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# BLOW UP FOR SOME NONLINEAR PARABOLIC PROBLEMS WITH CONVECTION UNDER DYNAMICAL BOUNDARY CONDITIONS

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**Abstract.** Blow up phenomena for solutions of nonlinear parabolic problems including a convection term  $\partial_t u = \operatorname{div}(a(x)\nabla u) + f(x, t, u, \nabla u)$  in a bounded domain under dissipative dynamical time lateral boundary conditions  $\sigma\partial_t u + \partial_\nu u = 0$  are investigated. It turns out that under natural assumptions in the proper superlinear case no global weak solutions can exist. Moreover, for certain classes of nonlinearities the blow up times can be estimated from above as in the reaction diffusion case [6]. Finally, as a model case including a sign change of the convection term, the occurrence of blow up is investigated for the one-dimensional equation  $\partial_t u = \partial_x^2 u - u\partial_x u + u^p$ .

## 1. INTRODUCTION

This contribution is devoted to the occurrence of blow up of solutions of nonlinear parabolic problems of the form

$$\begin{cases} \partial_t u = \operatorname{div}(a(x)\nabla u) + f(x, t, u, \nabla u) & \text{in } \bar{\Omega} \text{ for } t > 0, \\ B_\sigma(u) := \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\bar{\Omega}), \end{cases} \quad (1)$$

in a bounded domain  $\Omega$  in  $\mathbb{R}^n$  with  $C^2$ -boundary, unless otherwise stated, and with outer normal unit vector field  $\nu : \partial\Omega \rightarrow \mathbb{R}^n$ . As a distinctive feature a dynamical boundary condition is imposed on the time lateral boundary, relating the time derivative to the outer normal derivative. Dealing with classical solutions, we always shall assume that

$$\sigma \in C^1(\partial\Omega) \quad (2)$$

and the dissipativity condition

$$\sigma \geq 0 \quad \text{on } \partial\Omega. \quad (3)$$

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Throughout,  $a$ ,  $f$  and  $\varphi$  are supposed to fulfill the conditions

$$a \in C^1(\overline{\Omega}), \quad a > 0, \quad (4)$$

$$f \in C^1(\overline{\Omega} \times [0, \infty) \times \mathbb{R} \times \mathbb{R}^n), \quad (5)$$

$$f(\cdot, \cdot, 0, 0) \geq 0 \quad \text{in } [0, \infty), \quad (6)$$

$$\varphi \in C(\overline{\Omega}), \quad \varphi \geq 0, \quad \varphi \neq 0. \quad (7)$$

The aim of the present contribution is to generalize the blow up results obtained in [6] for the reaction diffusion case to the present one including a gradient term. In Section 2 the dependence of blow up time of classical solutions on the initial data and the coefficient  $\sigma$  is investigated, while the nonexistence of weak global solutions is shown in Section 3 in the proper superlinear case. Section 4 presents upper bounds for the blow up time for a nonlinearity growing at least as a power of  $u$  or bounded from below by an exponential function. The final Section 5 summarizes some results for the one-dimensional model problem with a convection term changing sign,

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_\sigma(u) = \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \quad (8)$$

in an interval  $\Omega = (-b, b)$  and for  $1 < p \in \mathbb{R}$ .

For further references and related topics, especially for local and global existence results, we refer to [1]–[6], [8, 9, 12, 13].

## 2. COMPARISON OF BLOW UP TIMES FOR CLASSICAL SOLUTIONS

Let

$$u_\sigma \in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$$

denote the maximal classical solution of Problem (1). Thus  $T = T(\sigma, \varphi)$  is the blow up time, that is the maximal existence time of  $u_\sigma$  with respect to the  $L^\infty$ -norm

$$T(\sigma, \varphi) = \inf \left\{ s > 0 \mid \lim_{t \nearrow s} \sup \{ |u_\sigma(x, t)| \mid x \in \overline{\Omega} \} = \infty \right\}.$$

As for standard boundary conditions, it is well-known that  $T(\sigma, \varphi)$  can be infinite, while some derivatives of  $u$  can explode in finite time, see e.g. [8, Section 2]. By the flow positivity result in [4],  $u_\sigma \geq 0$ , and, using the comparison principle l.c.,  $T(\sigma, \varphi)$  is seen to decrease with respect to the initial data  $\varphi$ .

**Theorem 2.1.** Suppose  $\varphi_1, \varphi_2 \in C(\overline{\Omega})$ . Then  $0 \leq \varphi_1 \leq \varphi_2$  implies

$$T(\sigma, \varphi_1) \geq T(\sigma, \varphi_2).$$

In order to show the increasing character of the blow up time with respect to the parameter  $\sigma$ , we impose the constraints

$$\partial_t f(x, t, \cdot, \cdot) \geq 0, \quad (9)$$

$$\varphi \in C^2(\bar{\Omega}), \quad \operatorname{div}(a(x)\nabla\varphi) + f(x, 0, \varphi, \nabla\varphi) \geq 0 \quad \text{in } \bar{\Omega}. \quad (10)$$

**Lemma 2.2.** Under (9) - (10), a solution  $u \in C^{2,1}(\bar{\Omega} \times [0, \tau])$  of (1) satisfies  $\partial_t u \geq 0$  in  $\bar{\Omega} \times [0, \tau]$ .

