Dynamic and static stability of a drop attached to an inhomogeneous plane wall

Julian F Scott

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Laboratoire de Mécanique des Fluides et d'Acoustique (LMFA), Université de Lyon, France

Abstract

This article concerns the stability of a drop on a wall for which the contact angle, θ_w, varies from place to place. Such a wall may allow unstable equilibria of the drop, i.e. ones for which small perturbations to equilibrium grow, making the equilibrium unrealisable in practice. This will be referred to as dynamic instability and is one of the two versions of instability considered. The other arises from consideration of potential energy, which is the sum of surface (liquid/gas, liquid/solid and solid/gas) components and the gravitational potential energy. Equilibria are extrema of the potential energy with respect to variations of drop geometry which preserve its volume. An equilibrium is said to be statically stable if it is a local minimum of the potential energy for volume-preserving perturbations of the drop. The relationship between static and dynamic stability is the main subject of this paper. The liquid flow is governed by the incompressible Navier-Stokes equations. To allow for the moving contact line, a Navier slip condition with slip length λ is used at the wall, as is a prescribed contact angle, $\theta_w = \theta_w(x, y)$, at the contact line, where x, y are Cartesian coordinates on the wall. The perturbation is assumed small, allowing linearization of the governing equations and, in the usual manner of stability analysis, complex modes having the time dependency e^{st} are introduced. This leads to an eigenvalue problem with eigenvalue s, the sign of whose real part determines dynamic stability/instability. A quite different eigenvalue problem, which describes static stability/instability is also derived. It is shown that, despite this difference, the conditions for dynamic and static instability are in fact the same. This conclusion is far from evident *a priori*, but should be good news for interested numerical analysts because determination of static stability is much less numerically costly than a dynamic stability study, whereas it is the latter which gives a true determination of stability.

Keywords: Instability, attached drop, inhomogeneous wall, moving contact lines.

1. Introduction

Progress in the fabrication of surfaces whose properties vary from point to point (see [1]) has led to an interest in the dynamics of liquid drops on such a surface. For instance, among the considerable literature on this subject, [2] concerns the slow spreading of a two-dimensional drop on a heterogeneous wall using lubrication theory, while [3] treats the three-dimensional case. A drop may approach equilibrium at large times, but only if the equilibrium is stable, i.e. departures from equilibrium, when small enough, decay with time. Equilibria which are unstable are not realised in practice. Thus, when studying equilibria, it is important to decide if a given equilibrium is stable or not.

Prior to considerations of their stability, it should be noted that the equilibria of the interfaces between immiscible fluids, such as those considered here, result from extrema of potential energy, which is the sum of gravitational and surface contributions (resulting from surface tension). Studies of such equilibria, both physical and mathematical, go back a very long way (see e.g. [4], especially the beginning of section 2, and [5]). [6] concerns computational approaches for heterogeneous surfaces.

For systems like the drop attached to an inhomogeneous wall considered here, there are two notions of stability (see [4], whose main topic is stability and which discusses both types). On the one hand, there is dynamic instability, defined earlier, for which the time-dependent problem predicts growth of the perturbation to equilibrium. On the other hand, there is another notion of stability, namely static stability, which arises from considerations of the potential energy defined above. The drop is said to be statically stable if the equilibrium provides a minimum of potential energy for small volume-preserving perturbations. Static stability is the main subject of section 2 of [4], while section 3 of that paper provides a review of work on dynamic stability. With the heterogeneous case in mind, [7] considers static stability, while the stability of 2D droplets has been studied in [8].

We are thus faced with two approaches to stability: dynamic and static. There is no obvious *a priori* relationship between the two. Indeed, there is no notion of potential energy in the usual theory of hydrodynamic stability, so dynamic and static stability are not always equivalent. An example in which the static and dynamic stability criteria differ is that of the Kelvin-Helmholtz instability of a horizontal interface with gravity and surface tension between homogeneous, inviscid fluids of different densities (see [9], at the beginning of section 3). In that case, the system is always statically stable for light over dense fluid, whereas it can be dynamically destabilised by flow. When static and dynamic instability criteria disagree, it is the dynamic one which gives a true indication of stability.

The main objective of this paper is to show that, perhaps surprisingly, the two types of stability are in fact equivalent for the problem considered here. From a numerical point of view, this is interesting because the determination of static stability/instability involves a two dimensional calculation, whereas dynamic stability calculations are intrinsically 3D, hence considerably more computer intensive. Furthermore, dynamic stability analysis involves additional parameters. In the present model, these are the Ohnesorge number, Oh, which measures the importance of viscosity, and the slip length, λ, which arises from the contact-line model described below. Such parameters do not influence static stability/instability and, when the static and dynamic approaches to stability can be shown to yield the same results, as here, the many numerical dynamic stability calculations involved in an exhaustive parametric study are unnecessary.

The model of drop dynamics uses the incompressible Navier-Stokes equations in the liquid and assumes an inviscid gas with constant pressure, p_a. The moving contact line is problematic because the usual model, i.e. the Navier-Stokes equations with a no-slip condition at the wall, implies an unacceptable singularity at the contact line (see e.g. [10, 11]). There is no agreed definitive model [1, 12], but one which is frequently used allows slip

at the wall via a Navier condition in which slip is proportional to shear rate, the constant of proportionality being referred to as the slip length and denoted λ (which is typically much smaller than the drop size). Such a model was used by [13] to treat the case of the slow spreading of a liquid droplet on a wall using matched asymptotic expansions, while [14] extended this approach to more general cases. More recently, models of this type have been employed in numerical studies (see e.g. [15, 16, 17, 18]). In this paper, we use a Navier condition at the wall and a prescribed contact angle θ_w. Inhomogeneity is expressed by variations of $\theta_w(x, y)$ with respect to the Cartesian coordinates x and y, which specify the location on the wall.

The paper is organised as follows. Section 2 describes the overall mathematical model, its linearization for small perturbations and the resulting energy equation. Section 3 is the heart of the paper. Section 3.1 derives an eigenvalue problem which provides a precise quantification of static stability/instability. Section 3.2 develops the theory of dynamic instability. Section 3.3 first shows that the dynamic stability characteristics of the equilibrium are independent of the Ohnesorge number, Oh. It then examines the limit $\mathrm{Oh} \to 0$ and concludes that the notions of dynamic and static stability are equivalent for the present problem. Finally, section 4 concerns the case of a two-dimensional drop without gravity and derives explicit stability conditions for that case.

2. Modelling

A liquid drop, attached to an inhomogeneous plane wall, is surrounded by gas which is supposed inviscid and of constant pressure, p_a. The liquid/gas interfacial surface tension, σ, is assumed constant, as are the liquid density and dynamic viscosity, ρ and μ. The problem is nondimensionalized using L, $\left(\rho L^3/\sigma\right)^{1/2}$, $\left(\sigma/\rho L\right)^{1/2}$, σ/L as length, time, velocity and pressure scales, where L is the length scale which gives a nondimensional drop volume equal to 1. The Navier-Stokes equations in the liquid give

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\nabla p + \mathrm{Oh}\,\nabla^2\mathbf{u} + \mathbf{G}\,, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0\,, \tag{2.2}$$

where $\mathrm{Oh} = \mu/\left(\sigma\rho L\right)^{1/2}$ is an Ohnesorge number, which measures the importance of viscosity, and $\mathbf{G}$ represents gravity. $\mathbf{G}$ is directed vertically downward and has norm $|\mathbf{G}| = \mathrm{Bo} = \rho g L^2/\sigma$ equal to the Bond number, a measure of the significance of gravity. We will later consider the limit $\mathrm{Oh} \to 0$, in which case we expect viscosity to be negligible over most of the drop, outside a thin boundary layer at the wall in which viscous effects are significant.

Using Cartesian coordinates, the spatial position vector is $\mathbf{x} = (x, y, z)$, where the wall lies at $z = 0$ and the drop in $z > 0$ (see figure 1). Note that there is no requirement that the wall be horizontal. The wall boundary conditions are

$$\lambda \frac{\partial u_x}{\partial z} = u_x, \quad \lambda \frac{\partial u_y}{\partial z} = u_y, \quad u_z = 0 \qquad z = 0, \tag{2.3}$$

where λ, taken constant, is the slip length, while the angle at which the liquid/gas interface meets the wall has the prescribed value $0 < \theta_w(x, y) < \pi$. The kinematic and stress boundary conditions

$$\frac{\partial F}{\partial t} + \mathbf{u}.\nabla F = 0, \tag{2.4}$$

$$\mathrm{Oh}\, \mathbf{n}.\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) = \mathbf{n}\left(p - p_a - \kappa\right), \tag{2.5}$$

apply at the liquid/gas interface, which is described by $F(\mathbf{x}, t) = 0$. Here, κ represents the interface curvature and $\mathbf{n}$ is a unit normal vector to the interface, directed from the liquid towards the gas. Note that, as usual when using the model (2.3), we have in mind that λ is small, though this is not assumed in what follows.

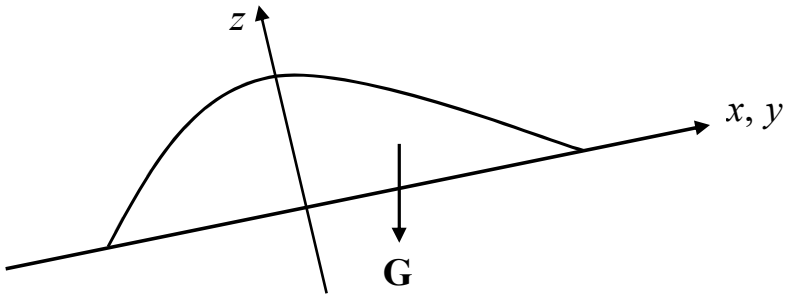

Figure 1: Schematic drop geometry, gravitational direction and Cartesian coordinate system.

Equilibrium of the drop implies $\mathbf{u} = 0$, hence constant $p - \mathbf{G}.\mathbf{x}$ according to (2.1). (2.5) requires constant $\kappa - \mathbf{G}.\mathbf{x}$ on the interface, which, together with the condition that the nondimensional drop volume equals 1 and the prescribed contact angle at the wall, determines possible equilibria. An alternative characterisation of equilibrium arises from considerations of energy. The nondimensional surface energy is

$$\int_{\tilde{S}_l} dS - \int_{\tilde{S}_w} \cos\theta_w \, dS \tag{2.6}$$

to within an additive constant, where $\tilde{S}_i$ and $\tilde{S}_w$ respectively represent the liquid/gas interface and the wetted area of the wall. Note that the second term in (2.6), which represents the surface energy of the wall, corresponds to the Young-Dupré equation for the contact angle. Adding in the gravitational potential energy gives the total potential energy as

$$\int_{\tilde{S}_i} dS - \int_{\tilde{S}_w} \cos\theta_w \, dS - \int_{\tilde{D}} \mathbf{G}.\mathbf{x} \, dv \,, \tag{2.7}$$

where $\tilde{D}$ is the interior of the drop. Equilibria are extrema of (2.7) under variations of drop geometry which maintain its volume equal to 1. An equilibrium will be referred to as statically stable if (2.7) is a local minimum for volume-preserving perturbations. Note that the equilibrium condition is independent of Oh and λ.

2.1 Linearization

Here we consider small perturbations to a given equilibrium. Let p' denote the difference between the perturbed and equilibrium pressures. Thus, (2.1) gives

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\nabla p' + \mathrm{Oh}\,\nabla^2\mathbf{u} \,, \tag{2.8}$$

which, given smallness of the perturbation, is linearized by dropping the convective term to obtain

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p' + \mathrm{Oh}\,\nabla^2\mathbf{u} \,. \tag{2.9}$$

(2.2) and (2.3) are already linear.

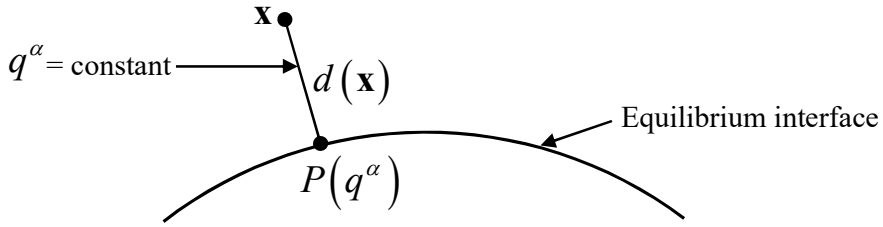

Figure 2: Coordinate system q^1, q^2, d. d is the distance from the equilibrium interface. q^1 and q^2 are the coordinates of the point P.

Let $d(\mathbf{x})$ be the signed distance from the equilibrium liquid/gas interface: $d(\mathbf{x}) = 0$ represents that interface, while $d(\mathbf{x}) < 0$ inside and $d(\mathbf{x}) > 0$ outside the equilibrium drop. Define coordinates q^1 and q^2 on the equilibrium interface. These coordinates are extended off that interface by attributing to a point $\mathbf{x}$ the values of q^α ($\alpha = 1, 2$) of the nearest point,

P, on the interface (see figure 2). Thus, q^1, q^2 and d provide a coordinate system for points near the equilibrium interface. It can be shown that constant q^1 and q^2 gives a straight line which intersects the surfaces of constant d orthogonally. Thus, $\nabla d.\nabla q^\alpha = 0$, while $|\nabla d| = 1$ follows from the definition of $d(\mathbf{x})$. The perturbed interface is represented by $d = \eta(q^\alpha, t)$, corresponding to $F(\mathbf{x},t) = d(\mathbf{x}) - \eta(q^\alpha(\mathbf{x}), t)$. $\eta(q^\alpha, t)$ expresses the perturbation in the interfacial position. It may be considered as a function defined on the equilibrium interface and extended to other locations by $\eta = \eta(q^\alpha(\mathbf{x}), t)$. Since $\nabla d.\nabla q^\alpha = 0$, $\nabla d.\nabla \eta = 0$.

The interfacial kinematic condition, (2.4), yields

$$\frac{\partial \eta}{\partial t} = \mathbf{u}.\nabla d - \mathbf{u}.\nabla \eta \tag{2.10}$$

on the perturbed interface. The second term on the right-hand side is second order in the perturbation and is therefore dropped, while $\nabla d = \mathbf{n}$ at leading order, hence the linearized kinematic condition

$$\frac{\partial \eta}{\partial t} = \mathbf{u}.\mathbf{n}\,. \tag{2.11}$$

Given the smallness of the perturbation, (2.11) is applied on the equilibrium interface, rather than the perturbed one.

The treatment of (2.5) is considerably more technical and the details are relegated to section A.1. The analysis relies on the differential geometry of the equilibrium interface and, in particular, its metric tensor, $g_{\alpha\beta}$, which gives the distance, ds, between two infinitesimally separated points on the interface as $ds^2 = g_{\alpha\beta} dq^\alpha dq^\beta$ (see e.g. [19], section 29), where, here and henceforth, the summation convention applies to repeated indices. It is found that

$$\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) = \mathbf{n}\left(\Delta\eta + \gamma(\mathbf{x})\eta + p'\right), \tag{2.12}$$

where γ is given by (A.7) and Δ is the Laplace-Beltrami operator, defined by

$$\Delta\eta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\alpha}\left(\sqrt{g}\, g^{\alpha\beta} \frac{\partial \eta}{\partial q^\beta}\right), \tag{2.13}$$

which is the natural extension of the 2D Laplace operator to curved surfaces and general coordinates (see [19], equation (43.8)). Here, g is the determinant of the matrix $g_{\alpha\beta}$ and $g^{\alpha\beta}$ is its inverse. The linearized kinematic and stress conditions, (2.11) and (2.12), apply on the equilibrium interface with $\mathbf{n}$ being the equilibrium normal vector.

It remains to express the condition of prescribed contact angle. Figure 3 shows the equilibrium contact line in the q^α plane. The equilibrium liquid/gas interface, henceforth denoted by S_i in physical space, is represented by Σ_i, the contact line by $\partial\Sigma_i$ and ν_α is the vector in the q^α plane which is orthogonal to $\partial\Sigma_i$, directed outwards from Σ_i and normalised using $g^{\alpha\beta}\nu_\alpha\nu_\beta = 1$. The contact-line condition is derived in section A.2 and takes the form

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\eta}{\partial q^\beta} = J(\mathbf{x})\eta\,, \tag{2.14}$$

where $J(\mathbf{x})$ is given by (A.45). Despite its recondite appearance, the left-hand side of (2.14) has a simple interpretation. It is the derivative of η at the contact line, taken in the direction tangential to S_i, normal to the contact line and outwards from S_i.

