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MONOGAMY OF ENTANGLEMENT BETWEEN CONES

GUILLAUME AUBRUN, ALEXANDER MÜLLER-HERMES, AND MARTIN PLÁVALA

ABSTRACT. A separable quantum state shared between parties A and B can be symmetrically extended to a quantum state shared between party A and parties B_1, \dots, B_k for every $k \in \mathbf{N}$. Quantum states that are not separable, i.e., entangled, do not have this property. This phenomenon is known as “monogamy of entanglement”. We show that monogamy is not only a feature of quantum theory, but that it characterizes the minimal tensor product of general pairs of convex cones \mathcal{C}_A and \mathcal{C}_B : The elements of the minimal tensor product $\mathcal{C}_A \otimes_{\min} \mathcal{C}_B$ are precisely the tensors that can be symmetrically extended to elements in the maximal tensor product $\mathcal{C}_A \otimes_{\max} \mathcal{C}_B^{\otimes_{\max} k}$ for every $k \in \mathbf{N}$. Equivalently, the minimal tensor product of two cones is the intersection of the nested sets of k -extendible tensors. It is a natural question when the minimal tensor product $\mathcal{C}_A \otimes_{\min} \mathcal{C}_B$ coincides with the set of k -extendible tensors for some finite k . We show that this is universally the case for every cone \mathcal{C}_A if and only if \mathcal{C}_B is a polyhedral cone with a base given by a product of simplices. Our proof makes use of a new characterization of products of simplices up to affine equivalence that we believe is of independent interest.

1. INTRODUCTION

Let V denote a finite-dimensional vector space over the reals and $\mathcal{C} \subseteq V$ a convex cone. We say that the cone \mathcal{C} is *proper* if it is closed, does not contain any line, and is not contained in any hyperplane. Any proper cone $\mathcal{C} \subseteq V$ can be described in terms of its dual $\mathcal{C}^* \subseteq V^*$, given by

$$\mathcal{C}^* = \{f \in V^* : f(x) \geq 0 \text{ for any } x \in \mathcal{C}\},$$

since by the bipolar theorem we have $(\mathcal{C}^*)^* = \mathcal{C}$. While any proper cone can be described from the inside by its extremal rays, the bipolar theorem describes the cone from the outside as an intersection of half-spaces. This duality is fundamental in the study of convex cones and it is a basic problem in convex analysis to switch between these two descriptions.

We will be interested in the two canonical tensor products of any pair of proper cones. Given proper cones $\mathcal{C}_A \subseteq V_A$, $\mathcal{C}_B \subseteq V_B$, we may form the *minimal tensor product*

$$\mathcal{C}_A \otimes_{\min} \mathcal{C}_B = \text{conv}\{x \otimes y : x \in \mathcal{C}_A, y \in \mathcal{C}_B\} \subseteq V_A \otimes V_B,$$

and the *maximal tensor product*

$$\mathcal{C}_A \otimes_{\max} \mathcal{C}_B = (\mathcal{C}_A^* \otimes_{\min} \mathcal{C}_B^*)^* = \{z \in V_A \otimes V_B : (f \otimes g)(z) \geq 0, f \in \mathcal{C}_A^*, g \in \mathcal{C}_B^*\}.$$

For proper cones \mathcal{C}_A and \mathcal{C}_B both tensor products are again proper cones. Moreover, the tensor products may be iterated and since they are associative we may define the k -fold versions $\mathcal{C}^{\otimes_{\min} k}$ and $\mathcal{C}^{\otimes_{\max} k}$ for any proper cone \mathcal{C} . The two tensor products reflect the aforementioned descriptions of convex cones: The minimal tensor product is described by specifying generating elements and the maximal

tensor product by an intersection of half-spaces. Moreover, their discrepancy can be seen as a general form of quantum entanglement [1].

A central feature of quantum physics is known as *monogamy of entanglement*. In our language, it can be described using the cones of positive semidefinite $d \times d$ matrices M_d^+ with complex entries, which is a proper cone inside the real vector space M_d^{sa} of Hermitian $d \times d$ matrices. A bipartite quantum state is given by a matrix $\rho_{AB} \in M_{d_A d_B}^+ \simeq (M_{d_A} \otimes M_{d_B})^+ \subsetneq M_{d_A}^+ \otimes_{\max} M_{d_B}^+$ for some integers $d_A, d_B \geq 2$ and it is called *entangled* if it does not belong to $M_{d_A}^+ \otimes_{\min} M_{d_B}^+$. The *monogamy theorem* [6, 16] asserts that a quantum state $\rho_{AB} \in (M_{d_A} \otimes M_{d_B})^+$ is entangled if and only if there exists an integer $k \geq 2$ for which no quantum state $\sigma_{AB_1 \dots B_k} \in (M_{d_A} \otimes M_{d_B}^{\otimes k})^+$ satisfies the equation

$$\rho_{AB} = \left(\text{Id}_{M_{d_A}^{\text{sa}}} \otimes \gamma_k^{\text{Tr}} \right) (\sigma_{AB_1 \dots B_k}),$$

where

$$\gamma_k^{\text{Tr}} = \frac{1}{k} \sum_{j=1}^k \text{Tr}^{\otimes(j-1)} \otimes \text{Id}_{M_{d_B}^{\text{sa}}} \otimes \text{Tr}^{\otimes(k-j)}$$

corresponds to (symmetrically) discarding all but one out of the k tensor factors labelled by B_1, \dots, B_k . Physically, this means that a bipartite quantum state is entangled if and only if it cannot be partially shared with arbitrarily many parties. Our first main result establishes a similar property for general minimal and maximal tensor products.

Theorem 1. *Consider proper cones $C_A \subseteq V_A$ and $C_B \subseteq V_B$ and a linear form ϕ in the interior of C_B^* . Then, we have*

$$(1) \quad C_A \otimes_{\min} C_B = \bigcap_{k \geq 1} \left(\text{Id}_{V_A} \otimes \gamma_k^\phi \right) \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right),$$

where, for an integer $k \geq 1$, the k th reduction map $\gamma_k^\phi : V_B^{\otimes k} \rightarrow V_B$ is defined as

$$\gamma_k^\phi = \frac{1}{k} \sum_{j=1}^k \phi^{\otimes(j-1)} \otimes \text{Id}_{V_B} \otimes \phi^{\otimes(k-j)}.$$

Our Theorem 1 shows that the monogamy property of quantum entanglement is actually a common feature of the most general forms of entanglement for any pair of cones. The question whether Theorem 1 holds has been asked in [11, §2.4.1]. Even for cones of positive semidefinite matrices our results are new: They show that entanglement cannot be partially shared with arbitrarily many parties even if we allow for unphysical quantum states represented in the maximal tensor product.

Equation (1) produces a decreasing sequence of outer approximations to the minimal tensor product, called the *extendability hierarchy*. A natural question is whether this sequence stops after finitely many steps. We answer this question in terms of combinatorial properties of the base $K_\phi = C_B \cap \phi^{-1}(1)$. We have the following theorem:

Theorem 2. *Consider a proper cone $C_B \subseteq V_B$ and a linear form ϕ in the interior of C_B^* . Then, for every integer $k \geq 1$, the following are equivalent:*

- (i) *For every proper cone $C_A \subseteq V_A$, we have the identity*

$$C_A \otimes_{\min} C_B = \left(\text{Id}_{V_A} \otimes \gamma_k^\phi \right) \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right);$$

(ii) The base K_ϕ is affinely equivalent to a Cartesian product of $\leq k$ simplices.

