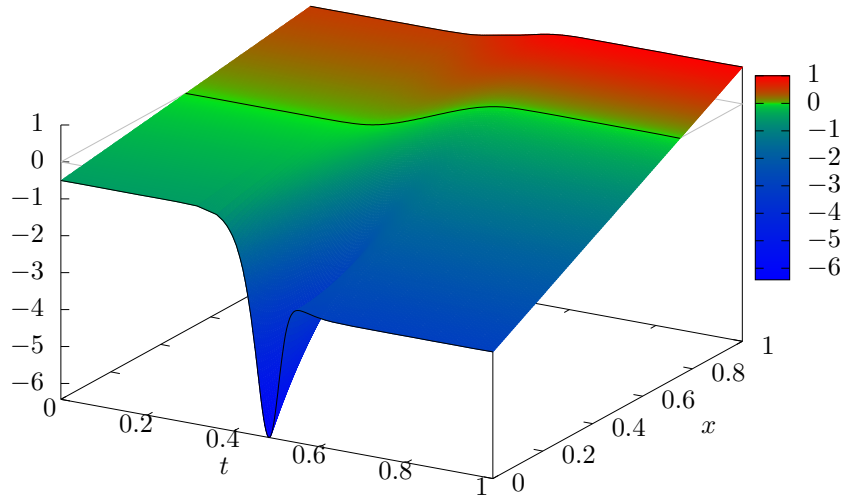
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(a) Controls.



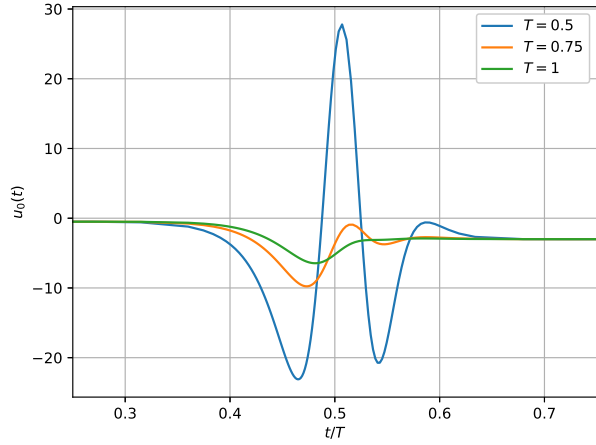
(b) Interface position.



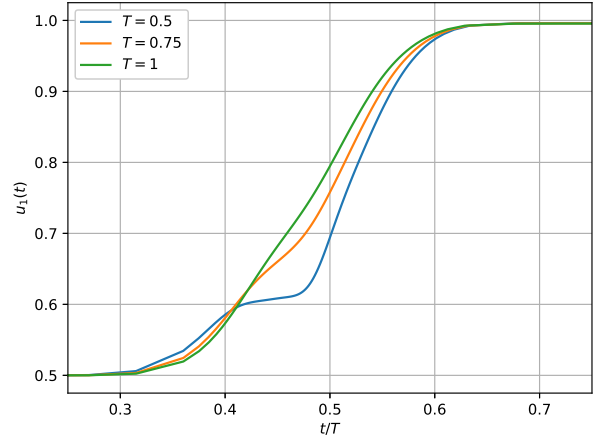
(c) Temperatures.

Figure 2: Numerical test with  $T = 1$ ,  $N = 7$ ,  $(b^0, v^0) = (1/2, 1)$ ,  $(\bar{b}, \bar{v}) = (3/4, 4)$ ,  $c_- = c_+ = 1$ .

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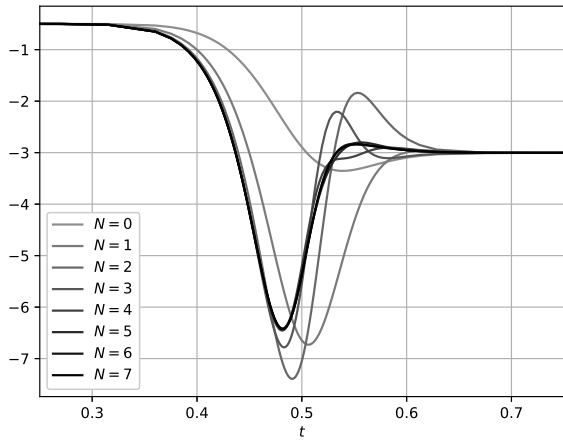


(a)  $u_0$  depending on  $T$ .

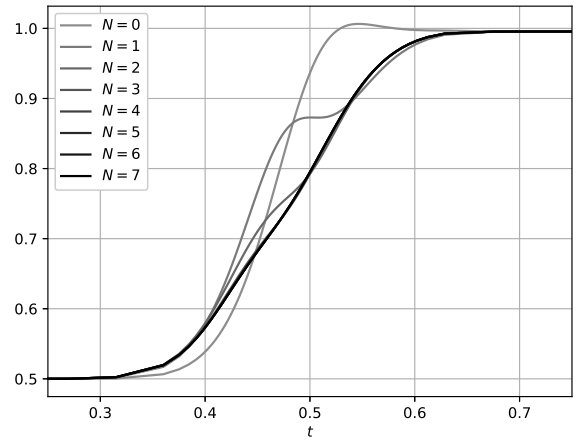


(b)  $u_1$  depending on  $T$ .

Figure 3: Dependence of the controls with respect to  $T$ . Test done with  $N = 4$ ,  $(b^0, v^0) = (1/2, 1)$ ,  $(\bar{b}, \bar{v}) = (3/4, 4)$ ,  $c_- = c_+ = 1$ .



(a)  $u_0$  depending on  $N$ .



(b)  $u_1$  depending on  $N$ .

Figure 4: Convergence of the controls with  $N$ . Test done with  $T = 1$ ,  $(b^0, v^0) = (1/2, 1)$ ,  $(\bar{b}, \bar{v}) = (3/4, 4)$ ,  $c_- = c_+ = 1$ .

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