*Proof.* Classical regularity results in [11] imply that  $u \in C^{2,2}(\bar{\Omega} \times [0, \tau])$ . Thus,  $y := \partial_t u \in C^{2,1}(\bar{\Omega} \times [0, \tau])$  is a solution of

$\partial_t y = \operatorname{div}(a(x)\nabla y) + f_t(x, t, u, \nabla u) + f_u(x, t, u, \nabla u)y + f_p(x, t, u, \nabla u) \cdot \nabla y$  in  $\bar{\Omega} \times [0, \tau]$  satisfying  $B_\sigma(y) = 0$  on  $\partial\Omega \times (0, \tau]$  and  $y(x, 0) = \operatorname{div}(a(x)\nabla\varphi) + f(x, 0, \varphi, \nabla\varphi) \geq 0$  in  $\bar{\Omega}$ . Then by (9) and [4, Cor. 2.4],  $y \geq 0$  in  $\bar{\Omega} \times [0, \tau]$ .  $\square$

**Theorem 2.3.** Suppose that Conditions (9)–(10) are fulfilled. Then  $0 \leq \sigma_1 \leq \sigma_2$  implies  $u_{\sigma_1} \geq u_{\sigma_2}$  in  $\bar{\Omega} \times [0, T(\sigma_1, \varphi))$  and

$$T(\sigma_1, \varphi) \leq T(\sigma_2, \varphi).$$

*Proof.* By Lemma 2.2 we have  $\partial_t u_\sigma \geq 0$  and  $\sigma_2 \partial_t u_{\sigma_1} + \partial_\nu u_{\sigma_1} \geq 0$ . Then the comparison principle from [4] yields  $u_{\sigma_1} \geq u_{\sigma_2}$ .  $\square$

Using Theorem 2.3 and the comparison principle l.c., we can compare solutions under different boundary conditions.

**Theorem 2.4.** Suppose that Conditions (9)–(10) are fulfilled. Then  $u_0 \geq u_\sigma \geq v \geq w$  in the domains of definition of  $u_0$ ,  $u_\sigma$  and  $v$  respectively, and

$$T(0, \varphi) \leq T(\sigma, \varphi) \leq T(\infty, \varphi) \leq T_0(\infty, \psi),$$

where  $v \in C(\bar{\Omega} \times [0, T(\infty, \varphi))) \cap C^{2,1}(\bar{\Omega} \times (0, T(\infty, \varphi)))$  is the maximal classical solution of

$$\begin{cases} \partial_t v = \operatorname{div}(a(x)\nabla v) + f(x, t, v, \nabla v) & \text{in } \bar{\Omega} \times (0, T(\infty, \varphi)), \\ v(\cdot, 0) = \varphi & \text{in } \bar{\Omega}, \\ v = \varphi & \text{on } \partial\Omega \times (0, T(\infty, \varphi)), \end{cases}$$

and  $w \in C(\bar{\Omega} \times [0, T_0(\infty, \psi))) \cap C^{2,1}(\bar{\Omega} \times (0, T_0(\infty, \psi)))$  is the maximal classical solution of

$$\begin{cases} \partial_t w = \operatorname{div}(a(x)\nabla w) + f(x, t, w, \nabla w) & \text{in } \bar{\Omega} \times (0, T_0(\infty, \psi)), \\ w(\cdot, 0) = \psi & \text{in } \bar{\Omega}, \\ w = 0 & \text{on } \partial\Omega \times (0, T_0(\infty, \psi)), \end{cases}$$

with  $\psi \in C(\bar{\Omega}), \psi = 0$  on  $\partial\Omega$  and  $\psi \leq \varphi$ .

Note that  $u_\sigma \geq w$  holds in the domain of definition of  $u_\sigma$  without (9)–(10) by the comparison principle under Dirichlet boundary conditions [14]. Under the additional hypothesis

$$f(x, t, u, 0) \leq h(u), \quad h \in C^1([0, \infty); [0, \infty)), \quad (11)$$

the maximal solution  $z \in C^1([0, t_0))$  of the ordinary IVP

$$\begin{cases} \dot{z} = h(z) & \text{in } [0, t_0), \\ z(0) = \|\varphi\|_\infty \end{cases} \quad (12)$$

yields a lower bound for the blow up time by the comparison principle from [4]. The solution  $u$  of Problem 1 satisfies

$$0 = \partial_t u - \operatorname{div}(a(x)\nabla u) - f(x, t, u, \nabla u) \leq \dot{z} - \operatorname{div}(a(x)\nabla z) - f(x, t, z, 0)$$

in  $\bar{\Omega}$  for  $t > 0$ . Thus, Theorem 2.1 permits to conclude the

**Theorem 2.5.** Under Condition (11),  $z \geq u_\sigma$  for  $0 \leq t < t_0$ , and, if in addition  $h(u) > 0$  for  $u > 0$ , then

$$T(\sigma, \varphi) \geq t_0 = \int_{\|\varphi\|_\infty}^{\infty} \frac{1}{h(\eta)} d\eta.$$