As a byproduct of our proof, we obtain a characterization of products of simplices as the only polytopes (up to affine equivalence) for which the operations “intersection” and “affine hull” commute when applied to the face lattice (see Figure 1 for an illustration). We denote by $\text{aff}(X)$ the affine hull of a subset X of a vector space, using the convention that $\text{aff}(\emptyset) = \emptyset$.

Theorem 3. *Let P be a polytope. The following are equivalent:*

- (i) *The polytope P is affinely equivalent to a Cartesian product of simplices;*
- (ii) *For every faces $(F_i)_{i \in I}$ of P , we have*

$$(2) \quad \text{aff} \left(\bigcap_{i \in I} F_i \right) = \bigcap_{i \in I} \text{aff}(F_i).$$

In the case of dimension 3, Theorem 3 means that the only polyhedra without stellations are the simplex, the cube and the triangular prism, extending an observation by Wenninger [15, p.35].

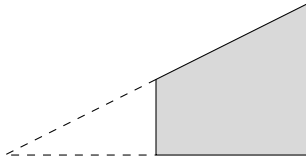


FIGURE 1. The planar case of Theorem 3: if a polygon is neither a triangle nor a parallelogram, it has disjoint edges whose affine hulls intersect.

Conventions and preliminaries. Given an integer $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$. All the vector spaces we consider are implicitly assumed to be finite-dimensional. If V is a vector space, the action of $f \in V^*$ on $x \in V$ is denoted as $f(x)$, $\langle f, x \rangle$ or $\langle x, f \rangle$. We always identify the double dual V^{**} with V itself. Similarly, for an integer $k \geq 2$ we always identify $(V^{\otimes k})^*$ with $(V^*)^{\otimes k}$.

Given vector spaces V_1, V_2 , the adjoint of a linear map $\Phi : V_1 \rightarrow V_2$ is denoted as $\Phi^* : V_2^* \rightarrow V_1^*$. If $X \subseteq V$, the affine hull $\text{aff}(X)$ is the set of elements of the form $\lambda_1 x_1 + \dots + \lambda_n x_n$ for an integer $n \geq 1$, elements x_1, \dots, x_n in X and real numbers $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 + \dots + \lambda_n = 1$. An element $y \in V$ belongs to $\text{aff}(X)$ if and only if every affine function $f : V \rightarrow \mathbf{R}$ which vanishes on X vanishes at y .

Structure of the paper. After reviewing the basic properties of the symmetric subspace, we discuss the basic properties of the generalized extendibility hierarchy (Section 2.1), a useful dual formulation (Section 2.2) and some examples (Section 2.3) including the case of quantum theory. In Section 3 we prove Theorem 1 by making use of a generalization of the de Finetti theorem due to Barrett and Leifer [3]. The proof of Theorem 2 is presented in Section 4. It relies on Theorem 3 which we prove in Section 5.

2. BASIC PROPERTIES OF THE EXTENDIBILITY HIERARCHY AND EXAMPLES

Let V be a finite-dimensional real vector space and $k \geq 1$ an integer. The symmetric group \mathfrak{S}_k acts naturally on $V^{\otimes k}$: if $\sigma \in \mathfrak{S}_k$, we define $U_\sigma : V^{\otimes k} \rightarrow V^{\otimes k}$ as the linear map such that

$$U_\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(k)}$$

for every $x_1, \dots, x_k \in V$. We define the *symmetric subspace* $\text{Sym}_k(V)$ as the subspace of $V^{\otimes k}$ which is invariant for this action

$$\text{Sym}_k(V) = \{x \in V^{\otimes k} : U_\sigma x = x \text{ for every } \sigma \in \mathfrak{S}_k\}.$$

The *symmetric projection* is the operator $P_{\text{Sym}_k(V)} : V^{\otimes k} \rightarrow V^{\otimes k}$ defined as

$$P_{\text{Sym}_k(V)} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} U_\sigma.$$

It is a projection whose range equals $\text{Sym}_k(V)$. If we identify $(V^{\otimes k})^*$ with $(V^*)^{\otimes k}$, we have the relation

$$P_{\text{Sym}_k(V)}^* = P_{\text{Sym}_k(V^*)}.$$

Moreover, if $C \subseteq V$ is a proper cone, then

$$P_{\text{Sym}_k(V)}(C^{\otimes_{\min} k}) = C^{\otimes_{\min} k} \cap \text{Sym}_k(V),$$

$$P_{\text{Sym}_k(V)}(C^{\otimes_{\max} k}) = C^{\otimes_{\max} k} \cap \text{Sym}_k(V).$$

2.1. The extendibility hierarchy. Consider proper cones $C_A \subseteq V_A$, $C_B \subseteq V_B$ and $\phi \in \text{int}(C_B^*)$. For an integer $k \geq 1$, denote by

$$(3) \quad \text{Ext}_k(C_A, C_B, \phi) = (\text{Id}_{V_A} \otimes \gamma_k^\phi) \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right)$$

the cone appearing in Theorem 1. Since $\gamma_k^\phi = (\text{Id}_{V_B} \otimes \phi^{\otimes(k-1)}) \circ P_{\text{Sym}_k(V_B)}$, an element $x \in V_A \otimes V_B$ belongs to $\text{Ext}_k(C_A, C_B, \phi)$ if and only if

$$\exists y_k \in (\text{Id}_{V_A} \otimes P_{\text{Sym}_k(V_B)}) \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right) : x = (\text{Id}_{V_A} \otimes \text{Id}_{V_B} \otimes \phi^{\otimes(k-1)}) (y_k).$$

Such an element y_k is called a *k-extension* of x and $\text{Ext}_k(C_A, C_B, \phi)$ is called the cone of *k-extendible elements*. Observe that

$$(\text{Id}_{V_A} \otimes P_{\text{Sym}_k(V_B)}) \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right) = (V_A \otimes \text{Sym}_k(V_B)) \cap \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right)$$

and therefore, whenever y_k is a *k-extension* of x , then $(\text{Id}_{V_A} \otimes \text{Id}_{V_B}^{\otimes(k-1)} \otimes \phi)(y_k)$ is a $(k-1)$ -extension of x . It follows that the sequence $\text{Ext}_k(C_A, C_B, \phi)$ is a decreasing sequence of cones. We also define

$$\text{Ext}_\infty(C_A, C_B, \phi) = \bigcap_{k \geq 1} \text{Ext}_k(C_A, C_B, \phi).$$

The first level of this hierarchy is the maximal tensor product itself, i.e., we have $\text{Ext}_1(C_A, C_B, \phi) = C_A \otimes_{\max} C_B$, and Theorem 1 asserts that $\text{Ext}_\infty(C_A, C_B, \phi) = C_A \otimes_{\min} C_B$.

