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Abstract. On digital planes (set of integer points between two parallel Euclidean planes), plane-probing algorithms initiate with a triangle, update one vertex at a time and approximate the plane on the fly. The L-algorithm is a plane-probing algorithm variant which takes into account a large neighborhood of points for its update process. We recall the framework of plane-probing algorithms, especially for the L-algorithm. We introduce the Delaunay property and prove that it is theoretically held by the L-algorithm. Lastly, we name a few consequences of such property, namely the research of minimal bases and an estimation for the locality. This technical report is provided as supplementary material to the DGMM2022 paper [11].

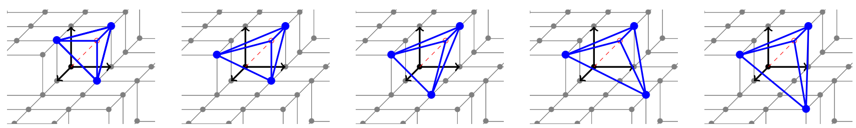
Keywords: Digital Plane Recognition · Plane-Probing Algorithm · Delaunay property

1 Introduction

Digital volumes are sets of voxels in the 3D Euclidean space. A digital surface is the boundary of a digital volume. In particular, digital plane can be viewed as a set integer points bounded in between two parallel planes in the space. Some geometrical properties of planes are translated into digital plane [2] and we are interested in studying the geometry of the digital surfaces, which leads to multiple applications such as recognizing local digital plane segments [16,6,8,3], and estimate differential quantities [5,4].

Plane-probing algorithm is first proposed in [9]. This type of algorithm probes some points in the digital surface and the output represent locally an approximation of the digital surface. A basic version of the plane-probing algorithm considers a tetrahedron whose apex is outside the surface and an triangle formed from three points that belongs to the surface. In Fig. 1, we show an example where the algorithm probes a digital plane. In this paper, we focus on the L-algorithm which fits into the general framework of plane-probing algorithm [10].

Plane probing algorithms return the exact normal on an infinite digital plane. However, it struggles on digital surfaces without additional information [10]. We



(a) (b) (c) (d) (e)

Fig. 1: The evolution (from left to right) of a tetrahedra-based plane-probing algorithm for normal $(1, 2, 5)$.

wish to estimate a minimal space in which the plane-probing algorithm provides a good normal estimation. In order to study the algorithm's locality,we study the relation between each point probed by the algorithm. The reduced 2D version of plane-probing algorithm probes a set of integer points that forms a delaunay triangulation of a digital segment [14]. In this paper, we aim to demonstrate the 3D extension of this property and to provide a theoretical bound for the effective space for probing, following the notations and workflow of [11].

In this paper, we start with introducing the general framework of plane-probing algorithms and the L-algorithm in sec. 2. The main theorem that mentions the Delaunay property is announced, in sec. 3, We then provide the conclusion to this paper before diving into the long and technical proofs. We complete the paper with two sections: In sec. 4, we prove the important lemma 2 leaving the technical details to sec. 5, which is arranged into three categories: projection-based-results (5.1), circumsphere-based-results (5.2), and closeness results (5.3).

2 The L-algorithm

A *digital plane* is an infinite digital set defined by a normal $\mathbf{N} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$, a shift value $\mu \in \mathbb{Z}$ and a thickness $\omega \in \mathbb{Z}$ as follows[13]:

$$\mathbf{P}_{\mu,\mathbf{N}} := \{\mathbf{x} \in \mathbb{Z}^3 \mid \mu \leq \mathbf{x} \cdot \mathbf{N} < \mu + \omega\}. \tag{1}$$

In this paper, we set $\omega := \|\mathbf{N}\|_1$ and we assume w.l.o.g. that $\mu = 0$ and that the components of $\mathbf{N}$ are positive, i.e., $\mathbf{N} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$. Given a digital plane $\mathbf{P} \in \{\mathbf{P}_{0,\mathbf{N}} \mid \mathbf{N} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}\}$ of unknown normal vector, a *plane-probing algorithm* computes the normal vector $\mathbf{N}$ of $\mathbf{P}$ by sparsely probing it with the predicate "is $\mathbf{x}$ in $\mathbf{P}$?". We describe below a plane-probing algorithm, called *L-algorithm* (see algorithm 1).

Initialization Let $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ be the canonical basis of $\mathbb{Z}^3$. Given a starting point $o \in \mathbf{P}$, let $\mathbf{q}$ be equal to $\mathbf{p} + \sum_k \mathbf{e}_k$ ($\mathbf{q}$ is by definition not in $\mathbf{P}$) and let $\mathbf{v}_k^{(0)}$ be equal to $\mathbf{q} - \mathbf{e}_k$ for all $k \in \mathbb{Z}/3\mathbb{Z}$. We define the initial triangle as $\mathbf{T}^{(0)} := (\mathbf{v}_k^{(0)})_{k \in \mathbb{Z}/3\mathbb{Z}}$ provided that $\mathbf{T}^{(0)} \subset \mathbf{P}$.

Algorithm 1: L-algorithm

Input: The predicate InPlane := "Is a point $\mathbf{x} \in \mathbf{P}$?", a point $o \in \mathbf{P}$

Output: A normal vector $\hat{\mathbf{N}}$ and a basis of the lattice $\{\mathbf{x} \mid \mathbf{x} \cdot \hat{\mathbf{N}} = \omega - 1\}$.

1 $\mathbf{q} \leftarrow o + \sum_k \mathbf{e}_k$; $(\mathbf{v}_k^{(0)})_{k\ \in \mathbb{Z}/3\mathbb{Z}} \leftarrow (\mathbf{q} - \mathbf{e}_k)_{k\ \in \mathbb{Z}/3\mathbb{Z}}$; `// initialization`

2 $i \leftarrow 0$;

3 **while** $\mathcal{N}_S^{(i)} \cap \{\mathbf{x} \mid \text{InPlane}(\mathbf{x})\} \neq \emptyset$ **do**

4   Let (k, α, β) be such that, for all $\mathbf{y} \in \mathcal{N}_S^{(i)} \cap \{\mathbf{x} \mid \text{InPlane}(\mathbf{x})\}$,

5     $\mathbf{v}_k^{(i)} + \alpha(\mathbf{q} - \mathbf{v}_{k+1}^{(i)}) + \beta(\mathbf{q} - \mathbf{v}_{k+2}^{(i)}) \leq_{\mathbf{T}} \mathbf{y}$; `// equation (3)`

6     $\mathbf{v}_k^{(i+1)} \leftarrow \mathbf{v}_k^{(i)} + \alpha(\mathbf{q} - \mathbf{v}_{k+1}^{(i)}) + \beta(\mathbf{q} - \mathbf{v}_{k+2}^{(i)})$; `// equation (4)`

7     $\forall l \in \{0, 1, 2\} \setminus k,\ \mathbf{v}_l^{(i+1)} \leftarrow \mathbf{v}_l^{(i)}$;

8     $i \leftarrow i + 1$;

9 $B \leftarrow \{\mathbf{v}_0^{(i)} - \mathbf{v}_1^{(i)}, \mathbf{v}_1^{(i)} - \mathbf{v}_2^{(i)}, \mathbf{v}_2^{(i)} - \mathbf{v}_0^{(i)}\}$;

10 Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the shortest and second shortest vectors of B ;

11 **return** $\mathbf{b}_1 \times \mathbf{b}_2, (\mathbf{b}_1, \mathbf{b}_2)$;

Candidate set At every step $i \in \mathbb{N}$, the triangle $\mathbf{T}^{(i)}$ represents the current approximation of the plane $\mathbf{P}$. The L-algorithm updates one vertex of $\mathbf{T}^{(i)}$ per iteration. That vertex is replaced by a point of $\mathbf{P}$ from a candidate set defined as follows:

$$\mathcal{N}_S^{(i)} := \left\{ \mathbf{v}_k^{(i)} + \alpha(\mathbf{q} - \mathbf{v}_{k+1}^{(i)}) + \beta(\mathbf{q} - \mathbf{v}_{k+2}^{(i)}) \mid k \in \mathbb{Z}/3\mathbb{Z},\ (\alpha, \beta) \in \mathbb{N}^2 \setminus (0,0) \right\}. \tag{2}$$

Order and update rule At every step $i \in \mathbb{N}$, let $\mathcal{H}_+^{(i)}$ (resp. $\mathcal{H}_-^{(i)}$) be the half-space lying above (resp. on or below) the plane incident to $\mathbf{T}^{(i)}$. Let us consider the ball $\mathcal{B}(\mathbf{T}^{(i)}, \mathbf{x})$ circumscribing $\mathbf{T}^{(i)}$ and a point $\mathbf{x} \in \mathcal{H}_+^{(i)}$. It induces a total preorder on $\mathcal{H}_+^{(i)}$ through the inclusion relation (see Appendix A). For any pair $\mathbf{x}, \mathbf{x}' \in \mathcal{H}_+^{(i)}$, we say that $\mathbf{x}'$ is *closer* to $\mathbf{T}^{(i)}$ than $\mathbf{x}$, denoted by $\mathbf{x}' \leq_{\mathbf{T}} \mathbf{x}$, if and only if $(\mathcal{B}(\mathbf{T}^{(i)}, \mathbf{x}') \cap \mathcal{H}_+^{(i)}) \subset (\mathcal{B}(\mathbf{T}^{(i)}, \mathbf{x}) \cap \mathcal{H}_+^{(i)})$.

The L-algorithm updates a vertex of $\mathbf{T}^{(i)}$ with a point of the set $\mathcal{N}_S^{(i)} \bigcap \mathbf{P}$ that is a closest one according to $\leq_{\mathbf{T}}$. More precisely, if $\mathcal{N}_S^{(i)} \bigcap \mathbf{P} \neq \emptyset$, there is an index $k \in \mathbb{Z}/3\mathbb{Z}$ and there are numbers $(\alpha, \beta) \in \mathbb{N}^2 \setminus (0,0)$ such that

$$\forall \mathbf{x} \in \mathcal{N}_S^{(i)} \cap \mathbf{P},\ \mathbf{v}_k^{(i)} + \alpha(\mathbf{q} - \mathbf{v}_{k+1}^{(i)}) + \beta(\mathbf{q} - \mathbf{v}_{k+2}^{(i)}) \leq_{\mathbf{T}} \mathbf{x}. \tag{3}$$

Note that the triple (k, α, β) may not be unique when several points are in a cospherical position. The update rule is then:

$$\begin{cases} \mathbf{v}_k^{(i+1)} := \mathbf{v}_k^{(i)} + \alpha(\mathbf{q} - \mathbf{v}_{k+1}^{(i)}) + \beta(\mathbf{q} - \mathbf{v}_{k+2}^{(i)}), \\ \mathbf{v}_{k+1}^{(i+1)} := \mathbf{v}_{k+1}^{(i)}, \\ \mathbf{v}_{k+2}^{(i+1)} := \mathbf{v}_{k+2}^{(i)}. \end{cases} \tag{4}$$

As shown in Algorithm 1, lines 5 to 7, equations (3) and (4) are used to update the current triangle.

Termination The algorithm terminates at a step n, when the neighborhood has an empty intersection with the plane, i.e., when $\mathcal{N}_S^{(n)} \bigcap \mathbf{P} = \emptyset$ (Algorithm 1, line 3). The number of steps, n, is less than or equal to $\omega - 3$, which is a tight bound reached for any normal of components $(1, 1, r)$ with $r \in \mathbb{N} \setminus \{0\}$. This result can be found in [10, Theorem 1]. Even if [10] only considers slightly different candidate sets, all mentioned results are valid for the larger candidate set we consider in this paper. In addition, if o is one of the least high points above $\mathbf{P}$, i.e., $o \cdot \mathbf{N} = 0$, $\mathbf{T}^{(n)}$ lines up with $\mathbf{P}$:

Theorem 1 ([10], Corollary 5). *If $o \cdot \mathbf{N} = 0$, the normal of $\mathbf{T}^{(n)}$ is equal to $\mathbf{N}$ and any two edges form a basis of the lattice of* upper leaning points*, i.e., the lattice $\{\mathbf{x} \in \mathbb{Z}^3 \mid \mathbf{x} \cdot \mathbf{N} = \omega - 1\}$.*

Invariants The update rule involves the following vectors:

$$\forall i \in \{0, \dots, n\},\ \forall k \in \{0, 1, 2\},\ \mathbf{m}_k^{(i)} := \mathbf{q} - \mathbf{v}_k^{(i)}. \tag{5}$$

It is thus not very surprising that the L-algorithm has these two properties:

Theorem 2 ([10], Lemma 3). *For all steps $i \in \{0, \dots, n\}$, $\det(\mathbf{m}_0^{(i)}, \mathbf{m}_1^{(i)}, \mathbf{m}_2^{(i)}) = 1$.*

This shows that, for all steps $i \in \{0, \dots, n\}$, $\left\{\mathbf{m}_0^{(i)}, \mathbf{m}_1^{(i)}, \mathbf{m}_2^{(i)}\right\}$ is a basis of $\mathbb{Z}^3$.