**Corollary 2.6.** Suppose that either  $h(u) = u^p$  with  $1 < p \in \mathbb{R}$  or  $h(u) = e^{qu}$  with  $1 \leq q \in \mathbb{R}$ . Then, under Condition (11), the blow up time of Problem (1) satisfies  $T(\sigma, \varphi) \geq T(0, \|\varphi\|_\infty)$  with

$$T(0, \|\varphi\|_\infty) = \frac{1}{(p-1)\|\varphi\|_\infty^{p-1}} \quad \text{or} \quad T(0, \|\varphi\|_\infty) = \frac{1}{q} e^{-q\|\varphi\|_\infty}$$

respectively.

Furthermore, the growth order can be determined for  $f(x, t, u, \nabla u) = u^p$ :

**Corollary 2.7.** Suppose that Conditions (9)–(10) are satisfied and that

$$f(x, t, u, \nabla u) = u^p, \quad \text{with } p > 1. \quad (13)$$

Then there exists a positive constant  $C$  such that

$$\|u_\sigma(\cdot, t)\|_\infty \leq \frac{C}{(T(\sigma, \varphi) - t)^{1/p-1}} \quad \text{for } t \in [0, T(\sigma, \varphi)).$$

*Proof.* Set  $u := u_\sigma$ ,  $\xi \in ]0, \frac{t_0}{2}]$  and  $\tilde{T} = \frac{\xi}{2}$ . We first show that

$$\exists \delta > 0 : \partial_t u \geq \delta u^p \quad \text{in } \bar{\Omega} \times [\xi, T(\sigma, \varphi)). \quad (14)$$

Since  $u(\cdot, \tilde{T}) \in C^2(\bar{\Omega})$ , standard regularity results [11] yield  $u \in C^{2,2}(\bar{\Omega} \times (0, T(\sigma, \varphi)))$ . By applying the strong minimum principle to

the solution  $y := \partial_t u \in C^{2,1}(\bar{\Omega} \times (0, T(\sigma, \varphi)))$  of

$$\begin{cases} \partial_t y = \operatorname{div}(a \nabla y) + p u^{p-1} y & \text{in } \bar{\Omega} \times [\tilde{T}, T(\sigma, \varphi)), \\ y(\cdot, T) = \partial_t u(\cdot, \tilde{T}) \geq 0 & \text{in } \bar{\Omega}, \\ B_\sigma(y) = 0 & \text{on } \partial\Omega \times [\tilde{T}, T(\sigma, \varphi)), \end{cases}$$

we can conclude that there exists  $c > 0$  such that

$$y := \partial_t u \geq c > 0 \text{ in } \bar{\Omega} \times [\xi, T(\sigma, \varphi)). \quad (15)$$

Now, define  $J$  in  $\bar{\Omega} \times [\xi, T(\sigma, \varphi))$  by  $J = \partial_t u - \delta u^p$ , where  $\delta > 0$  is sufficiently small such that  $J(\cdot, \xi) \geq 0$  in  $\bar{\Omega}$  thanks to (15).  $J \in C^{2,1}(\bar{\Omega} \times [\xi, T(\sigma, \varphi)))$  fulfills

$$\partial_t J - \operatorname{div}(a \nabla J) - p u^{p-1} J = \delta p(p-1) u^{p-2} a \|\nabla u\|^2 \geq 0 \text{ in } \bar{\Omega} \times [\xi, T(\sigma, \varphi))$$

and  $J$  satisfies the boundary condition  $B_\sigma(J) = 0$ . Finally, the comparison principle implies  $J \geq 0$  in  $\bar{\Omega} \times [\xi, T(\sigma, \varphi))$ .

The remaining part of the proof is identical to the proof of [6, Thm. 2.9], but for the reader's convenience we repeat the argument here. Since  $u \leq z$ , it suffices to estimate  $\|u(\cdot, t)\|_\infty$  for  $t \in [\frac{t_0}{2}, T(\sigma, \varphi))$ . For each  $x \in \bar{\Omega}$  the integral

$$\int_t^T \frac{\partial_t u(x, s)}{u^p(x, s)} ds = \int_{u(x, t)}^{u(x, T)} \frac{1}{\eta^p} d\eta$$

converges as  $T \nearrow T(\sigma, \varphi)$ . We conclude

$$\frac{u^{1-p}(x, T(\sigma, \varphi)) - u^{1-p}(x, t)}{1-p} \geq \delta (T(\sigma, \varphi) - t),$$

and at a blow up point  $x$ ,

$$u(x, t) \leq (p-1)^{\frac{1}{1-p}} \max \left\{ \delta^{\frac{1}{1-p}}, \left( \frac{2T(\sigma, \varphi)}{t_0} - 1 \right)^{\frac{1}{p-1}} \right\} \frac{1}{(T(\sigma, \varphi) - t)^{1/p-1}}.$$