2.2. The dual of the extendibility hierarchy. The adjoint of the k th reduction map γ_k^ϕ is the map $(\gamma_k^\phi)^* : V_B^* \rightarrow (V_B^*)^{\otimes k}$ which for $\psi \in V_B^*$ reads as

$$(\gamma_k^\phi)^*(\psi) = P_{\text{Sym}_k(V_B^*)}(\psi \otimes \phi^{\otimes(k-1)}).$$

Let $\zeta \in V_A^* \otimes V_B^*$. It follows immediately from (3) that

$$(4) \quad \zeta \in \text{Ext}_k(\mathbf{C}_A, \mathbf{C}_B, \phi)^* \iff (\text{Id}_{V_A} \otimes (\gamma_k^\phi)^*)(\zeta) \in \mathbf{C}_A^* \otimes_{\min} (\mathbf{C}_B^*)^{\otimes_{\min} k}.$$

In particular, we can write down a statement which is dual to Theorem 1.

Theorem 4. *Let $\mathbf{C}_A \subseteq V_A$ and $\mathbf{C}_B \subseteq V_B$ be proper cones. For every $x \in \text{int}(\mathbf{C}_A \otimes_{\max} \mathbf{C}_B)$ and $y \in \text{int}(\mathbf{C}_B)$, there is an integer $k \geq 1$ such that*

$$(\text{Id}_{V_A} \otimes P_{\text{Sym}_k(V_B)}) (x \otimes y^{\otimes(k-1)}) \in \mathbf{C}_A \otimes_{\min} \mathbf{C}_B^{\otimes_{\min} k}.$$

Proof. By Theorem 1 applied to \mathbf{C}_A^* and \mathbf{C}_B^* , we have

$$\mathbf{C}_A^* \otimes_{\min} \mathbf{C}_B^* = \text{Ext}_\infty(\mathbf{C}_A^*, \mathbf{C}_B^*, y) = \bigcap_{k \in \mathbf{N}} \text{Ext}_k(\mathbf{C}_A^*, \mathbf{C}_B^*, y).$$

Using the bipolar theorem in the form $(\mathbf{C}^*)^* = \overline{\mathbf{C}}$ whenever \mathbf{C} is a not-necessarily-closed convex cone, this is equivalent to the dual equation

$$\mathbf{C}_A \otimes_{\max} \mathbf{C}_B = \left(\bigcap_{k \in \mathbf{N}} \text{Ext}_k(\mathbf{C}_A^*, \mathbf{C}_B^*, y) \right)^* = \overline{\bigcup_{k \in \mathbf{N}} \text{Ext}_k(\mathbf{C}_A^*, \mathbf{C}_B^*, y)^*}.$$

Since $x \in \text{int}(\mathbf{C}_A \otimes_{\max} \mathbf{C}_B)$, there is an integer $k \geq 1$ such that $x \in \text{Ext}_k(\mathbf{C}_A^*, \mathbf{C}_B^*, y)^*$. By (4), this means that $(\text{Id}_{V_A} \otimes (\gamma_k^\phi)^*)(x) \in \mathbf{C}_A \otimes_{\min} \mathbf{C}_B^{\otimes_{\min} k}$, which is the desired conclusion. \square

2.3. Examples.

Simplicial cones. We say that a cone $\mathbf{C} \subseteq V$ is *simplicial* (or *classical*) if any of its bases is a simplex. Since γ_1^ϕ is the identity operator for any nonzero linear form ϕ , the case $k = 1$ of Theorem 2 says that a cone \mathbf{C}_B is simplicial if and only if $\mathbf{C}_A \otimes_{\min} \mathbf{C}_B = \mathbf{C}_A \otimes_{\max} \mathbf{C}_B$ for every proper cone \mathbf{C}_A . This result can be traced back to [12] and is much simpler to prove than the case $k > 1$ of Theorem 2. A strengthening of the $k = 1$ case was recently obtained [1]: given two proper cones $\mathbf{C}_A, \mathbf{C}_B$, we have $\mathbf{C}_A \otimes_{\min} \mathbf{C}_B = \mathbf{C}_A \otimes_{\max} \mathbf{C}_B$ if and only if either \mathbf{C}_A or \mathbf{C}_B is simplicial.

Cone over a square. Consider the following vectors in \mathbf{R}^3

$$(5) \quad x_{+0} = (1, 1, 0), \quad x_{-0} = (1, -1, 0), \quad x_{0+} = (1, 0, 1), \quad x_{0-} = (1, 0, -1)$$

and let \mathbf{C}_B be the cone they generate (see Figure 2).

The dual cone \mathbf{C}_B^* is generated by the following linear forms, identified with vectors in \mathbf{R}^3

$$\psi_{++} = \frac{1}{2}(1, 1, 1), \quad \psi_{+-} = \frac{1}{2}(1, 1, -1), \quad \psi_{-+} = \frac{1}{2}(1, -1, 1), \quad \psi_{--} = \frac{1}{2}(1, -1, -1).$$

Consider the linear form $\phi = (1, 0, 0)$. The corresponding base K_ϕ of \mathbf{C}_B is the square whose vertices are the vectors (5). The second reduction map satisfies

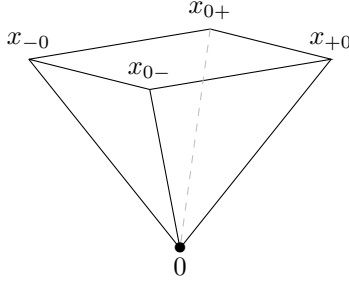


FIGURE 2. The cone \mathbf{C}_B over a square

$\gamma_2^\phi(u \otimes v) = \frac{u+v}{2}$ for every $u, v \in K_\phi$. One can check that it can be rewritten, for $y \in \mathbf{R}^3 \otimes \mathbf{R}^3$, as

$$\begin{aligned} 2\gamma_2^\phi(y) &= (\psi_{++} \otimes \psi_{+-} + \psi_{+-} \otimes \psi_{++})(y) \cdot x_{+0} \\ &\quad + (\psi_{-+} \otimes \psi_{--} + \psi_{--} \otimes \psi_{-+})(y) \cdot x_{-0} \\ &\quad + (\psi_{++} \otimes \psi_{-+} + \psi_{-+} \otimes \psi_{++})(y) \cdot x_{0+} \\ &\quad + (\psi_{+-} \otimes \psi_{--} + \psi_{--} \otimes \psi_{+-})(y) \cdot x_{0-}. \end{aligned}$$

Let $\mathbf{C}_A \subseteq V_A$ be an arbitrary cone. For any $z \in \mathbf{C}_A \otimes_{\max} \mathbf{C}_B \otimes_{\max} \mathbf{C}_B$, this formula shows that $(\text{Id}_A \otimes \gamma_2^\phi)(z)$ belongs to $\mathbf{C}_A \otimes_{\min} \mathbf{C}_B$, since it produces a decomposition into a sum of 4 terms (with nonnegative weights) of the form $x_A \otimes x_B$ with $x_A \in \mathbf{C}_A$ and x_B one of the vectors in (5).

We also observe that the choice of the linear form ϕ is crucial. If we choose $\phi' \in \text{int}(\mathbf{C}_B^*)$ which is not proportional to ϕ , then the corresponding base $K_{\phi'}$ is a quadrilateral which is not a parallelogram. Theorem 2 then implies that the extendibility hierarchy for ϕ' does not stop after finitely many steps.