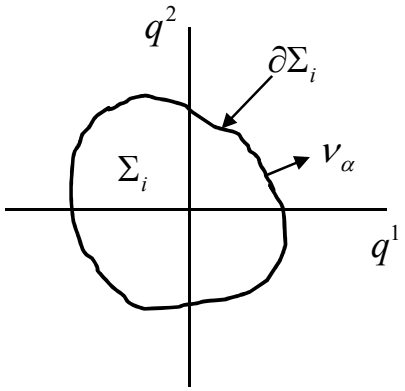

Figure 3: The equilibrium interface, Σ_i, and contact line, $\partial\Sigma_i$, in the q^α plane. ν_α is normal to $\partial\Sigma_i$.

The linearized problem consists of the incompressibility condition and momentum equation, (2.2) and (2.9), within the equilibrium drop, (2.3) at the wall, the kinematic and stress conditions, (2.11) and (2.12), on the equilibrium interface and (2.14) at the equilibrium contact line. In principle, initial conditions are also required. The condition that the perturbed and equilibrium drops have the same nondimensional volume (namely 1) is (to first order)

$$\int_{S_i}\eta dS = 0\,, \tag{2.15}$$

where the integral is taken over the equilibrium interface, S_i. This condition constrains the initial η, but if it is satisfied initially, integration of (2.2) over the drop, followed by use of the divergence theorem, $u_z = 0$ at the wall and (2.11) at the interface, shows that it continues to be satisfied at subsequent times (an expression of drop volume conservation).

2.2 Energetics

Based on the linearized problem, it is shown in section A.4 that

$$\frac{dE}{dt} = -2\,\mathrm{Oh}\int_D \mathbf{e}:\mathbf{e}\,dv - \lambda\,\mathrm{Oh}\int_{S_w}\left(\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2\right)dS\,, \tag{2.16}$$

where D, S_w are the equilibrium drop volume and wetted wall area, $\mathbf{e} = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)/2$ is the strain-rate tensor,

$$E = \frac{1}{2}\int_D |\mathbf{u}|^2\,dv + E_s[\eta]\,, \tag{2.17}$$

$$E_s[\eta] = \frac{1}{2}\left(\int_{S_i}\left(g^{\alpha\beta}\frac{\partial\eta}{\partial q^\alpha}\frac{\partial\eta}{\partial q^\beta} - \gamma\eta^2\right)dS - \oint_C J\eta^2 ds\right) \tag{2.18}$$

and C is the equilibrium contact line. (2.16) describes the evolution of the total drop mechanical energy, E. Both terms on the right-hand side are negative or zero and express dissipation of energy. The first has the familiar form for volumetric viscous dissipation (see e.g. [20], equation (3.4.5) for the dimensional version, where the second term in brackets is zero because here the flow is incompressible), while the other is due to slip at the wall. Turning attention to (2.17), the first term on the right-hand side evidently represents the kinetic energy of the drop. The second, given by (2.18), combines surface and gravitational potential energies, the gravitational contribution appearing via γ (see (A.7)). For the small, volume-preserving perturbations considered here (recall that η satisfies (2.15), so the perturbation is indeed volume preserving), $E_s[\eta]$ is the difference between the perturbed and equilibrium values of (2.7), correct to second order in the perturbation. The first-order difference is zero because (2.7) is an extremum for volume-preserving perturbations to equilibrium, while $E_s[\eta]$ is the second-order difference. If E_s is positive for all nonzero η satisfying (2.15), the equilibrium is statically stable.

3. Stability

3.1 Static stability

With the eventual aim of characterising static stability using eigenvalues, consider the problem of finding extrema of $E_s[\omega]$ among $\omega(q^\alpha)$ on S_i for which

$$\int_{S_i}\omega dS = 0\,, \qquad \int_{S_i}\omega^2 dS = 1\,. \tag{3.1}$$

It is shown in section A.5, using the calculus of variations, that

$$\Delta\omega + (\gamma + \zeta)\omega + \phi = 0 \tag{3.2}$$

and

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\omega}{\partial q^\beta} = J\omega, \tag{3.3}$$

on C, where ϕ and ζ are constants representing Lagrange multipliers for the constraints (3.1). The first of equations (3.1), together with (3.2) and (3.3), form an eigenvalue problem whose eigenvalue is ζ and whose unknowns are $\omega\left(q^\alpha\right)$ and the constant ϕ. The second of equations (3.1) provides a normalisation condition. Allowing for possible multiplicity, let us denote the eigenvalues by ζ_n and the corresponding ω and ϕ by ω_n and ϕ_n. Note that, unlike the dynamic stability eigenvalue problem of the next section, the present one is independent of Oh and λ. Also unlike the dynamic problem, the one considered here only involves real quantities.

It is shown in section A.6 that the ω_n can be chosen orthonormal in the sense that

$$\int_{S_i}\omega_n\omega_m dS = \delta_{nm}, \tag{3.4}$$

where δ_{nm} is the Kronecker delta. Since each of the ω_n satisfies the first of conditions (3.1), it is reasonable to assume that the eigenfunctions ω_n form a complete set for functions on S_i which satisfy (2.15). In that case, we can write any such function as

$$\eta = \sum_n c_n\omega_n\,. \tag{3.5}$$

Section A.6 also shows that

$$E_s[\eta] = \frac{1}{2}\sum_n \zeta_n c_n^2\,, \tag{3.6}$$

which describes the second-order variations of the potential energy for small perturbations to equilibrium, hence characterising static stability. If all the ζ_n are positive, $E_s[\eta] > 0$ for any nonzero η satisfying (2.15). In that case the equilibrium is statically stable. On the other hand, if one or more ζ_n are negative it is said to be statically unstable. Finally, when $\zeta = 0$ is an eigenvalue and all others are positive, the equilibrium will be referred to as statically neutral. Note that $\zeta_n = 0$ arises if there are equilibria infinitesimally close to the given one. An example is provided by a drop on a homogeneous wall, for which neighbouring equilibria can be obtained by small displacements parallel to the wall. However, when the wall is inhomogeneous, as we have in mind here, we expect all ζ_n to be nonzero in general. This does not mean that the special case $\zeta = 0$ is unimportant, indeed it is required for static neutrality.

3.2 Dynamic stability

The fact that the linearized problem contains no time-dependent coefficients allows complex solutions of the form

$$\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x})e^{st}, \quad p' = \tilde{p}(\mathbf{x})e^{st}, \quad \eta = \tilde{\eta}(q^{\alpha})e^{st}. \tag{3.7}$$

Such solutions are often referred to as modes in the hydrodynamic stability community. The idea here is that the general solution can be expressed as a linear combination of modes, each of which can be studied separately. (2.9) and (2.2) imply

$$s\tilde{\mathbf{u}} = -\nabla\tilde{p} + \mathrm{Oh}\,\nabla^2\tilde{\mathbf{u}}, \tag{3.8}$$

$$\nabla.\tilde{\mathbf{u}} = 0. \tag{3.9}$$

(2.3) gives the wall condition

$$\lambda\frac{\partial\tilde{u}_x}{\partial z} = \tilde{u}_x, \quad \lambda\frac{\partial\tilde{u}_y}{\partial z} = \tilde{u}_y, \quad \tilde{u}_z = 0 \qquad z = 0. \tag{3.10}$$

(2.11) and (2.12) provide the linearized kinematic and interfacial stress conditions

$$s\tilde{\eta} = \tilde{\mathbf{u}}.\mathbf{n}, \tag{3.11}$$

$$\mathrm{Oh}\,\mathbf{n}.\left(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T\right) = \mathbf{n}\left(\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}\right), \tag{3.12}$$

on the equilibrium interface. (2.14) yields

$$\nu_{\alpha}g^{\alpha\beta}\frac{\partial\tilde{\eta}}{\partial q^{\beta}} = J\tilde{\eta} \tag{3.13}$$

at the equilibrium contact line. Finally, (2.15) implies

$$\int_{S_i}\tilde{\eta}dS = 0. \tag{3.14}$$

(3.8)-(3.14) form an eigenvalue problem with eigenvalue $s = s_r + is_i$ which determines the modes. For each mode, s_i is an oscillation frequency, whereas the sign of s_r controls the exponential growth/decay of the mode. If $s_r > 0$, the mode is growing, while $s_r < 0$ for a decaying mode. If one or more eigenvalues can be found for which $s_r > 0$, a general perturbation grows, indicating instability of the given equilibrium. On the other hand, if $s_r < 0$ for all modes, the equilibrium is stable. The remaining possibility, referred to as neutral stability, is when one or more modes have $s_r = 0$, while the others are decaying. We use the terms dynamic stability, instability and neutrality to distinguish them from the corresponding static classification introduced earlier. Dynamic instability/stability determines whether small perturbations grow or decay, which is the true test of stability.

The complex eigenfunctions, $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\eta}$, determine the spatial structure (amplitude and phase) of modes. Note that for modes with real s, the eigenfunctions can be chosen real, in which case the entire modal solution is real. On the other hand, if $s_i \neq 0$, the complex conjugate of the modal solution is also a solution. Thus, for each complex mode, there is a conjugate mode. A physically meaningful perturbation must be real and this requires a suitable linear combination of modes and their conjugates. Note that there may be several modes associated with a given eigenvalue, i.e. it admits of two or more linearly independent choices of eigenfunctions.

It is shown in section A.7 that

$$s^* \int_D |\tilde{\mathbf{u}}|^2 \, dv + s\tilde{E}_s[\tilde{\eta}] = -2\,\mathrm{Oh} \int_D \tilde{\mathbf{e}} : \tilde{\mathbf{e}}^* dv - \lambda\,\mathrm{Oh} \int_{S_w} \left(\left| \frac{\partial \tilde{u}_x}{\partial z} \right|^2 + \left| \frac{\partial \tilde{u}_y}{\partial z} \right|^2 \right) dS, \quad (3.15)$$

where $\tilde{\mathbf{e}} = \left(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T \right)/2$,

$$\tilde{E}_s[\tilde{\eta}] = \int_{S_i} \left(g^{\alpha\beta} \frac{\partial \tilde{\eta}}{\partial q^\alpha} \frac{\partial \tilde{\eta}^*}{\partial q^\beta} - \gamma |\tilde{\eta}|^2 \right) dS - \oint_C J |\tilde{\eta}|^2 \, ds, \quad (3.16)$$

and * denotes complex conjugation. There are clear similarities between equations (3.15) and (2.16). Note however that, though they are closely related mathematically, both in their structure and derivation, (2.16) describes the time evolution of total energy in physical space, whereas (3.15) refers to a particular complex mode. Like the right-hand side of (2.16), that of (3.15) has two terms, both of which are real and either negative or zero. The left-hand side of (3.15) corresponds with that of (2.16), the analogy with the kinetic and potential energy terms of (2.17) being obvious. Symmetry of $g^{\alpha\beta}$ implies that $\tilde{E}_s$ is real and $\tilde{E}_s = 2\left(E_s[\tilde{\eta}_r] + E_s[\tilde{\eta}_i] \right)$, where $\tilde{\eta} = \tilde{\eta}_r + i\tilde{\eta}_i$. The real and imaginary parts of (3.15) give

$$s_r \left(\int_D |\tilde{\mathbf{u}}|^2 \, dv + \tilde{E}_s \right) = -2\,\mathrm{Oh} \int_D \tilde{\mathbf{e}} : \tilde{\mathbf{e}}^* dv - \lambda\,\mathrm{Oh} \int_{S_w} \left(\left| \frac{\partial \tilde{u}_x}{\partial z} \right|^2 + \left| \frac{\partial \tilde{u}_y}{\partial z} \right|^2 \right) dS, \quad (3.17)$$

$$s_i \left(\int_D |\tilde{\mathbf{u}}|^2 \, dv - \tilde{E}_s \right) = 0. \quad (3.18)$$

Suppose static stability, so both $\tilde{E}_s$ and the kinetic energy term on the left-hand side of (3.17) are positive, while the right-hand side of (3.17) is negative. Thus, $s_r < 0$, hence all modes are decaying. It follows that static stability implies dynamic stability. This might have been expected because the total energy, E, decays due to dissipation according to (2.16). In the case of a mode and a statically stable drop, both the potential and kinetic contributions to E are increasing functions of the modal amplitude, so that amplitude must also decay. On the other hand, static instability does not exclude the possibility that, as in the statically stable case, the bracketed term on the left-hand side of (3.17) be positive, hence $s_r \leq 0$. If this is true

of all modes, the drop would be dynamically stable or neutral, despite being statically unstable. The elimination of this possibility is the main motivation for the remaining analysis of this section.

Consider a mode with $s_i \neq 0$, thus

$$\int_D \left|\tilde{\mathbf{u}}\right|^2 dv = \tilde{E}_s \tag{3.19}$$

according to (3.18), hence

$$2s_r \int_D \left|\tilde{\mathbf{u}}\right|^2 dv = -2\,\mathrm{Oh} \int_D \tilde{\mathbf{e}} : \tilde{\mathbf{e}}^* dv - \lambda\,\mathrm{Oh} \int_{S_w} \left(\left|\frac{\partial \tilde{u}_x}{\partial z}\right|^2 + \left|\frac{\partial \tilde{u}_y}{\partial z}\right|^2 \right) dS \tag{3.20}$$

from (3.17). With the aim of deriving a contradiction, suppose that $s_r \geq 0$. The left-hand side of (3.20) is positive or zero, while the right-hand side cannot be positive. Thus, both must be zero, hence $\tilde{\mathbf{e}} = 0$. The real and imaginary parts of $\tilde{\mathbf{u}}$ then consist of a rotation and a translation, which is only consistent with (3.10) if $\tilde{\mathbf{u}} = 0$. When $\tilde{\mathbf{u}} = 0$, $\tilde{p}$ is a constant according to (3.8), $\tilde{\eta} = 0$ from the linearized kinematic condition (3.11) (because $s \neq 0$) and $\tilde{p} = 0$ from (3.12). It is in the nature of an eigenvalue problem that the eigenfunctions should not all be zero, so we have a contradiction. The conclusion is that $s_i \neq 0$ implies $s_r < 0$. It follows that modes are either decaying or have real $s \geq 0$. The former being unimportant from a stability standpoint, most attention is devoted to the latter. Note that growing modes have real $s > 0$.

Consider a mode with $s = 0$. The left-hand side of (3.15) is zero, hence $\tilde{\mathbf{e}} = 0$ for the mode in question. The reasoning following (3.20) leads to $\tilde{\mathbf{u}} = 0$, hence constant $\tilde{p} = \tilde{p}_0$ according to (3.8). However, we can no longer conclude from (3.11) that $\tilde{\eta} = 0$ because $s = 0$. Instead, (3.12) gives

$$\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}_0 = 0\,. \tag{3.21}$$

Thus, a mode with $s = 0$ has $\tilde{\mathbf{u}} = 0$, $\tilde{p} = \tilde{p}_0$ and $\tilde{\eta}$, $\tilde{p}_0$ which satisfy (3.13), (3.14) and (3.21). Note that $\tilde{\eta}$ and $\tilde{p}_0$ cannot both be zero because that would make all eigenfunctions zero, which is not allowed. Given (3.13), (3.14) and (3.21) and taking η, ϕ equal to either the real or imaginary parts of $\tilde{\eta}$, $\tilde{p}_0$, (2.15), (3.2) and (3.3) are satisfied with $\zeta = 0$. Thus, $\zeta = 0$ is an eigenvalue of (2.15), (3.2) and (3.3). The converse also holds, i.e. if $\zeta = 0$ is an eigenvalue, there is a corresponding mode with $s = 0$, $\tilde{\mathbf{u}} = 0$, $\tilde{p} = \phi$ and $\tilde{\eta} = \eta$. It follows that there is a precise correspondence between zero-ζ eigenfunctions and modes with $s = 0$. Since the eigenvalue problem for ζ is independent of Oh and λ, so are $s = 0$ modes.

Next, consider a mode with $s \neq 0$. $\tilde{\mathbf{u}}$ cannot be identically zero, otherwise (3.11) gives $\tilde{\eta} = 0$, hence (3.12) yields $\tilde{p} = 0$. This makes all eigenfunctions zero, which is not allowed. Thus, $\tilde{\mathbf{u}} = 0$ is characteristic of $s = 0$ modes.

3.3 The effects of varying Oh

In this section, we vary Oh, but keep everything else, in particular the equilibrium, the same. As Oh varies, the eigenvalues s move around in the complex plane. Let us consider the possibility that the nature of the equilibrium, dynamically stable, unstable or neutral, changes as Oh varies over positive values. Recall that only modes with real $s \geq 0$ are significant for instability and that the number of $s = 0$ modes is independent of Oh. Thus, a change in stability character is only possible if an $s > 0$ mode approaches $s = 0$ as $\mathrm{Oh} \to \mathrm{Oh}_c$ from either above or below, where $\mathrm{Oh}_c > 0$ is the value at which the supposed change in stability character occurs. It is shown in section A.8 that this cannot happen. The essential reason is that the number of $s = 0$ modes is constant, but the mathematical reasoning is lengthy and technical, hence its relegation to the appendix. The conclusion is that the character of the equilibrium does not depend on Oh, so any value of Oh should give the same result. We henceforth consider the limit $\mathrm{Oh} \to 0$.