Theorem 3 ([10], Lemma 5). *For all steps $i \in \{0, \dots, n\}$, for all $k \in \{0, 1, 2\}$, $\mathbf{m}_k \cdot \mathbf{N} > 0$.*

From Theorem 3, we understand that the algorithm always replaces a vertex with a *higher* candidate point in direction $\mathbf{N}$. That property is a key point in the proof of Theorem 1. In addition, we can derive the following small lemma we will use in sec. 4.

Lemma 1. *For all steps $i \in \{0, \dots, n\}$, with $\mathbf{p}^{(i)} := \mathbf{q} - \sum_k \mathbf{m}_k^{(i)}$, we have $\mathbf{p}^{(i)} \cdot \mathbf{N} \geq 0$.*

Proof. By definition $\mathbf{p}^{(0)} = o$ and o is assumed to belong to $\mathbf{P}$. As a consequence, $\mathbf{p}^{(0)} \cdot \mathbf{N} \geq 0$.

For any $i \in \{1, \dots, n-1\}$, there is (k, α, β) such that $\mathbf{m}_k^{(i+1)} = \mathbf{m}_k^{(i)} - \alpha\mathbf{m}_{k+1}^{(i)} - \beta\mathbf{m}_{k+2}^{(i)}$, $\mathbf{m}_{k+1}^{(i+1)} = \mathbf{m}_{k+1}^{(i)}$ and $\mathbf{m}_{k+2}^{(i+1)} = \mathbf{m}_{k+2}^{(i)}$ by (4). Then, we remark that $\mathbf{p}^{(i+1)} \cdot \mathbf{N} - \mathbf{p}^{(i)} \cdot \mathbf{N} = \alpha\mathbf{m}_{k+1}^{(i)} \cdot \mathbf{N} + \beta\mathbf{m}_{k+2}^{(i)} \cdot \mathbf{N}$, which is strictly positive by Theorem 3. We can conclude by induction. □

3 Main theorem

By convenience, let $\mathbf{T}^{(-1)}$ be the degenerated triangle whose three vertices are all at o. At every step $i \in \mathbb{N}$, let $\mathcal{B}^{(i)}$ be the ball uniquely determined by the four distinct points of $\mathbf{T}^{(i-1)} \cup \mathbf{T}^{(i)}$. In experiments, we observe that the L-algorithm verifies the following property:

Property 1 (Delaunay property for plane-probing algorithms). For all steps $i \in \{0, \ldots, n\}$, the ball $\mathcal{B}^{(i)}$ does not contain any points of $\mathbf{P}$ in its interior.

The main purpose of this paper is to prove the following theorem:

Theorem 4. *The L-algorithm verifies the Delaunay property (Property 1).*

The proof of Theorem 4 requires the following lemma whose proof is postponed to the sec. 4.

Lemma 2. *For all $i \in \{0, \ldots, n-1\}$, if the interior of $\mathcal{B}^{(i)}$ contains no point of $\mathbf{P}$, then the interior of $\mathcal{B}^{(i+1)}$ contains no point of $\mathbf{P} \cap \mathcal{H}_+^{(i)}$.*

Proof. of Theorem 4

Base case $\mathcal{B}^{(0)}$, which passes through all the vertices of a unit cube, contains no integer point in its interior and as a consequence, no point of $\mathbf{P}$.

Induction step We assume that $\mathcal{B}^{(i)}$ contains no point of $\mathbf{P}$ in its interior for any $i \in \{0, \ldots, n-1\}$ and we want to show that no point of $\mathbf{P}$ lies in the interior of $\mathcal{B}^{(i+1)}$.

By definition, the boundary of both $\mathcal{B}^{(i)}$ and $\mathcal{B}^{(i+1)}$ passes through the vertices of $\mathbf{T}^{(i)}$ and there is a point $\mathbf{x}^\star$, chosen by the algorithm, lying in $\mathcal{H}_+^{(i)}$ and such that $\mathbf{x}^\star = \mathbf{T}^{(i+1)} \setminus \mathbf{T}^{(i)}$.

First, we can safely discard the points of $\mathbf{P}$ that are located in $\mathcal{H}_-^{(i)}$. Indeed, $\mathbf{x}^\star \in \mathcal{H}_+^{(i)}$ (by definition) and $\mathbf{x}^\star \notin \mathcal{B}^{(i)}$ (by hypothesis) together imply that $\left(\mathcal{B}^{(i)} \cap \mathcal{H}_+^{(i)}\right) \subset \left(\mathcal{B}^{(i+1)} \cap \mathcal{H}_+^{(i)}\right)$, thus $\left(\mathcal{B}^{(i+1)} \cap \mathcal{H}_-^{(i)}\right) \subset \left(\mathcal{B}^{(i)} \cap \mathcal{H}_-^{(i)}\right)$ (see Appendix A remark 2). We conclude that the interior of $\left(\mathcal{B}^{(i+1)} \cap \mathcal{H}_-^{(i)}\right)$ contains no point of $\mathbf{P}$, because it is included in the interior of $\left(\mathcal{B}^{(i)} \cap \mathcal{H}_-^{(i)}\right)$, itself included in the interior of $\mathcal{B}^{(i)}$, which is assumed to contain no point of $\mathbf{P}$.

Second, regarding the points of $\mathbf{P}$ that are located in $\mathcal{H}_+^{(i)}$, by Lemma 2, we know that none of them are in the interior of $\mathcal{B}^{(i+1)}$, which concludes. □

One of the consequences of theorem 4 is the following result:

Corollary 1. *The final triangle $\mathbf{T}^{(n)}$ have acute or right angles.*

Proof. By theorem 4, the circumsphere $\mathcal{B}^{(n)}$ does not contain any point of $\mathbf{P}$. In particular, the circumcircle passing by $\mathbf{T}^{(n)}$ does not contain the points $\mathbf{v}_n^{(k)} + (\mathbf{v}_n^{(k+1)} - \mathbf{v}_n^{(k)}) + (\mathbf{v}_n^{(k+2)} - \mathbf{v}_n^{(k)})$, for all $k \in \mathbb{Z}/3\mathbb{Z}$. By Lemma 13, this implies that the final triangle has three acute or right angles. □

That geometrical result has another consequence that requires the following definition:

Definition 1. *Let L be a two-rank integral lattice. A basis $(\mathbf{x}, \mathbf{y})$ of L is minimal if and only if $\|\mathbf{x}\|, \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} + \mathbf{y}\|$.*

Such a basis is said *minimal* because this definition matches with the Minkowski's minima [12, Theorem 7].

Corollary 2. *The two shortest edges of the final triangle* $\mathbf{T}^{(n)}$ *form a minimal basis of the lattice of* upper leaning points, *i.e., the lattice* $\{\mathbf{x} \in \mathbb{Z}^3 \mid \mathbf{x} \cdot \mathbf{N} = \omega - 1\}$.

Proof. We know by Theorem 1 that any two edges of the final triangle form a basis of the lattice of upper leaning points. We show below that the fact that all angles are acute or right (Corollary 1) implies that the two shortest edges form a *minimal* basis.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be respectively equal to $(\mathbf{v}_1^{(n)} - \mathbf{v}_0^{(n)})$, $(\mathbf{v}_2^{(n)} - \mathbf{v}_0^{(n)})$ and $(\mathbf{v}_2^{(n)} - \mathbf{v}_1^{(n)})$ and assume w.l.o.g. that $\mathbf{x}$ and $\mathbf{y}$ are the two shortest vectors, i.e., $\|\mathbf{x}\|, \|\mathbf{x}\| \leq \|\mathbf{z}\|$. On one hand, since $-\mathbf{z} = \mathbf{x} - \mathbf{y}$, we have by definition

$$\|\mathbf{x}\|, \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|.$$

On the other hand, since $\mathbf{x} \cdot \mathbf{y} \geq 0$, it is obvious that

$$\|\mathbf{x}\|, \|\mathbf{y}\|, \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} + \mathbf{y}\|.$$

Putting all together, we have $\|\mathbf{x}\|, \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} + \mathbf{y}\|$, which means by definition that the basis $(\mathbf{x}, \mathbf{y})$ is minimal. □

4 Proof of Lemma 2

This section is dedicated to the proof of Lemma 2. For a fixed step i, we partition the points of $\mathcal{H}_+^{(i)}$ into different categories according to their position and we treat each case with distinct lemmas (lemma 3, lemma 4 and lemma 6). Since we now focus on a step $i \in \{0, \dots, n-1\}$, for sake of simplicity, we drop the exponent (i) in the notations of this section.

4.1 Outline of the proof

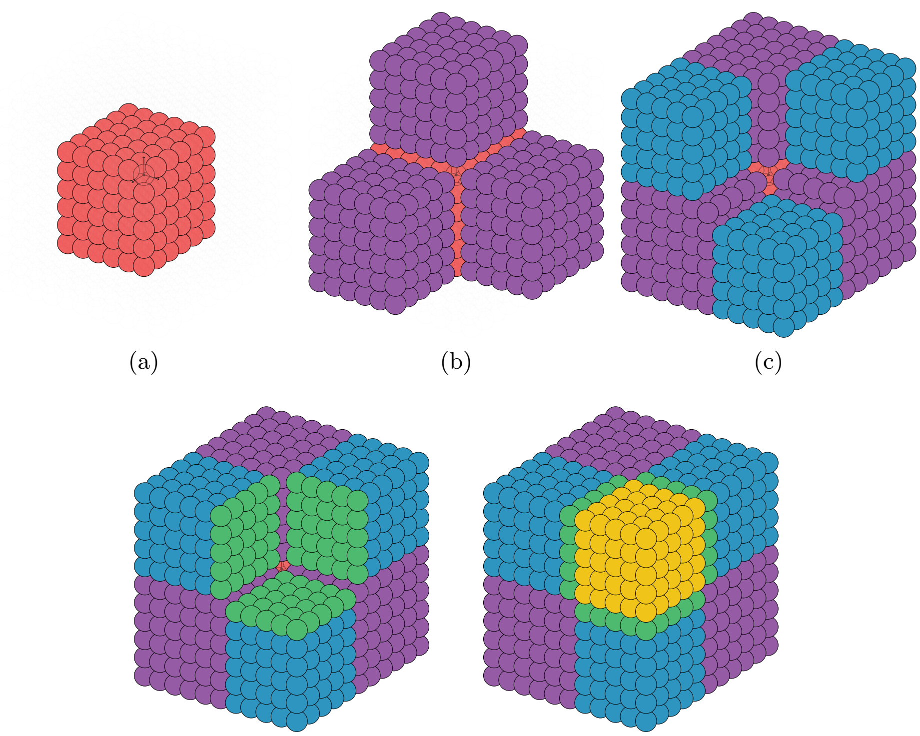
Let $\mathbf{p} := \mathbf{q} - \sum_k \mathbf{m}_k$. We conveniently describe any integer point $\mathbf{y} \in \mathbb{Z}^3$ as a linear combination of $\mathbf{m}_0$, $\mathbf{m}_1$ and $\mathbf{m}_2$, which form a basis of $\mathbb{Z}^3$(by Theorem 2), i.e. $\mathbf{y} := \mathbf{p} + \sum_k c_k \mathbf{m}_k$, with $c_k \in \mathbb{Z}$ for all $k \in \{0, 1, 2\}$ By construction, the supporting plane of $\mathcal{H}_+$ is defined by the vertices $\{\mathbf{p} + \mathbf{m}_0 + \mathbf{m}_1, \mathbf{p} + \mathbf{m}_1 + \mathbf{m}_2, \mathbf{p} + \mathbf{m}_0 + \mathbf{m}_2\}$. Integer points $\mathbf{y} := \mathbf{p} + \sum_k c_k \mathbf{m}_k$ on such plane are such that $\sum_k c_k = 2$. Hence, we have $\mathbf{y} \in \mathcal{H}_+ \Leftrightarrow \sum_k c_k \geq 3$. In this section, we assume that $\mathbf{y} \in \mathcal{H}_+$.