The quantum case. Consider an operator $\rho_{AB} \in (\mathbf{M}_{d_1} \otimes \mathbf{M}_{d_2})^+$ and an integer $k \geq 1$. We say that ρ_{AB} is k -max-extendible if there exists an operator $\sigma_{AB_1 \dots B_k} \in \mathbf{M}_{d_1}^+ \otimes_{\max} (\mathbf{M}_{d_2}^+)^{\otimes_{\max} k}$ such that

$$\rho_{AB} = \left(\text{Id}_{\mathbf{M}_{d_1}^{\text{sa}}} \otimes \gamma_k^{\text{Tr}} \right) (\sigma_{AB_1 \dots B_k}).$$

If moreover the operator $\sigma_{AB_1 \dots B_k}$ can be chosen to be positive semidefinite, we say that ρ_{AB} is k -PSD-extendible. An operator which is k -PSD-extendible is also k -max-extendible, and the following proposition shows that the converse is false.

Proposition 5 (Proved in Appendix 5). *There is an operator $\rho_{AB} \in (\mathbf{M}_3 \otimes \mathbf{M}_3)^+$ which is 2-max-extendible but not 2-PSD-extendible.*

The usual monogamy theorem [6, 16] states that an operator which is k -PSD-extendible for every integer $k \geq 1$ is separable; the stronger version given by Theorem 1 is that an operator which is k -max-extendible for every $k \geq 1$ is separable.

Finally, let us mention the following consequence of Theorem 4. To our knowledge, this result does not appear in the quantum information literature.

Corollary 6. *Let $\rho_{AB} \in (M_{d_1} \otimes M_{d_2})^+$ be of full rank. Then there exists an integer $k \geq 1$ such that the operator on $M_{d_1} \otimes M_{d_2}^{\otimes k}$ defined as*

$$\frac{1}{k} \sum_{i=1}^k \rho_{AB_i} = \left(\text{Id}_{M_{d_1}^{\text{sa}}} \otimes P_{\text{Sym}_k(M_{d_2}^{\text{sa}})} \right) \left(\rho_{AB} \otimes \mathbf{1}_{d_2}^{\otimes(k-1)} \right)$$

is fully separable, i.e., belongs to $M_{d_1}^+ \otimes_{\min} (M_{d_2}^+)^{\otimes_{\min} k}$.

3. PROOF OF THEOREM 1

One direction is easy: if $x \in C_A \otimes_{\min} C_B$, we may write $x = \sum_i a_i \otimes b_i$ with $a_i \in C_A$ and $b_i \in C_B$. We may assume by rescaling a_i that $\phi(b_i) = 1$. The formula $y_k = \sum_i a_i \otimes b_i^{\otimes k}$ shows that x is k -extendible for every integer k .

The other direction of the proof relies on a specific choice of extensions. We say that $(y_k)_{k \geq 0}$ is a *compatible sequence* (for (C_A, C_B, ϕ)) if the following conditions hold:

- (i) We have $y_0 \in C_A \setminus \{0\}$;
- (ii) For every $k \geq 1$, we have $y_k \in \left(V_A \otimes \text{Sym}_k(V_B) \right) \cap \left(C_A \otimes_{\max} C_B^{\otimes_{\max} k} \right)$;
- (iii) For every $k \geq 1$, we have

$$y_{k-1} = \left(\text{Id}_{V_A} \otimes \text{Id}_{V_B}^{\otimes(k-1)} \otimes \phi \right) (y_k).$$

Iterating the last property shows that whenever $(y_k)_{k \geq 0}$ is a compatible sequence, then y_k is a k -extension of y_1 for every $k \geq 1$.

Our proof strategy is to first show in Lemma 7 that for every $x \in \text{Ext}_{\infty}(C_A, C_B, \phi)$ the extensions can be chosen such that they form a compatible sequence. Then we use a representation of compatible sequences given by Proposition 8 to finish the proof.

Lemma 7. *Let $x \in \text{Ext}_{\infty}(C_A, C_B, \phi)$. Then there exists a compatible sequence $(y_k)_{k \geq 0}$ such that $y_1 = x$.*

Proof. We necessarily have $y_1 = x$ and $y_0 = (\text{Id}_{V_A} \otimes \phi)(x)$. For every $k \geq 2$, let x_k be an arbitrary k -extension of x . The compatibility can be enforced by a compactness argument. For $k \leq n$, the vector

$$y_{k,n} = (\text{Id}_{V_A} \otimes \text{Id}_{V_B^{\otimes k}} \otimes \phi^{\otimes(n-k)})(x_n)$$

is a k -extension of x . Since for every integer k the set of k -extensions of x is compact, by a diagonal extraction process (see, e.g., [13, Theorem I.24]), we may find an increasing function $g : \mathbf{N} \rightarrow \mathbf{N}$ such that the limit $y_k = \lim_{n \rightarrow \infty} y_{k,g(n)}$ exists for every $k \geq 2$. The sequence $(y_k)_{k \geq 0}$ is compatible, as needed. \square

Proposition 8. *Let $C_A \subseteq V_A$, $C_B \subseteq V_B$ be proper cones and $\phi \in \text{int}(C_B^*)$. Let $(y_k)_{k \geq 0}$ be a compatible sequence. Then there exists a couple (π, α) , where*

- (i) π is a Borel probability measure on K_{ϕ} ,
- (ii) $\alpha : K_{\phi} \rightarrow V_A$ is a Borel map such that $\pi(\{\alpha \in C_A\}) = 1$,
- (iii) for every integer $k \geq 0$,

$$(6) \quad y_k = \int_{K_{\phi}} \alpha(\omega) \otimes \omega^{\otimes k} d\pi(\omega).$$

Moreover, if (π', α') is another couple satisfying these three conditions, then $\pi = \pi'$ and $\pi(\{\alpha = \alpha'\}) = 1$.

The result of Proposition 8 is very similar to (and could be deduced from) a theorem by Barrett and Leifer [3] which is a version of the de Finetti theorem for arbitrary cones (see also [5] for related results). This extends the argument given in [4] for the quantum case. For the reader's convenience, we will present a new simpler proof of Proposition 8 using directly the classical de Finetti theorem.

Let us recall the statement of the de Finetti theorem. Let F be a finite set and $(Y_n)_{n \geq 1}$ be a sequence of F -valued random variables which is *exchangeable* (i.e., for any integer $n \geq 1$ and any permutation $\sigma \in \mathfrak{S}_n$, the tuples (Y_1, \dots, Y_n) and $(Y_{\sigma(1)}, \dots, Y_{\sigma(n)})$ have the same distribution). The de Finetti theorem asserts that there is a unique Borel probability measure π on the set $\text{Prob}(F)$ of probability measures on F such that for every $n \geq 1$ and $x_1, \dots, x_n \in F^n$,

$$\mathbf{P}(X_1 = x_1, \dots, X_n = x_n) = \int_{\text{Prob}(F)} p(x_1) \dots p(x_n) \, d\pi(p).$$

We need a small extension of this statement. Assume that W is an additional random variable taking values in a finite set E and such that the sequence (X_n) is exchangeable conditionally on W . Then there is a unique family $(\pi_i)_{i \in E}$ of positive Borel measures on $\text{Prob}(F)$ such that, for $i \in E$

$$(7) \quad \mathbf{P}(W = i, X_1 = x_1, \dots, X_n = x_n) = \int_{\text{Prob}(F)} p(x_1) \dots p(x_n) \, d\pi_i(p).$$

We necessarily have $\pi = \sum_{i \in E} \pi_i$. For $i \in E$, denote by α_i the Radon–Nikodym derivative of π_i with respect to π . We can rewrite (7) as

$$(8) \quad \mathbf{P}(W = i, X_1 = x_1, \dots, X_n = x_n) = \int_{\text{Prob}(F)} \alpha_i(p) p(x_1) \dots p(x_n) \, d\pi(p).$$

We also use the following simple lemma.