3.3.1 Viscous versus inviscid modes

As $\mathrm{Oh} \to 0$, $s \neq 0$ modes are of two types. Some approach an inviscid limit and will be discussed later. On the other hand, there are modes which continue to be significantly affected by viscosity as $\mathrm{Oh} \to 0$. The two types are distinguished by the importance, or otherwise, of the viscous term in (3.8), the inviscid modes making this term negligible. As we shall see later, inviscid modes are irrotational, hence the need for essentially viscous modes to express the vorticity of a general flow inside the drop. Consider such a mode. To keep the viscous term in (3.8), s must go to zero as $s = O(\mathrm{Oh})$. Smallness of s as $\mathrm{Oh} \to 0$ reflects the slowness of viscous diffusion of vorticity when Oh is small. Recalling that $\tilde{\mathbf{u}}$ is not identically zero (since $s \neq 0$), we normalise the mode such that $\tilde{\mathbf{u}} = O(1)$, hence $\tilde{p} = O(\mathrm{Oh})$ in order that the pressure-gradient term appear in (3.8) with the others. The left-hand side of (3.12) is $O(\mathrm{Oh})$, as is the pressure term. To make $\Delta\tilde{\eta} + \gamma\tilde{\eta} = O(\tilde{\eta})$ of the same order, $\tilde{\eta} = O(\mathrm{Oh})$ so $\tilde{E}_s[\tilde{\eta}] = O(\mathrm{Oh}^2)$ from (3.16). Because $\tilde{\mathbf{u}} = O(1)$,

$$\int_D |\tilde{\mathbf{u}}|^2 \, dv = O(1) \tag{3.22}$$

dominates $\tilde{E}_s[\tilde{\eta}] = O(\mathrm{Oh}^2)$ at small Oh. Thus, (3.18) implies $s_i = 0$. The right-hand side of (3.17) is negative or zero. If it were zero, $\tilde{\mathbf{e}} = 0$ and the reasoning following (3.20) leads to $\tilde{\mathbf{u}} = 0$, which is not the case because $s \neq 0$. Thus, the right-hand side of (3.17) is negative, while the integral on the left-hand side is positive and dominates the term $\tilde{E}_s[\tilde{\eta}]$. It follows that s is real and negative, implying decay of modes for which viscosity remains significant

in the limit $\mathrm{Oh} \to 0$. Because they move towards $s = 0$, we expect such modes to become increasingly dense on the negative, real s-axis as Oh decreases. However, because they decay, they are unimportant from a stability point of view. Thus, we focus on modes which approach an inviscid limit as $\mathrm{Oh} \to 0$.

3.3.2 Inviscid modes

Given $s \neq 0$, let $\mathbf{v} = s^{-1}\tilde{\mathbf{u}}$ and $\psi = s^{-2}\left(\tilde{p} - \tilde{p}_0\right)$, where the constant $\tilde{p}_0$ is such that

$$\int_D \psi \, dv = 0 . \tag{3.23}$$

(3.8)-(3.12) give

$$\mathbf{v} = -\nabla\psi + \varepsilon\nabla^2\mathbf{v} , \tag{3.24}$$

$$\nabla.\mathbf{v} = 0 \tag{3.25}$$

with

$$\lambda\frac{\partial v_x}{\partial z} = v_x , \quad \lambda\frac{\partial v_y}{\partial z} = v_y , \quad v_z = 0 \tag{3.26}$$

on the wall and

$$\tilde{\eta} = \mathbf{v}.\mathbf{n} , \tag{3.27}$$

$$\varepsilon\sigma\mathbf{n}.\left(\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^T\right) = \mathbf{n}\left(\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}_0 + \sigma\psi\right) \tag{3.28}$$

on the interface, where $\varepsilon = s^{-1}\,\mathrm{Oh}$ and $\sigma = s^2$ (not to be confused with surface tension). (3.13) and (3.14) apply as before.

In order that the mode approach an inviscid limit, the viscous term in (3.24) must be negligible, hence $\varepsilon \to 0$ as $\mathrm{Oh} \to 0$. When $\varepsilon \to 0$, (3.24) has the limiting form

$$\mathbf{v} = -\nabla\psi , \tag{3.29}$$

and (3.25) implies

$$\nabla^2\psi = 0 . \tag{3.30}$$

In the viscous problem there are three boundary conditions, given by (3.26), at the wall. However, only one is possible in the inviscid limit considered here. Mathematically, this is because (3.30) only allows one condition per boundary point. The inviscid condition is the usual one of wall impermeability, i.e. $v_z = 0$. Thus, (3.29) gives

$$\frac{\partial\psi}{\partial z} = 0 \tag{3.31}$$

as the wall boundary condition. (3.27), (3.28) and (3.29) yield

$$\mathbf{n}.\nabla\psi = -\tilde{\eta}\,, \tag{3.32}$$

$$\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}_0 + \sigma\psi = 0 \tag{3.33}$$

at the interface. (3.13), (3.14), (3.23) and (3.30)-(3.33) form the inviscid eigenvalue problem, which has eigenvalue σ and unknowns $\psi(\mathbf{x})$, $\tilde{\eta}(q^\alpha)$ and $\tilde{p}_0$. The corresponding s, $\tilde{\mathbf{u}}$ and $\tilde{p}$ follow from $s = \pm\sigma^{1/2}$, $\tilde{\mathbf{u}} = -s\nabla\psi$ and $\tilde{p} = \tilde{p}_0 + \sigma\psi$. Given the choice of sign, the number of inviscid modes for a given eigenvalue $\sigma \neq 0$ is twice the multiplicity of σ. Note that (3.29) shows that inviscid modes are irrotational, as stated earlier.

The case $s = 0$ was excluded from the above analysis just prior to equation (3.23), however it is in fact allowed for by the above inviscid problem. $s = 0$ modes satisfy (3.13), (3.14) and (3.21), hence (3.33) holds with $\sigma = 0$, while ψ is determined by (3.23) and (3.30)-(3.32). Thus, if there are modes with $s = 0$, $\sigma = 0$ is an eigenvalue of the inviscid problem, whose eigenfunctions, $\tilde{\eta}$, $\tilde{p}_0$, provide the corresponding $s = 0$ modes. We saw earlier that $s = 0$ modes correspond with $\zeta = 0$ eigenfunctions. Here, we see that this correspondence extends to $\sigma = 0$ inviscid eigenfunctions. Note that the number of $s = 0$ modes equals the $\sigma = 0$ multiplicity.

It is shown in section A.9 that the eigenvalues σ are real. If $\sigma < 0$, the resulting pair of inviscid modes are purely oscillatory, whereas $\sigma > 0$ yields one decaying and one growing mode. Thus, the existence of an eigenvalue with $\sigma > 0$ implies inviscid instability. Since σ is real, the unknowns, ψ, $\tilde{\eta}$ and $\tilde{p}_0$, can be chosen real and this is supposed in what follows. Given real ψ, $\tilde{\eta}$ and $\tilde{p}_0$, $\tilde{p} = \tilde{p}_0 + \sigma\psi$ yields real $\tilde{p}$, while $\tilde{\mathbf{u}} = -s\nabla\psi$ implies that $\tilde{\mathbf{u}}$ is real if $\sigma > 0$ and purely imaginary when $\sigma < 0$. Note that $\sigma = 0$ gives $\tilde{\mathbf{u}} = 0$ and $\tilde{p} = \tilde{p}_0$, as it should do since it represents $s = 0$.

Denote the eigenvalues (allowing for possible multiplicity and zero σ) by σ_k and the corresponding real ψ, $\tilde{\eta}$ and $\tilde{p}_0$ by ψ_k, $\tilde{\eta}_k$ and $\tilde{p}_{0k}$. Section A.9 shows that ψ_k, $\tilde{\eta}_k$ and $\tilde{p}_{0k}$ can be chosen such that

$$\int_D \nabla\psi_k . \nabla\psi_l dS = \delta_{kl} \tag{3.34}$$

and that the matrix

$$\tilde{E}_{kl} = \int_{S_i} \left(g^{\alpha\beta} \frac{\partial\tilde{\eta}_k}{\partial q^\alpha} \frac{\partial\tilde{\eta}_l}{\partial q^\beta} - \gamma\tilde{\eta}_k\tilde{\eta}_l \right) dS - \oint_C J\tilde{\eta}_k\tilde{\eta}_l ds \tag{3.35}$$

is given by

$$\tilde{E}_{kl} = -\sigma_k\delta_{kl}\,. \tag{3.36}$$

Let $\tilde{\eta}$ be any nonzero, possibly complex, function on S_i which satisfies (3.14). Define ψ as the solution of (3.23) and (3.30)-(3.32). Such a solution exists thanks to (3.14). Assuming the $\tilde{\eta}_k$ form a complete set for functions satisfying (3.14),

$$\tilde{\eta} = \sum_k \tilde{c}_k \tilde{\eta}_k \,, \tag{3.37}$$

hence, according to (3.23) and (3.30)-(3.32) and their equivalents for ψ_k and $\tilde{\eta}_k$,

$$\psi = \sum_k \tilde{c}_k \psi_k \,. \tag{3.38}$$

Let

$$R\left[\tilde{\eta}\right] = -\frac{\tilde{E}_s\left[\tilde{\eta}\right]}{\int_D \left|\nabla \psi\right|^2 dS}, \tag{3.39}$$

in which the denominator is positive, otherwise ψ is constant and $\tilde{\eta} = 0$ from (3.32), which is incompatible with the requirement that $\tilde{\eta}$ be nonzero. (3.16) and (3.34)-(3.36) imply

$$\int_D \left|\nabla \psi\right|^2 dS = \sum_k \left|\tilde{c}_k\right|^2 , \tag{3.40}$$

$$\tilde{E}_s\left[\tilde{\eta}\right] = \sum_{k,l} \tilde{E}_{kl} \tilde{c}_k \tilde{c}_l^* = -\sum_k \sigma_k \left|\tilde{c}_k\right|^2 , \tag{3.41}$$

hence

$$R\left[\tilde{\eta}\right] = \frac{\sum_k \sigma_k \left|\tilde{c}_k\right|^2}{\sum_k \left|\tilde{c}_k\right|^2} . \tag{3.42}$$

We suppose that the σ_k are bounded above, otherwise there would be growing inviscid modes with arbitrarily large growth rates. Let σ_s be the supremum of the set of σ_k. It follows from (3.42) that $\max R\left[\tilde{\eta}\right] = \sigma_s$, where the maximum is taken over nonzero $\tilde{\eta}$ which satisfy (3.14). Although we suppose an upper bound for σ_k, there is no obvious reason for a lower bound. Indeed, we expect an infinity of negative σ_k, extending to $\sigma = -\infty$ and representing oscillatory inviscid modes of higher and higher oscillation frequencies.

If all the ζ_n are positive or zero (static stability or neutrality), (3.6) indicates that $E_s\left[\eta\right] \geq 0$ for any η satisfying (2.15). This is the case for both $\tilde{\eta}_r$ and $\tilde{\eta}_i$ according to (3.14), hence $\tilde{E}_s\left[\tilde{\eta}\right] = 2\left(E_s\left[\tilde{\eta}_r\right] + E_s\left[\tilde{\eta}_i\right]\right) \geq 0$. It follows from (3.39) that $R\left[\tilde{\eta}\right] \leq 0$ for any nonzero $\tilde{\eta}$ satisfying (3.14), hence $\sigma_s = \max R\left[\tilde{\eta}\right] \leq 0$. Thus, $\sigma \leq 0$ for all eigenvalues when

the equilibrium is statically stable or neutral. The neutral case is distinguished from the stable one by the existence of $\zeta = 0$ as an eigenvalue. As we saw above, this is equivalent to $\sigma = 0$. Thus, static stability gives $\sigma_k < 0$ for all k, while static neutrality gives a $\sigma = 0$ eigenvalue, all others being negative. On the other hand, if some ζ_n is negative (static instability), $\tilde{E}_s[\eta_n] = 2E_s[\eta_n] < 0$ from (3.6). Choosing $\tilde{\eta} = \eta_n$, $R[\tilde{\eta}] > 0$ from (3.39), hence $\sigma_s = \max R[\tilde{\eta}] > 0$. It follows that there is at least one positive σ_k. Combining this result with the earlier one that $\sigma \le 0$ for all eigenvalues when the equilibrium is statically stable or neutral, we conclude that the existence of a positive σ_k is equivalent to static instability.

To summarise, the static stability characteristics of the equilibrium can be determined either from the eigenvalues ζ_n, as described in section 3.1, or from the σ_k. It is the correspondence between static stability/instability and the σ_k which leads to equivalence of the static and dynamic approaches at small Oh. Static stability occurs when all σ_k are negative, static instability if one or more of them are positive and neutrality when one or more is zero and the others are negative.

3.3.3 Consequences for stability

Static instability implies a positive eigenvalue σ, hence a growing inviscid mode. This mode is the $\mathrm{Oh} \to 0$ limit of a growing viscous mode, corresponding to dynamic instability at small Oh. On the other hand, supposing dynamic instability, one possibility would be that a growing viscous mode, which, as we saw earlier, has real $s > 0$, approaches $s = 0$ as $\mathrm{Oh} \to 0$. However, it is shown in section A.10 that this cannot occur. Thus, any growing viscous mode must approach a growing inviscid one. Such a mode has $\sigma > 0$, hence static instability. It follows that static and dynamic instability are equivalent at small Oh. The same is therefore true of static and dynamic stability/neutrality. Stability and neutrality are distinguished by the existence of zero ζ_n in the static approach and zero s in the dynamic one. The correspondence between modes with $s = 0$ and eigenvalues $\zeta_n = 0$ leads to the conclusion that, for small Oh, the character, stable/unstable/neutral, of the equilibrium is the same for the static and dynamic approaches. Given that the static character is independent of Oh and it has been shown that the dynamic character is also independent of Oh, this result holds without the requirement of small Oh. It follows that the static and dynamic approaches to stability agree, no matter what parameters are chosen, which is the main conclusion of this paper. Because the static problem is independent of both Oh and λ, the same is true of the dynamic stability character of the equilibrium. However, that character does depend on $\theta_w(x, y)$, $\mathbf{G}$ and the choice of equilibrium.

3.3.4 Further details of the inviscid limit

In the $\mathrm{Oh} \to 0$ analysis given above, we have frequently referred to the inviscid limit of viscous modes which admit of such a limit, but without giving details of how the limit is

approached. Viscous modes with $s=0$ are straightforward because they are also inviscid ones. Here, we consider other (i.e. $s\neq 0$) inviscid modes and attempt to construct corresponding viscous ones for small Oh in the usual way by adding a boundary layer at the wall. This layer is thin, of thickness $O\left(\mathrm{Oh}^{1/2}\right)$, and is described by the rescaled coordinate $\xi = z/\mathrm{Oh}^{1/2}$. The viscous term in (3.8) can be approximated as

$$\mathrm{Oh}\,\nabla^2\tilde{\mathbf{u}} = \frac{\partial^2\tilde{\mathbf{u}}}{\partial\xi^2} + \mathrm{Oh}\left(\frac{\partial^2\tilde{\mathbf{u}}}{\partial x^2}+\frac{\partial^2\tilde{\mathbf{u}}}{\partial y^2}\right) \sim \frac{\partial^2\tilde{\mathbf{u}}}{\partial\xi^2}, \tag{3.43}$$

hence

$$s\tilde{\mathbf{u}} = -\nabla\tilde{p} + \frac{\partial^2\tilde{\mathbf{u}}}{\partial\xi^2}, \tag{3.44}$$

where s has its inviscid value. (3.9) yields

$$\frac{\partial\tilde{u}_z}{\partial\xi} = -\mathrm{Oh}^{1/2}\left(\frac{\partial\tilde{u}_x}{\partial x}+\frac{\partial\tilde{u}_y}{\partial y}\right), \tag{3.45}$$

which, with $\tilde{u}_z = 0$ at $\xi = 0$, implies $\tilde{u}_z = O\left(\mathrm{Oh}^{1/2}\right)$. Thus, $\tilde{u}_z$ is negligible compared with the other velocity components within the layer. The component of (3.44) normal to the wall gives

$$\frac{\partial\tilde{p}}{\partial\xi} = \mathrm{Oh}^{1/2}\left(\frac{\partial^2\tilde{u}_z}{\partial\xi^2} - s\tilde{u}_z\right) = O\left(\mathrm{Oh}\right). \tag{3.46}$$

The projection of (3.44) parallel to the wall gives

$$s\tilde{\mathbf{u}}_{\|} = -\nabla_{\|}\tilde{p} + \frac{\partial^2\tilde{\mathbf{u}}_{\|}}{\partial\xi^2}, \tag{3.47}$$

where the subscript $\|$ denotes wall-parallel projection. The first two equations of (3.10), which do not appear in the inviscid problem, give

$$\lambda\,\mathrm{Oh}^{-1/2}\frac{\partial\tilde{\mathbf{u}}_{\|}}{\partial\xi} = \tilde{\mathbf{u}}_{\|} \qquad \xi = 0. \tag{3.48}$$

(3.46) indicates that, as usual, the variations of pressure across the boundary layer are asymptotically negligible. Thus, $\nabla_{\|}\tilde{p}$ can be taken to be independent of ξ in (3.47).