We consider several cases:

1. all coefficients $\{c_k\}$ are strictly positive (see Lemma 3),
2. one coefficient is zero and the others are strictly positive (such points are taken into account in the algorithm),

3. one coefficient is strictly negative and the others are strictly positive (see Lemma 4 and Lemma 5),
4. one coefficient is strictly positive and the others are negative (strictly negative or null) (see Lemma 6 and Lemma 7).

To check that any $\mathbf{y} \in \mathcal{H}_+$ is in one of the previous cases, it is enough to consider the partition of $\mathbb{Z}^3$ into eight octants depending on the signs of the coefficients and with a convention for null coefficients (see Fig. 2). The *negative* octant, in red, does not intersect $\mathcal{H}_+$ and is therefore discarded. The *positive* octant is itself divided into two regions, the interior, in yellow, corresponds to item 1, whereas the boundary faces, in green, corresponds to item 2. Among the last six octants, three of them, in blue, correspond to item 3, whereas the other three, in purple, correspond to item 4.



(a) (b) (c)

(d) (e)

Fig. 2: The discrete space $\mathbb{Z}^3$ (intersected with the box $[-5, 5]^3$ for the illustration) is partitioned into five regions. The yellow, green, blue and purple regions respectively correspond to cases 1, 2, 3 and 4. The red one is discarded because none of its point lies in $\mathcal{H}_+$.

In the next lemmas, we use for sake of clarity the bar notation whenever a scalar product with $\mathbf{N}$ is required, i.e., $\overline{(\mathbf{y})}$ instead of $\mathbf{y} \cdot \mathbf{N}$ for any vector $\mathbf{y} \in \mathbb{Z}^3$.

Lemma 1 ensures that $\overline{\mathbf{p}} \geq 0$. Since $\forall k$, $\overline{\mathbf{m}}_k > 0$ by Theorem 3, all points of the form $\mathbf{p} + \sum_k c_k \mathbf{m}_k$ with positive coefficients are such that $\overline{\mathbf{p}} + \sum_k c_k \overline{\mathbf{m}}_k > 0$. That is why we will only check if $\overline{\mathbf{p}} + \sum_k c_k \overline{\mathbf{m}}_k < \omega$, whenever we want to determine whether such a point is $\mathbf{P}$ or not.

Lemma 3. *Let $\mathbf{y}$ be equal to $\mathbf{p} + \sum_k c_k \mathbf{m}_k = \mathbf{q} + \sum_k (c_k - 1)\mathbf{m}_k$, with $c_k \in \mathbb{Z}$ for all $k \in \{0, 1, 2\}$. If $c_0, c_1, c_2 > 0$, then $\mathbf{y} \notin \mathbf{P}$.*

Proof. Since $(c_k - 1) \geq 0$ for all $k \in \{0, 1, 2\}$ (by hypothesis), $\overline{\mathbf{q}} = \omega$ (by definition) and $\overline{\mathbf{m}}_k > 0$ (by Theorem 3), it follows that $\overline{\mathbf{y}} \geq \omega$, thus $\mathbf{y} \notin \mathbf{P}$. $\square$

Let Σ be the set of all permutations over $\{0, 1, 2\}$.

Lemma 4. *Let $\mathbf{y} = \mathbf{p} + \sum_k c_k \mathbf{m}_k$ such that $\sum_k c_k \geq 3$, If there is a permutation $\sigma \in \Sigma$ such that $c_{\sigma(0)} < 0$ and $c_{\sigma(1)}, c_{\sigma(2)} > 0$ and $\mathbf{y} \in \mathbf{P}$, then $\mathbf{p} + 2\mathbf{m}_{\sigma(1)} + \mathbf{m}_{\sigma(2)} \in \mathbf{P}$ or $\mathbf{p} + \mathbf{m}_{\sigma(1)} + 2\mathbf{m}_{\sigma(2)} \in \mathbf{P}$ (The two points can both be in the plane $\mathbf{P}$).*

If, in addition, $-c_{\sigma(0)} < \min(c_{\sigma(1)}, c_{\sigma(2)}) - 1$, then $\mathbf{p} + (c_{\sigma(0)} + c_{\sigma(1)})\mathbf{m}_{\sigma(1)} + (c_{\sigma(0)} + c_{\sigma(2)})\mathbf{m}_{\sigma(2)} \in \mathbf{P}$.

Proof. We assume w.l.o.g. that σ is the identity, i.e. $\sigma(0) = 0$, $\sigma(1) = 1$ and $\sigma(2) = 2$.

Since $\mathbf{y} \in \mathbf{P}$, we have

$$\overline{\mathbf{y}} = \overline{\mathbf{p}} + \sum_k c_k \overline{\mathbf{m}}_k = \overline{\mathbf{q}} + \sum_k (c_k - 1)\overline{\mathbf{m}}_k < \omega.$$

Since $\overline{\mathbf{q}} = \omega$, the last inequality is equivalent to $\sum_k (c_k - 1)\overline{\mathbf{m}}_k < 0$.

With h set to $\min(\overline{\mathbf{m}}_1, \overline{\mathbf{m}}_2)$ and noticing that $c_0 < 0 \Leftrightarrow -(c_0 - 1) > 1$, we equivalently have

$$\frac{(c_1 + c_2 - 2)h}{-(c_0 - 1)} \leq \frac{(c_1 - 1)\overline{\mathbf{m}}_1 + (c_2 - 1)\overline{\mathbf{m}}_2}{-(c_0 - 1)} < \overline{\mathbf{m}}_0.$$

In addition, we have

$$\sum_k c_k \geq 3 \Leftrightarrow c_1 + c_2 - 2 \geq -c_0 + 1,$$

which means that $h < \overline{\mathbf{m}}_0$.

We conclude that if $h = \overline{\mathbf{m}}_1$ (resp. $h = \overline{\mathbf{m}}_2$), $\overline{\mathbf{p}} + 2\overline{\mathbf{m}}_1 + \overline{\mathbf{m}}_2$ (resp. $\overline{\mathbf{p}} + \overline{\mathbf{m}}_1 + 2\overline{\mathbf{m}}_2$) is strictly smaller than $\overline{\mathbf{p}} + \sum_k \overline{\mathbf{m}}_k = \overline{\mathbf{q}} = \omega$ and thus, the point $\mathbf{p} + 2\mathbf{m}_1 + \mathbf{m}_2$ (resp. $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2$) is in $\mathbf{P}$.

For the second part, we similarly derive from $\sum_k (c_k - 1)\overline{\mathbf{m}}_k < 0$:

$$\frac{(\min(c_1, c_2) - 1)}{-(c_0 - 1)}(\overline{\mathbf{m}}_1 + \overline{\mathbf{m}}_2) \leq \frac{(c_1 - 1)\overline{\mathbf{m}}_1 + (c_2 - 1)\overline{\mathbf{m}}_2}{-(c_0 - 1)} < \overline{\mathbf{m}}_0.$$

Since we assume $(\min(c_1, c_2) - 1) > -c_0$, we have $\frac{(\min(c_1,c_2)-1)}{-(c_0-1)} \geq 1$ and it follows that $(\overline{\mathbf{m}}_1 + \overline{\mathbf{m}}_2) < \overline{\mathbf{m}}_0$.

As a consequence,

$$\overline{\mathbf{p}} + (c_0 + c_1)\overline{\mathbf{m}}_1 + (c_0 + c_2)\overline{\mathbf{m}}_2 < \overline{\mathbf{p}} + \sum_k c_k \overline{\mathbf{m}}_k = \overline{\mathbf{y}} < \omega,$$

which concludes. □

Recall that Λ_σ is the set $\{\mathbf{v}_{\sigma(0)} + \alpha\mathbf{m}_{\sigma(1)} + \beta\mathbf{m}_{\sigma(2)} \mid \alpha, \beta \in \mathbb{N}, \alpha + \beta \geq 1\}$.

The proof of the following three lemmas requires a lot of technical details. They are cited as auxiliary lemmas which we will introduce later in sec. 5 for the sake of readability. Here, we also introduce the notation (see Fig. 3 in sec. 5):

$$\forall k \in \mathbb{Z}/3\mathbb{Z},\ \mathbf{d}_k := \mathbf{m}_{k+1} - \mathbf{m}_{k+2}.$$

Lemma 5. *Let $\mathbf{y} = \mathbf{p} + \sum_k c_k\mathbf{m}_k$ such that $\sum_k c_k \geq 3$, and let $\mathbf{x}^\star \in \Lambda_\sigma$ be a point such that $\forall \mathbf{x} \in \Lambda_\sigma$, $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{x}$ (i.e. $\mathbf{x}^\star \in \arg\min_{\leq_{\mathbf{T}}}\{\mathbf{x} \in \Lambda_\sigma\}$). Suppose that there is a permutation $\sigma \in \Sigma$ such that $c_{\sigma(0)} < 0$ and $c_{\sigma(1)}, c_{\sigma(2)} > 0$. If $\mathbf{y} \in \mathbf{P}$ and if the interior of $\mathcal{B}$ contains no point of $\mathbf{P}$, then $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{y}$.*

Proof. We assume w.l.o.g. that σ is the identity. We also assume w.l.o.g. that $c_1 \leq c_2$ and consider three separate cases:

(i) $c_1 - 1 \leq c_2 \leq -c_0$,
(ii) $(c_1 - 1) \leq -c_0 < c_2$,
(iii) $-c_0 < (c_1 - 1) < c_2$.

Since $\mathbf{y} \in \mathbf{P}$, either $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2$ or $\mathbf{p} + 2\mathbf{m}_1 + \mathbf{m}_2$ is in $\mathbf{P}$ by Lemma 4. We assume w.l.o.g. that $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \in \mathbf{P}$ in the first two cases.

We have a stronger result in the third case. Indeed, $\mathbf{y} \in \mathbf{P}$ implies $\mathbf{p} + (c_0 + c_1)\mathbf{m}_1 + (c_0 + c_2)\mathbf{m}_2 \in \mathbf{P}$ by Lemma 4, which on its turn implies both $\mathbf{p}+\mathbf{m}_1+2\mathbf{m}_2 \in \mathbf{P}$ and $\mathbf{p}+2\mathbf{m}_1+\mathbf{m}_2 \in \mathbf{P}$ because $(c_0+c_1) \geq 2$ and $(c_0+c_2) \geq 2$.

Let $\mathbf{u} := -\mathbf{m}_0 + \mathbf{m}_1 + \mathbf{m}_2$. The first step of the proof is to show the following results:

$$\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \in \mathbf{P} \Rightarrow \begin{cases} \mathbf{d}_1 \cdot \mathbf{m}_2 \geq 0, & (6)\\ \mathbf{m}_2 \cdot \mathbf{u} \geq 0, & (7)\\ \mathbf{d}_1 \cdot \mathbf{u} \geq 0, & (8)\\ (-\mathbf{d}_2) \cdot \mathbf{u} \geq 0, & (9)\end{cases}$$

and

$$\mathbf{p} + 2\mathbf{m}_1 + \mathbf{m}_2 \in \mathbf{P} \Rightarrow \mathbf{m}_1 \cdot \mathbf{u} \geq 0. \tag{10}$$

Those results are used in a second step to complete the proof: (6), (7), (8), (9) are used in cases (i) and (ii), while (7) and (10) are used in case (iii).

First step: If $\mathbf{p}+\mathbf{m}_1+2\mathbf{m}_2$ is in $\mathbf{P}$, so is $\mathbf{p}+2\mathbf{m}_2$ (we have $\overline{\mathbf{p}} \geq 0$ by Theorem 1, then $0 < \overline{\mathbf{p}} + 2\overline{\mathbf{m}}_2 < \overline{\mathbf{p}} + \overline{\mathbf{m}}_1 + 2\overline{\mathbf{m}}_2 < \omega$ by Theorem 3). As $\mathcal{B}$ does not contain any point of $\mathbf{P}$ by hypothesis, $\mathbf{p}+2\mathbf{m}_2 \notin \mathcal{B}$. By rewriting

$$\mathbf{p} + 2\mathbf{m}_2 = \mathbf{v}_0 - \mathbf{d}_0 = \mathbf{v}_2 + \mathbf{d}_1 - \mathbf{d}_0 \,,$$

we can then apply Lemma 13 with the two vectors $(-\mathbf{d}_0), \mathbf{d}_1$ and the point $\mathbf{v}_2$ as origin to get $(-\mathbf{d}_0) \cdot \mathbf{d}_1 \geq 0$. From that, we finally get (6) by Lemma 10.