Lemma 9. *Let $K \subseteq \mathbf{R}^n$ be a convex body, i.e., a compact convex set with non-empty interior. There is a sequence $(\Delta_k)_{k \geq 1}$ of simplices such that $\bigcap \Delta_k = K$.*

Proof. Without loss of generality, assume that $0 \in K$. It suffices to show that for every $x \in \mathbf{R}^n \setminus K$, there is a simplex Δ containing K but not x . This can be seen as follows: by the Hahn–Banach separation theorem, there is a linear form f_1 such that $f_1(x) > B := \max_K f_1$. We may find linear forms f_2, \dots, f_{n+1} such that $0 \in \text{int conv}(f_1, \dots, f_{n+1})$. For $A > 0$, the set

$$\{x \in \mathbf{R}^n : f_1(x) \leq B, f_i(x) \leq A \text{ for } i \in \{2, \dots, n+1\}\}$$

is a simplex which does not contain x ; for A large enough, it contains K as needed. \square

Proof of Proposition 8. Suppose first that the cones C_A and C_B are simplicial. Without loss of generality, we may then identify C_A and C_B with the cone of positive measures on some finite sets E and F , and K_ϕ with the set of probability measures on F . Up to rescaling, we may also assume that y_0 is a probability measure on E . In that case, the conclusion is equivalent to the conditional de Finetti theorem as in (8).

Consider now the general case. We use repeatedly the fact that the sequence $(y_k)_{k \geq 0}$ is a compatible sequence for (C, C', ϕ) whenever C and C' are proper cones

containing respectively C_A and C_B , and such that $\phi \in \text{int}(C'^*)$. In particular, choosing C and C' to be simplicial cones, we obtain the existence of a positive Borel measure $\pi = \pi_{C,C'}$ on $\Delta := C' \cap \phi^{-1}(1)$ and a Borel map $\alpha = \alpha_{C,C'}$ such that for every $k \geq 0$

$$y_k = \int_{\Delta} \alpha(\omega) \otimes \omega^{\otimes k} d\pi(\omega).$$

To finish the proof, it suffices to show that (i) the measure π is supported on K_ϕ and (ii) the function α is π -almost surely C_A -valued; the uniqueness property for (C_A, C_B) then follows from the uniqueness property for (C, C') .

Let Δ_1 be another simplex in the affine hyperplane $\phi^{-1}(1)$ such that $K_\phi \subseteq \Delta_1$ and Δ_2 a third simplex such that $\Delta \cup \Delta_1 \subseteq \Delta_2$. By the uniqueness property for $(C, \text{cone}(\Delta_2))$, it follows that $\pi = \pi_{C, \text{cone}(\Delta_1)} = \pi_{C, \text{cone}(\Delta_2)}$. In particular, the measure π is supported in $\Delta \cap \Delta_1$. Applying Lemma 9 proves that π is supported on K_ϕ .

Similarly, let C_1 be another simplicial cone such that $C_A \subseteq C_1$ and C_2 a third simplicial cone such that $C \cup C_1 \subseteq C_2$. By the uniqueness property for (C_2, C') , we have $\alpha = \alpha_{C_1, C'} = \alpha_{C_2, C'}$ π -a.s. It follows that α takes π -a.s. values in $C \cap C_1$. Applying Lemma 9 proves that α takes π -a.s. values in C_A . \square

Proof of Theorem 1. By Lemma 7, given an arbitrary element $x \in \text{Ext}_\infty(C_A, C_B, \phi)$, there is a compatible sequence $(y_k)_{k \geq 0}$ with $y_1 = x$. Equation (6) applied with $k = 1$ shows that $x = y_1 \in C_A \otimes_{\min} C_B$. \square

Remark 1. Theorem 1 can be extended to the case of more than two factors. Given proper cones $C_A, C_{B_1}, \dots, C_{B_d}$ and for every $i \in [d]$ a linear form $\phi_i \in \text{int}(C_{B_i}^*)$, the minimal tensor product $C_A \otimes_{\min} C_{B_1} \otimes_{\min} \dots \otimes_{\min} C_{B_d}$ is equal to

$$\bigcap_{k_1, \dots, k_d \geq 1} \left(\text{Id}_{V_A} \otimes \gamma_{k_1}^{\phi_1} \otimes \dots \otimes \gamma_{k_d}^{\phi_d} \right) \left(C_A \otimes_{\max} C_{B_1}^{\otimes_{\max} k_1} \otimes_{\max} \dots \otimes_{\max} C_{B_d}^{\otimes_{\max} k_d} \right).$$

The proof of this result is similar to the proof of Theorem 1. The corresponding classical de Finetti theorem is the following. Suppose that $(X_n^i)_{n \in \mathbf{N}, i \in [d]}$ is a family of random variables such that, for every finitely supported permutations $\sigma_1, \dots, \sigma_d$ of the integers, $(X_{\sigma_i(n)}^i)_{n \in \mathbf{N}, i \in [d]}$ has the same distribution as $(X_n^i)_{n \in \mathbf{N}, i \in [d]}$. Then (X_n^i) is distributed as a mixture of i.i.d. variables, each element in the mixture being distributed as an independent tuple $(Y_i)_{i \in [d]}$. We omit details.

4. PROOF OF THEOREM 2

Fix a proper cone $C \subseteq V$ and $\phi \in \text{int}(C^*)$. The k th reduction map $\gamma_k^\phi : V^{\otimes k} \rightarrow V$ can be viewed as a tensor, namely the unique element of $(V^*)^{\otimes k} \otimes V$ such that

$$(9) \quad \langle \gamma_k^\phi, x_1 \otimes \dots \otimes x_k \otimes \psi \rangle = \frac{\psi(x_1) + \dots + \psi(x_k)}{k}.$$

for every $x_1, \dots, x_k \in K_\phi$ and $\psi \in V^*$.

We say that the k th reduction map γ_k^ϕ is *entanglement-breaking* if, as a tensor, it belongs to $(C^*)^{\otimes_{\min} k} \otimes_{\min} C$. We use the following basic property of entanglement-breaking maps (see for instance [2, Proposition 2.2]). Given proper cones $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ and a linear map $\Phi : V_1 \rightarrow V_2$, the following are equivalent:

- (i) The map Φ , as a tensor in $V_1^* \otimes V_2$, belongs to $C_1^* \otimes_{\min} C_2$;

(ii) For every proper cone $C_A \subseteq V_A$, we have

$$(\text{Id}_{V_A} \otimes \Phi)(C_A \otimes_{\max} C_1) \subseteq C_A \otimes_{\min} C_2.$$

Using this equivalence for the cones $C_1 = C^{\otimes_{\max} k}$, $C_2 = C$ and the map $\Phi = \gamma_k^\phi$, we may reformulate Theorem 2 as follows.