Assuming that s does not lie on the negative real axis, the solution of (3.47) which does not grow exponentially with ξ is

$$\tilde{\mathbf{u}}_{\|} = -s^{-1}\nabla_{\|}\tilde{p}\left(1 + Ae^{-s^{1/2}\xi}\right), \tag{3.49}$$

where A is an unknown constant and $s^{1/2}$ is a principal value (so the exponential decays, rather than grows, as ξ increases). Matching to the inviscid flow outside the layer, where the exponential is negligible, $-s^{-1}\nabla_{\|}\tilde{p} = \tilde{\mathbf{u}}_{\|}^{inv}$, where $\tilde{\mathbf{u}}_{\|}^{inv}$ is the wall velocity of the given inviscid mode. Thus,

$$\tilde{\mathbf{u}}_{\|} = \tilde{\mathbf{u}}_{\|}^{inv}\left(1 + Ae^{-s^{1/2}\xi}\right). \tag{3.50}$$

Applying (3.48),

$$A = -\frac{1}{1 + \lambda s^{1/2}\,\mathrm{Oh}^{-1/2}}, \tag{3.51}$$

which, with (3.50), describes the flow inside the boundary layer.

In the strict mathematical limit $\mathrm{Oh} \to 0$, $A \to 0$ according to (3.51), hence the boundary layer disappears and the viscous mode approaches the inviscid one, even close to the wall. However, it is usually the case that λ is small and there are then three distinct regions of small Oh. When $\lambda^2 << \mathrm{Oh} << 1$, $A \sim -1$, which yields a classical boundary layer having $\tilde{\mathbf{u}} = 0$ at the wall. If $\mathrm{Oh} = O\left(\lambda^2\right)$, which would usually be very small, A departs from -1 and the layer is no longer a classical one (there is slip everywhere on the wall, not just near the contact line). Finally, when $\mathrm{Oh} << \lambda^2$, the boundary layer disappears. None of these changes inside the boundary layer affects the flow outside the layer, which approaches the inviscid limit provided only that $\mathrm{Oh} << 1$.

Decaying inviscid modes arise when $\sigma > 0$ and have $s = -\sigma^{1/2}$. Such negative, real values of s were excluded just prior to (3.49) and their treatment is somewhat more subtle. The solution of (3.47) is

$$\tilde{\mathbf{u}}_{\|} = -s^{-1}\nabla_{\|}\tilde{p}\left(1 + B\cos|s|^{1/2}\,\xi + C\sin|s|^{1/2}\,\xi\right) \tag{3.52}$$

and can be seen to have oscillatory, rather than decaying exponential, dependence on ξ unless $B = C = 0$. These oscillations are viscous in origin. (3.48) gives

$$\lambda|s|^{1/2}\,C = \mathrm{Oh}^{1/2}\left(1 + B\right). \tag{3.53}$$

In the strict mathematical limit $\mathrm{Oh} \to 0$, we can choose $B = 0$, hence $C \to 0$ according to (3.53). This eliminates the viscous oscillations with respect to ξ, so we have the inviscid mode as a possible limit, even close to the wall, as for the other types of mode described above. However, if λ and Oh are small and $\mathrm{Oh} \geq O\left(\lambda^2\right)$, (3.53) does not allow both B and C to be small. Thus, despite Oh being small, the viscous mode is not close to the inviscid

one, either near the wall or in the drop interior. It is similar to the modes which retain viscous effects as $\mathrm{Oh} \to 0$ and which were discussed earlier. In any case, because the mode in question is decaying, it is irrelevant from a stability point of view.

4. The 2D drop

This section concerns a toy problem: a two-dimensional drop without gravity, for which everything depends only on the coordinates x and z, for instance the contact angle has the form $\theta_w = \theta_w(x)$. The third dimension is suppressed completely and, although this is hardly a realistic representation of a drop, the intention is to illustrate the previous analyses via a full analytical treatment of stability. Note that, because the perturbation, like the equilibrium, is constrained to be 2D, we are not considering the stability of a 2D drop to 3D perturbations.

The previous analysis of the 3D case carries over to 2D with some minor modifications. The length scale used for nondimensionalisation is now such that the drop area is 1. The velocity vector, $\mathbf{u}$, only has x and z components, so the second equation in (2.3) is dropped, while neglect of gravity means that $\mathbf{G} = 0$. Otherwise, the governing equations are (2.1)-(2.5) as before.

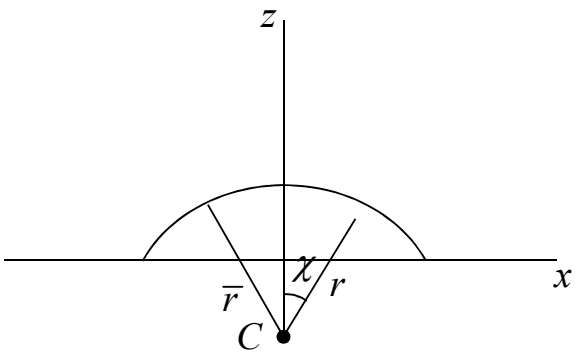

Figure 4: The 2D equilibrium drop interface and associated polar coordinates.

Equilibria are extrema of the surface energy, (2.6), under drop-area preserving perturbations, where the surface integrals in (2.6) are replaced by line integrals and $\tilde{S}_i$ and $\tilde{S}_w$ are now lines in the x-z plane representing the liquid-gas interface and wetted part of the wall. At equilibrium, the interface is a circular arc which meets the wall, $z = 0$, at $x = x_\pm$, where

$$x_+ - x_- = \frac{2\sin\bar{\theta}}{\left(\bar{\theta} - \sin\bar{\theta}\cos\bar{\theta}\right)^{1/2}} \tag{4.1}$$

and

$$\bar{\theta} = \theta_w(x_\pm) \tag{4.2}$$

is the equilibrium contact angle, which has the same value for both contact lines. (4.1) and (4.2) provide three equations for the three unknowns x_+, x_- and $\bar{\theta}$ which determine possible equilibria. The radius of the equilibrium drop is $\bar{r} = \left(\bar{\theta} - \sin\bar{\theta}\cos\bar{\theta}\right)^{-1/2}$. Figure 4 shows the equilibrium interface, which has radius $\bar{r}$ and is centred at C. Polar coordinates, r and χ, are defined as indicated. The equilibrium interface is given by $r = \bar{r}$, $-\bar{\theta} \le \chi \le \bar{\theta}$, while the perturbed interface lies at $r = \bar{r} + \eta(\chi, t)$.

Linearization follows the lines of section 2.1. Thus, (2.2) and (2.9) hold inside the equilibrium drop, while the first and third of equations (2.3) apply at the wall. The interfacial conditions are (2.11) and (2.12), with $\gamma = \bar{r}^{-2}$ and

$$\Delta\eta = \frac{1}{\bar{r}^2}\frac{\partial^2\eta}{\partial\chi^2}. \tag{4.3}$$

The contact-line condition, (2.14), becomes

$$\frac{1}{\bar{r}}\frac{\partial\eta}{\partial\chi} = \pm J_\pm\eta \qquad \chi = \pm\bar{\theta}, \tag{4.4}$$

where

$$J_\pm = \frac{\cot\bar{\theta}}{\bar{r}} \mp \frac{1}{\sin\bar{\theta}}\theta_w'(\bar{x}_\pm) \tag{4.5}$$

and $\theta_w'(x)$ is the derivative of $\theta_w(x)$. Finally, (2.15) becomes

$$\int_{-\bar{\theta}}^{\bar{\theta}} \eta d\chi = 0. \tag{4.6}$$

Based on the linearized problem, an energy equation corresponding to (2.16) can be derived. The volume integrals in (2.16) and (2.17) become surface integrals over the drop interior, while the surface integral in (2.16) becomes a line integral. The 2D equivalent of (2.18) is

$$E_s[\eta] = \frac{1}{2}\left(\frac{1}{\bar{r}}\int_{-\bar{\theta}}^{\bar{\theta}}\left(\left(\frac{\partial\eta}{\partial\chi}\right)^2 - \eta^2\right)d\chi - \left(J_+\eta_+^2 + J_-\eta_-^2\right)\right), \tag{4.7}$$

where $\eta_\pm$ are the contact-line values of η.

Static stability is treated as in section 3.1. Extrema of (4.7) are looked for subject to the contraints (4.6) and the 2D equivalent of (3.1). The result is the eigenvalue problem (4.4), (4.6) and

$$\frac{1}{\bar{r}^2}\left(\frac{d^2\eta}{d\chi^2} + \eta\right) + \zeta\eta + \phi = 0. \tag{4.8}$$

This problem depends on the values of $J_{\pm}$. Thus, the eigenvalues, ζ_n, vary as the $J_{\pm}$ are varied. Let us determine the condition for a zero eigenvalue.

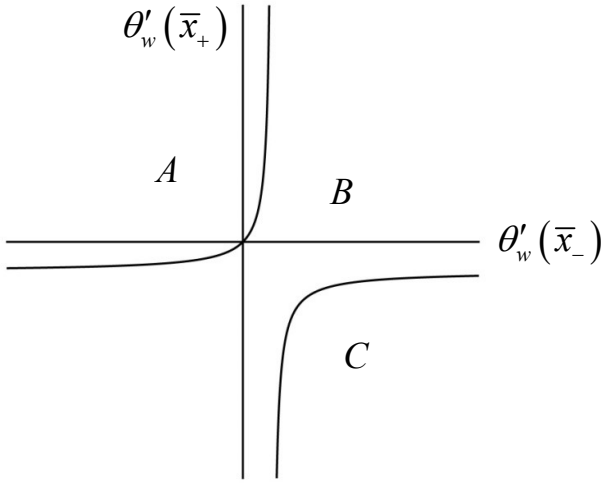

Figure 5: Curves of zero ζ in the $\theta_w'(\bar{x}_-)$-$\theta_w'(\bar{x}_+)$ plane for a fixed value of $\bar{\theta}$.

Setting $\zeta = 0$ in (4.8), its solution is

$$\eta = -\phi\bar{r}^2 + B_1\cos\chi + B_2\sin\chi\,. \tag{4.9}$$

Applying (4.4),

$$\begin{gathered}\mp B_1\sin\bar{\theta} + B_2\cos\bar{\theta} = \\ \left(-\frac{\bar{r}}{\sin\bar{\theta}}\theta_w'(\bar{x}_\pm) \pm \cot\bar{\theta}\right)\left(-\phi\bar{r}^2 + B_1\cos\bar{\theta} \pm B_2\sin\bar{\theta}\right),\end{gathered} \tag{4.10}$$

where (4.5) has been used. (4.6) gives

$$-\phi\bar{r}^2\bar{\theta} + B_1\sin\bar{\theta} = 0\,. \tag{4.11}$$

Using (4.11) to eliminate ϕ, (4.10) becomes

$$\begin{gathered}\left(\bar{\theta} - \sin\bar{\theta}\cos\bar{\theta} \pm \bar{r}\theta_w'(\bar{x}_\pm)\left(\sin\bar{\theta} - \bar{\theta}\cos\bar{\theta}\right)\right)B_1 \\ -\bar{r}\theta_w'(\bar{x}_\pm)\bar{\theta}\sin\bar{\theta}B_2 = 0\end{gathered}, \tag{4.12}$$

which provides two equations for B_1 and B_2. There is a nonzero solution if

$$\begin{gathered}\left(\bar{\theta} - \sin\bar{\theta}\cos\bar{\theta}\right)\left(\theta_w'(\bar{x}_+) - \theta_w'(\bar{x}_-)\right) - \\ 2\bar{r}\left(\sin\bar{\theta} - \bar{\theta}\cos\bar{\theta}\right)\theta_w'(\bar{x}_+)\theta_w'(\bar{x}_-) = 0\end{gathered}, \tag{4.13}$$

which is the condition for a zero eigenvalue of (4.4), (4.6) and (4.8).

Figure 5 shows the curves in the $\theta'_w(\overline{x}_-)$-$\theta'_w(\overline{x}_+)$ plane resulting from (4.13) for a fixed value of $\overline{\theta}$. These curves separate three regions, A, B and C. It can be shown that all the ζ_n are positive in region A, which is thus the statically stable region. One of the ζ_n is negative in region B and two are negative in region C. Thus, the curve forming the boundary A/B represents static neutrality and the equilibrium is statically unstable in regions B and C. Note that there is always an infinity of positive ζ_n, whereas there are at most two negative ones.

Without going into the details, the analyses of sections 3.2 and 3.3 carry through much as for the 3D case. As before, the main conclusion is that the static and dynamic stability characteristics of the equilibrium coincide. Thus, the drop is dynamically stable in region A, neutrally stable on A/B and unstable in regions B and C. There is one growing mode in region B, which is joined by a second in region C.

Note that gravity has been neglected here, so any instability is due to surface energy and inhomogeneity. The latter allows the decrease of the former as the departure from equilibrium increases, leading to static instability.

5. Conclusions

In this paper, we have analysed and compared the static and dynamic stability characteristics of a drop on an inhomogeneous wall. This has been done using a drop model which is rather general because it allows for arbitrary contact-angle distributions, $\theta_w(x,y)$, and also for gravity.

Section 2 describes the model. The liquid flow inside the drop is governed by the incompressible Navier-Stokes equations, whereas the gas outside the drop is supposed inviscid and of constant pressure. The boundary conditions at the liquid/gas interface are the usual ones, namely convection of the interface by the flow and a condition on the interfacial stress involving surface tension, whereas, to allow for the moving contact line, those at the wall are a combination of a Navier slip condition on the wetted area and a prescribed contact angle $\theta_w(x,y)$ at the contact line. Equilibrium can be expressed in two ways. The first uses the governing equations and boundary conditions, assuming zero flow velocity. The second is based on the potential energy, (2.7), which is the sum of superficial and gravitational energies and is an extremum for volume-preserving perturbations to equilibrium. When the equilibrium is a local minimum with respect to such perturbations, it is referred to as statically stable.

Assuming a small perturbation to a given equilibrium, the governing equations and boundary conditions are linearized and an energy equation derived. The difference of potential energies between the perturbed drop and the equilibrium one is given by (2.18) as a second-order functional, $E_s[\eta]$, of the interfacial displacement, η. The equilibrium is statically stable if $E_s[\eta]>0$ for all nonzero η satisfying (2.15), which is the volume preservation condition for small perturbations.

Section 3 is the heart of the paper and is devoted to stability, both static and dynamic, of a given equilibria. Section 3.1 provides a quantification of static stability and instability via an eigenvalue problem, which is independent of the Ohnesorge number, Oh, and the slip length, λ, and quite different from that of the dynamic stability theory which is developed later. The eigenvalues, ζ, are real and determine the nature, statically stable or unstable, of the equilibrium. If $\zeta > 0$ for all eigenvalues, it is statically stable, whereas a single $\zeta < 0$ means it is statically unstable. The other possibility, that there are one or more zero eigenvalues and that the rest are positive, is referred to as static neutrality.

Section 3.2 describes dynamic (to distinguish it from static) stability theory. The equilibrium is said to be dynamically unstable if perturbations grow and stable otherwise. The theory focuses on particular solutions (modes) of the linearized problem which can be complex and have exponential time dependence, e^{st}, where $s = s_r + is_i$. This leads to an eigenvalue problem with eigenvalue s. Eigenvalues with $s_r > 0$ imply a growing mode. As a result, the general perturbation, which is a sum over modes, is also growing. This is dynamic instability and just one mode with $s_r > 0$ is sufficient. It means that the equilibrium is unrealisable in practice. On the other hand, $s_r < 0$ for all modes implies that perturbations decay, i.e. the equilibrium is dynamically stable. Dynamic neutrality means that there are modes with $s_r = 0$, while all others decay.