□

### 3. NONEXISTENCE OF WEAK GLOBAL SOLUTIONS

In this section we generalize a result on nonexistence from [2] to the following convection problem on a bounded domain  $\Omega$  with Lipschitz boundary  $\partial\Omega$ :

$$\begin{cases} \partial_t u = \operatorname{div}(a(x) \nabla u) + m(x, t) f(u) & \text{in } \Omega \times (0, T), \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = \varphi & \end{cases} \quad (16)$$

In addition to (2)–(7) we require here that

$$m \in L^\infty(\Omega \times (0, T)), \quad m \geq 0 \quad (17)$$

$$f(s) > 0 \text{ for } s > 0, \quad f(s), f'(s) \geq 0 \text{ for all } s \in \mathbb{R}, \quad (18)$$

$$\int_{s_0}^{\infty} \frac{d\xi}{f(\xi)} < \infty \text{ for some positive } s_0. \quad (19)$$

Recall that  $u \in \mathfrak{B} := L^2((0, T), H^1(\Omega))$ ,  $T > 0$ , is called a *weak solution*  $u$  of Problem (16) in  $(0, T)$  if  $\partial_t u \in L^2((0, T), L^2(\Omega))$ , trace  $\partial_t u \in L^2((0, T), L^2(\partial\Omega))$ ,  $f(u(\cdot)) \in L^2((0, T), H^1(\Omega))$  and

$$\begin{aligned} \int_0^\tau \left( \int_\Omega \partial_t u \psi \, dx + \oint_{\partial\Omega} \sigma a \partial_t u \psi \, ds + \int_\Omega a(\nabla u, \nabla \psi) \, dx \right) dt \\ = \int_0^\tau \left( \int_\Omega m f \psi \, dx \right) dt \end{aligned}$$

for all  $\tau \in (0, T)$  and for all  $\psi \in \mathfrak{B}$ , and  $\varphi \in L^2(\Omega) \times L^2(\partial\Omega)$  with  $\|u(\cdot, t) - \varphi\|_{L^2(\Omega)}, \|\text{trace } u(\cdot, t) - \varphi\|_{L^2(\partial\Omega)} \rightarrow 0$  as  $t \rightarrow 0$ .

We shall prove that all solutions blow up in finite time if the ODE  $\dot{z} = f(z)$  has no global solution for positive initial value. For that purpose we first state the following lemma that can be shown exactly in the same way as in [2].

**Lemma 3.1.** Let  $u$  be a weak solution of Problem (16) in  $(0, T)$  and  $h$  be a weak solution of the homogeneous Problem (16) with  $f \equiv 0$  and  $h(x, 0) \leq \varphi(x)$  a.e. in  $\Omega$ . Then  $h \leq u$ . Especially, if  $\varphi \geq 0$  then  $u \geq 0$ .

**Theorem 3.2.** Suppose  $\int_\Omega \varphi \, dx + \oint_{\partial\Omega} \sigma \varphi \, ds > 0$  and

$$\int_0^t \left( \int_\Omega m(x, \tau) \, dx \right) d\tau \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Then Problem (16) has no global weak solution.

*Proof.* Suppose that  $u \in \mathfrak{B}_\infty := L^2((0, \infty), H^1(\Omega))$  is a weak solution of Problem (16) in  $(0, \infty)$ . By [1], let  $h \in \mathfrak{B}_\infty$  denote the unique solution of the homogeneous linear problem

$$\begin{cases} \partial_t u = \text{div}(a \nabla u) & \text{in } \Omega \times \mathbb{R}^+, \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot, 0) = \varphi & \text{in } \Omega \end{cases} \quad (20)$$

Lemma 3.1 shows that  $u \geq h$ . By the spectral expansion results [1], [2]

$$h(x, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \left( \int_\Omega \varphi \psi_n \, dx + \oint_{\partial\Omega} \sigma \varphi \psi_n \, ds \right) \psi_n(x) + h_0$$

with

$$h_0 := \frac{\int_{\Omega} \varphi \, dx + \oint_{\partial\Omega} \sigma \varphi \, ds}{|\Omega| + \oint_{\partial\Omega} \sigma \, ds},$$

where  $\psi_n$  and  $\lambda_n$ ,  $i \in \mathbb{N}$  denote all the eigenfunctions and eigenvalues of the problem

$$-\operatorname{div}(a\nabla\psi) = \lambda\psi \text{ in } \Omega, \quad \psi_\nu = \sigma\lambda\psi \text{ on } \partial\Omega.$$

Since  $\lambda_n > 0$  for  $n > 0$  we deduce that for all  $x \in \Omega$

$$\lim_{t \rightarrow \infty} h(x, t) = h_0.$$

By assumption  $h_0 > 0$ . Consequently, using higher regularity properties of  $h$  and the strong minimum principle from [4], for any  $\epsilon > 0$  there exists  $t_0$  such that  $u \geq h_0 - \epsilon$  in  $\bar{\Omega} \times (t_0, \infty)$  and  $f(u) > c_0 > 0$  for  $t \geq t_0$ .