We can similarly get (7) and (10). To explain why, we focus on the case where $\mathbf{p}+\mathbf{m}_1+2\mathbf{m}_2$ is assumed to be in $\mathbf{P}$ because the other case is symmetric. Note first that $\mathbf{p}+\mathbf{m}_0 \in \mathbf{P}$ (same arguments as above for $\mathbf{p}+2\mathbf{m}_2$). As a consequence, both $\mathbf{p}+\mathbf{m}_1+2\mathbf{m}_2$ and $\mathbf{p}+\mathbf{m}_0$ are not in $\mathcal{B}$ by hypothesis. We can then apply Lemma 14 with the two vectors $\mathbf{m}_2, (-\mathbf{u})$ and the point $\mathbf{v}_0$ as origin (with $\mathbf{v}_0 + \mathbf{m}_2 - \mathbf{u} = \mathbf{v}_1$ on the boundary of $\mathcal{B}$) to get $\mathbf{m}_2 \cdot (-\mathbf{u}) \leq 0$ and thus (7).

By Lemma 11, (8) is a simple consequence of (7). It remains (9), whose proof is separated into two distinct cases.

If $(-\mathbf{d}_1 \cdot \mathbf{d}_2) \geq 0$, we straightforwardly have by Lemma 11 (9). Otherwise, i.e., if $(-\mathbf{d}_1 \cdot \mathbf{d}_2) < 0$, the point

$$\mathbf{p} + 2\mathbf{m}_2 = \mathbf{v}_0 - \mathbf{d}_1 + \mathbf{d}_2 = \mathbf{q} - \mathbf{u}$$

is necessarily in $\mathcal{B}$ by Lemma 13. Moreover, since no point of $\mathcal{B}$ belongs to $\mathbf{P}$, we deduce that $\mathbf{p}+2\mathbf{m}_2$ is not in $\mathbf{P}$.

We have therefore $\overline{\mathbf{q}} - \overline{\mathbf{u}} \geq \omega \Leftrightarrow \overline{\mathbf{u}} \leq 0$. It follows that $\overline{\mathbf{v}}_2 < \omega \Rightarrow \overline{\mathbf{v}}_2 + \overline{\mathbf{u}} < \omega$, which means that the point

$$\mathbf{v}_2 + \mathbf{u} = \mathbf{v}_0 + \mathbf{m}_1 = \mathbf{p} + 2\mathbf{m}_1 + \mathbf{m}_2$$

is in $\mathbf{P}$. In this case, we have (10) and as a consequence, (9) by Lemma 12.

Second step:

(i) One can check that

$$\begin{aligned}\mathbf{y} &= \mathbf{p} + \sum_k c_k \mathbf{m}_k \\ &= \mathbf{v}_0 + c_0\mathbf{m}_0 + (c_1-1)\mathbf{m}_1 + (c_2-1)\mathbf{m}_2 \\ &= \mathbf{v}_0 + (-c_0-c_1+1)(\mathbf{d}_1) + (-c_0-c_2+1)(-\mathbf{d}_2) + \big(\sum_k c_k - 2\big)\mathbf{u}.\end{aligned}$$

Let $\mathbf{w} := (-c_0-c_1+1)(\mathbf{d}_1) + (-c_0-c_2+1)(-\mathbf{d}_2) + \big(\sum_k c_k - 2\big)\mathbf{u}$. All its coefficients, i.e., $(-c_0-c_1+1)$, $(-c_0-c_2+1)$, $(\sum_k c_k - 2)$, are positive. Since we also have (8) and (9), we can apply Lemma 18 to show that $\delta^0_{\mathbf{T}}(\mathbf{m}_2, \mathbf{w}) \geq 0$, which means that $\mathbf{v}_0 + \mathbf{m}_2 \leq_{\mathbf{T}} \mathbf{v}_0 + \mathbf{w}$, where $\mathbf{v}_0 + \mathbf{w} = \mathbf{y}$. By transitivity, we have $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{v}_0 + \mathbf{m}_2 \leq_{\mathbf{T}} \mathbf{y}$.

(ii)

$$\begin{aligned}\mathbf{y} &= \mathbf{v}_0 + c_0\mathbf{m}_0 + (c_1-1)\mathbf{m}_1 + (c_2-1)\mathbf{m}_2 \\ &= \mathbf{v}_0 + (-c_0-c_1+1)(\mathbf{d}_1) + (c_0+c_2-1)(\mathbf{m}_2) + (c_1-1)(\mathbf{u}).\end{aligned}$$

As in (i), all coefficients, i.e., $(-c_0-c_1+1)$, (c_0+c_2-1), (c_1-1), are positive. From that and (6), (7), (8), (9), we can use Lemma 19 to get $\mathbf{x}^\star \leq_\mathbf{T} \mathbf{v}_0 + \mathbf{m}_2 \leq_\mathbf{T} \mathbf{y}$.

(iii)

$$\begin{aligned}\mathbf{y} &= \mathbf{v}_0 + c_0\mathbf{m}_0 + (c_1-1)\mathbf{m}_1 + (c_2-1)\mathbf{m}_2 \\ &= \mathbf{v}_0 + (c_0+c_1-1)(\mathbf{m}_1) + (c_0+c_2-1)(\mathbf{m}_2) + (-c_0)(\mathbf{u}).\end{aligned}$$

As in the previous cases, all coefficients, i.e., (c_0+c_1-1), (c_0+c_2-1), $(-c_0)$, are positive. From that and (7), (10), we can use Lemma 20 to get $\mathbf{x}^\star \leq_\mathbf{T} \mathbf{y}$. □

Lemma 6. *Let* $\mathbf{y} = \mathbf{p} + \sum_k c_k\mathbf{m}_k$ *such that* $\sum_k c_k \geq 3$. *If there is a permutation* $\sigma \in \Sigma$ *such that* $c_{\sigma(0)}, c_{\sigma(1)} \leq 0$, *then* $\mathbf{y} \in \mathbf{P}$ *implies both:*

- $\mathbf{p} + \mathbf{m}_{\sigma(0)} + 2\mathbf{m}_{\sigma(2)} \in \mathbf{P}$ *or* $\mathbf{p} + \mathbf{m}_{\sigma(1)} + 2\mathbf{m}_{\sigma(2)} \in \mathbf{P}$,
- $\mathbf{p} + 2\mathbf{m}_{\sigma(2)} \in \mathbf{P}$.

Proof. We assume w.l.o.g. that σ is the identity.

Since $\mathbf{y} \in \mathbf{P}$, we have

$$\overline{\mathbf{y}} = \overline{\mathbf{p}} + \sum_k c_k\overline{\mathbf{m}}_k = \overline{\mathbf{q}} + \sum_k (c_k-1)\overline{\mathbf{m}}_k < \omega.$$

Since $\overline{\mathbf{q}} = \omega$, the last inequality is equivalent to $\sum_k (c_k-1)\overline{\mathbf{m}}_k < 0$.

With h set to $\max(\overline{\mathbf{m}}_0, \overline{\mathbf{m}}_1)$ and noting that $(c_2-1) \geq 2$ (since $\sum_k c_k \geq 3$ and $c_0, c_1 \leq 0$), we equivalently have

$$\overline{\mathbf{m}}_2 \leq \frac{-(c_0-1)\overline{\mathbf{m}}_0 - (c_1-1)\overline{\mathbf{m}}_1}{c_2-1} < \frac{(-c_0-c_1+2)h}{c_2-1}.$$

In addition, we have

$$\sum_k c_k \geq 3 \Leftrightarrow c_2 - 1 \geq -c_0 - c_1 + 2,$$

which means that $\overline{\mathbf{m}}_2 < h$.

We conclude that if $h = \overline{\mathbf{m}}_0$ (resp. $h = \overline{\mathbf{m}}_1$), $\overline{\mathbf{p}} + \overline{\mathbf{m}}_1 + 2\overline{\mathbf{m}}_2$ (resp. $\overline{\mathbf{p}} + \overline{\mathbf{m}}_0 + 2\overline{\mathbf{m}}_2$) is strictly smaller than $\overline{\mathbf{p}} + \sum_k \overline{\mathbf{m}}_k = \overline{\mathbf{q}} = \omega$ and thus, the point $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2$ (resp. $\mathbf{p} + \mathbf{m}_0 + 2\mathbf{m}_2$) is in $\mathbf{P}$. *A fortiori* and whatever h is, $\mathbf{p} + 2\mathbf{m}_2 \in \mathbf{P}$. □

Lemma 7. *Suppose that the interior of* $\mathcal{B}$ *contains no point of* $\mathbf{P}$. *Let* $\mathbf{y} = \mathbf{p} + \sum_k c_k\mathbf{m}_k \in \mathbf{P}$ *such that* $\sum_k c_k \geq 3$, *If there is a permutation* $\sigma \in \Sigma$ *such that* $c_{\sigma(0)}, c_{\sigma(1)} \leq 0$ *and* $c_{\sigma(2)} > 0$, *then, for all points* $\mathbf{x}^\star \in \Lambda_\sigma$ *such that* $\forall \mathbf{x} \in \Lambda_\sigma, \mathbf{x}^\star \leq_\mathbf{T} \mathbf{x}$, *we also have* $\mathbf{x}^\star \leq_\mathbf{T} \mathbf{y}$.

Proof. We assume w.l.o.g. that σ is the identity.

Since $\mathbf{y} \in \mathbf{P}$, $\mathbf{p} + 2\mathbf{m}_2 \in \mathbf{P}$ by Lemma 6. That point, which is also at $\mathbf{v}_0 - \mathbf{d}_0 = \mathbf{v}_1 + \mathbf{d}_1$, is not in $\mathcal{B}$ by hypothesis and we can apply Lemma 13 with the two vectors $(-\mathbf{d}_0), \mathbf{d}_1$ and the point $\mathbf{v}_2$ as origin to get $(-\mathbf{d}_0) \cdot \mathbf{d}_1 \geq 0$.

Furthermore, either $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \in \mathbf{P}$ or $\mathbf{p} + 2\mathbf{m}_1 + \mathbf{m}_2 \in \mathbf{P}$ by Lemma 6. We assume below w.l.o.g. that $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \in \mathbf{P}$.

One can check that

$$\mathbf{y} = \mathbf{p} + \sum_k c_k \mathbf{m}_k = \mathbf{v}_0 + c_0\mathbf{m}_0 + (c_1 - 1)\mathbf{m}_1 + (c_2 - 1)\mathbf{m}_2$$
$$= \mathbf{v}_0 + (-c_1 + 1)(-\mathbf{d}_0) + (-c_0)\mathbf{d}_1 + (\sum_k c_k - 2)\mathbf{m}_2.$$

Let $\mathbf{w} := (-c_1 + 1)(-\mathbf{d}_0) + (-c_0)\mathbf{d}_1 + (\sum_k c_k - 2)\mathbf{m}_2$. All coefficients, i.e., $(-c_1 + 1)$, $(-c_0)$, $(\sum_k c_k - 2)$, are positive. We can thus apply Lemma 21 to show that $\delta^0_{\mathbf{T}}(\mathbf{m}_2, \mathbf{w}) \geq 0$. Equivalently, we have $\mathbf{v}_0 + \mathbf{m}_2 \leq_{\mathbf{T}} \mathbf{v}_0 + \mathbf{w}$, where $\mathbf{v}_0 + \mathbf{w} = \mathbf{y}$ and $\mathbf{v}_0 + \mathbf{m}_2 = \mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2$.

To end, we have by transitivity $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \leq_{\mathbf{T}} \mathbf{y}$ because $\mathbf{p} + \mathbf{m}_1 + 2\mathbf{m}_2 \in \Lambda_\sigma$. □

Now we have all the material required to prove Lemma 2:

Proof. of Lemma 2 For all $i \in \{0, \ldots, n-1\}$, the interior of $\mathcal{B}^{(i)}$ is assumed to contain no point of $\mathbf{P}$.

Let $\mathbf{x}^\star$ be the point chosen by the algorithm at step i, i.e., $\mathbf{x}^\star = \mathbf{T}^{(i+1)} \setminus \mathbf{T}^{(i)}$. We want to show that $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{y}$, for all $\mathbf{y} \in \mathbf{P} \cap \mathcal{H}^{(i)}_+$. Let $\mathbf{y}$ be denoted as $\mathbf{p}^{(i)} + \sum_k c_k \mathbf{m}^{(i)}_k$. Note that $\sum_k c_k \geq 3$ because $\mathbf{y} \in \mathcal{H}^{(i)}_+$ and that the coefficients cannot be all strictly positive by Lemma 3, because $\mathbf{y} \in \mathbf{P}$.