Section 3.3 concerns the effects of varying Oh, while keeping everything else, in particular the equilibrium, constant. The first, and perhaps most important, result is that the dynamic character, stable, unstable or neutral, of the equilibrium does not change as $\mathrm{Oh} > 0$ varies. Thus, the character of the equilibrium does not depend on Oh and any value of $\mathrm{Oh} > 0$ should give the same result. This leads to consideration of the limit $\mathrm{Oh} \to 0$.

Modes are of two types as $\mathrm{Oh} \to 0$. For some, viscosity remains significant despite the small Ohnesorge number. However, these modes are shown to be decaying and so are unimportant from a stability point of view. On the other hand, others approach an inviscid limit. The limit is described by an inviscid eigenvalue problem with eigenvalue $\sigma = s^2$, which is shown to be real. $\sigma > 0$ implies a growing inviscid mode which corresponds to the limit of a viscous mode which grows for small Oh. On the other hand $\sigma < 0$ results in a pair of oscillatory inviscid modes, which result from the limit of decaying viscous modes as $\mathrm{Oh} \to 0$. Finally, inviscid modes with $\sigma = 0$ correspond precisely with viscous modes having $s = 0$, which are independent of Oh.

Perhaps surprisingly, there is a close relationship between the inviscid eigenvalues σ and static stability/instability/neutrality. It is found that the existence of a positive σ leads to static instability, whereas the equilibrium is statically stable if all σ are negative. Static neutrality arises when one or more σ are zero and the remaining ones negative. Combining these results with the connection, described in the previous paragraph, between the $\mathrm{Oh} \to 0$ limit of viscous modes and the eigenvalues σ, it has been shown that the notions of static

and dynamic stability/instability/neutrality are equivalent at small Oh. Given that the dynamic character, stable, unstable or neutral, of the equilibrium does not change as $\mathrm{Oh} > 0$ varies and that the static character is independent of Oh, this result holds for any value of Oh.

Thus, despite their quite different definitions and eigenvalue problems, dynamic and static stability, instability and neutrality are found to be equivalent for the present problem. This conclusion is far from evident *a priori*, but, as noted earlier, should be of interest to numerical analysts because determination of static stability is considerably less numerically costly than a dynamic stability study, whereas it is the latter which provides a true determination of stability. It would be interesting to know if the conclusions arrived at here remain valid for other models of the moving contact line.

Finally, the way in which viscous modes approach inviscid ones via boundary layers as $\mathrm{Oh} \to 0$ was addressed towards the end of section 3.3 and the case of a 2D drop was the subject of section 4. Although the 2D case is hardly realistic, it provides an illustration of the 3D methodology for which full analytical results concerning stability are derived.

Among the open questions related to the present work are the following. Do the conclusions arrived at here, that the static and dynamic stability characteristics of equilibria coincide, hold for other contact-line models? Secondly, to make the present results useful, a numerical method for obtaining statically stable equilibria which does not rely on solution of the full 3D dynamical problem is needed. Such a method might involve finding local minima of the potential energy, (2.7), subject to fixed drop volume. One possibility is the "basin-hopping" algorithm (see e.g. [21]).

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Appendix A: Mathematical details

A.1 Derivation of (2.12)

The normal unit vector, $\mathbf{n}$, is originally only defined on the perturbed interface, $d=\eta$, but can be extended using

$$\mathbf{n}=\frac{\nabla F}{\left|\nabla F\right|}. \tag{A.1}$$

The interface curvature follows from

$$\kappa=\nabla\mathbf{.n} \tag{A.2}$$

evaluated at the interface. Since $F=d-\eta$, $\left|\nabla d\right|=1$ and $\nabla d\mathbf{.}\nabla\eta=0$,

$$\left|\nabla F\right|^2=1+\left|\nabla\eta\right|^2. \tag{A.3}$$

The second term on the right-hand side being of second order, it is neglected, hence $\mathbf{n}=\nabla F$ according to (A.1). Thus, (A.2) gives

$$\begin{aligned}\kappa&=\nabla^2F\\&=\nabla^2d-\nabla^2\eta\end{aligned} \tag{A.4}$$

correct to first order when evaluated at the interface.

For the equilibrium, we set $\eta=0$ in (A.4) and $\mathbf{u}=0$ in (2.5) to derive

$$p_e\big|_{d=0}=p_a+\nabla^2d\big|_{d=0} \tag{A.5}$$

for the equilibrium pressure, p_e, at the equilibrium interface, $d=0$. Making the small displacement $\eta\mathbf{n}$ to arrive at the perturbed interface and given that $p_e-\mathbf{G.x}$ is constant,

$$\begin{aligned}p_e\big|_{d=\eta}&=p_a+\nabla^2d\big|_{d=0}+\eta\mathbf{G.n}\\&=p_a+\nabla^2d\big|_{d=\eta}+\gamma\eta\end{aligned}\ , \tag{A.6}$$

correct to first order, where

$$\gamma=\mathbf{n.}\left(\mathbf{G}-\nabla\left(\nabla^2d\right)\right). \tag{A.7}$$

On the other hand, applying (2.5) and (A.4) to the perturbed interface

$$\begin{aligned}\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{u}+\left(\nabla\mathbf{u}\right)^{T}\right) &= \mathbf{n}\left(p\big|_{d=\eta}-p_a-\nabla^2 d\big|_{d=\eta}+\nabla^2\eta\right)\\ &= \mathbf{n}\left(p'\big|_{d=\eta}+p_e\big|_{d=\eta}-p_a-\nabla^2 d\big|_{d=\eta}+\nabla^2\eta\right)\end{aligned}. \tag{A.8}$$

Using (A.6),

$$\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{u}+\left(\nabla\mathbf{u}\right)^{T}\right)=\mathbf{n}\left(\nabla^2\eta+\gamma\eta+p'\right). \tag{A.9}$$

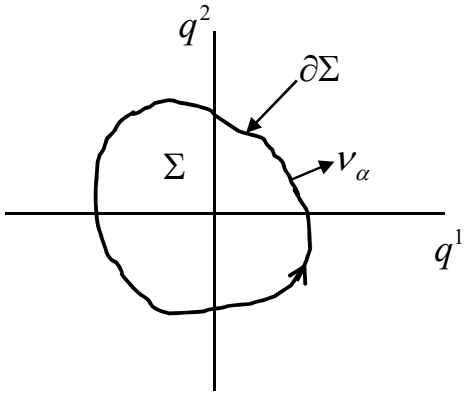

Figure A.1: The region Σ and its boundary $\partial\Sigma$ in the q^α plane.

It remains to express the term $\nabla^2\eta$ in terms of $\eta\left(q^\alpha\right)$. To this end, let Σ be a region in the q^α plane representing part of the equilibrium interface and V the volume in physical space for which $q^\alpha\in\Sigma$ and $0<d<\delta$, where δ is an infinitesimal constant (see figures A.1 and A.2). The divergence theorem gives

$$\int_V \nabla^2\eta dv=\int_{\partial V}\frac{\partial\eta}{\partial n}dS\,. \tag{A.10}$$

The normal vectors of the parts of ∂V with constant d are directed parallel to ∇d. Since $\nabla d.\nabla\eta=0$, $\partial\eta/\partial n=0$. Thus, these parts of ∂V do not contribute to (A.10). The remainder of ∂V gives

$$\delta\oint_{\partial S}\mathbf{N}.\nabla\eta ds \tag{A.11}$$

for the right-hand side of (A.10), where ∂S is the curve on the equilibrium interface corresponding to the boundary $\partial\Sigma$ in the q^α plane, ds is elementary arc length along ∂S and $\mathbf{N}$ is the unit vector, tangential to the equilibrium interface and orthogonal to ∂S, which is directed outwards from S (see figure A.2). Thus, (A.10) implies

$$\int_S \nabla^2\eta dS=\oint_{\partial S}\mathbf{N}.\nabla\eta ds\,, \tag{A.12}$$

where S corresponds to Σ.

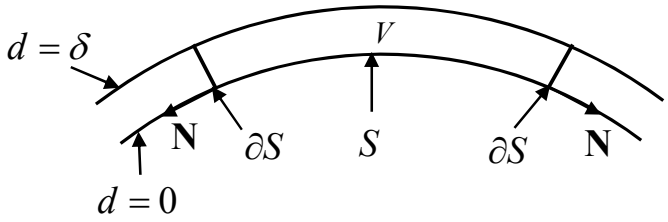

Figure A.2: The volume V, surface S, boundary, ∂S, of S, and unit vector $\mathbf{N}$. The latter is tangential to the equilibrium interface, orthogonal to ∂S and directed outwards from S.

From here until the end of this section, we restrict attention to the equilibrium interface. A point on the interface has position vector $\mathbf{x}\left(q^{\alpha}\right)$, whose derivatives, $\partial\mathbf{x}/\partial q^{1}$ and $\partial\mathbf{x}/\partial q^{2}$, are tangential to the interface and yield the metric tensor via

$$g_{\alpha\beta} = \frac{\partial\mathbf{x}}{\partial q^{\alpha}} . \frac{\partial\mathbf{x}}{\partial q^{\beta}} . \tag{A.13}$$

The components of this tensor form a symmetric, positive-definite matrix. Since $\mathbf{N}$ is tangential and $\partial\mathbf{x}/\partial q^{1}$, $\partial\mathbf{x}/\partial q^{2}$ span the space of such vectors,

$$\mathbf{N} = \mu^{\alpha}\frac{\partial\mathbf{x}}{\partial q^{\alpha}} . \tag{A.14}$$

It is convenient to define $\nu_{\alpha} = g_{\alpha\beta}\mu^{\beta}$, hence $\mu^{\alpha} = g^{\alpha\beta}\nu_{\beta}$, where $g^{\alpha\beta}$ is the inverse of the matrix $g_{\alpha\beta}$. Thus,

$$\mathbf{N} = \nu_{\alpha} g^{\alpha\beta}\frac{\partial\mathbf{x}}{\partial q^{\beta}} . \tag{A.15}$$

$\mathbf{N}.\mathbf{N} = 1$, (A.13), (A.15) and the definition of $g^{\alpha\beta}$ imply

$$g^{\alpha\beta}\nu_{\alpha}\nu_{\beta} = 1 . \tag{A.16}$$

An infinitesimal displacement dq^{α} in the q^{α} plane produces the displacement $d\mathbf{x} = dq^{\alpha}\partial\mathbf{x}/\partial q^{\alpha}$ on S_i, hence

$$dq^{\alpha} = d\mathbf{x}.\nabla q^{\alpha} = dq^{\beta}\frac{\partial\mathbf{x}}{\partial q_{\beta}}.\nabla q^{\alpha} . \tag{A.17}$$

Since (A.17) holds for any choice of dq^{α},

$$\frac{\partial \mathbf{x}}{\partial q^\alpha}.\nabla q^\beta = \delta_\alpha^\beta, \tag{A.18}$$

where δ_α^β is the Kronecker delta. Writing

$$\nabla \eta = \frac{\partial \eta}{\partial q^\alpha} \nabla q^\alpha \tag{A.19}$$

and using (A.15) and (A.18),

$$\mathbf{N}.\nabla \eta = \nu_\alpha g^{\alpha\beta} \frac{\partial \eta}{\partial q^\beta}. \tag{A.20}$$

Let dq^α represent an infinitesimal displacement along the curve $\partial\Sigma$, in the sense, anticlockwise, indicated by the arrow in figure A.1. The corresponding displacement, $d\mathbf{x} = dq^\alpha \partial \mathbf{x} / \partial q^\alpha$, along ∂S is perpendicular to $\mathbf{N}$, hence, using (A.13), (A.15) and the definition of $g^{\alpha\beta}$,

$$\nu_\alpha dq^\alpha = 0. \tag{A.21}$$

This result shows that ν_α provides a normal vector to $\partial\Sigma$ in the q^α plane, as indicated in figure A.1, and implies

$$dq^1 = -\tau \nu_2, \quad dq^2 = \tau \nu_1, \tag{A.22}$$

where $\tau \neq 0$ is infinitesimal. Thus, the vector $b_\alpha = \tau \nu_\alpha$ has components $b_1 = dq^2$ and $b_2 = -dq^1$. Given that the displacement along $\partial\Sigma$ is anticlockwise, b_α, which is normal to $\partial\Sigma$, is directed outwards from Σ. Next, consider the infinitesimal displacement b_α in the q^α plane. This produces $d\mathbf{x} = b_\alpha \partial \mathbf{x} / \partial q^\alpha$ in physical space. (A.15) and $b_\alpha = \tau \nu_\alpha$ give

$$\mathbf{N} = \tau^{-1} b_\alpha g^{\alpha\beta} \frac{\partial \mathbf{x}}{\partial q^\beta}, \tag{A.23}$$

hence

$$\begin{aligned} d\mathbf{x}.\mathbf{N} &= \tau^{-1} b_\alpha b_\gamma g^{\alpha\beta} \frac{\partial \mathbf{x}}{\partial q^\beta} . \frac{\partial \mathbf{x}}{\partial q^\gamma} \\ &= \tau^{-1} b_\alpha b_\gamma g^{\alpha\beta} g_{\beta\gamma} \\ &= \tau^{-1} b_\alpha b_\gamma \delta_\gamma^\alpha \\ &= \tau^{-1} b_\alpha b_\alpha \end{aligned} . \tag{A.24}$$

Because b_α is directed outwards from Σ, $d\mathbf{x}$ takes us from ∂S to a location on S_i just outside S. By definition, $\mathbf{N}$ is a normal vector directed outwards from S. Thus, $d\mathbf{x}.\mathbf{N} > 0$,

hence, since $b_\alpha b_\alpha > 0$, (A.24) gives $\tau > 0$. Given that b_α is directed outwards from Σ, $\nu_\alpha = \tau^{-1} b_\alpha$ is an outwardly directed normal vector to $\partial\Sigma$, as indicated in figure A.1.

The components of $g^{\alpha\beta}$ are

$$g^{11} = g^{-1} g_{22}, \quad g^{22} = g^{-1} g_{11}, \quad g^{12} = g^{21} = -g^{-1} g_{12}, \tag{A.25}$$

where g is the determinant of the matrix $g_{\alpha\beta}$, which is positive because $g_{\alpha\beta}$ is positive definite. (A.16), (A.22), (A.25) and $\tau > 0$ imply

$$\tau = \left(\frac{g_{\alpha\beta} dq^\alpha dq^\beta}{g} \right)^{1/2}. \tag{A.26}$$

Since $ds = \left(g_{\alpha\beta} dq^\alpha dq^\beta \right)^{1/2}$, (A.20), (A.22) and (A.26) give

$$\oint_{\partial S} \mathbf{N}.\nabla \eta ds = \oint_{\partial\Sigma} \sqrt{g} \frac{\partial \eta}{\partial q^\beta} \left(g^{1\beta} dq^2 - g^{2\beta} dq^1 \right). \tag{A.27}$$

Using Green's theorem,

$$\oint_{\partial S} \mathbf{N}.\nabla \eta ds = \int_\Sigma \frac{\partial}{\partial q^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial \eta}{\partial q^\beta} \right) dq^1 dq^2 . \tag{A.28}$$

Consider a rectangular area element, $dq^1 dq^2$, in the q^α plane. This corresponds to a small parallelogram on S with sides $\mathbf{a}_1 dq^1$ and $\mathbf{a}_2 dq^2$, where $\mathbf{a}_\alpha = \partial \mathbf{x} / \partial q^\alpha$. The area of this parallelogram is $dS = \left| \mathbf{a}_1 \times \mathbf{a}_2 \right| dq^1 dq^2 = \left(\left| \mathbf{a}_1 \right|^2 \left| \mathbf{a}_2 \right|^2 - \left(\mathbf{a}_1 . \mathbf{a}_2 \right)^2 \right)^{1/2} dq^1 dq^2$ according to the Lagrange identity. Employing (A.13), $\left| \mathbf{a}_1 \right|^2 \left| \mathbf{a}_2 \right|^2 - \left(\mathbf{a}_1 . \mathbf{a}_2 \right)^2 = g_{11} g_{22} - g_{12}^2 = g$, hence $dS = \sqrt{g} dq^1 dq^2$, which is the relation between elementary areas in physical space and the q^α plane. Using this result and (A.28), (A.12) yields

$$\int_\Sigma \left(\sqrt{g} \nabla^2 \eta - \frac{\partial}{\partial q^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial \eta}{\partial q^\beta} \right) \right) dq^1 dq^2 = 0. \tag{A.29}$$

Finally, letting Σ shrink down to approach a point, $\nabla^2 \eta = \Delta \eta$, hence (A.9) gives (2.12).