Let  $\psi \in \mathfrak{B}$  be a test function in the weak formulation of (16) on the interval  $(t_0, \tau)$ . By Green's formula and the dynamical boundary conditions

$$\int_{\Omega} a \Delta u \psi \, dx = - \int_{\Omega} (\nabla(a\psi), \nabla u) \, dx - \oint_{\partial\Omega} \sigma a \psi \partial_t u \, ds$$

and

$$\begin{aligned} \int_{t_0}^{\tau} \left( \int_{\Omega} \partial_t u \psi \, dx + \oint_{\partial\Omega} \sigma a \partial_t u \psi \, ds \right) dt = \\ - \int_{t_0}^{\tau} \int_{\Omega} a (\nabla u, \nabla \psi) \, dx \, dt + \int_{t_0}^{\tau} \int_{\Omega} m f \psi \, dx \, dt. \end{aligned}$$

Set

$$g(s) := \int_{s_0}^s \frac{d\xi}{f(\xi)},$$

choose  $\psi = \frac{1}{f(u)}$  and get by using (18)

$$\begin{aligned} \int_{\Omega} g(u(\tau, x)) \, dx - \int_{\Omega} g(u(t_0, x)) \, dx + \oint_{\partial\Omega} \sigma a g(u(\tau, x)) \, ds - \oint_{\partial\Omega} \sigma a g(u(t_0, x)) \, ds \\ = \int_{t_0}^{\tau} \int_{\Omega} a \frac{f'(u)}{(f(u))^2} \|\nabla u\|^2 \, dx \, dt + \int_{t_0}^{\tau} \int_{\Omega} m \, dx \, dt \geq \int_{t_0}^{\tau} \int_{\Omega} m \, dx \, dt. \end{aligned}$$

But this is impossible since the r.h.s. tends to infinity whereas the l.h.s. remains bounded as  $\tau \rightarrow \infty$ .  $\square$



## 4. UPPER BOUNDS FOR THE BLOW UP TIME

In this section, we derive upper bounds for the blow up time by comparing the present convection case with the reaction diffusion case. We assume that the nonlinearity  $f$  grows as a power of  $u$  or is an exponential function. Note that the latter case has not been considered in [6]. Let  $\lambda$  denote the minimal eigenvalue of  $-\operatorname{div}(a\nabla)$  in  $H_0^1(\Omega)$  and  $\psi \in H_0^1(\Omega)$  the eigenfunction belonging to  $\lambda$  satisfying  $0 < \psi \leq 1$  in  $\Omega$  and  $1 \in \psi(\Omega)$ . Then we can state the following

**Theorem 4.1.** Let  $\alpha > 0$ ,  $T > 0$  and  $p > 1$  denote real constants and let

$$u \in C(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T))$$

be a maximal solution of

$$\begin{cases} \partial_t u \geq \operatorname{div}(a(x)\nabla u) + \alpha u^p & \text{in } \bar{\Omega} \text{ for } t > 0, \\ u \geq 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\bar{\Omega}) \end{cases} \quad (21)$$

If

$$\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{p-1}} |\Omega| < \int_{\Omega} \varphi \psi \, dx$$

then  $u$  blows up in finite time  $T$  satisfying

$$T \leq \frac{1}{\lambda(p-1)} \ln \left( \frac{\alpha}{\alpha - \lambda |\Omega|^{p-1} \left(\int_{\Omega} \varphi \psi \, dx\right)^{1-p}} \right) =: t_1.$$

*Proof.* Introduce  $M(t) = \int_{\Omega} u \psi \, dx$ . Then

$$\dot{M} \geq \int_{\Omega} \operatorname{div}(a\nabla u) \psi \, dx + \alpha \int_{\Omega} u^p \psi \, dx.$$

Green's formula and the behaviour of  $\psi$  and  $\partial_\nu \psi$  on the boundary imply

$$\int_{\Omega} \operatorname{div}(a\nabla u) \psi \, dx = - \int_{\Omega} a(\nabla \psi, \nabla u) \, dx \geq \int_{\Omega} \operatorname{div}(a\nabla \psi) u \, dx.$$

Thus

$$\dot{M} \geq \int_{\Omega} \operatorname{div}(a\nabla \psi) u \, dx + \alpha \int_{\Omega} u^p \psi \, dx = -\lambda M + \alpha \int_{\Omega} u^p \psi \, dx.$$

Hölder's inequality leads to

$$\dot{M} \geq -\lambda M + \alpha |\Omega|^{1-p} M^p. \quad (22)$$

This is the same differential inequality as in the reaction diffusion case, and we can follow the proof of Theorem 4.1 in [6] in order to show that  $u$  blows up at  $T \leq t_1$ .  $\square$

**Corollary 4.2.** Suppose the above conditions on  $p$ ,  $\alpha$  and  $\varphi$  to be satisfied and, in addition,  $f(x, t, u, \nabla u) \geq u^p$  for  $u \geq 0$ . Then the maximal solution of Problem 1 blows up in finite time  $T \leq t_1$ , with  $t_1$  as in Thm. 4.1.

Spectral comparison used in the special case where  $h$  grows exponentially yields to a similar upper bound. Here  $\int_{\Omega} e^u \psi dx$  plays the same role as the first Fourier coefficient in the above case.