Theorem 2'. *Let $C \subseteq V$ be a proper cone, $\phi \in \text{int}(C^*)$ and $k \geq 1$ an integer. Set $K_\phi = C \cap \phi^{-1}(1)$. The following are equivalent:*

- (i) *The k th reduction map γ_k^ϕ is entanglement-breaking;*
- (ii) *The base K_ϕ is affinely isomorphic to the Cartesian product of $\leq k$ simplices.*

If P is a polytope, we denote by $\mathcal{V}(P)$ the set of its vertices and by $\mathcal{F}(P)$ the set of its facets (=faces of codimension 1). We will start by proving the easy direction:

Proof that (ii) implies (i) in Theorem 2'. A basic observation is that given two polytopes P and Q , the faces of the Cartesian product $P \times Q$ have the form $F \times G$, where F is face of P and G is face of Q . Consider simplices $\Delta_1, \dots, \Delta_k$, possibly 0-dimensional, and the polytope $\Pi = \Delta_1 \times \dots \times \Delta_k$. Its vertices have the form (v_1, \dots, v_k) for $v_i \in \mathcal{V}(\Delta_i)$ and its facets have the form

$$(10) \quad \Delta_1 \times \dots \times \Delta_{i-1} \times F \times \Delta_{i+1} \times \dots \times \Delta_k,$$

for $i \in [k]$ and $F \in \mathcal{F}(\Delta_i)$. Assume that K_ϕ is affinely isomorphic to Π . We may label the vertices of K_ϕ as

$$\{x_{v_1, \dots, v_k} : v_i \in \mathcal{V}(\Delta_i)\}$$

and define, for every $i \in [k]$ and $v \in \mathcal{V}(\Delta_i)$, an affine map $\psi_i^v : \text{aff}(K_\phi) \rightarrow \mathbf{R}$ by

$$\psi_i^v(x_{v_1, \dots, v_k}) = \begin{cases} 1 & \text{if } v_i = v \\ 0 & \text{otherwise.} \end{cases}$$

We may extend ψ_i^v to a linear map defined on V , which we still denote by ψ_i^v , and which is an element of C^* . We observe that for every $j \in [k]$,

$$(11) \quad \phi = \sum_{v \in \mathcal{V}(\Delta_j)} \psi_j^v$$

since both sides evaluate to 1 on any vertex of K_ϕ . Let $i \in [k]$, $v \in \mathcal{V}(\Delta_i)$ and $z_1, \dots, z_k \in K_\phi$. We compute, using the fact that $P_{\text{Sym}_k(V)}^* = P_{\text{Sym}_k(V^*)}$ and (11) for every $j \neq i$,

$$\begin{aligned} & \left\langle \sum_{v_1 \in \mathcal{V}(\Delta_1)} \dots \sum_{v_k \in \mathcal{V}(\Delta_k)} P_{\text{Sym}_k(V^*)}(\psi_1^{v_1} \otimes \dots \otimes \psi_k^{v_k}) \otimes x_{v_1, \dots, v_k}, z_1 \otimes \dots \otimes z_k \otimes \psi_i^v \right\rangle \\ &= \sum_{v_1 \in \mathcal{V}(\Delta_1)} \dots \sum_{v_k \in \mathcal{V}(\Delta_k)} \mathbf{1}_{\{v_i=v\}} \langle \psi_1^{v_1} \otimes \dots \otimes \psi_k^{v_k}, P_{\text{Sym}_k(V)}(z_1 \otimes \dots \otimes z_k) \rangle \\ &= \langle \phi^{\otimes(i-1)} \otimes \psi_i^v \otimes \phi^{\otimes(k-i)}, P_{\text{Sym}_k(V)}(z_1 \otimes \dots \otimes z_k) \rangle \\ &= \frac{\psi_i^v(z_1) + \dots + \psi_i^v(z_k)}{k} \\ &= \langle \gamma_k^\phi, z_1 \otimes \dots \otimes z_k \otimes \psi_i^v \rangle. \end{aligned}$$

Since elements of the form $z_1 \otimes \cdots \otimes z_k \otimes \psi_i^v$ span the space $V^{\otimes k} \otimes V^*$, we conclude that

$$\gamma_k^\phi = \sum_{v_1 \in \mathcal{V}(\Delta_1)} \cdots \sum_{v_k \in \mathcal{V}(\Delta_k)} P_{\text{Sym}_k(V^*)}(\psi_1^{v_1} \otimes \cdots \otimes \psi_k^{v_k}) \otimes x_{v_1, \dots, v_k}$$

and therefore the map γ_k^ϕ is entanglement-breaking. \square

The second half of the proof of Theorem 2' requires a couple of preparatory lemmas and some terminology about polytopes. Let P denote a polytope. The *avoiding set* of a vertex $x \in \mathcal{V}(P)$ is given by

$$\text{Av}(x) = \{F \in \mathcal{F}(P) : x \notin F\},$$

i.e., the set of facets not containing x . We say that a tuple $(F_1, \dots, F_k, x) \in \mathcal{F}(P)^k \times \mathcal{V}(P)$ is *admissible* if $\text{Av}(x) \subseteq \{F_1, \dots, F_k\}$.

Lemma 10. *If γ_k^ϕ is entanglement-breaking, then K_ϕ is a polytope.*

Proof. By assumption, there exist an integer $N \geq 1$, elements x_1, \dots, x_N in K_ϕ and $(f_i^j)_{i \in [N], j \in [k]}$ in \mathbb{C}^* such that

$$(12) \quad \gamma_k^\phi = \sum_{i=1}^N f_i^1 \otimes \cdots \otimes f_i^k \otimes x_i.$$

For every $y \in K_\phi$, we have

$$y = \gamma_k^\phi(y^{\otimes k}) = \sum_{i=1}^N \left(\prod_{j=1}^k f_i^j(y) \right) x_i.$$

It follows that $K_\phi \subseteq \text{conv}\{x_1, \dots, x_N\}$ and therefore $K_\phi = \text{conv}\{x_1, \dots, x_N\}$. In particular, K_ϕ is a polytope. \square

We now assume that γ_k^ϕ is entanglement-breaking and hence that K_ϕ is a polytope. To every facet $F \in \mathcal{F}(K_\phi)$ we associate an element $\psi_F \in \mathbb{C}^*$ with the property that $F = K_\phi \cap \ker \psi_F$ (this determines ψ_F up to a positive scalar).

Lemma 11. *If γ_k^ϕ is entanglement-breaking, then γ_k^ϕ belongs to the cone generated by elements of the form $\psi_{F_1} \otimes \cdots \otimes \psi_{F_k} \otimes x$, where $(F_1, \dots, F_k, x) \in \mathcal{F}(K_\phi)^k \times \mathcal{V}(K_\phi)$ is admissible.*

Proof. For every facet $F \in \mathcal{F}(K_\phi)$, pick an arbitrary element x_F in the relative interior of F . With this we define the *vertex-facet tensor* as

$$(13) \quad \omega_k = \sum_{F \in \mathcal{F}(K_\phi)} x_F^{\otimes k} \otimes \psi_F \in V^{\otimes k} \otimes V^*.$$

Since $\psi_F(x_F) = 0$, it follows from (9) that $\langle \gamma_k^\phi, \omega_k \rangle = 0$.