1. if one coefficient is zero and the others are strictly positive, then $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{y}$ by the design of the algorithm,
2. if one coefficient is strictly negative and the others are strictly positive, then there exist a permutation σ and a point $\mathbf{x} \in \Lambda_\sigma$ such that $\mathbf{x} \leq_{\mathbf{T}} \mathbf{y}$ by Lemma 5 . Then $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{x}$ by the design of the algorithm.
3. if one coefficient is strictly positive and the others are strictly negative or null, then , similarly, there exist a permutation σ and a point $\mathbf{x} \in \Lambda_\sigma$ such that $\mathbf{x} \leq_{\mathbf{T}} \mathbf{y}$ by Lemma 7 . Then $\mathbf{x}^\star \leq_{\mathbf{T}} \mathbf{x}$ by the design of the algorithm.

Since there is no other possibility, the proof is complete. □

5 Technical details

5.1 Projection-based results

Let us first introduce the following notations (k is taken modulo 3, see Fig. 3):

$$\forall i \in \{0, \ldots, n\}, \begin{cases} \forall k,\ \mathbf{d}^{(i)}_k := \mathbf{m}^{(i)}_{k+1} - \mathbf{m}^{(i)}_{k+2}, \\ \forall k,\ \hat{\mathbf{N}}^{(i)}_k := \mathbf{m}^{(i)}_{k+1} \times \mathbf{m}^{(i)}_{k+2}, \\ \sum_{k \in \{0,1,2\}} \hat{\mathbf{N}}^{(i)}_k =: \hat{\mathbf{N}}((\mathbf{T}^{(i)})). \end{cases} \tag{11}$$

Note that the following equality also holds for the estimated normal vector, which is normal to the current triangle:

$$\forall i \in \{0, \ldots, n\},\ \forall k,\ \hat{\mathbf{N}}((\mathbf{T}^{(i)})) = \mathbf{d}^{(i)}_k \times \mathbf{d}^{(i)}_{k+1}.$$

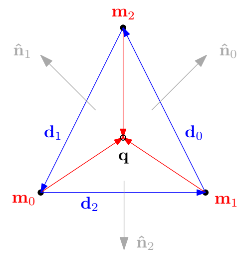


Fig. 3: Definitions of (11).

Lemma 8. *For all steps* $i \in \{0, \ldots, n\}$, $\forall k$, $\hat{\mathbf{N}}_k^{(i)} \cdot \hat{\mathbf{N}}_{k+1}^{(i)} \geq 0$ *and* $\hat{\mathbf{N}}_k^{(i)} \cdot \hat{\mathbf{N}}((\mathbf{T}^{(i)})) > 0$.

Proof. Note that the second inequality is a simple consequence of the first one. We now focus on the first one and prove it by induction.

Base case: When $i = 0$, the traingle $\mathbf{T}^{(0)}$ and $\mathbf{q}$ forms a trirectangular tetrahedron. We have $\forall k$, $\hat{\mathbf{N}}_k^{(0)} \cdot \hat{\mathbf{N}}_{k+1}^{(0)} = 0$ and $\hat{\mathbf{N}}_k^{(0)} \cdot \hat{\mathbf{N}}((\mathbf{T}^{(0)})) > 0$.

Induction case: we now assume that for any $i \in \{0, \ldots, n-1\}$, $\forall k$, $\hat{\mathbf{N}}_k^{(i)} \cdot \hat{\mathbf{N}}_{k+1}^{(i)} \geq 0$ and $\hat{\mathbf{N}}_k^{(i)} \cdot \hat{\mathbf{N}}((\mathbf{T}^{(i)})) > 0$. By the update rule (see 4), we straightforwardly have:

$$\hat{\mathbf{N}}_{\tau(0)}^{(i+1)} = \hat{\mathbf{N}}_{\tau(0)}^{(i)}, \quad \hat{\mathbf{N}}_{\tau(1)}^{(i+1)} = \hat{\mathbf{N}}_{\tau(1)}^{(i)} + \alpha \hat{\mathbf{N}}_{\tau(0)}^{(i)}, \quad \hat{\mathbf{N}}_{\tau(2)}^{(i+1)} = \hat{\mathbf{N}}_{\tau(2)}^{(i)} + \beta \hat{\mathbf{N}}_{\tau(0)}^{(i)},$$

and

$$\begin{aligned}
\hat{\mathbf{N}}_{\tau(0)}^{(i+1)} \cdot \hat{\mathbf{N}}_{\tau(1)}^{(i+1)} &= \hat{\mathbf{N}}_{\tau(0)}^{(i)} \cdot \hat{\mathbf{N}}_{\tau(1)}^{(i)} + \alpha \|\hat{\mathbf{N}}_{\tau(0)}^{(i)}\|^2, \\
\hat{\mathbf{N}}_{\tau(1)}^{(i+1)} \cdot \hat{\mathbf{N}}_{\tau(2)}^{(i+1)} &= \hat{\mathbf{N}}_{\tau(1)}^{(i)} \cdot \hat{\mathbf{N}}_{\tau(2)}^{(i)} + \alpha(\hat{\mathbf{N}}_{\tau(0)}^{(i)} \cdot \hat{\mathbf{N}}_{\tau(2)}^{(i)}) + \beta(\hat{\mathbf{N}}_{\tau(1)}^{(i)} \cdot \hat{\mathbf{N}}_{\tau(0)}^{(i)}) + \alpha\beta\big(\hat{\mathbf{N}}_{\tau(0)}^{(i)}\big)^2, \\
\hat{\mathbf{N}}_{\tau(2)}^{(i+1)} \cdot \hat{\mathbf{N}}_{\tau(0)}^{(i+1)} &= \hat{\mathbf{N}}_{\tau(2)}^{(i)} \cdot \hat{\mathbf{N}}_{\tau(0)}^{(i)} + \beta \|\hat{\mathbf{N}}_{\tau(0)}^{(i)}\|^2.
\end{aligned}$$

The Induction hypothesis implies that $\forall k$, $\hat{\mathbf{N}}_k^{(i+1)} \cdot \hat{\mathbf{N}}_{k+1}^{(i+1)} \geq 0$ and $\hat{\mathbf{N}}_k^{(i+1)} \cdot \hat{\mathbf{N}}((\mathbf{T}^{(i+1)})) > 0$ because it is equal to $\|\hat{\mathbf{N}}_k^{(i+1)}\|^2 > 0$ plus some other positive terms. $\square$

From now on, we omit once again the exponent (i) for clarity. We go on with this purely geometrical result (see Fig. 4):

Lemma 9.
Let $\mathbf{d}$ *and* $\mathbf{d}'$ *be two vectors that span a plane of normal* $\mathbf{N} := \mathbf{d}' \times \mathbf{d}$. *Let* $\mathbf{m}$ *be another vector that projects along* $\mathbf{N}$ *into the interior of the convex combination of* $\mathbf{d}$ *and* $\mathbf{d}'$, *i.e.* $(\mathbf{N} \times \mathbf{d}) \cdot \mathbf{m} < 0$ *and* $(\mathbf{N} \times \mathbf{d}') \cdot \mathbf{m} > 0$. *If* $\mathbf{d} \cdot \mathbf{d}' \geq 0$, *then* $\mathbf{d} \cdot \mathbf{m} > 0$ *and* $\mathbf{d}' \cdot \mathbf{m} > 0$.

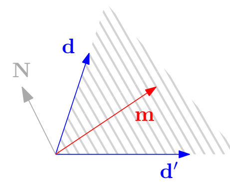


Fig. 4: Illustration of Lemma 9. Note that $\mathbf{m}$ do not belong to the span of $\mathbf{d}$ and $\mathbf{d}'$. However, it projects along $\mathbf{N}$ into the interior of the convex combination of $\mathbf{d}$ and $\mathbf{d}'$ (hatched area).

Proof. We first expand $(\mathbf{N}\times\mathbf{d})\cdot\mathbf{m} < 0$, which is equivalent to $(\mathbf{d}\times\mathbf{m})\cdot(\mathbf{d}\times\mathbf{d}') > 0$, using the scalar quadruple product rule:

$$\|\mathbf{d}\|^2\mathbf{d}'\cdot\mathbf{m} - (\mathbf{d}\cdot\mathbf{d}')\mathbf{d}\cdot\mathbf{m} > 0. \tag{12}$$

We then similarly expand $(\mathbf{d}'\times\mathbf{m})\cdot(\mathbf{d}\times\mathbf{d}') < 0$:

$$(\mathbf{d}\cdot\mathbf{d}')\mathbf{d}'\cdot\mathbf{m} - \|\mathbf{d}'\|^2\mathbf{d}\cdot\mathbf{m} < 0. \tag{13}$$

If $\mathbf{d}\cdot\mathbf{d}' = 0$, we can conclude.

If not, by the hypothesis, we suppose that $\mathbf{d}\cdot\mathbf{d}' > 0$ in the following.

We now derive lower and upper bounds for $\mathbf{d}'\cdot\mathbf{m}$, respectively from (12) and (13):

$$\frac{(\mathbf{d}\cdot\mathbf{d}')}{\|\mathbf{d}\|^2}\mathbf{d}\cdot\mathbf{m} < \mathbf{d}'\cdot\mathbf{m} < \frac{\|\mathbf{d}'\|^2}{(\mathbf{d}\cdot\mathbf{d}')}\mathbf{d}\cdot\mathbf{m}. \tag{14}$$

Multiplying both sides by $\|\mathbf{d}\|^2$ and $(\mathbf{d}\cdot\mathbf{d}')$ leads to:

$$\|\mathbf{d}\cdot\mathbf{d}'\|^2\mathbf{d}\cdot\mathbf{m} < \|\mathbf{d}'\|^2\|\mathbf{d}\|^2\mathbf{d}\cdot\mathbf{m} \Leftrightarrow \Big(\|\mathbf{d}\cdot\mathbf{d}'\|^2 - \|\mathbf{d}'\|^2\|\mathbf{d}\|^2\Big)(\mathbf{d}\cdot\mathbf{m}) < 0.$$

Since $\|\mathbf{d}\cdot\mathbf{d}'\|^2 \leq \|\mathbf{d}'\|^2\|\mathbf{d}\|^2$, we conclude that $\mathbf{d}\cdot\mathbf{m} > 0$. In addition, since $\mathbf{d}\cdot\mathbf{m} > 0$ and $\mathbf{d}\cdot\mathbf{d}' > 0$, it follows from (14) that $\mathbf{d}'\cdot\mathbf{m} > 0$. □

That last lemma can be related to the current triangle (see Fig. 5 and the following lemmas).

Lemma 10. *For all k, if $\mathbf{d}_k\cdot\mathbf{d}_{k+1} \leq 0$, then $\mathbf{d}_{k+1}\cdot\mathbf{m}_{k+2} > 0$ and $\mathbf{d}_k\cdot\mathbf{m}_{k+2} < 0$.*

Proof. We use Lemma 9, with $\mathbf{d}, \mathbf{d}', \mathbf{m}$ respectively set to $(-\mathbf{d}_k)$, $\mathbf{d}_{k+1}$ and $\mathbf{m}_{k+2}$. Note that the normal $\mathbf{d}_k\times\mathbf{d}_{k+1}$ is by definition equal to $\hat{\mathbf{N}}((\mathbf{T}))$. Note also that Lemma 8 implies (see section B):

$$\begin{aligned}\big(\hat{\mathbf{N}}((\mathbf{T}))\times(-\mathbf{d}_k)\big)\cdot\mathbf{m}_{k+2} < 0,\\ \big(\hat{\mathbf{N}}((\mathbf{T}))\times\mathbf{d}_{k+1}\big)\cdot\mathbf{m}_{k+2} > 0,\end{aligned}$$

which the projection criterion of Lemma 9.

Since we assume in addition that $(-\mathbf{d}_k)\cdot\mathbf{d}_{k+1} \geq 0$, we conclude by Lemma 9 that $(-\mathbf{d}_k)\cdot\mathbf{m}_{k+2} > 0$ and $\mathbf{d}_{k+1}\cdot\mathbf{m}_{k+2} > 0$. □

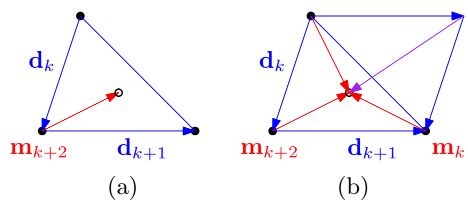


Fig. 5: Illustration of Lemma 10 in (a) and Lemma 11 in (b).