**Theorem 4.3.** Let  $u \in C(\overline{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T))$  be a maximal solution of

$$\begin{cases} \partial_t u \geq \operatorname{div}(a(x)\nabla u) + \alpha e^{pu} & \text{in } \overline{\Omega} \text{ for } t > 0, \\ u \geq 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \quad (23)$$

If

$$\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{p-1}} |\Omega| < \int_{\Omega} e^{\varphi} \psi dx, \quad (24)$$

then

$$T \leq \frac{1}{\lambda(p-1)} \ln \left( \frac{\alpha}{\alpha - \lambda |\Omega|^{p-1} \left(\int_{\Omega} e^{\varphi} \psi dx\right)^{1-p}} \right) =: t_2.$$

*Proof.* Set  $M(t) = \int_{\Omega} e^{u(\cdot, t)} \psi dx$ . Cor. 7.3 in [6] yields  $u \geq 0$  in  $\overline{\Omega} \times (0, T)$ , thus

$$\dot{M} \geq \int_{\Omega} \psi \partial_t u dx \geq \int_{\Omega} \operatorname{div}(a\nabla u) \psi dx + \alpha \int_{\Omega} \psi e^{pu} dx.$$

By following the proof of Thm. 4.1, we obtain

$$\dot{M} \geq \int_{\Omega} \operatorname{div}(a\nabla \psi) u dx + \alpha \int_{\Omega} e^{pu} \psi dx \geq -\lambda \int_{\Omega} e^u \psi dx + \alpha \int_{\Omega} (e^u \psi)^p dx.$$

Hölder's inequality,  $\int_{\Omega} (e^u \psi)^p dx \geq |\Omega|^{1-p} \left(\int_{\Omega} e^u \psi dx\right)^p$  leads to the ODIE (22), that permits to conclude as above.  $\square$

These last results are obviously valid in the case where  $a \equiv 1$ . In particular, we deduce an upper bound for the blow up time for Problem (23), which extends the results obtained in [6] to an exponential type nonlinearity. Furthermore, in the case  $p = 1$ , i.e.  $f(x, t, u, \nabla u) \geq e^u$ , all solutions blow up in finite time independently of the size of  $\Omega$  for positive nonvanishing initial values, see [2].

**Remark 4.4.** The above results can be extended to the case where  $a$  depends on  $x$  and  $u$  with  $a \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$ ,  $a > 0$  and  $\frac{\partial a}{\partial u} \geq 0$ , since

in this case,  $\operatorname{div}(a\nabla u) = (\nabla a, \nabla u) + \frac{\partial a}{\partial u} |\nabla u|^2 + a\Delta u$  with controllable sign of the quadratic gradient term.

Energy type techniques apply also to the present case. Let us illustrate this in the following classical case. Let  $E : H^1(\Omega) \cap L^{p+1}(\Omega) \rightarrow \mathbb{R}$  be the energy functional defined by

$$E(u) = \frac{1}{2} \int_{\Omega} a \|\nabla u\|_2^2 dx - \frac{1}{p+1} \int_{\Omega} u^{p+1} dx$$

and assume that the initial value  $\varphi$  belongs to  $H_0^1(\Omega) \cap C(\overline{\Omega})$ .

**Lemma 4.5.** If  $u \in C(\overline{\Omega} \times [0, \tau]) \cap C^{2,1}(\overline{\Omega} \times (0, \tau])$  is a solution of

$$\begin{cases} \partial_t u = \operatorname{div}(a\nabla u) + \alpha u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C_0(\overline{\Omega}), \end{cases} \quad (25)$$

then the function  $t \mapsto E(u(\cdot, t))$  is decreasing.

An estimation of the  $L^2$ -norm of the solution of the Dirichlet problem (25) yields an upper bound for the blow up time. Furthermore, by comparison principles the next result involving the maximal solution of Problem (1) can be deduced. Since the proofs are quite similar to the ones in [6], we omit the details here.

**Corollary 4.6.** Suppose  $\tilde{\varphi} \in H_0^1(\Omega) \cap C(\overline{\Omega})$ ,  $E(\tilde{\varphi}) \leq 0$  and  $\tilde{\varphi} \leq \varphi$  and, in addition,  $f(u) \geq u^p$  for  $u \geq 0$ . Then the maximal solution  $u$  of Problem 1 blows up in finite time  $T$  satisfying

$$T \leq \frac{p+1}{(p-1)^2} |\Omega|^{\frac{p-1}{2}} \|\tilde{\varphi}\|_2^{1-p} - \|\tilde{\varphi}\|_{\infty}^{-2p} [-E(\tilde{\varphi})] |\Omega|^{-1} \frac{p+1}{p(p-1)}.$$

**Remark 4.7.** Note that the blow up time under dynamical boundary condition with  $\sigma > 0$  can be strictly greater than the one under the Neumann boundary condition, see the example given in [6, 2.10]. An example for a finite blow up time under dynamical boundary condition with  $\sigma > 0$  and global existence under Dirichlet boundary condition is given in [2, p.63].