A.2 Derivation of (2.14)

The condition that the liquid/gas interface meets the wall at angle $\theta_w(x, y)$ is

$$n_z = \cos \theta_w \tag{A.30}$$

at the contact line, where n_z is the z-component of $\mathbf{n}$. As we saw earlier, $\mathbf{n} = \nabla d - \nabla \eta$ correct to first order in the perturbation. Thus,

$$n_z = \frac{\partial d}{\partial z} - \frac{\partial \eta}{\partial z}. \tag{A.31}$$

Let (x, y) be a point on the equilibrium contact line and $\bar{\theta} = \theta_w(x, y)$ the associated contact angle (see figure A.3). Setting $\eta = 0$ in (A.31), (A.30) gives

$$\frac{\partial d}{\partial z}(x, y, z = 0) = \cos\bar{\theta}. \tag{A.32}$$

Since $d(x, y, z = 0) = 0$, the point $(x + dx, y + dy)$ lies on the perturbed contact line, $d(x, y, z = 0) = \eta$, provided

$$\frac{\partial d}{\partial x} dx + \frac{\partial d}{\partial y} dy = \eta. \tag{A.33}$$

Applying (A.30) and (A.31) at $(x + dx, y + dy)$,

$$\frac{\partial d}{\partial z}(x + dx, y + dy, z = 0) - \frac{\partial \eta}{\partial z} = \cos\theta_w(x + dx, y + dy). \tag{A.34}$$

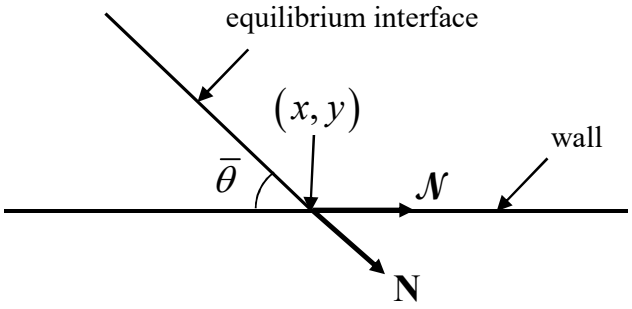

Figure A.3: The equilibrium interface close to its contact line, which intersects the plane of the figure at right angles. $\mathbf{N}$ and $\mathcal{N}$ are unit vectors normal to the contact line and directed outwards from the drop. The former is tangential to S_i, the latter to the wall.

Subtracting (A.32) and recalling that $\bar{\theta} = \theta_w(x, y)$,

$$\frac{\partial^2 d}{\partial x \partial z} dx + \frac{\partial^2 d}{\partial y \partial z} dy - \frac{\partial \eta}{\partial z} = -\sin\bar{\theta}\left(\frac{\partial \theta_w}{\partial x} dx + \frac{\partial \theta_w}{\partial y} dy\right). \tag{A.35}$$

Taking the displacement (dx, dy) perpendicular to the equilibrium contact line, it lies in the direction of the vector $\left(\frac{\partial d}{\partial x}, \frac{\partial d}{\partial y}\right)$ and (A.33) implies

$$(dx, dy) = \eta \left(\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 \right)^{-1} \left(\frac{\partial d}{\partial x}, \frac{\partial d}{\partial y}\right). \tag{A.36}$$

With this displacement,

$$\frac{\partial^2 d}{\partial x \partial z} dx + \frac{\partial^2 d}{\partial y \partial z} dy = \frac{1}{2} \eta \left(\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 \right)^{-1} \frac{\partial}{\partial z} \left(\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 \right) \tag{A.37}$$

and

$$\frac{\partial \theta_w}{\partial x} dx + \frac{\partial \theta_w}{\partial y} dy = \eta \left(\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 \right)^{-1/2} \boldsymbol{\mathcal{N}} . \nabla \theta_w , \tag{A.38}$$

where

$$\boldsymbol{\mathcal{N}} = \left(\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 \right)^{-1/2} \left(\frac{\partial d}{\partial x}, \frac{\partial d}{\partial y}\right) \tag{A.39}$$

is a unit vector, normal to the equilibrium contact line and tangential to the wall, which is directed outwards from the wetted region (see figure A.3). Thus, $\boldsymbol{\mathcal{N}} . \nabla \theta_w$ is the normal derivative of $\theta_w(x, y)$ at the contact line. Since $|\nabla d| = 1$,

$$\left(\frac{\partial d}{\partial x}\right)^2 + \left(\frac{\partial d}{\partial y}\right)^2 = 1 - \left(\frac{\partial d}{\partial z}\right)^2 , \tag{A.40}$$

hence (A.37) and (A.38) give

$$\frac{\partial^2 d}{\partial x \partial z} dx + \frac{\partial^2 d}{\partial y \partial z} dy = -\eta \left(1 - \left(\frac{\partial d}{\partial z}\right)^2 \right)^{-1} \frac{\partial d}{\partial z} \frac{\partial^2 d}{\partial z^2} , \tag{A.41}$$

$$\frac{\partial \theta_w}{\partial x} dx + \frac{\partial \theta_w}{\partial y} dy = \eta \left(1 - \left(\frac{\partial d}{\partial z}\right)^2 \right)^{-1/2} \boldsymbol{\mathcal{N}} . \nabla \theta_w . \tag{A.42}$$

Using (A.32) and (A.35),

$$\frac{\partial \eta}{\partial z} = \eta\left(\boldsymbol{\mathcal{N}}.\nabla \theta_w - \frac{\cos\overline{\theta}}{\sin^2\overline{\theta}} \frac{\partial^2 d}{\partial z^2} \right). \tag{A.43}$$

Let $\mathbf{N}$ be the unit vector shown in figure A.3, which is tangential to the equilibrium interface and normal to the contact line. Keeping (x, y) constant, $d\eta = dz\partial\eta/\partial z$ is the change in η for the increment dz. $\nabla d.\nabla\eta = 0$ implies that the component of $\nabla\eta$ normal to the interface is zero. Thus, only the component of displacement, $-dz\mathbf{N}\sin\overline{\theta}$, parallel to the interface produces a change in η, hence $d\eta = -dz\sin\overline{\theta}\mathbf{N}.\nabla\eta$. It follows that $\partial\eta/\partial z = -\sin\overline{\theta}\mathbf{N}.\nabla\eta$ so (A.43) gives

$$\mathbf{N}.\nabla\eta = J\eta \tag{A.44}$$

as the contact-line condition, where

$$J = \frac{1}{\sin\overline{\theta}}\left(\frac{\cos\overline{\theta}}{\sin^2\overline{\theta}} \frac{\partial^2 d}{\partial z^2} - \boldsymbol{\mathcal{N}}.\nabla\theta_w \right). \tag{A.45}$$

Taking the surface S, used in section A.1 and shown in figure A.2, to be the entire equilibrium interface, ∂S is the contact line and $\mathbf{N}$ is the vector defined above. Using (A.20), (A.44) yields (2.14). Note that, as stated following (2.14), its left-hand side is the derivative, $\mathbf{N}.\nabla\eta$, of η, taken tangential to S_i, normal to the contact line and outwards from S_i.

A.3 A frequently used identity

The following identity will often be used in subsequent sections:

$$\int_{S_i} f_1 \Delta f_2 dS = \oint_C f_1 g^{\alpha\beta} \nu_\alpha \frac{\partial f_2}{\partial q^\beta} ds - \int_{S_i} g^{\alpha\beta} \frac{\partial f_1}{\partial q^\alpha} \frac{\partial f_2}{\partial q^\beta} dS, \tag{A.46}$$

where f_1 and f_2 are any functions defined on the equilibrium liquid/gas interface S_i and C is the equilibrium contact line. This identity can be derived as follows. Using $dS = \sqrt{g}\,dq^1 dq^2$ and (2.13),

$$\int_{S_i} f_1 \Delta f_2 dS = \int_{\Sigma_i} f_1 \frac{\partial}{\partial q^\alpha}\left(\sqrt{g} g^{\alpha\beta} \frac{\partial f_2}{\partial q^\beta} \right) dq^1 dq^2, \tag{A.47}$$

where Σ_i represents the entire equilibrium interface in the q^α plane. Writing

$$f_1 \frac{\partial}{\partial q^\alpha}\left(\sqrt{g} g^{\alpha\beta} \frac{\partial f_2}{\partial q^\beta} \right) = \frac{\partial}{\partial q^\alpha}\left(f_1 \sqrt{g} g^{\alpha\beta} \frac{\partial f_2}{\partial q^\beta} \right) - \sqrt{g} g^{\alpha\beta} \frac{\partial f_1}{\partial q^\alpha} \frac{\partial f_2}{\partial q^\beta}, \tag{A.48}$$

the first term can be treated using the two-dimensional divergence theorem in the q^α plane:

$$\int_{\Sigma_i} \frac{\partial}{\partial q^\alpha}\left(f_1\sqrt{g}g^{\alpha\beta}\frac{\partial f_2}{\partial q^\beta}\right) dq^1dq^2 = \oint_{\partial\Sigma_i} f_1\sqrt{g}g^{\alpha\beta}\frac{\partial f_2}{\partial q^\beta}\frac{\nu_\alpha}{\sqrt{\nu_\delta\nu_\delta}}ds_q\,, \qquad \text{(A.49)}$$

where ds_q is elementary arc length on $\partial\Sigma_i$. Transforming to physical space, $\partial\Sigma_i$ becomes the equilibrium contact line, C, and it can be shown (using (A.22), (A.26) and $ds = \left(g_{\alpha\beta}dq^\alpha dq^\beta\right)^{1/2}$) that $\sqrt{g}ds_q / \sqrt{\nu_\delta\nu_\delta} = ds$, where ds is elementary arc length on C. Thus,

$$\oint_{\partial\Sigma_i} f_1\sqrt{g}g^{\alpha\beta}\frac{\partial f_2}{\partial q^\beta}\frac{\nu_\alpha}{\sqrt{\lambda_\delta\lambda_\delta}}ds_q = \oint_C f_1 g^{\alpha\beta}\nu_\alpha\frac{\partial f_2}{\partial q^\beta}ds\,. \qquad \text{(A.50)}$$

Again using $dS = \sqrt{g}dq^1dq^2$,

$$\int_{\Sigma_i}\sqrt{g}g^{\alpha\beta}\frac{\partial f_1}{\partial q^\alpha}\frac{\partial f_2}{\partial q^\beta}dq^1dq^2 = \int_{S_i} g^{\alpha\beta}\frac{\partial f_1}{\partial q^\alpha}\frac{\partial f_2}{\partial q^\beta}dS\,. \qquad \text{(A.51)}$$

(A.47)-(A.51) give (A.46).

An application of (A.46) is the following. Let $\tilde{\eta}$ be a function, possibly complex, on S_i such that (3.13) holds on C. It follows from (3.16) and (A.46) that

$$\int_{S_i}\tilde{\eta}\left(\Delta\tilde{\eta}^* + \gamma\tilde{\eta}^*\right)dS = -\tilde{E}_s\left[\tilde{\eta}\right], \qquad \text{(A.52)}$$

where * denotes complex conjugation.

A.4 Derivation of the energy equation, (2.16)

Writing $\boldsymbol{\sigma} = -p'\mathbf{I} + 2\,\mathrm{Oh}\,\mathbf{e}$, where $\mathbf{I}$ is the identity tensor and $\mathbf{e} = \left(\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^T\right)/2$ the strain-rate tensor, $\boldsymbol{\sigma}$ is the perturbation of the stress tensor. (2.2) and (2.9) give

$$\frac{\partial\mathbf{u}}{\partial t} = \nabla\mathbf{.}\boldsymbol{\sigma}\,, \qquad \text{(A.53)}$$

while (2.12) implies

$$\boldsymbol{\sigma}\mathbf{.n} = \mathbf{n}\left(\Delta\eta + \gamma\eta\right) \qquad \text{(A.54)}$$

on the interface. Employing (2.2) and the definitions of $\boldsymbol{\sigma}$ and $\mathbf{e}$, the identity $\mathbf{u}\mathbf{.}\left(\nabla\mathbf{.}\boldsymbol{\sigma}\right) = \nabla\mathbf{.}\left(\mathbf{u}\mathbf{.}\boldsymbol{\sigma}\right) - \boldsymbol{\sigma}\mathbf{:}\nabla\mathbf{u}$ yields $\mathbf{u}\mathbf{.}\left(\nabla\mathbf{.}\boldsymbol{\sigma}\right) = \nabla\mathbf{.}\left(\mathbf{u}\mathbf{.}\boldsymbol{\sigma}\right) - 2\,\mathrm{Oh}\,\mathbf{e}\mathbf{:}\mathbf{e}$. Using this result, scalar multiplying (A.53) by $\mathbf{u}$ and integrating over the drop, D, the divergence theorem leads to

$$\frac{1}{2}\frac{d}{dt}\int_D |\mathbf{u}|^2\,dv = \int_{\partial D}\mathbf{u}\mathbf{.}\boldsymbol{\sigma}\mathbf{.n}dS - 2\,\mathrm{Oh}\int_D \mathbf{e}\mathbf{:}\mathbf{e}dv\,. \qquad \text{(A.55)}$$

(2.3) and the definitions of $\boldsymbol{\sigma}$ and $\mathbf{e}$ give

$$\begin{aligned}\int_{S_w} \mathbf{u}.\boldsymbol{\sigma}.\mathbf{n}\,dS &= 2\,\mathrm{Oh}\int_{S_w} \mathbf{u}.\mathbf{e}.\mathbf{n}\,dS \\ &= -2\,\mathrm{Oh}\int_{S_w}\left(u_x e_{xz} + u_y e_{yz}\right)dS \\ &= -\,\mathrm{Oh}\int_{S_w}\left(u_x\frac{\partial u_x}{\partial z} + u_y\frac{\partial u_y}{\partial z}\right)dS \\ &= -\lambda\,\mathrm{Oh}\int_{S_w}\left(\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2\right)dS\end{aligned} \quad . \tag{A.56}$$

as the wall contribution to the surface integral in (A.55). Note that S_w is the equilibrium wetted area of the wall.

The interfacial contribution to (A.55) follows from (2.11) and (A.54) as

$$\int_{S_i} \mathbf{u}.\boldsymbol{\sigma}.\mathbf{n}\,dS = \int_{S_i}\frac{\partial \eta}{\partial t}\left(\Delta\eta + \gamma\eta\right)dS\,. \tag{A.57}$$

Applying (A.46),

$$\int_{S_i}\frac{\partial \eta}{\partial t}\Delta\eta\,dS = \oint_C \frac{\partial \eta}{\partial t}g^{\alpha\beta}\nu_\alpha\frac{\partial \eta}{\partial q^\beta}\,ds - \int_{S_i} g^{\alpha\beta}\frac{\partial^2\eta}{\partial q^\alpha \partial t}\frac{\partial \eta}{\partial q^\beta}\,dS\,. \tag{A.58}$$

Using (2.14),

$$\begin{aligned}\oint_C \frac{\partial \eta}{\partial t}g^{\alpha\beta}\nu_\alpha\frac{\partial \eta}{\partial q^\beta}\,ds &= \oint_C J\frac{\partial \eta}{\partial t}\eta\,ds \\ &= \frac{1}{2}\frac{d}{dt}\oint_C J\eta^2\,ds\end{aligned} \quad . \tag{A.59}$$

Symmetry of $g^{\alpha\beta}$ implies

$$\int_{S_i} g^{\alpha\beta}\frac{\partial^2\eta}{\partial q^\alpha \partial t}\frac{\partial \eta}{\partial q^\beta}\,dS = \frac{1}{2}\frac{d}{dt}\int_{S_i} g^{\alpha\beta}\frac{\partial \eta}{\partial q^\alpha}\frac{\partial \eta}{\partial q^\beta}\,dS\,. \tag{A.60}$$

Finally,

$$\int_{S_i}\frac{\partial \eta}{\partial t}\gamma\eta\,dS = \frac{1}{2}\frac{d}{dt}\int_{S_i}\gamma\eta^2\,dS\,. \tag{A.61}$$

(A.57)-(A.61) give

$$\int_{S_i} \mathbf{u}.\boldsymbol{\sigma}.\mathbf{n}\,dS = -\frac{dE_s}{dt} \tag{A.62}$$

as the interfacial contribution to (A.55), where E_s is given by (2.18). Combining (A.55), (A.56) and (A.62) yields (2.16).