Likewise,

Lemma 11. *For all k, if $\mathbf{d}_k \cdot \mathbf{d}_{k+1} \leq 0$, then $\mathbf{d}_k \cdot (\mathbf{d}_k + \mathbf{m}_k) > 0$ and $\mathbf{d}_{k+1} \cdot (\mathbf{d}_k + \mathbf{m}_k) < 0$.*

Proof. We use Lemma 9, with $\mathbf{d}, \mathbf{d}', \mathbf{m}$ respectively set to $(-\mathbf{d}_k)$, $\mathbf{d}_{k+1}$ and $-(\mathbf{d}_k + \mathbf{m}_k)$. Note that the normal is equal to $\hat{\mathbf{N}}((\mathbf{T}))$ and the projection criterion is implied by Lemma 8 (see section B):

$$\big(\hat{\mathbf{N}}((\mathbf{T})) \times (-\mathbf{d}_k)\big) \cdot \big(-(\mathbf{d}_k + \mathbf{m}_k)\big) < 0,$$
$$\big(\hat{\mathbf{N}}((\mathbf{T})) \times \mathbf{d}_{k+1}\big) \cdot \big(-(\mathbf{d}_k + \mathbf{m}_k)\big) > 0.$$

From Lemma 9, we thus have $\mathbf{d}_k \cdot (\mathbf{d}_k + \mathbf{m}_k) > 0$ and $\mathbf{d}_{k+1} \cdot (\mathbf{d}_k + \mathbf{m}_k) > 0$. □

Finally,

Lemma 12. *For all k, if $\mathbf{m}_k \cdot (\mathbf{d}_k + \mathbf{m}_k) \geq 0$, then $\mathbf{d}_{k+1} \cdot \mathbf{m}_k < 0$ and $\mathbf{d}_{k+1} \cdot (\mathbf{d}_k + \mathbf{m}_k) < 0$. Similarly, if $\mathbf{m}_{k+1} \cdot (\mathbf{d}_k + \mathbf{m}_k) \geq 0$, then $\mathbf{d}_k \cdot \mathbf{m}_{k+1} > 0$ and $\mathbf{d}_k \cdot (\mathbf{d}_k + \mathbf{m}_k) > 0$.*

Proof. We focus on the first part, because the proof of the second part is quite similar.

We use Lemma 9, with $\mathbf{d}, \mathbf{d}', \mathbf{m}$ respectively set to $(\mathbf{d}_k + \mathbf{m}_k)$, $\mathbf{m}_k$ and $(-\mathbf{d}_{k+1})$. Note that the normal is equal to $\mathbf{m}_k \times (\mathbf{d}_k + \mathbf{m}_k) = \hat{\mathbf{N}}_{k+1} + \hat{\mathbf{N}}_{k+2}$ and the projection criterion is implied by Lemma 8 (see section B):

$$\big((\hat{\mathbf{N}}_{k+1} + \hat{\mathbf{N}}_{k+2}) \times (\mathbf{d}_k + \mathbf{m}_k)\big) \cdot (-\mathbf{d}_{k+1}) < 0,$$
$$\big((\hat{\mathbf{N}}_{k+1} + \hat{\mathbf{N}}_{k+2}) \times \mathbf{m}_k\big) \cdot (-\mathbf{d}_{k+1}) > 0.$$

From Lemma 9, we thus have $\mathbf{d}_{k+1} \cdot \mathbf{m}_k < 0$ and $\mathbf{d}_{k+1} \cdot (\mathbf{d}_k + \mathbf{m}_k) < 0$, which concludes. □

5.2 Circumsphere-based results

In this section, we show some general circumsphere-based results.

Lemma 13. *Let two non-zero vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ and a sphere passing through the origin o, as well as through $o + \mathbf{u}$ and $o + \mathbf{w}$. The point $o + \mathbf{u} + \mathbf{w}$ lies in the sphere if and only if $\mathbf{u} \cdot \mathbf{w} \leq 0$.*

Proof. Since o, $o+\mathbf{u}$, $o+\mathbf{w}$ and $o+\mathbf{u}+\mathbf{w}$ is a planar parallelogram, we can focus on the intersection between the sphere and the plane containing the parallelogram, which is a circle (see Fig. 6-(a)). If the point $o+\mathbf{u}+\mathbf{w}$ lies in (resp. outside of) the closed disk, we know that there exists a real number $s \geq 1$ (resp. $s < 1$) such that $o+s\mathbf{u}$ lies on the circle. Then, $\pi = \angle(o+\mathbf{u})(o+s\mathbf{u}+\mathbf{w})(o+\mathbf{w}) + \angle(o+\mathbf{u})(o)(o+\mathbf{w}) \leq 2\angle(o+\mathbf{u})(o)(o+\mathbf{w})$ (resp. $>$ for the final inequality). That is to say, $\mathbf{u}\cdot\mathbf{w} \leq 0$ (resp. $>$). □

Lemma 14. *Let two non-zero vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ and a closed ball whose border passes through the origin o and the point $o+\mathbf{u}+\mathbf{w}$. If $o+\mathbf{u}$ and $o+\mathbf{w}$ do not lie in the interior the ball, then $\mathbf{u}\cdot\mathbf{w} \leq 0$.*

Proof. Let us prove the statement by contradiction. We focus on the plane including $o, o+\mathbf{u}$, $o+\mathbf{w}$ (and $o+\mathbf{u}+\mathbf{w}$). In this plane, if $\mathbf{u}\cdot\mathbf{w} > 00$, one half of the disk of diameter $[o, o+\mathbf{u}+\mathbf{w}]$ contains $o+\mathbf{u}$, whereas the other contains $o+\mathbf{w}$. Furthermore, any other disk whose border passes through o and $o+\mathbf{u}+\mathbf{w}$ must include one of the previous halves, thus one of the two points. Since any ball whose border passes through o and $o+\mathbf{u}+\mathbf{w}$ covers such a disk, the result follows (see Fig. 6-(b)). □

Lemma 15. *Let a non-zero vector $\mathbf{u} \in \mathbb{R}^3$ and a closed ball whose border passes through the origin o and the point $o+\mathbf{u}$. No point $o+\delta\mathbf{u}$ such that $\delta > 1$ lies in the ball.*

Proof. The intersection between the ball and the ray starting from o in direction $\mathbf{u}$ is the segment $[o, o+\mathbf{u}]$, which is equal, by convexity, to the set $\{o+\delta'\mathbf{u}\}_{0\leq\delta'\leq 1}$. The points $o+\delta\mathbf{u}$ such that $\delta > 1$ do not lie in that set and therefore do not lie in the ball (see Fig. 6-(c)). □

Lemma 16. *Let two vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ and a sphere (or the circle) passing through the origin o, $o+\mathbf{u}$ and $o+\mathbf{w}$. No point of the convex combination of $-\mathbf{u}$ and $-\mathbf{w}$ issued from o lies in the interior of the sphere (or circle).*

Proof. The points o, $o+\mathbf{u}$, $o+\mathbf{w}$ and any points of form $o-a\mathbf{u}-b\mathbf{w}$ with $(a+b) > 0$ share the same plane, thus the proof can be restricted to the circumcircle case.

We denote C the circle that passes through the origin o, $o+\mathbf{u}$ and $o+\mathbf{w}$ as well as $\tilde{C}$ the circle that passes through the origin o, $o-\mathbf{u}$ and $o-\mathbf{w}$ (see Fig. 6-(d)).

First, we prove that o is the only contact point between the two circles C and $\tilde{C}$. In fact, if there exists a vector $\mathbf{v} \neq 0$ such that $o+\mathbf{v}$ is the second contact point, then by symmetry we know that $o-\mathbf{v}$ is also a contact point. It is impossible for two circles to have three contact point which are aligned. Therefore, we prove that o is the only contact point between the two circles.

Since the two circles share a unique contact point, there exist a straight line L that separates the two circles. All points of the convex combination of $-\mathbf{u}$ and $-\mathbf{w}$ lies in the half plane that includes $\tilde{C}$, thus none of them are in the circle C. □

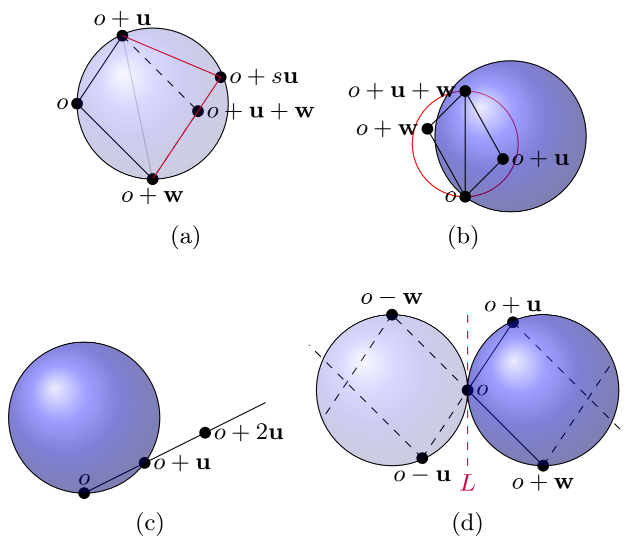


Fig. 6: Illustrations for (a) lemma 13, (b) lemma 14, (c) lemma 15, and (d) lemma 16.

5.3 Closeness results

In this subsection, we demonstrated some technical lemmas that are used in the proof of lemma 5 and lemma 7.

We recall the notation $\delta^0_{\mathbf{T}}(\cdot,\cdot)$ used in [15] with the relation

$$\delta^0_{\mathbf{T}}(\mathbf{x},\mathbf{y}) \geq 0 \Leftrightarrow \mathbf{x} \leq_{\mathbf{T}} \mathbf{y}, \tag{15}$$

and the identity

$$\delta^0_{\mathbf{T}}(\mathbf{z},\mathbf{z}'+\mathbf{z}'') = \delta^0_{\mathbf{T}}(\mathbf{z},\mathbf{z}') + \delta^0_{\mathbf{T}}(\mathbf{z},\mathbf{z}'') + \left(2\mathbf{z}'\cdot\mathbf{z}''\right)\det\left[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{z}\right]. \tag{16}$$

Lemma 17. *Let* $\mathbf{u} := -\mathbf{m}_0+\mathbf{m}_1+\mathbf{m}_2$*. If* $\mathbf{d}_1\cdot\mathbf{u}\geq 0$ *(resp.* $(-\mathbf{d}_2)\cdot\mathbf{u}\geq 0$*), then* $\delta^0_{\mathbf{T}}(\mathbf{m}_1,a\mathbf{u})\geq 0$ *(resp.* $\delta^0_{\mathbf{T}}(\mathbf{m}_2,a\mathbf{u})\geq 0$*) for all* $a\in\mathbb{N}$*.*

Proof. The lemma is trivially true for $a=0$ and we can safely assume that $a\geq 1$.

Base case: Using Lemma 13 with the vectors $\mathbf{d}_1$, $\mathbf{u}$ and the origin set to $\mathbf{v}_2$, $\mathbf{d}_1\cdot\mathbf{u}\geq 0$ implies that the sphere passing through $\mathbf{T}$ and $\mathbf{v}_2+\mathbf{u}=\mathbf{v}_0+\mathbf{m}_1$ does not include $\mathbf{v}_2+\mathbf{d}_1+\mathbf{u}=\mathbf{v}_0+\mathbf{u}$ in its interior. That means that $\delta^0_{\mathbf{T}}(\mathbf{m}_1,\mathbf{u})\geq 0$ and we can similarly show that $\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{u})\geq 0$ if $(-\mathbf{d}_2)\cdot\mathbf{u}\geq 0$.

Induction step: Let $\mathbf{m}$ be either $\mathbf{m}_1$ or $\mathbf{m}_2$. We now assume that for some α such that $1\leq\alpha<a$, $\delta^0_{\mathbf{T}}(\mathbf{m},\alpha\mathbf{u})\geq 0$ and we want to show that $\delta^0_{\mathbf{T}}(\mathbf{m},(\alpha+1)\mathbf{u})\geq 0$.

By (16), we have

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m},(\alpha+1)\mathbf{u}) = \;& \delta^0_{\mathbf{T}}(\mathbf{m},\alpha\mathbf{u}) + \delta^0_{\mathbf{T}}(\mathbf{m},\mathbf{u}) \\ &+ 2\alpha(\mathbf{u}\cdot\mathbf{u})\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}].\end{aligned}$$

Since $\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}] = \det[\mathbf{m}_0,\mathbf{m}_1,\mathbf{m}_2]$, which is equal to 1 by Theorem 2, the whole sum is strictly positive due to the induction hypothesis and the base case. □

Lemma 18. *Let* $\mathbf{u} := -\mathbf{m}_0+\mathbf{m}_1+\mathbf{m}_2$ *and* $\mathbf{w} := a(\mathbf{d}_1)+b(-\mathbf{d}_2)+c(\mathbf{u})$, *with* $a,b,c\geq 0$. *If* $\mathbf{d}_1\cdot\mathbf{u}\geq 0$ *and* $(-\mathbf{d}_2)\cdot\mathbf{u}\geq 0$, *then* $\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{w})\geq 0$.