## 5. A PROBLEM INCLUDING A SIGN CHANGE OF THE CONVECTION TERM

In this section, we consider the following one-dimensional problem including a sign change of the gradient term

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_{\sigma}(u) := \sigma \partial_t u + \partial_{\nu} u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \quad (26)$$

with  $1 < p \in \mathbb{R}$  and  $\Omega = (-b, b)$  with  $b > 0$ . Moreover, we assume that  $\sigma > 0$  is constant. We present some results on blow up phenomena that depend on the value of  $p$ , and on the growth behaviour of the solutions when approaching the blow up time. For more details and results on the blow up set we refer to a forthcoming paper [7].

In fact, for  $1 < p \leq 2$ , the solutions of Problem (26) do not blow up and exist globally. Let the initial value  $\varphi$  satisfy  $0 < \sigma \leq \|\varphi\|_\infty^{2-p}$ , and let  $\alpha$  and  $A$  be some positive constants satisfying

$$A \geq \|\varphi\|_\infty e^{\alpha\sigma}, \quad \frac{1}{\sigma} \geq \alpha \geq \|\varphi\|_\infty^{p-2}.$$

Then the function  $y$  defined by

$$y(x, t) = Ae^{\alpha(x + \frac{t}{\sigma})}$$

is a globally bounded upper solution of Problem (26) with boundary condition  $\sigma \partial_t u + \partial_\nu u = 0$  on  $\partial\Omega \times (0, +\infty)$ . For  $p > 2$  the situation is different.

**Theorem 5.1.** For  $p > 2$ , the classical solutions  $u$  of (26) blow up in finite time if  $\varphi > 0$  in  $\bar{\Omega}$ .

*Proof.* We can follow an idea from [10] as follows. Suppose that  $u$  is a classical solution of Problem (26) with maximal existence time  $T$ . Introduce a symmetric hump function  $\tilde{\varphi}$  satisfying

$$\tilde{\varphi} \in H_0^1((-l, l)) \cap C^2([-l, l]), \quad (27)$$

$$0 < \tilde{\varphi}(x) = \tilde{\varphi}(-x) \text{ in } [-l, l], \quad (28)$$

$$\tilde{\varphi}'(x) > 0 \text{ in } [-l, 0), \tilde{\varphi}'(x) < 0 \text{ in } (0, l] \quad (29)$$

$$\Delta \tilde{\varphi} + \tilde{\varphi}^p \geq 0 \text{ in } [-l, l] \quad (30)$$

where  $l \in (0, b)$ . Let  $\tilde{u}$  be the maximal classical solution of the Dirichlet problem

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} + \tilde{u}^p & \text{in } [-l, l] \times (0, \infty), \\ \tilde{u}(\pm l, \cdot) = 0 & \text{for } t > 0, \\ \tilde{u}(\cdot, 0) = \tilde{\varphi} & \text{in } [-l, l]. \end{cases} \quad (31)$$

Under (27), if  $\tilde{\varphi}$  has the non-positive energy

$$E(\tilde{\varphi}) = \frac{1}{2} \int_{-l}^l \tilde{\varphi}'^2 dx - \frac{1}{p+1} \int_{-l}^l \tilde{\varphi}^{p+1} dx \leq 0, \quad (32)$$

it is well known by [6] that  $\tilde{u}$  blows up in finite time  $\tilde{T}$ , depending on  $\sigma$  and  $\tilde{\varphi}$ . It is not possible to compare the classical maximal solutions  $u$  and  $\tilde{u}$  by using the classical comparison principle of [6] due to the

changing sign of the gradient term. However, we can modify  $\tilde{u}$  into a subsolution  $y$  of Problem (26) defined on a suitable network corresponding to a subdivision of the interval  $\Omega$ . This subsolution  $y$  blows up at the latest as  $\tilde{u}$  for  $t = \tilde{T}$ . Set

$$m(t) = \tilde{u}(0, t) = \max_{-l \leq x \leq l} \tilde{u}(x, t) \quad \text{for } t > 0 \quad \text{and} \quad r(t) = \int_0^t m(\tau) d\tau.$$

Since  $p > 2$ , Thm. 2.9 from [6] yields  $r_0 := r(\tilde{T}) < \infty$ . Introduce the sets

$$\begin{aligned} R_0 &= \{0 < x \leq b \mid 0 < t < \tilde{T}\}, \\ R_1 &= \{r(t) - r_0 < x < 0 \mid 0 < t < \tilde{T}\}, \\ R_2 &= \{r(t) - r_0 - b \leq x < r(t) - r_0 \mid 0 < t < \tilde{T}\} \end{aligned}$$

and the function  $y$  defined by

$$y(x, t) = \begin{cases} \tilde{u}(x, t) & \text{in } R_0, \\ m(t) & \text{in } R_1, \\ \tilde{u}(x - r(t) + r_0, t) & \text{in } R_2. \end{cases}$$