A.5 Derivation of (3.2) and (3.3)

Using Lagrange multipliers for the constraints (3.1), we look for extrema of

$$Q[\omega]=\frac{1}{2}\left(\int_{S_i}\left(g^{\alpha\beta}\frac{\partial\omega}{\partial q^\alpha}\frac{\partial\omega}{\partial q^\beta}-(\gamma+\zeta)\omega^2-2\phi\omega\right)dS-\oint_C J\omega^2 ds\right). \qquad \text{(A.63)}$$

Symmetry of $g^{\alpha\beta}$ implies

$$\delta Q=\int_{S_i}\left(g^{\alpha\beta}\frac{\partial\delta\omega}{\partial q^\alpha}\frac{\partial\omega}{\partial q^\beta}-\delta\omega\left((\gamma+\zeta)\omega+\phi\right)\right)dS-\oint_C J\omega\delta\omega ds \qquad \text{(A.64)}$$

for the variation of $Q[\omega]$ due to the infinitesimal variation, $\delta\omega$, of ω. Using (A.46),

$$\delta Q=-\int_{S_i}\delta\omega\left(\Delta\omega+(\gamma+\zeta)\omega+\phi\right)dS+\oint_C\delta\omega\left(\nu_\alpha g^{\alpha\beta}\frac{\partial\omega}{\partial q^\beta}-J\omega\right)ds\,. \qquad \text{(A.65)}$$

The condition for an extremum is $\delta Q=0$ for any $\delta\omega$. Thus, we obtain (3.2) and (3.3). The constraints (3.1) also need to be imposed. Of these, the first contributes to the eigenvalue problem, whereas the second provides a normalisation condition.

A.6 Derivation of (3.4) and (3.6)

The first of equations (3.1), together with (3.2) and (3.3), give

$$\int_{S_i}\omega_n dS=0\,, \qquad \text{(A.66)}$$

$$\Delta\omega_n+(\gamma+\zeta_n)\omega_n+\phi_n=0 \qquad \text{(A.67)}$$

on S_i and

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\omega_n}{\partial q^\beta}=J\omega_n \qquad \text{(A.68)}$$

on C. Multiplying (A.67) by ω_m, integrating over S_i and using (A.66) with n replaced by m,

$$\int_{S_i}\omega_m\left(\Delta\omega_n+(\gamma+\zeta_n)\omega_n\right)dS=0\,. \qquad \text{(A.69)}$$

Using (A.46),

$$\int_{S_i}\left(\left(\gamma+\zeta_n\right)\omega_n\omega_m - g^{\alpha\beta}\frac{\partial\omega_n}{\partial q^\alpha}\frac{\partial\omega_m}{\partial q^\beta}\right)dS + \oint_C J\omega_n\omega_m ds = 0\,. \tag{A.70}$$

Permuting n and m and subtracting, symmetry of $g^{\alpha\beta}$ yields

$$\left(\zeta_n-\zeta_m\right)\int_{S_i}\omega_n\omega_m dS = 0\,. \tag{A.71}$$

If $\zeta_n \neq \zeta_m$, (A.71) gives orthogonality of ω_n and ω_m. In the case of a degenerate eigenvalue, the associated eigenfunctions can be orthogonalised, so (3.4) holds for all $n \neq m$. It also applies when $n = m$, thanks to the normalisation resulting from the second equation of (3.1).

Using (3.5) in (2.18),

$$E_s\left[\eta\right] = \frac{1}{2}\sum_{n,m}E_{nm}c_n c_m\,, \tag{A.72}$$

where

$$E_{nm} = \int_{S_i}\left(g^{\alpha\beta}\frac{\partial\omega_n}{\partial q^\alpha}\frac{\partial\omega_m}{\partial q^\beta} - \gamma\omega_n\omega_m\right)dS - \oint_C J\omega_n\omega_m ds\,. \tag{A.73}$$

Employing (3.4) and (A.70),

$$E_{nm} = \zeta_n\int_{S_i}\omega_n\omega_m dS = \zeta_n\delta_{nm}\,, \tag{A.74}$$

hence (A.72) gives (3.6).

A.7 Derivation of (3.15)

Let $\tilde{\boldsymbol{\sigma}} = -\tilde{p}\mathbf{I} + 2\,\mathrm{Oh}\,\tilde{\mathbf{e}}$, where $\tilde{\mathbf{e}} = \left(\nabla\tilde{\mathbf{u}} + \left(\nabla\tilde{\mathbf{u}}\right)^T\right)/2$. (3.8) and (3.9) imply

$$s\tilde{\mathbf{u}} = \nabla.\tilde{\boldsymbol{\sigma}}\,, \tag{A.75}$$

while (3.12) gives

$$\tilde{\boldsymbol{\sigma}}.\mathbf{n} = \mathbf{n}\left(\Delta\tilde{\eta} + \gamma\tilde{\eta}\right) \tag{A.76}$$

on the interface. Taking the complex conjugate of (A.75), scalar multiplying by $\tilde{\mathbf{u}}$ and integrating over the drop D, $\tilde{\mathbf{u}}.\left(\nabla.\tilde{\boldsymbol{\sigma}}^*\right) = \nabla.\left(\tilde{\mathbf{u}}.\tilde{\boldsymbol{\sigma}}^*\right) - 2\,\mathrm{Oh}\,\tilde{\mathbf{e}}:\tilde{\mathbf{e}}^*$ and the divergence theorem lead to

$$s^*\int_D\left|\tilde{\mathbf{u}}\right|^2 dv = \int_{\partial D}\tilde{\mathbf{u}}.\tilde{\boldsymbol{\sigma}}^*.\mathbf{n}dS - 2\,\mathrm{Oh}\int_D\tilde{\mathbf{e}}:\tilde{\mathbf{e}}^* dv\,. \tag{A.77}$$

(3.10) and the definitions of $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\mathbf{e}}$ give

$$\begin{aligned}\int_{S_w} \tilde{\mathbf{u}}.\tilde{\boldsymbol{\sigma}}^*.\mathbf{n}dS &= 2\,\mathrm{Oh}\int_{S_w} \tilde{\mathbf{u}}.\tilde{\mathbf{e}}^*.\mathbf{n}dS \\ &= -2\,\mathrm{Oh}\int_{S_w}\left(\tilde{u}_x\tilde{e}^*_{xz}+\tilde{u}_y\tilde{e}^*_{yz}\right)dS \\ &= -\,\mathrm{Oh}\int_{S_w}\left(\tilde{u}_x\frac{\partial\tilde{u}^*_x}{\partial z}+\tilde{u}_y\frac{\partial\tilde{u}^*_y}{\partial z}\right)dS \\ &= -\lambda\,\mathrm{Oh}\int_{S_w}\left(\left|\frac{\partial\tilde{u}_x}{\partial z}\right|^2+\left|\frac{\partial\tilde{u}_y}{\partial z}\right|^2\right)dS\end{aligned} \tag{A.78}$$

as the wall contribution to the surface integral in (A.77).

Using (3.11), (A.52) and (A.76),

$$\int_{S_i} \tilde{\mathbf{u}}.\tilde{\boldsymbol{\sigma}}^*.\mathbf{n}dS = -s\tilde{E}_s\left[\tilde{\eta}\right] \tag{A.79}$$

for the interfacial contribution to (A.77). (3.15) follows from (A.77)-(A.79).

A.8 Nonexistence of $s\neq 0$ modes such that $s\to 0$ as $\mathrm{Oh}\to\mathrm{Oh}_c>0$

In order to derive a contradiction, suppose an $s\neq 0$ mode which approaches $s=0$ as $\mathrm{Oh}\to\mathrm{Oh}_c>0$. The limit would be an $s=0$ mode. Unless $\zeta=0$ is an eigenvalue there are no such modes and we already have a contradiction. The case in which $\zeta=0$ is an eigenvalue is more complicated and is treated below.

Let ω_n be one of the $\zeta=0$ eigenfunctions introduced in section 3.1. Thus, ω_n, which is real and corresponds to an $s=0$ mode, satisfies

$$\int_{S_i}\omega_n dS = 0, \tag{A.80}$$

$$\Delta\omega_n+\gamma\omega_n+\phi_n=0 \tag{A.81}$$

with constant ϕ_n, and

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\omega_n}{\partial q^\beta}=J\omega_n \tag{A.82}$$

on the contact line. Define $\mathbf{v}_n(\mathbf{x})$ and $\chi_n(\mathbf{x})$ via

$$\int_D \chi_n dv = 0, \tag{A.83}$$

$$\mathrm{Oh}_c\,\nabla^2\mathbf{v}_n=\nabla\chi_n, \tag{A.84}$$

$$\nabla.\mathbf{v}_n=0, \tag{A.85}$$

$$\lambda \frac{\partial v_{nx}}{\partial z} = v_{nx}, \quad \lambda \frac{\partial v_{ny}}{\partial z} = v_{ny}, \qquad v_{nz} = 0 \qquad\qquad z = 0 \tag{A.86}$$

and

$$\mathbf{v}_n.\mathbf{n} = \omega_n, \tag{A.87}$$

$$\mathbf{n}.\left(\nabla \mathbf{v}_n + \left(\nabla \mathbf{v}_n\right)^T\right).\mathbf{t} = 0 \tag{A.88}$$

on S_i, where $\mathbf{t}$ is any tangent vector to S_i. Taking two independent choices for $\mathbf{t}$, (A.88) gives two boundary conditions. (A.83)-(A.88) can be interpreted as follows. (A.84) and (A.85) mean that $\mathbf{v}_n(\mathbf{x})$ and $\chi_n(\mathbf{x})$ are the velocity and pressure of a steady, incompressible Stokes flow within D. This flow is subject to the Navier conditions (A.86) on the wall and (A.87), (A.88) on S_i. (A.87) specifies the normal component of velocity as ω_n, while (A.88) means that the tangential components of the surface force are zero. (A.80) is required for a solution. This follows from integration of (A.85) over D, use of the divergence theorem, $v_{nz} = 0$ on the wall and (A.87) on S_i. (A.83) makes the solution for $\chi_n(\mathbf{x})$, which would otherwise be only determined up to an additive constant, unique.

Consider a mode with $s \neq 0$ and let $\mathbf{v} = s^{-1}\tilde{\mathbf{u}}$, $\chi = s^{-1}\left(\tilde{p} - \tilde{p}_0\right)$, where the constant $\tilde{p}_0$ is such that

$$\int_D \chi dv = 0. \tag{A.89}$$

Equations (3.8)-(3.14) give

$$s\mathbf{v} = -\nabla\chi + \mathrm{Oh}\,\nabla^2\mathbf{v}, \tag{A.90}$$

$$\nabla.\mathbf{v} = 0 \tag{A.91}$$

with

$$\lambda \frac{\partial v_x}{\partial z} = v_x, \quad \lambda \frac{\partial v_y}{\partial z} = v_y, \qquad v_z = 0 \tag{A.92}$$

on the wall,

$$\tilde{\eta} = \mathbf{v}.\mathbf{n}, \tag{A.93}$$

$$s\,\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^T\right) = \mathbf{n}\left(\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}_0 + s\chi\right), \tag{A.94}$$

on the equilibrium interface,

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\tilde{\eta}}{\partial q^\beta}=J\tilde{\eta}, \tag{A.95}$$

at the equilibrium contact line and

$$\int_{S_i}\tilde{\eta}dS=0. \tag{A.96}$$

The normal and tangential components of (A.94) give

$$s\,\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^T\right).\mathbf{n}=\Delta\tilde{\eta}+\gamma\tilde{\eta}+\tilde{p}_0+s\chi \tag{A.97}$$

and

$$\mathbf{n}.\left(\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^T\right).\mathbf{t}=0, \tag{A.98}$$

where $\mathbf{t}$ is any tangent vector to S_i.

Using (A.91), (A.90) can be rewritten as

$$s\mathbf{v}=\nabla.\hat{\boldsymbol{\sigma}}, \tag{A.99}$$

where $\hat{\boldsymbol{\sigma}}=-\chi\mathbf{I}+2\,\mathrm{Oh}\,\hat{\mathbf{e}}$ and $\hat{\mathbf{e}}=\left(\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^T\right)/2$. (A.94) gives

$$s\hat{\boldsymbol{\sigma}}.\mathbf{n}=\mathbf{n}\left(\Delta\tilde{\eta}+\gamma\tilde{\eta}+\tilde{p}_0\right) \tag{A.100}$$

on S_i. Scalar multiplying (A.99) by $\mathbf{v}_n$ and integrating over D,

$$s\int_D\mathbf{v}_n.\mathbf{v}dv=\int_D\mathbf{v}_n.\left(\nabla.\hat{\boldsymbol{\sigma}}\right)dv. \tag{A.101}$$

Using $\mathbf{v}_n.\left(\nabla.\hat{\boldsymbol{\sigma}}\right)=\nabla.\left(\mathbf{v}_n.\hat{\boldsymbol{\sigma}}\right)-2\,\mathrm{Oh}\,\hat{\mathbf{e}}_n:\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}_n=\left(\nabla\mathbf{v}_n+\left(\nabla\mathbf{v}_n\right)^T\right)/2$, and the divergence theorem,

$$s\int_D\mathbf{v}_n.\mathbf{v}dv=\int_{\partial D}\mathbf{v}_n.\hat{\boldsymbol{\sigma}}.\mathbf{n}dS-2\,\mathrm{Oh}\int_D\hat{\mathbf{e}}_n:\hat{\mathbf{e}}dv. \tag{A.102}$$

The contribution of S_w to the surface integral can be evaluated using $\hat{\boldsymbol{\sigma}}=-\chi\mathbf{I}+2\,\mathrm{Oh}\,\hat{\mathbf{e}}$, $\hat{\mathbf{e}}=\left(\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^T\right)/2$ and $v_z=v_{nz}=0$. Thus,

$$\int_{S_w}\mathbf{v}_n.\hat{\boldsymbol{\sigma}}.\mathbf{n}dS=-\mathrm{Oh}\int_{S_w}\left(v_{nx}\frac{\partial v_x}{\partial z}+v_{ny}\frac{\partial v_y}{\partial z}\right)dS. \tag{A.103}$$

(A.80), (A.87) and (A.100) imply

$$\int_{S_i} \mathbf{v}_n.\hat{\boldsymbol{\sigma}}.\mathbf{n} dS = s^{-1}\int_{S_i} \omega_n \left(\Delta\tilde{\eta} + \gamma\tilde{\eta}\right) dS\,. \tag{A.104}$$

Using (A.46), (A.82), (A.95) and symmetry of $g^{\alpha\beta}$,

$$\int_{S_i} \omega_n \Delta\tilde{\eta} dS = \oint_C J\omega_n\tilde{\eta} ds - \int_{S_i} g^{\alpha\beta}\frac{\partial\omega_n}{\partial q^\alpha}\frac{\partial\tilde{\eta}}{\partial q^\beta} dS = \int_{S_i}\tilde{\eta}\Delta\omega_n dS\,, \tag{A.105}$$

hence

$$\int_{S_i} \mathbf{v}_n.\hat{\boldsymbol{\sigma}}.\mathbf{n} dS = s^{-1}\int_{S_i}\tilde{\eta}\left(\Delta\omega_n + \gamma\omega_n\right) dS\,, \tag{A.106}$$

which is zero according to (A.81) and (A.96). Thus, (A.102) and (A.103) give

$$s\int_D \mathbf{v}_n.\mathbf{v} dv = -2\,\mathrm{Oh}\int_D \hat{\mathbf{e}}_n : \hat{\mathbf{e}}\, dv - \mathrm{Oh}\int_{S_w}\left(v_{nx}\frac{\partial v_x}{\partial z} + v_{ny}\frac{\partial v_y}{\partial z}\right) dS \tag{A.107}$$

for any $n \in N$, where N denotes the set of n for which $\zeta_n = 0$.

Suppose the given $s \neq 0$ mode approaches $s = 0$ as $\mathrm{Oh} \to \mathrm{Oh}_c$ and is normalised using

$$\int_{S_i} |\tilde{\eta}|^2 dS = 1\,. \tag{A.108}$$

In the limit $\mathrm{Oh} \to \mathrm{Oh}_c$, $s \to 0$, $\tilde{\eta} \to \tilde{\eta}_c$, $\mathbf{v} \to \mathbf{v}_c$, $\chi \to \chi_c$ and $\tilde{p}_0 \to \tilde{p}_{0c}$, where, according to (A.95)-(A.97), $\tilde{\eta}_c$ and $\tilde{p}_{0c}$ satisfy the $s = 0$ problem, (3.13), (3.14) and (3.21). (A.89)-(A.93) and (A.98) give

$$\int_D \chi_c dv = 0\,, \tag{A.109}$$

$$\mathrm{Oh}_c \nabla^2\mathbf{v}_c = \nabla\chi_c\,, \tag{A.110}$$

$$\nabla.\mathbf{v}_c = 0\,, \tag{A.111}$$

$$\lambda\frac{\partial v_{cx}}{\partial z} = v_{cx}\,, \quad \lambda\frac{\partial v_{cy}}{\partial z} = v_{cy}\,, \quad v_{cz} = 0 \qquad z = 0 \tag{A.112}$$

and

$$\mathbf{v}_c.\mathbf{n} = \tilde{\eta}_c\,, \tag{A.113}$$

$$\mathbf{n}.\left(\nabla\mathbf{v}_c + \left(\nabla\mathbf{v}_c\right)^T\right).\mathbf{t} = 0 \tag{A.114}$$

on S_i. Given $\tilde{\eta}_c$, (A.109)-(A.114) determine $\mathbf{v}_c$ and χ_c. Using (A.86), (A.107) has the limiting form

$$2\int_D \hat{\mathbf{e}}_n : \hat{\mathbf{e}}_c dv + \lambda \int_{S_w} \left(\frac{\partial v_{nx}}{\partial z}\frac{\partial v_{cx}}{\partial z} + \frac{\partial v_{ny}}{\partial z}\frac{\partial v_{cy}}{\partial z} \right) dS = 0 \qquad \text{(A.115)}$$

for $n \in N$, where $\hat{\mathbf{e}}_c = \left(\nabla \mathbf{v}_c + (\nabla \mathbf{v}_c)^T \right)/2$.