Given a tuple $(F_1, \dots, F_k, x) \in \mathcal{F}(K_\phi)^k \times \mathcal{V}(K_\phi)$, we have

$$\langle \psi_{F_1} \otimes \cdots \otimes \psi_{F_k} \otimes x, \omega_k \rangle = \sum_{F \in \mathcal{F}(K_\phi)} \psi_F(x) \prod_{j=1}^k \psi_{F_j}(x_F) = \sum_{F \in \text{Av}(x)} \psi_F(x) \prod_{j=1}^k \psi_{F_j}(x_F).$$

This quantity is nonnegative and vanishes if and only if, for every $F \in \text{Av}(x)$, there is $j \in [k]$ such that $\psi_{F_j}(x_F) = 0$. By definition of x_F , we have $\psi_{F_j}(x_F) = 0$ if and only if $F = F_j$. We conclude that

$$(14) \quad \langle \psi_{F_1} \otimes \cdots \otimes \psi_{F_k} \otimes x, \omega_k \rangle \geq 0$$

with equality if and only if (F_1, \dots, F_k, x) is admissible.

Since $\mathbf{C}^* = \text{cone}\{\psi_F : F \in \mathcal{F}(K_\phi)\}$, we may expand the decomposition (12) in the form

$$\gamma_k^\phi = \sum_{F_1 \in \mathcal{F}(K_\phi)} \cdots \sum_{F_k \in \mathcal{F}(K_\phi)} \sum_{x \in \mathcal{V}(K_\phi)} \lambda_{F_1, \dots, F_k, x} \psi_{F_1} \otimes \cdots \otimes \psi_{F_k} \otimes x$$

for some $\lambda_{F_1, \dots, F_k, x} \geq 0$. Since $\langle \gamma_k^\phi, \omega_k \rangle = 0$, we conclude from (14) that $\lambda_{F_1, \dots, F_k, x} = 0$ whenever (F_1, \dots, F_k, x) is not admissible. This finishes the proof. \square

Proof that (i) implies (ii) in Theorem 2'. Assume that the map γ_k^ϕ is entanglement-breaking. By Lemma 11 the base K_ϕ is a polytope. We now show that

$$(15) \quad \text{aff} \left(\bigcap_{j=1}^n F_j \right) = \bigcap_{j=1}^n \text{aff}(F_j),$$

for any integer $n \geq 1$ and $F_1, \dots, F_n \in \mathcal{F}(K_\phi)$. Since any face is an intersection of facets, (15) implies that K_ϕ satisfies the condition (ii) from Theorem 3 and therefore is affinely equivalent to a product of simplices.

Note that $\text{aff}(\bigcap F_j) \subseteq \bigcap \text{aff}(F_j)$. Conversely, consider $a \in \bigcap \text{aff}(F_j)$ and any $h \in V^*$ which vanishes on $\bigcap F_j$. For every $j \in [n]$, we have $\psi_{F_j}(a) = 0$ since $a \in \text{aff}(F_j)$. For any admissible $(G_1, \dots, G_k, y) \in \mathcal{F}(K_\phi)^k \times \mathcal{V}(K_\phi)$, we have

$$(16) \quad \langle \psi_{G_1} \otimes \cdots \otimes \psi_{G_k} \otimes y, a^{\otimes k} \otimes h \rangle = \psi_{G_1}(a) \cdots \psi_{G_k}(a) h(y) = 0.$$

To prove (16), observe that if $h(y) \neq 0$, then $y \notin \bigcap F_j$ and hence there is an index $j \in [n]$ such that $y \notin F_j$. This means that $F_j \in \text{Av}(y)$ and therefore $F_j = G_i$ for some $i \in [k]$. We obtain that $\psi_{G_i}(a) = \psi_{F_j}(a) = 0$, proving (16).

By Lemma 11, it follows that $\langle \gamma_k^\phi, a^{\otimes k} \otimes h \rangle = 0$. By (9), we have $h(a) = \langle \gamma_k^\phi, a^{\otimes k} \otimes h \rangle$ and therefore $h(a) = 0$. Since every $h \in V^*$ which vanishes on $\bigcap F_j$ vanishes at a , we have $a \in \text{span}(\bigcap F_j)$. We conclude the proof of (15) by observing that $a \in \text{aff}(K_\phi) \cap \text{span}(\bigcap F_j) = \text{aff}(\bigcap F_j)$.

Theorem 3 tells us that K_ϕ is affinely isomorphic to $\Pi = \Delta_1 \times \cdots \times \Delta_l$ for some integer $l \geq 1$ and nontrivial simplices $\Delta_1, \dots, \Delta_l$; we still need to show that $l \leq k$. By Lemma 11, there is at least one admissible tuple $(F_1, \dots, F_k, x) \in \mathcal{F}(K_\phi)^k \times \mathcal{V}(K_\phi)$. In particular, $\text{card Av}(x) \leq k$. On the other hand, we observe from (10) that the polytope Π has the property that the avoiding set of any vertex contains exactly l elements. It follows that $l \leq k$. \square

Further remarks: We finish this section with two remarks about concepts appearing in the previous proof.

(a) It is easy to see that if K is the product of k nontrivial simplices, then $\text{card}(\text{Av}(x)) = k$ for every vertex $x \in \mathcal{V}(K)$. For $k = 1$ the converse holds as well. It would be interesting to determine which polytopes K have the property that $\text{card}(\text{Av}(x)) = 2$ for every vertex $x \in \mathcal{V}(K)$. It was observed by Martin Winter [9] that this class of polytopes is closed under taking free joins. Therefore, it contains examples that are not products of simplices, e.g., the free join of two squares.

(b) The vertex-facet tensor ω_k from (13) is an interesting object encoding certain combinatorial properties of the polytope K_ϕ . For example, we have the following proposition:

Proposition 12. *Consider a polyhedral cone \mathbf{C} with base $K_\phi = \phi^{-1}(1) \cap \mathbf{C}$ for some functional $\phi \in \text{int}(\mathbf{C}^*)$ and an integer $k \geq 1$. Then, the following are equivalent:*

(i) *We have*

$$\omega_k \in \text{int}(\mathbf{C}^{\otimes_{\max} k} \otimes_{\max} \mathbf{C}^*);$$

(ii) *For every vertex $x \in \mathcal{V}(K_\phi)$, we have $\text{card}(\text{Av}(x)) > k$.*

Proof. For any proper cone $\mathbf{C}_0 \subseteq V_0$, the interior of \mathbf{C}_0^* is the set of linear forms $f \in V_0^*$ such that $f(x) > 0$ for every $x \in \mathbf{C}_0 \setminus \{0\}$. Using this observation for the cone $\mathbf{C}_0 = (\mathbf{C}^*)^{\otimes_{\min} k} \otimes_{\min} \mathbf{C}$ shows that (i) is equivalent to the statement

$$\forall F_1, \dots, F_k \in \mathcal{F}(K_\phi), \forall x \in \mathcal{V}(K_\phi), \exists F \in \mathcal{F}(K_\phi) \setminus \{F_1, \dots, F_k\} : \psi_F(x) > 0,$$

which is equivalent to (ii). \square

Since $\langle \gamma_k^\phi, \omega_k \rangle = 0$ always holds, Proposition 12 implies directly that γ_k^ϕ is not entanglement-breaking whenever $\text{card}(\text{Av}(x)) > k$ for every vertex $x \in \mathcal{V}(K_\phi)$.