Clearly,  $y(\cdot, t)$  and  $\partial_x y(\cdot, t)$  are continuous on  $\partial R_0 \cap \partial R_1$  and  $\partial R_1 \cap \partial R_2$  and  $y$  satisfies a classical Kirchhoff law. Furthermore, thanks to (28)–(30) we can prove that  $\partial_t y - \Delta y + y \partial_x y - y^p \leq 0$  in  $R_0$ ,  $R_1$  and  $R_2$  for  $0 < t < \tilde{T}$ . Then, by applying the comparison principle related to networks from [3] or [5], we can conclude that  $y \leq u$  in  $\overline{R_0} \cup \overline{R_1} \cup \overline{R_2} \times (0, \tilde{T})$ , which proves that  $u$  blows up in finite time  $T \leq \tilde{T}$ .  $\square$

For  $p > 3$ , the growth order of the solutions of Problem (26) amounts to  $\frac{-1}{p-1}$  when  $t$  approaches the blow up time, under initial data fulfilling

$$\varphi \in C^2([-b, b]), \quad \varphi'' - \varphi \varphi' + \varphi^p \geq 0 \quad \text{in } [-b, b]. \quad (33)$$

**Theorem 5.2.** Suppose  $p > 3$  and Condition (33) satisfied, then there exists a positive constant  $C$  such that

$$\|u_\sigma(\cdot, t)\|_\infty \leq \frac{C}{(T(\sigma, \varphi) - t)^{1/p-1}} \quad \text{for } t \in [0, T(\sigma, \varphi)).$$

*Proof.* Suppose  $\xi \in ]0, \frac{t_0}{2}]$  with  $t_0 = \frac{\|\varphi\|_\infty^{1-p}}{p-1}$ . Set  $p$  and  $q$  such that

$$p > q > 3 \quad \text{and} \quad p > q + \frac{1}{3} \quad (34)$$

and set  $u := u_\sigma \in C^{2,1}([-b, b] \times [0, T(\sigma, \varphi)))$  the maximal solution of Problem (8). Then, we can prove that there exists  $\delta > 0$  and  $1 < M \in \mathbb{R}$  such that

$$\partial_t u \geq \delta \exp(-Mt)(u^p + u^q) \quad \text{in } \overline{\Omega} \times [\xi, T(\sigma, \varphi)). \quad (35)$$

Set  $\tilde{T} = \frac{\xi}{2}$ . In the same way as in Cor. 2.7, we show by the strong minimum principle that there exists a positive constant  $c$  such that

$$y := \partial_t u \geq c > 0 \text{ in } [-b, b] \times [\xi, T(\sigma, \varphi)). \quad (36)$$

Now, set  $d(t) = \exp(-Mt)$  with  $M > 1$  and  $k(u) = u^p + u^q$  and introduce

$$J = \partial_t u - \delta d(t)k(u),$$

where  $\delta > 0$  is sufficiently small such that

$$J(\cdot, \xi) \geq 0 \text{ in } [-b, b] \quad (37)$$

in the view to (36).  $J$  satisfies the boundary condition

$$B_\sigma(J) = B_\sigma(y) - \delta dk'(u)B_\sigma(u) - \sigma \delta d'k(u) = \sigma \delta \exp(-Mt)k(u) \geq 0.$$

Moreover,  $J \in C^{2,1}([-b, b] \times [\xi, T(\sigma, \varphi)))$  fulfills

$$\partial_t J - \Delta J + u \partial_x J - (pu^{p-1} - \partial_x u)J = \delta dH(u) \text{ in } \bar{\Omega} \times [\xi, T(\sigma, \varphi)),$$

where

$$H(u) := pu^{p-1}k(u) - k'(u)u^p + k''(u)(\partial_x u)^2 - \frac{d'}{d}k(u) - k(u)\partial_x u.$$

In order to show that  $H(u) \geq 0$ , we prove the following inequality

$$\begin{aligned} |\partial_x u|(u^p + u^q) &\leq M(u^p + u^q) + (p - q)u^{p+q-1} \\ &+ (p(p - 1)u^{p-2} + q(q - 1)u^{q-2})(\partial_x u)^2. \end{aligned} \quad (38)$$

(38) is obvious when  $|\partial_x u| \leq M$ . Now, we suppose that  $|\partial_x u| > M$ . In the case where  $u^2 \leq q(q - 1)|\partial_x u|$ , (38) is fulfilled because  $u^q|\partial_x u| \leq q(q - 1)u^{q-2}(\partial_x u)^2$  and  $u^p|\partial_x u| \leq p(p - 1)u^{p-2}(\partial_x u)^2$  since  $p > q$ . If  $u^2 > q(q - 1)|\partial_x u|$ , then we have  $u > 1$  since  $M > 1 > \frac{1}{q(q-1)}$ , and by (34), we are led to

$$(u^p + u^q)|\partial_x u| \leq \frac{u^{p+2} + u^{q+2}}{q(q - 1)} \leq \frac{2}{q(q - 1)}u^{p+q-1} \leq (p - q)u^{p+q-1},$$

which leads to (38). Now, the comparison principle implies  $J \geq 0$  in  $[-b, b] \times [\xi, T(\sigma, \varphi))$ . Finally, the estimate of  $u$  when approaching  $T(\sigma, \varphi)$  is shown exactly in the same way as for Cor. 2.7.  $\square$

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