Proof. By (16), we have

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{w}) = \;& \delta^0_{\mathbf{T}}\big(\mathbf{m}_2,a\mathbf{d}_1+b(-\mathbf{d}_2)\big) + \delta^0_{\mathbf{T}}(\mathbf{m}_2,c\mathbf{u}) \\ &+ 2\Big(\big(a\mathbf{d}_1+b(-\mathbf{d}_2)\big)\cdot c\mathbf{u}\Big)\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}_2].\end{aligned}$$

One can easily check that $\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}_2] = \det[\mathbf{m}_0,\mathbf{m}_1,\mathbf{m}_2]$, which is equal to 1 by Theorem 2.

Furthermore,

- $\delta^0_{\mathbf{T}}\big(\mathbf{m}_2,a\mathbf{d}_1+b(-\mathbf{d}_2)\big)\geq 0$ by Lemma 16 (consider the convex combination $\{\mathbf{v}_0+a\mathbf{d}_1-b\mathbf{d}_2 | a+b>0\}$), and the sphere passing by $\mathbf{v}_0+\mathbf{m}_2$ and the three vertices of $\mathbf{T}$.
- since $(-\mathbf{d}_2)\cdot\mathbf{u}\geq 0$, $\delta^0_{\mathbf{T}}(\mathbf{m}_2,c\mathbf{u})\geq 0$ by Lemma 17,
- $\big(a\mathbf{d}_1+b(-\mathbf{d}_2)\big)\cdot c\mathbf{u}\geq 0$ because of the hypotheses.

Since the three terms are positive, so is the whole sum. □

Lemma 19. *Let* $\mathbf{u} := -\mathbf{m}_0+\mathbf{m}_1+\mathbf{m}_2$ *and* $\mathbf{w} := a(\mathbf{d}_1)+b(\mathbf{m}_2)+c(\mathbf{u})$, *with* $a,b,c\geq 0$. *If* $\mathbf{m}_2\cdot\mathbf{u}\geq 0, \mathbf{d}_1\cdot\mathbf{u}\geq 0, (-\mathbf{d}_2)\cdot\mathbf{u}\geq 0$ *and* $\mathbf{d}_1\cdot\mathbf{m}_2\geq 0$, *then* $\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{w})\geq 0$.

Proof. By (16), we have

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{w}) = \;& \delta^0_{\mathbf{T}}(\mathbf{m}_2,a\mathbf{d}_1+b\mathbf{m}_2) + \delta^0_{\mathbf{T}}(\mathbf{m}_2,c\mathbf{u}) \\ &+ 2\Big(\big(a\mathbf{d}_1+b\mathbf{m}_2\big)\cdot c\mathbf{u}\Big)\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}_2].\end{aligned}$$

One can easily check that $\det[\mathbf{d}_2,-\mathbf{d}_1,\mathbf{m}_2] = \det[\mathbf{m}_0,\mathbf{m}_1,\mathbf{m}_2]$, which is equal to 1 by Theorem 2.

In addition, we use (16) again to decompose the first term and finally get

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m}_2,\mathbf{w}) = \;& \delta^0_{\mathbf{T}}\big(\mathbf{m}_2,a\mathbf{d}_1\big) + \delta^0_{\mathbf{T}}\big(\mathbf{m}_2,b\mathbf{m}_2\big) + \delta^0_{\mathbf{T}}(\mathbf{m}_2,c\mathbf{u}) \\ &+ 2ab\big(\mathbf{d}_1\cdot\mathbf{m}_2\big) + 2\big(a\mathbf{d}_1+b\mathbf{m}_2\big)\cdot c\mathbf{u}.\end{aligned}$$

We can now prove that each term of the sum is positive:

- we can use Lemma 15 to show that the first two terms are positive or null,

– since $(-\mathbf{d}_2) \cdot \mathbf{u} \geq 0$, $\delta^0_{\mathbf{T}}(\mathbf{m}_2, c\mathbf{u}) \geq 0$ by Lemma 17,
– all scalar products of the last two terms are positive or null due to the hypotheses.

□

Lemma 20. *Let* $\mathbf{u} := -\mathbf{m}_0 + \mathbf{m}_1 + \mathbf{m}_2$*,* $\mathbf{w} := a(\mathbf{m}_1) + b(\mathbf{m}_2) + c(\mathbf{u})$*, with* $a, b, c \geq 0$*. Let* Λ *be the set* $\{\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 \mid \alpha, \beta \in \mathbb{N}, \alpha + \beta \geq 1\}$ *and* $\mathbf{w}' \in \Lambda$ *be such that* $\forall \mathbf{w}'' \in \Lambda, \delta^0_{\mathbf{T}}(\mathbf{w}', \mathbf{w}'') \geq 0$*. If* $\mathbf{m}_1 \cdot \mathbf{u} \geq 0$ *and* $\mathbf{m}_2 \cdot \mathbf{u} \geq 0$*, then* $\delta^0_{\mathbf{T}}(\mathbf{w}', \mathbf{w}) \geq 0$*.*

Proof. By (16), we have

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{w}', \mathbf{w}) = \; & \delta^0_{\mathbf{T}}\big(\mathbf{w}', a\mathbf{m}_1 + b\mathbf{m}_2)\big) + \delta^0_{\mathbf{T}}(\mathbf{w}', c\mathbf{u}) \\ & + 2\Big(\big(a\mathbf{m}_1 + b\mathbf{m}_2)\big) \cdot c\mathbf{u}\Big) \det[\mathbf{d}_2, -\mathbf{d}_1, \mathbf{w}'].\end{aligned}$$

Let $\mathbf{w}' = a'\mathbf{m}_1 + b'\mathbf{m}_2$. One can easily check that

$$\det[\mathbf{d}_2, -\mathbf{d}_1, \mathbf{w}'] = (a' + b') \det[\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2] = (a' + b') \geq 1.$$

The first and third terms of the sum are obviously positive due to the hypotheses. To show that the whole sum is positive, it remains to show that the second term is also positive.

By Lemma 12, $\mathbf{m}_2 \cdot \mathbf{u} \geq 0 \Rightarrow \mathbf{d}_1 \cdot \mathbf{u} > 0$. From the last inequality, we have by Lemma 17, $\delta^0_{\mathbf{T}}(\mathbf{m}_1, c\mathbf{u}) \geq 0$, which means that $\mathbf{v}_0 + \mathbf{m}_1 \leq_{\mathbf{T}} \mathbf{v}_0 + c\mathbf{u}$. However, since $\mathbf{v}_0 + \mathbf{w}' \leq_{\mathbf{T}} \mathbf{v}_0 + \mathbf{m}_1$, we have by transitivity $\mathbf{v}_0 + \mathbf{w}' \leq_{\mathbf{T}} \mathbf{v}_0 + c\mathbf{u}$, i.e., $\delta^0_{\mathbf{T}}(\mathbf{w}', c\mathbf{u}) \geq 0$. □

Lemma 21. *Let* $\mathbf{w} := a(-\mathbf{d}_0) + b(\mathbf{d}_1) + c(\mathbf{m}_2)$*, with* $a, b, c \geq 0$*. If* $(-\mathbf{d}_0) \cdot \mathbf{d}_1 \geq 0$*, then* $\delta^0_{\mathbf{T}}(\mathbf{m}_2, \mathbf{w}) \geq 0$*.*

Proof. By (16), we have

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m}_2, \mathbf{w}) = \; & \delta^0_{\mathbf{T}}\big(\mathbf{m}_2, a(-\mathbf{d}_0) + b(\mathbf{d}_1)\big) + \delta^0_{\mathbf{T}}(\mathbf{m}_2, c\mathbf{m}_2) \\ & + 2\Big(\big(a(-\mathbf{d}_0) + b\mathbf{d}_1\big) \cdot c\mathbf{m}_2\Big) \det[\mathbf{d}_2, -\mathbf{d}_1, \mathbf{m}_2].\end{aligned}$$

One can easily check that $\det[\mathbf{d}_2, -\mathbf{d}_1, \mathbf{m}_2] = \det[\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2]$, which is equal to 1 by Theorem 2.

In addition, we use (16) again to decompose the first term and finally get

$$\begin{aligned}\delta^0_{\mathbf{T}}(\mathbf{m}_2, \mathbf{w}) = \; & \delta^0_{\mathbf{T}}\big(\mathbf{m}_2, a(-\mathbf{d}_0)\big) + \delta^0_{\mathbf{T}}\big(\mathbf{m}_2, b(\mathbf{d}_1)\big) + \delta^0_{\mathbf{T}}(\mathbf{m}_2, c\mathbf{m}_2) \\ & + 2ab\big((-\mathbf{d}_0) \cdot \mathbf{d}_1\big) + 2\big(a(-\mathbf{d}_0) + b\mathbf{d}_1\big) \cdot c\mathbf{m}_2,\end{aligned}$$

Note first that $\forall \mathbf{u} \in \mathbb{Z}^3, \delta^0_{\mathbf{T}}\big(\mathbf{u}, (-\mathbf{d}_0)\big) = (-\mathbf{d}_0) \cdot \mathbf{d}_1 \geq 0$.

We can now prove that each term of the sum is positive:

– we can use Lemma 15 to show that the first three terms are positive or null. (For the first one, we have to notice that $(-\mathbf{d}_0) \cdot \mathbf{d}_1 \geq 0$ implies $\delta^0_{\mathbf{T}}\big(\mathbf{m}_2, (-\mathbf{d}_0)\big) \geq 0$ by lemma 13).

– $\big((-\mathbf{d}_0)\cdot\mathbf{d}_1\big) \geq 0$ by hypothesis,
– for the sign of $\big(a(-\mathbf{d}_0)+b\mathbf{d}_1\big)\cdot c\mathbf{m}_2$, it is enough to note that the hypothesis also implies $\mathbf{m}_2\cdot(-\mathbf{d}_0) > 0$ and $\mathbf{m}_2\cdot\mathbf{d}_1 > 0$ (by Lemma 10). As a consequence, $\big(a(-\mathbf{d}_0)+b\mathbf{d}_1\big)\cdot c\mathbf{m}_2$ develops into two positive scalar products and is therefore positive.

□

6 Bound on the maximum distance

In ths section, we focus on the last step, where $i = n$. We will demonstrate how to obtain an upper bound for the magnitude of last three vectors $\mathbf{m}_k^{(n)}$, which we name it as the max distance.

Suppose that the l_2-norm of the vectors $\mathbf{v}_1^{(n)} - \mathbf{v}_0^{(n)}$ and $\mathbf{v}_2^{(n)} - \mathbf{v}_0^{(n)}$ are smaller than the l_2-norm of the vector $\mathbf{v}_1^{(n)} - \mathbf{v}_2^{(n)}$. We note w.l.o.g. the three length as $a \leq b \leq c$. The corollary 2 states that the L-algorithm returns a minimal basis of the lattice of upper leaning points. Then we have the following relation [12]:

$$a^2 \leq \frac{2}{\sqrt{3}} vol(L) \tag{17}$$

$$b \leq \sqrt{\frac{2}{3}} vol(L) \tag{18}$$

By corollary 1, the last triange $\mathbf{T}^{(n)} = \mathbf{T} = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ is acute or straight.

Using law of cosines, we know that the length c of the longest side of $\mathbf{T}$ is bounded as: [3]

$$c \leq \sqrt{\frac{2}{\sqrt{3}} vol(L) + \frac{2}{3} vol(L)^2} = \sqrt{\frac{2}{\sqrt{3}} \|\mathbf{N}\|_2 + \frac{2}{3}\|\mathbf{N}\|_2^2} \tag{19}$$

The orthographic projection of $\mathbf{m}_k^{(n)}$ with the direction of viewing $\mathbf{N}$ can be written as:

$$p_{\mathbf{N}}(\mathbf{m}_k^{(n)}) = \mathbf{m}_k^{(n)} - (\mathbf{m}_k^{(n)} \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|_2}) \frac{\mathbf{N}}{\|\mathbf{N}\|_2}. \tag{20}$$

[3] $vol(L) = \|\mathbf{N}\|_2$ [12, p27].