Since $\tilde{\eta}_c$ and $\tilde{p}_{0c}$ satisfy the $s = 0$ problem, (3.13), (3.14) and (3.21), $\tilde{\eta}_c$ can be expressed as a linear combination of the $\zeta = 0$ eigenfunctions of (3.1), (3.2) and (3.3) , i.e.

$$\tilde{\eta}_c = \sum_{n \in N} c_n \omega_n , \qquad \text{(A.116)}$$

where the coefficients c_n may be complex and, using (3.4) and (A.108),

$$\sum_{n \in N} |c_n|^2 = 1 . \qquad \text{(A.117)}$$

Comparing (A.83)-(A.88) with (A.109)-(A.114), we see that

$$\mathbf{v}_c = \sum_{n \in N} c_n \mathbf{v}_n , \qquad \chi_c = \sum_{n \in N} c_n \chi_n . \qquad \text{(A.118)}$$

Using the first of equations (A.118) in (A.115),

$$\sum_{m \in N} A_{nm} c_m = 0 \qquad n \in N , \qquad \text{(A.119)}$$

where

$$A_{nm} = 2\int_D \hat{\mathbf{e}}_n : \hat{\mathbf{e}}_m \, dv + \lambda \int_{S_w} \left(\frac{\partial v_{nx}}{\partial z}\frac{\partial v_{mx}}{\partial z} + \frac{\partial v_{ny}}{\partial z}\frac{\partial v_{my}}{\partial z} \right) dS \qquad \text{(A.120)}$$

is a real, square, symmetric matrix defined for $n, m \in N$. Let $\overline{c}_n$ be real and $\overline{\mathbf{v}} = \sum_{n \in N} \overline{c}_n \mathbf{v}_n$, then

$$\sum_{n,m \in N} A_{nm} \overline{c}_n \overline{c}_m = 2\int_D \overline{\mathbf{e}} : \overline{\mathbf{e}} \, dv + \lambda \int_{S_w} \left(\left(\frac{\partial \overline{v}_x}{\partial z} \right)^2 + \left(\frac{\partial \overline{v}_y}{\partial z} \right)^2 \right) dS , \qquad \text{(A.121)}$$

where $\overline{\mathbf{e}} = \left(\nabla \overline{\mathbf{v}} + (\nabla \overline{\mathbf{v}})^T \right)/2$. (A.121) is obviously positive or zero. If it were zero, then $\overline{\mathbf{e}} = 0$, hence $\overline{\mathbf{v}}$ is a combination of a translation and a rotation. On the other hand, (A.86) implies

$$\lambda \frac{\partial \overline{v}_x}{\partial z} = \overline{v}_x , \quad \lambda \frac{\partial \overline{v}_y}{\partial z} = \overline{v}_y , \quad \overline{v}_z = 0 \qquad z = 0 , \qquad \text{(A.122)}$$

thus $\overline{\mathbf{v}} = 0$, hence $\overline{c}_n = 0$. We conclude that A_{nm} is positive definite. It follows from (A.119) that $c_n = 0$, which is incompatible with (A.117). This contradiction means $s \neq 0$ modes cannot approach $s = 0$ as $\mathrm{Oh} \to \mathrm{Oh}_c > 0$.

A.9 Some properties of inviscid modes

Given (3.30), $\nabla.\left(\psi \nabla \psi^*\right) - \left|\nabla \psi\right|^2 = \psi \nabla^2 \psi^* = 0$. Integrating over D, the divergence theorem, (3.31) and (3.32) give

$$\int_{S_i} \psi \tilde{\eta}^* dS = -\int_D \left|\nabla \psi\right|^2 dv \,. \tag{A.123}$$

On the other hand, (3.14) and (3.33) implies

$$\sigma \int_{S_i} \psi \tilde{\eta}^* dS = -\int_{S_i} \tilde{\eta}^* \left(\Delta \tilde{\eta} + \gamma \tilde{\eta}\right) dS \,. \tag{A.124}$$

Using (3.13) and (A.46),

$$\int_{S_i} \tilde{\eta}^* \Delta \tilde{\eta} dS = \oint_C J \left|\tilde{\eta}\right|^2 ds - \int_{S_i} g^{\alpha\beta} \frac{\partial \tilde{\eta}^*}{\partial q^\alpha} \frac{\partial \tilde{\eta}}{\partial q^\beta} dS \,. \tag{A.125}$$

(A.123)-(A.125) yield

$$\sigma \int_D \left|\nabla \psi\right|^2 dS = \oint_C J \left|\tilde{\eta}\right|^2 ds - \int_{S_i} \left(g^{\alpha\beta} \frac{\partial \tilde{\eta}^*}{\partial q^\alpha} \frac{\partial \tilde{\eta}}{\partial q^\beta} - \gamma \left|\tilde{\eta}\right|^2 \right) dS \,. \tag{A.126}$$

The integral on the left-hand side cannot be zero, otherwise ψ is constant and $\tilde{\eta} = 0$ from (3.32). (3.23) implies $\psi = 0$, hence $\tilde{p}_0 = 0$ from (3.33). Thus, all unknowns would be zero, which is not allowed for an eigenvalue problem. Symmetry of $g^{\alpha\beta}$ makes the right-hand side of (A.126) real. We conclude that σ is real. For each eigenvalue σ_k, ψ_k, $\tilde{\eta}_k$ and $\tilde{p}_{0k}$ are chosen real from here on. Because the integral on the left-hand side of (A.126) is positive, the ψ_k can be normalised such that (3.34) holds when $k = l$.

(3.30) implies $\nabla.\left(\psi_k \nabla \psi_l\right) - \nabla \psi_k . \nabla \psi_l = \psi_k \nabla^2 \psi_l = 0$. Integrating over D, the divergence theorem, (3.31) and (3.32) give

$$\int_{S_i} \psi_k \tilde{\eta}_l dS = -\int_D \nabla \psi_k . \nabla \psi_l dv \,, \tag{A.127}$$

while (3.14) and (3.33) imply

$$\sigma_k \int_{S_i} \psi_k \tilde{\eta}_l dS = -\int_{S_i} \tilde{\eta}_l \left(\Delta \tilde{\eta}_k + \gamma \tilde{\eta}_k\right) dS \,. \tag{A.128}$$

Using (3.13), (A.46) and symmetry of $g^{\alpha\beta}$,

$$\int_{S_i} \tilde{\eta}_l \Delta \tilde{\eta}_k dS = \oint_C J \tilde{\eta}_k \tilde{\eta}_l ds - \int_{S_i} g^{\alpha\beta} \frac{\partial \tilde{\eta}_k}{\partial q^\alpha} \frac{\partial \tilde{\eta}_l}{\partial q^\beta} dS \,. \tag{A.129}$$

(A.127)-(A.129) yield

$$\sigma_k \int_D \nabla \psi_k . \nabla \psi_l dS = \oint_C J \tilde{\eta}_k \tilde{\eta}_l ds - \int_{S_i} \left(g^{\alpha\beta} \frac{\partial \tilde{\eta}_k}{\partial q^\alpha} \frac{\partial \tilde{\eta}_l}{\partial q^\beta} - \gamma \tilde{\eta}_k \tilde{\eta}_l \right) dS \,. \tag{A.130}$$

Thus,

$$\sigma_k \int_D \nabla \psi_k . \nabla \psi_l dS = -\tilde{E}_{kl} \,, \tag{A.131}$$

where the matrix $\tilde{E}_{kl}$ is given by (3.35) and is symmetric. Using symmetry of $\tilde{E}_{kl}$, permutation of k, l and subtraction gives

$$\left(\sigma_k - \sigma_l\right) \int_D \nabla \psi_k . \nabla \psi_l dS = 0 \,. \tag{A.132}$$

When $\sigma_k \neq \sigma_l$, (A.132) implies

$$\int_D \nabla \psi_k . \nabla \psi_l dS = 0 \,. \tag{A.133}$$

If $\sigma_k = \sigma_l$ is a degenerate eigenvalue, its eigenfunctions can be orthogonalised such that (A.133) applies for $k \neq l$. Thus, (A.133) holds for all $k \neq l$. Given the normalisation referred to above, we obtain (3.34). (3.36) follows from (3.34) and (A.131).

A.10 Nonexistence of $s \neq 0$ modes such that $s \to 0$ as $\mathrm{Oh} \to 0$

Modes which continue to be affected by viscosity as $\mathrm{Oh} \to 0$ are discussed in the main text. A mode of this type has $s = O(\mathrm{Oh})$ and hence approaches $s = 0$ as $\mathrm{Oh} \to 0$, but it is decaying and hence unimportant from a stability point of view. Here, we consider modes of the other type, i.e. those which approach an inviscid limit.

The analysis given here has many similarities with that of section A.8. One significant difference is that $\mathbf{v}_n$ and η_n are replaced by $\mathbf{v}_k = \nabla \psi_k$ and $\tilde{\eta}_k$, where ψ_k, $\tilde{\eta}_k$ are $\sigma = 0$ eigenfunctions of the inviscid problem (3.13), (3.14), (3.23) and (3.30)-(3.33). As in section A.8, given an $s \neq 0$ mode, let $\mathbf{v} = s^{-1}\tilde{\mathbf{u}}$ and $\chi = s^{-1}\left(\tilde{p} - \tilde{p}_0\right)$, where the constant $\tilde{p}_0$ is determined by (A.89). (A.90)-(A.107), with $\mathbf{v}_n$, η_n and $\hat{\mathbf{e}}_n$ replaced by $\mathbf{v}_k = \nabla \psi_k$, $\tilde{\eta}_k$ and $\hat{\mathbf{e}}_k = \left(\nabla \mathbf{v}_k + \left(\nabla \mathbf{v}_k\right)^T\right)/2$, follow as before.

Letting $\psi = s^{-1}\chi$, (A.89)-(A.96) give

$$\int_D \psi dv = 0 \,, \tag{A.134}$$

$$\mathbf{v} = -\nabla\psi + s^{-1}\,\mathrm{Oh}\,\nabla^2\mathbf{v}\,, \tag{A.135}$$

$$\nabla.\mathbf{v} = 0 \tag{A.136}$$

with

$$\lambda\frac{\partial v_x}{\partial z} = v_x\,, \quad \lambda\frac{\partial v_y}{\partial z} = v_y\,, \quad v_z = 0 \tag{A.137}$$

on the wall,

$$\tilde{\eta} = \mathbf{v}.\mathbf{n}\,, \tag{A.138}$$

$$s\,\mathrm{Oh}\,\mathbf{n}.\left(\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^T\right) = \mathbf{n}\left(\Delta\tilde{\eta} + \gamma\tilde{\eta} + \tilde{p}_0 + s^2\psi\right), \tag{A.139}$$

on the equilibrium interface,

$$\nu_\alpha g^{\alpha\beta}\frac{\partial\tilde{\eta}}{\partial q^\beta} = J\tilde{\eta}\,, \tag{A.140}$$

at the equilibrium contact line and

$$\int_{S_i}\tilde{\eta}dS = 0\,. \tag{A.141}$$

(A.107), with $\mathbf{v}_n$, $\hat{\mathbf{e}}_n$ replaced by $\mathbf{v}_k$, $\hat{\mathbf{e}}_k$, gives

$$\int_D \mathbf{v}_k.\mathbf{v}dv = -2s^{-1}\,\mathrm{Oh}\int_D \hat{\mathbf{e}}_k : \hat{\mathbf{e}}dv - s^{-1}\,\mathrm{Oh}\int_{S_w}\left(v_{kx}\frac{\partial v_x}{\partial z} + v_{ky}\frac{\partial v_y}{\partial z}\right)dS\,, \tag{A.142}$$

where $\hat{\mathbf{e}} = \left(\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^T\right)/2$. Recalling that $\mathbf{v}_k = \nabla\psi_k$, where ψ_k is a $\sigma = 0$ inviscid eigenfunction, (A.142) applies for all k for which $\sigma_k = 0$. We denote the set of those k by K.

Suppose an $s \neq 0$ mode approaches an inviscid limit with $s = 0$ as $\mathrm{Oh} \to 0$. In order that the viscous term in (A.135) be negligible in the limit, $s^{-1}\,\mathrm{Oh} \to 0$. Normalising the mode using (A.108), $\tilde{\eta} \to \tilde{\eta}_c$, $\mathbf{v} \to \mathbf{v}_c$, $\tilde{p}_0 \to \tilde{p}_{0c}$ and $\psi \to \psi_c$. (A.134) implies

$$\int_D \psi_c dv = 0\,. \tag{A.143}$$

(A.135) and $s^{-1}\,\mathrm{Oh} \to 0$ give

$$\mathbf{v}_c = -\nabla\psi_c\,. \tag{A.144}$$

Thus,

$$\nabla^2 \psi_c = 0 \tag{A.145}$$

according to (A.136). As discussed following equation (3.30), the inviscid problem only allows one wall boundary condition, rather than the three expressed by (A.137) for the viscous problem. The first two equations of (A.137) drop out in the inviscid limit, leaving $v_z = 0$, hence

$$\frac{\partial \psi_c}{\partial z} = 0 \tag{A.146}$$

on the wall. (A.138)-(A.141) yield

$$\frac{\partial \psi_c}{\partial n} = -\tilde{\eta}_c, \tag{A.147}$$

$$\Delta \tilde{\eta}_c + \gamma \tilde{\eta}_c + \tilde{p}_{0c} = 0, \tag{A.148}$$

on the equilibrium interface,

$$v_\alpha g^{\alpha\beta} \frac{\partial \tilde{\eta}_c}{\partial q^\beta} = J \tilde{\eta}_c, \tag{A.149}$$

at the equilibrium contact line and

$$\int_{S_i} \tilde{\eta}_c dS = 0. \tag{A.150}$$

(A.148)-(A.150) show that $\tilde{\eta}_c$ and $\tilde{p}_{0c}$ satisfy the $s = 0$ problem, (3.13), (3.14) and (3.21), while (A.143) and (A.145)-(A.147) correspond to (3.23) and (3.30)-(3.32) and determine ψ_c given $\tilde{\eta}_c$. Since $s^{-1}\,\mathrm{Oh} \to 0$, (A.142) implies

$$\int_D \mathbf{v}_k . \mathbf{v}_c dv = 0 \tag{A.151}$$

for all $k \in K$.

That $\tilde{\eta}_c$ and $\tilde{p}_{0c}$ satisfy the $s = 0$ problem, (3.13), (3.14) and (3.21), indicates that $\tilde{\eta}_c$ is a $\sigma = 0$ eigenfunction, hence

$$\tilde{\eta}_c = \sum_{k \in K} \tilde{c}_k \tilde{\eta}_k . \tag{A.152}$$

(A.143), (A.145)-(A.147) and the corresponding equations for ψ_k and $\tilde{\eta}_k$ imply

$$\psi_c = \sum_{k \in K} \tilde{c}_k \psi_k \tag{A.153}$$

so

$$\mathbf{v}_c = -\sum_{k\in K} \tilde{c}_k \nabla \psi_k \qquad \text{(A.154)}$$

according to (A.144). Scalar multiplying by $\nabla\psi_l$, where $l \in K$, integrating over D and using (3.34), $\mathbf{v}_l = \nabla\psi_l$ and (A.151),

$$\tilde{c}_l = -\int_D \mathbf{v}_c . \nabla\psi_l dv = -\int_D \mathbf{v}_c . \mathbf{v}_l dv = 0 . \qquad \text{(A.155)}$$

This result means that $\tilde{\eta}_c = 0$, which is incompatible with the normalisation (A.108). Thus, we have a contradiction and conclude that $s \neq 0$ modes which approach an inviscid limit with $s = 0$ as $\mathrm{Oh} \to 0$ do not exist.