5. PROOF OF THEOREM 3

Before proving Theorem 3, we recall some facts about polytopes. Given a polytope P and a vertex $v \in \mathcal{V}(P)$, we denote by P/v the *vertex figure* of P at v (see [8, Chapter 2.1] for definition). A d -polytope P is said to be *simple* if any vertex is contained in exactly d facets, or equivalently if all its vertex figures are $(d-1)$ -simplices. A polytope P is said to be *2-level* if for every facet $F \in \mathcal{F}(P)$, all vertices not in F lie in the same translate of $\text{aff}(F)$. Since the complement of any facet in the graph of P is connected [14, p. 475], P is 2-level if and only if any face disjoint from any facet $F \in \mathcal{F}(P)$ lies in a translate of $\text{aff}(F)$.

We use the following characterization of products of simplices. For characterizations of products of simplices up to combinatorial equivalence, see [17].

Theorem 13 (Kaibel–Wolff, [10]). *A polytope is affinely equivalent to a product of simplices if and only if it is simple and 2-level.*

A polytope $P \subseteq \mathbf{R}^n$ is said to be a *0/1-polytope* if its vertices are a subset of $\{0, 1\}^n$. Kaibel and Wolff proved in [10] that a simple 0/1-polytope is a product of simplices. Since any 2-level polytope is affinely equivalent to a 0/1-polytope (see [7]), this implies Theorem 13.

Proof of Theorem 3. The fact that a simplex satisfies (ii) is a consequence of the following observation: if $(x_i)_{i \in I}$ are affinely independent, then for every $J_1, J_2 \subseteq I$, we have

$$\text{aff}(\{x_i : i \in J_1 \cap J_2\}) = \text{aff}(\{x_i : i \in J_1\}) \cap \text{aff}(\{x_i : i \in J_2\}).$$

By induction, to prove (i) \implies (ii), it suffices to prove that the class of polytopes satisfying (ii) is closed under products. Let P and Q be polytopes satisfying (ii) and consider a family $(F_i)_{i \in I}$ of faces of $P \times Q$. There are faces $(G_i)_{i \in I}$ of P

and $(H_i)_{i \in I}$ of Q such that $F_i = G_i \times H_i$ for every $i \in I$. Using the relation $\text{aff}(A \times B) = \text{aff}(A) \times \text{aff}(B)$ whenever A, B are subsets of vector spaces, we obtain

$$\begin{aligned} \text{aff}\left(\bigcap_{i \in I} F_i\right) &= \text{aff}\left(\bigcap_{i \in I} G_i \times \bigcap_{i \in I} H_i\right) = \text{aff}\left(\bigcap_{i \in I} G_i\right) \times \left(\bigcap_{i \in I} H_i\right) \\ &= \left(\bigcap_{i \in I} \text{aff}(G_i)\right) \times \left(\bigcap_{i \in I} \text{aff}(H_i)\right) = \bigcap_{i \in I} \text{aff}(G_i) \times \text{aff}(H_i) = \bigcap_{i \in I} \text{aff}(F_i) \end{aligned}$$

and the implication (i) \implies (ii) follows.

Conversely, let P be an n -polytope satisfying (ii). We first show that P is simple. Consider a vertex $x \in \mathcal{V}(P)$ and facets $F_1, \dots, F_{n-1} \in \mathcal{F}(P)$ containing x . The set $\bigcap \text{aff}(F_i)$ is the nonempty intersection of $n-1$ affine hyperplanes, hence has dimension ≥ 1 . By (2), the face $\bigcap F_i$ has dimension ≥ 1 . By the 1-1 correspondence between faces of P containing x and faces of P/x (see [8, Proposition 2.4]), the $(n-1)$ -polytope P/x is dual- $(n-1)$ -neighbourly, i.e., any $n-1$ facets have a common point. Since a d -polytope which is dual- k -neighbourly for $k > \lfloor d/2 \rfloor$ is a simplex (see [8, p. 123]), it follows that P/x is a simplex for every $x \in \mathcal{V}(P)$ and therefore P is simple.

Let $F \in \mathcal{F}(P)$ and G be a face of P such that $F \cap G = \emptyset$. By (2), we have $\text{aff}(F) \cap \text{aff}(G) = \emptyset$ and therefore G lies in a translate of $\text{aff}(F)$. This means that P is 2-level. By Theorem 13, it follows that P is affinely equivalent to a product of simplices. \square

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APPENDIX A. PROOF OF PROPOSITION 5

We denote by $(|1\rangle, |2\rangle, |3\rangle)$ the canonical basis of \mathbf{C}^3 . As common in the quantum information theory literature, we write $|ij\rangle$ as a shortcut for $|i\rangle \otimes |j\rangle$. Consider the operators

$$\begin{aligned} X_{\alpha, \beta, \gamma} &= \alpha \sum_i |ii\rangle\langle ii| + \beta \sum_{i \neq j} |ij\rangle\langle ij| + \gamma \sum_{i \neq j} |ii\rangle\langle jj| \\ &= \begin{pmatrix} \alpha & & & \gamma & & & \gamma \\ & \beta & & & & & \\ & & \beta & & & & \\ \gamma & & & \beta & \alpha & & \gamma \\ & & & & \beta & & \\ & & & & & \beta & \\ \gamma & & & & & & \alpha \end{pmatrix}. \end{aligned}$$

with parameters $\alpha, \beta, \gamma \in \mathbf{R}$. This operator is positive semidefinite if $\beta \geq 0$ and $\alpha \geq \gamma \geq 0$. Set $\eta = 1 - \sqrt{2}/2$. Consider the operator

$$Y = X_{1,\eta,1} = \frac{1}{4}X_{4,1,2\sqrt{2}+1} + \frac{3-2\sqrt{2}}{4}X_{0,1,1} \in (\mathbf{M}_3 \otimes \mathbf{M}_3)^+$$

Observe the following facts.

- The operator $X_{4,1,2\sqrt{2}+1}$ is 2-PSD-extendible, an extension being given by $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$, with

$$|\psi_1\rangle = \sqrt{2}|111\rangle + |212\rangle + |221\rangle + |313\rangle + |331\rangle,$$

$$|\psi_2\rangle = \sqrt{2}|222\rangle + |323\rangle + |332\rangle + |121\rangle + |112\rangle,$$

$$|\psi_3\rangle = \sqrt{2}|333\rangle + |131\rangle + |113\rangle + |212\rangle + |221\rangle.$$

- The operator $X_{0,1,1}$ is 2-max-extendible, since its partial transpose is 2-PSD-extendible, an extension being given by $|\psi\rangle\langle\psi|$, with

$$|\psi\rangle = |123\rangle + |132\rangle + |213\rangle + |231\rangle + |312\rangle + |321\rangle.$$

- By the two previous points, the operator Y is 2-max-extendible as a positive combination of 2-max-extendible operators.
- To show that Y is not 2-PSD-extendible, consider the operator $W = X_{1,\eta,-2\eta}$. It is easy to check numerically that the operator

$$W_2 = (\text{Id}_{\mathbf{M}_3} \otimes P_{\text{Sym}_2(\mathbf{M}_3)}) (W \otimes \mathbf{1}_3)$$

is positive definite. Assume that Y is 2-PSD-extendible, with extension $Y_2 \in (\mathbf{M}_3^{\otimes 3})^+$. We would then have

$$\text{Tr}(Y_2 W_2) = \text{Tr}(Y W) = 3 \cdot 1 + 6 \cdot \eta^2 + 6 \cdot (-2\eta) = 0,$$

showing that $Y_2 = 0$, leading to a contradiction.

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