Moreover, at the final step n, the length of $p_{\mathbf{N}}(\mathbf{m}_k^{(n)})$ is bounded by $\sqrt{\frac{2}{\sqrt{3}}\|\mathbf{N}\|_2 + \frac{2}{3}\|\mathbf{N}\|_2^2}$, and we deduce a bound for the lenght of $\mathbf{m}_k^{(n)}$.

$$\begin{aligned}
\|p_{\mathbf{N}}(\mathbf{m}_n^{(i)})\|^2 &= \|\mathbf{m}_k^{(n)}\|^2 - 2(\mathbf{m}_k^{(i)} \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|_2})^2 + (\mathbf{m}_k^{(n)} \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|_2})^2(\frac{\mathbf{N}}{\|\mathbf{N}\|_2})^2 \\
&= \|\mathbf{m}_k^{(n)}\|^2 - (\mathbf{m}_k^{(i)} \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|_2})^2 \\
&\leq \frac{2}{\sqrt{3}}\|\mathbf{N}\|_2 + \frac{2}{3}\|\mathbf{N}\|_2^2 \\
\Rightarrow \|\mathbf{m}_k^{(n)}\|^2 &\leq (\mathbf{m}_k^{(n)} \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|_2})^2 + \frac{2}{\sqrt{3}}\|\mathbf{N}\|_2 + \frac{2}{3}\|\mathbf{N}\|_2^2
\end{aligned}$$

In addition, we have $\mathbf{m}_k^{(n)} \cdot \mathbf{N} = 1$ [10]. Hence,

$$\|\mathbf{m}_k^{(n)}\|^2 < \frac{2}{3}\|\mathbf{N}\|_2^2 + \frac{2}{\sqrt{3}}\|\mathbf{N}\|_2 + \frac{1}{\|\mathbf{N}\|_2^2}.$$

Finally we obtain the max distance,

$$Dist_{max}^{(n)} = \max_k\{\|\mathbf{m}_k^{(n)}\|\} \leq \sqrt{\frac{2}{3}\|\mathbf{N}\|_2^2 + \frac{2}{\sqrt{3}}\|\mathbf{N}\|_2 + \frac{1}{\|\mathbf{N}\|_2^2}} \tag{21}$$

7 Conclusion

A large proportion of this paper is dedicated to prove that the L-algorithm verifies the delaunay property. We invoke several geometry properties related to projections and spheres in order to proceed to the proof by recurrence. By proving that L-algorithm verifies the delaunay property, a direct consequence is that the output triangle contains a minimal basis for the 2D Lattice. Even if the L-algorithm's complexity is not as good as the optimized version of R-algorithm, it is the only variant of the plane-probing algorithms that has a theoretical proof on outputting a minimal basis. We also prove that such minimal basis provides a raw estimation of the probed space by the algorithm.

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A Relation between the preorder and the intersection of balls and half-spaces

In this section, we show that that $\leq_{\mathbf{T}}$ is a total preorder on $\mathcal{H}_+$. For any pair $\mathbf{x}, \mathbf{y} \subset \mathcal{H}_+$, we remind that $\mathbf{y} \leq_{\mathbf{T}} \mathbf{x}$ if and only if $(\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_+) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_+)$. (The step i is fixed thus we ignore the exponent i.)

- **Reflexivity**: the sphere defined by $\mathbf{T}$ and $x \in \mathcal{H}_+$ is unique, thus $x \leq_{\mathbf{T}} x$.
- **Transitivity**: it is induced from the transitivity of the order $\subseteq$.
- **Totality**: the power of a point with respect to a sphere is negative (resp. zero or positive) if it is inside of (resp. on or outside of) the reference sphere.

All intersection of point of two spheres have null power to each spheres. The locus of the point that has equal power to the two spheres is a plane (three dimensional extension of the theorem of the chordal [7][1]). For any $\mathbf{x}, \mathbf{y} \in \mathcal{H}_+$, we deduce that the intersection of the two spheres $\partial\mathcal{B}(\mathbf{T}, \mathbf{x})$ and $\partial\mathcal{B}(\mathbf{T}, \mathbf{y})$ should belong to the radical plane, which includes the triangle $\mathbf{T}$. In other words, the two spheres have no intersection point in $\mathcal{H}_+$ (nor in $\mathcal{H}_-$). Therefore, we have either $(\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_+) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_+)$ or $(\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_+) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_+)$.

Remark 1. The order is *not antisymmetric* because there exists co-spherical cases ($\mathbf{x} \neq \mathbf{y}$ but $\mathcal{B}(\mathbf{T}, \mathbf{x}) = \mathcal{B}(\mathbf{T}, \mathbf{y})$).

Remark 2. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^3$, if $(\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_+) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_+)$ then $(\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_-) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_-)$.

Proof. For any pair $\mathbf{x}', \mathbf{y}' \subset \mathcal{H}_-$, we denote $\mathbf{y}' \preceq_{\mathbf{T}} \mathbf{x}'$ if and only if $(\mathcal{B}(\mathbf{T}, \mathbf{y}') \cap \mathcal{H}_-) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}') \cap \mathcal{H}_-)$. As for $\leq_{\mathbf{T}}$, note that $\preceq_{\mathbf{T}}$ is a total preorder. Let us now consider two points $\mathbf{x}' \in (\partial\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_-)$ and $\mathbf{y}' \in (\partial\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_-)$ (both points lie on the boundary of either $\mathcal{B}(\mathbf{T}, \mathbf{x})$ and $\mathcal{B}(\mathbf{T}, \mathbf{y})$ in $\mathcal{H}_-$). Note that, by construction, $\mathcal{B}(\mathbf{T}, \mathbf{y}') = \mathcal{B}(\mathbf{T}, \mathbf{y})$ and $\mathcal{B}(\mathbf{T}, \mathbf{x}') = \mathcal{B}(\mathbf{T}, \mathbf{x})$.

Since the relation $\preceq_{\mathbf{T}}$ is total, we have either $\mathbf{y}' \preceq_{\mathbf{T}} \mathbf{x}'$ or $\mathbf{x}' \preceq_{\mathbf{T}} \mathbf{y}'$. As the second case, implies the remark statement by definition, we focus below on the first case. By definition, $\mathbf{y}' \preceq_{\mathbf{T}} \mathbf{x}'$ implies $(\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_-) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_-)$. Since we assume $\mathbf{y} \leq_{\mathbf{T}} \mathbf{x}$, we also have $(\mathcal{B}(\mathbf{T}, \mathbf{y}) \cap \mathcal{H}_+) \subseteq (\mathcal{B}(\mathbf{T}, \mathbf{x}) \cap \mathcal{H}_+)$ by definition. If we take the union of both sides of the inclusion, we have $\mathcal{B}(\mathbf{T}, \mathbf{y}) \subseteq \mathcal{B}(\mathbf{T}, \mathbf{x})$. If $\mathcal{B}(\mathbf{T}, \mathbf{y}) = \mathcal{B}(\mathbf{T}, \mathbf{x})$, the overall remark statement is trivially true. If $\mathcal{B}(\mathbf{T}, \mathbf{y}) \subset \mathcal{B}(\mathbf{T}, \mathbf{x})$, we have a contradiction as both balls are constructed from the same triangle $\mathbf{T}$. $\square$

B Derivations

In this section, we detail some elements of the technical proofs which are implied by Lemma 8. In lemma 10:

$$\begin{aligned}
\big(\hat{\mathbf{N}}((\mathbf{T})) \times (-\mathbf{d}_k)\big) \cdot \mathbf{m}_{k+2} =& \big(\hat{\mathbf{N}}((\mathbf{T})) \times (-\mathbf{m}_{k+1})\big) \cdot \mathbf{m}_{k+2} \\
=& - \big(\mathbf{m}_{k+1} \times \mathbf{m}_{k+2}\big) \cdot \hat{\mathbf{N}}((\mathbf{T})) \\
=& - \hat{\mathbf{N}}_k \cdot \hat{\mathbf{N}}((\mathbf{T})) < 0.
\end{aligned}$$

And,

$$\begin{aligned}
\big(\hat{\mathbf{N}}((\mathbf{T})) \times (\mathbf{d}_{k+1})\big) \cdot \mathbf{m}_{k+2} =& \big(\hat{\mathbf{N}}((\mathbf{T})) \times (-\mathbf{m}_k)\big) \cdot \mathbf{m}_{k+2} \\
=& - \big(\mathbf{m}_k \times \mathbf{m}_{k+2}\big) \cdot \hat{\mathbf{N}}((\mathbf{T})) \\
=& \hat{\mathbf{N}}_{k+1} \cdot \hat{\mathbf{N}}((\mathbf{T})) > 0.
\end{aligned}$$

In lemma 11:

$$\begin{aligned}\big(\hat{\mathbf{N}}((\mathbf{T}))\times(-\mathbf{d}_k)\big)\cdot(\mathbf{d}_k+\mathbf{m}_k) =&\big((-\mathbf{d}_k)\times(\mathbf{d}_k+\mathbf{m}_k)\big)\cdot\hat{\mathbf{N}}((\mathbf{T}))\\ =&\big((-\mathbf{d}_k)\times\mathbf{m}_k\big)\cdot\hat{\mathbf{N}}((\mathbf{T}))\\ =&\big((-\mathbf{m}_{k+1}+\mathbf{m}_{k+2})\times\mathbf{m}_k\big)\cdot\hat{\mathbf{N}}((\mathbf{T}))\\ =&(\hat{\mathbf{N}}_{k+2}+\hat{\mathbf{N}}_{k+1})\cdot\hat{\mathbf{N}}((\mathbf{T}))>0\end{aligned}$$

And,

$$\begin{aligned}\big(\hat{\mathbf{N}}((\mathbf{T}))\times\mathbf{d}_{k+1}\big)\cdot(\mathbf{d}_k+\mathbf{m}_k) =&\big(\hat{\mathbf{N}}((\mathbf{T}))\times\mathbf{d}_{k+1}\big)\cdot(-\mathbf{d}_{k+1}+\mathbf{m}_{k+1})\\ =&\big(\hat{\mathbf{N}}((\mathbf{T}))\times\mathbf{d}_{k+1}\big)\cdot(\mathbf{m}_{k+1})\\ =&\big(\mathbf{d}_{k+1}\times(\mathbf{m}_{k+1})\big)\cdot\hat{\mathbf{N}}((\mathbf{T}))\\ =&\big(((\mathbf{m}_{k+2}-\mathbf{m}_k)\times(\mathbf{m}_{k+1})\big)\cdot\hat{\mathbf{N}}((\mathbf{T}))\\ =&(-\hat{\mathbf{N}}_{k+2}-\hat{\mathbf{N}}_k))\cdot\hat{\mathbf{N}}((\mathbf{T}))<0\end{aligned}$$

For lemma 12:

$$\begin{aligned}\big((\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\times(\mathbf{d}_k+\mathbf{m}_k)\big)\cdot(-\mathbf{d}_{k+1}) =&\big((\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\times(\mathbf{m}_{k+1}-\mathbf{d}_{k+1})\big)\cdot(-\mathbf{d}_{k+1})\\ =&\big((\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\times(\mathbf{m}_{k+1})\big)\cdot(-\mathbf{d}_{k+1})\\ =&\big((\mathbf{m}_{k+1})\times(-\mathbf{d}_{k+1})\big)\cdot(\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\\ =&(-\hat{\mathbf{N}}_{k+2}-\hat{\mathbf{N}}_k)\cdot(\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\\ =&-\big(\hat{\mathbf{N}}_{k+2}\cdot\hat{\mathbf{N}}_{k+1}+\underbrace{\|\hat{\mathbf{N}}_{k+2}\|^2}_{>0}+\hat{\mathbf{N}}_k\cdot\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_k\cdot\hat{\mathbf{N}}_{k+2}\big)<0.\end{aligned}$$

And,

$$\begin{aligned}\big((\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\times\mathbf{m}_k\big)\cdot(-\mathbf{d}_{k+1}) =&\big(\mathbf{m}_k\times(-\mathbf{d}_{k+1})\big)\cdot(\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\\ =&\big(\mathbf{m}_k\times(-\mathbf{m}_{k+2})\big)\cdot(\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\\ =&\hat{\mathbf{N}}_{k+1}\cdot(\hat{\mathbf{N}}_{k+1}+\hat{\mathbf{N}}_{k+2})\\ =&\big(\underbrace{\|\hat{\mathbf{N}}_{k+1}\|^2}_{>0}+\hat{\mathbf{N}}_{k+1}\cdot\hat{\mathbf{N}}_{k+2}\big)>0.\end{aligned}$$